THE DEGREE OF THE SECANT VARIETY AND THE JOIN OF MONOMIAL CURVES

K. RANESTAD*

ABSTRACT. A monomial curve is a curve parametrized by monomials. The degree of the secant variety of a monomial curve is given in terms of the sequence of exponents of the monomials defining the curve. Likewise, the degree of the join of two monomial curves is given in terms of the two sequences of exponents.

CONTENTS

1. INTRODUCTION

A monomial curve C is the image of an injective morphism of $f : \mathbb{P}^1 \to \mathbb{P}^r$ defined by monomials. After ordering the monomials by ascending degree it is therefore given by

$$(s : t) \mapsto (s^d : s^{d-a_1}t^{a_1} : \ldots : s^{d-a_{r-1}}t^{a_{r-1}} : t^d)$$

where $a_1 < a_2 < \ldots < a_r = d$. So this latter sequence completely determines C. We define the first secant variety $SecC$ to be the closure of the union of lines that meet C in two distinct points. The first aim of this note is to compute the degree of this secant variety as a subvariety of $\mathbb{P}^r$. A simple wellknown argument using a general projection $\pi : C \to \overline{C} \subset \mathbb{P}^2$ shows that this degree is given by the formula

$$\deg SecC = \binom{d-1}{2} - \delta_p - \delta_q$$

where δ_p and δ_q are the genus contributions or equivalently, $2\delta_p$ and $2\delta_q$ are the Milnor numbers of the cusps at $p = \pi([1 : 0 \ldots : 0])$ and $q = \pi([0 : \ldots : 0 : 1])$ on $\overline{C}$. To compute δ_p and δ_q given C, we find the characteristic terms of the Puiseux expansion of $\overline{C}$ at the cusps. From the characteristic terms of the Puiseux expansion the genus contribution is

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computed by an algorithm due to Chisini and Enriques, eventually refined and given a closed form by Casas-Alvero.

Given two curves C and D in $\mathbb{P}^r$ we define their join $Join(C,D)$ to be the closure of the union of lines that meet C and D in two distinct points. We consider the join of two monomial curves C and D: In the notation of the previous section we ask that the two curves are defined by

$$C : (s:t) \mapsto (s^{d_C} : s^{d_C-a_1}t^{a_1} : \ldots : s^{d_C-a_{r-1}}t^{a_{r-1}} : t^{d_C})$$

where $a_1 < a_2 < \ldots < a_r = d_C$, and

$$D : (s:t) \mapsto (s^d_D : s^{d_D-b_1}t^{b_1} : \ldots : s^{d_D-b_{r-1}}t^{b_{r-1}} : t^{d_D})$$

where $b_1 < b_2 < \ldots < b_r = d_B$. Again the two sequences

$$a_1 < a_2 < \ldots < a_r = d_C, \quad b_1 < b_2 < \ldots < b_r = d_B$$

determine the two curves completely, and our second goal is to compute the degree of the join of C and D as a subvariety of $\mathbb{P}^r$. In this case the general projection of the two curves to $\mathbb{P}^2$ gives the formula

$$\deg Join(C,D) = d_C \cdot d_D - I_s(C,D),$$

where $I_s(C,D)$ is the sum of the intersection multiplicities in $\mathbb{P}^2$ of the two curves at the images of intersection points between the two curves in $\mathbb{P}^r$. An algorithm computing the sum of intersection multiplicity $I_s(C,D)$ is given in section 4.

A closed formula for the degree of the join in terms of the sequences $a_1 < a_2 < \ldots < a_r = d_C$ and $b_1 < b_2 < \ldots < b_r = d_B$, like the formula for the degree of the secant variety, would have been prefered. So far we can only give an explicit algorithm for its computation.

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2. The multiplicity sequence of a plane curve singularity

A crucial ingredient in the two algorithms below is the multiplicity sequence of a plane curve singularity. Given a point p in the plane and a sequence of blowups at simple points $(p = p_0, p_1, p_2, ..., p_s)$, such that all exceptional divisors lie over p, i.e. is mapped to p by the natural map to the original plane, and such that the strict transform of the curve is smooth. The multiplicities $m_0(C)$, (resp. $m_i(C), i > 0$) of C at p (respectively its strict transforms at p_i), form the multiplicity sequence of C at p with respect to the sequence of blowups. Equivalently, the multiplicity sequence coincides with the sequence of intersection numbers of the strict transform of the C with the exceptional divisor of each blow up. The multiplicity sequence may contain 1's, but these would not appear in a blowup that provides a minimal resolution of the singularity. In the latter case we say that the multiplicity sequence is minimal. Note that by the unicity of a minimal resolution of a plane curve singularity, the minimal multiplicity sequence is unique. Both minimal and nonminimal cases will however occur in our setting.

We shall use the multiplicity sequence of plane singularities with given Puiseux series. Consider the parameterized affine plane curve

$$C : t \mapsto t^m, t^{k_1} + t^{k_2} + \cdots .$$

This plane curve has a cusp at the origin, where $t = 0$. The multiplicity sequence is computed from the sequence $m, k_1, k_2, \ldots$ as described in a result of Enriques and Chisini [1] Theorem 8.4.12. We give this algorithm. The genus contribution may be computed from the multiplicity sequence. Casas-Alvero have found a closed form for this genus contribution which we present below.

2.1. **Algorithm computing the multiplicity sequence.** Consider the strictly increasing sequence

$$m < k_1 < k_2 < \ldots$$

Step 1. The gcd-sequence and characteristic terms. (This step is not neccessary to compute the multiplicity sequence, but clearifies the role of the different terms k_i.) Let $g_0 = m$ and $g_i = gcd\{m, k_1, \ldots, k_i\}$ for $i > 0$. The g_i form the *gcd*-sequence of $m, k_1, \ldots, k_r$:

$$g_0 \geq g_1 \geq g_2 \geq g_3 \ldots .$$

Clearly, in the *gcd*-sequence, $g_i = 1$ for some i, since otherwise the parametrization is not $1 : 1$. The *characteristic* terms in the sequence $t^{k_1} + t^{k_2} + \cdots$ are the terms

$$k_{i_1}, ..., k_{i_s}$$

where $i_1 = \min\{i | g_i < m\}$, $i_2 = \min\{i | g_i < g_{i_1}\}$ etc. Thus $m = g_0 > g_{i_1}$ and

$$g_{i_1} > ... > g_{i_s} = 1.$$

In particular the number of characteristic terms is finite and bounded by the number of prime factors in m.

Step 2. The multiplicity sequence. Given the sequence

$$m < k_1 < k_2 < \ldots$$

let $\kappa_i = k_i - k_{i-1}$ where $k_0 = 0$ and $i = 1, 2, ...$ We call

$$\kappa_1, \kappa_2, \ldots$$

the *difference sequence* of the cusp. We will only need a finite number of terms in this sequence. In fact the difference sequence of the finite sequence of characteristic terms form will do. So we assume we have a difference sequence with s terms.

Apply the Euclidean algorithm successively to the elements of the difference sequence: Let

$$\kappa_i = e_{i,1} r_{i,1} + r_{i,2}$$
$$r_{i,1} = e_{i,2} r_{i,2} + r_{i,3}$$
$$\ldots$$
$$r_{i,w(i)-1} = e_{i,w(i)} r_{i,w(i)}$$

with $0 \leq r_{i,j+1} < r_{i,j}$ and $r_{1,1} = m, r_{i,1} = r_{i-1,w(i-1)}, i > 1$. Note that

$$r_{i+1,1} = g_i = \gcd(m, k_1, ..., k_i).$$

The *multiplicity sequence* is $e_{i,j}$ times the multiplicity $r_{i,j}$, with $1 \leq i \leq s$ and $1 \leq j \leq w(i)$.

As a common convention we write the sequence in the order it is computed, and with repetitions in stead of the numbers $e_{i,j}$. To separate the output between each subroutine i above, we sometimes use a semicolon. Note that the overall sequence anyway is nonincreasing. The genus contribution or δ-invariant of the sequence is given by

$$\delta = \sum_{i,j} e_{i,j} \binom{r_{i,j}}{2}.$$

This sum is given a closed form in terms of the original sequence and its gcd-sequence by the following result due to Casas-Alvero:

Proposition 2.1. [2] *Given the sequence*

$$m < k_1 < k_2 < \dots$$

Let

$$g_0 \geq g_1 \geq g_2 \geq g_3 \cdots$$

be its gcd-sequence. Then the δ-invariant of the sequence is

$$\delta = \frac{1}{2}(\sum_{i \geq 1} k_i(g_{i-1} - g_i) - m + 1.$$

Proof. See [2] page 194 exercise 5.6. □

3. The degree of the secant variety of a monomial curve

Let $C \subset \mathbb{P}^r$ be a monomial curve defined by the sequence of positive integers $a_1 < a_2 < \dots < a_r = d$ as above. Consider the secant variety $SecC$ of C. This is a threefold, so its degree is counted by the intersection of this variety with a general codimension three subspace, or equivalently by the number of ordinary double points of the general projection $\pi : C \to \mathbb{P}^2$. For a general projection the only other singularities on $\overline{C} = \pi(C)$ are possible cusps at the image of the points $\pi(p)$ and $\pi(q)$ where $p = (1 : \dots : 0)$ and $q = (0 : \dots : 1)$ in $\mathbb{P}^r$. The formula for the arithmetic genus of a plane curve of degree d and the computation of the genus contribution at these cups provides a formula for the degree of $SecC$.

Proposition 3.1. *Let $C \subset \mathbb{P}^r$ be a monomial curve defined by the sequence of positive integers $a_1 < a_2 < \dots < a_r = d$. Let $b_i = d - a_{r-i}$, for $i = 1, ..., r-1$ and $b_r = d$. Let $g_i = gcd(a_1, ..., a_i)$ and $h_i = gcd(b_1, ..., b_i)$, then*

$$\deg SecC = \binom{d-1}{2} - \frac{1}{2}(\sum_i a_{i+1}(g_i - g_{i+1}) - a_1 + \sum_i b_{i+1}(h_i - h_{i+1}) - b_1) - 1.$$

Proof. The arithmetic genus $p(C)$ for a curve C on a smooth surface S is given by the adjunction formula [3] on the surface:

$$2p(C) - 2 = C \cdot C + C \cdot K_S$$

where K_S is the canonical divisor on S. If C has multiplicity m at a point q on S, and $S' \to S$ is the blowup of S at q, then the adjunction formula on S' says

$$2p(C') - 2 = C' \cdot C' + C' \cdot K_{S'} =$$

$$= (C^* - mE) \cdot (C^* - mE) + (C^* - mE) \cdot (K_S + E) = 2p(C) - 2 - m^2 + m$$

where E is the exceptional divisor and C^* is the total transform and C' is the strict transform of C(cf. [3] chapter V). Thus

$$p(C') = p(C) - \binom{m}{2}.$$

After resolving all singularities on $\overline{C} \subset \mathbb{P}^2$ by a series of blow ups centered at singular points of $\overline{C}$ or its strict transform, the arithmetic genus of the strict transform C' is 0 since it is a rational curve. At the ordinary double points the difference between the arithmetic genus of the curve and its strict transform after blowing up the point is $\binom{2}{2} = 1$. The points $\pi(p)$ and $\pi(q)$ are the only other singularities on $\overline{C}$. The contribution δ_p is by definition the difference between the arithmetic genus of $\overline{C}$ and a smooth strict transform C' that is isomorphic to $\overline{C}$ outside the point p. Likewise for δ_q. Since $K_{\mathbb{P}^2} \cong -3L$, where L is a line in the plane, the arithmetic genus of $\overline{C}$ is given by $2p(\overline{C}) - 2 = d_C(d_C - 3)$. Adding all genus contributions we get the formula:

$$\deg SecC = \binom{d-1}{2} - \delta_p - \delta_q.$$

It remains therefore to give an explicit computation of the genus contributions at p and q. As explained above the genus contribution of a plane curve singularity is determined by the *multiplicity sequence* of the singularity. The first term in this sequence is the multiplicity of the curve in the singular point, the next term is the multiplicity of the strict transform at the singular point on the exceptional divisor (if the strict transform is not already smooth) etc. The algorithms 2.1 computes this multiplicity sequence from the exponents of the Puiseux expansion, so the genus contribution is nothing but the δ invariant of the sequence of exponents in the Puiseux expansion.

The parametrization of the cusp at p is given by

$$x = t^{a_1} + b_{13}t^{a_3} + ... + b_{1r}t^{a_r}, y = t^{a_2} + b_{23}t^{a_3} + ... + b_{2r}t^{a_r}.$$

To compute the multiplicity sequence of the Puiseux expansion we need only the characteristic terms in a Puiseux expansion of our curve at $\pi(p)$.

Lemma 3.2. *The characteristic terms in the Puiseux expansion of $\overline{C}$ at p coincides with the characteristic terms in the Puiseux expansion*

$$x = t^{a_1}, y = t^{a_2} + b_{23}t^{a_3} + ... + b_{2r}t^{a_r}.$$

Proof. To start with, the exponents a_i are coprime. The ideal in $k[t]$ generated by x and y therefore has finite codimension as a vectorspace, and there is an N_0 such that t^N is in the ideal when $N \geq N_0$. So it is enough to prove the statement of the lemma modulo

t^{N_0}. We therefore reparameterize $\overline{C}$ by substituting t with $t + ut^{a_3-a_1+1}$ for suitable u to cancel the coefficient of t^{a_3}. In the new parametrization we get:

$$x = t^{a_1} + b'_{14}t^{a'_4} + ... + b'_{1r}t^{a'_{r'}}, y = t^{a_2} + b_{23}t^{a_3} + ... + b_{2r}t^{a_r} + c_1t^{b_1} + ...$$

where $a'_4 > a_3$, and all new exponents appearing are of the form $a_i + k(a_3 - a_1)$ for some positive integer k. Compare the greatest common divisors g_i of a_1 and the i lowest exponents of t occuring in y, before and after the reparametrization. The only difference is a possible repetition of some terms, so the characteristic terms remain the same. Now, we may reparameterize until x has only one term with exponent less than N and we are done. □

Since non-characteristic terms do not contribute to the δ-invariant the proposition follows from 2.1. □

Example 3.3. Consider the monomial curve C given by the sequence $(0, 30, 45, 55, 78)$. At $p = (1 : 0)$, we may compute the δ-invariant from the Puiseux expansion with exponents $m = 30$, $(a_3, a_4, a_5) = (45, 55, 78)$ The gcd-sequence is $(30, 15, 5, 1)$ and the δ-invariant is

$$\delta_p = \frac{1}{2}(45(30 - 15) + 55(15 - 5) + 78(5 - 1) - 30 + 1) = 754.$$

At $q = (0 : 1)$ we compute the δ-invariant from the Puiseux expansion with exponents $m = 23$ and $(a_3, a_4, a_5) = (33, 48, 78)$. Since m is prime and coprime to 33, the only characteristic term is 33 with gcd-sequence $(23, 1)$. The δ-invariant is

$$\delta_q = \frac{1}{2}(33(23 - 1) - 23 + 1) = 352$$

The degree of the secant variety of C is

$$\deg SecC = \binom{77}{2} - \delta_p - \delta_q = 2926 - 754 - 352 = 1820$$

4. The degree of the join of two monomial curves

Consider the join of two monomial curves C and D in $\mathbb{P}^r$ defined by

$$C : (s : t) \mapsto (s^{d_C} : s^{d_C-a_1}t^{a_1} : \ldots : s^{d_C-a_{r-1}}t^{a_{r-1}} : t^{d_C})$$

where $a_1 < a_2 < \ldots < a_r = d_C$, and

$$D : (s : t) \mapsto (s^d_D : s^{d_D-b_1}t^{b_1} : \ldots : s^{d_D-b_{r-1}}t^{b_{r-1}} : t^{d_D})$$

where $b_1 < b_2 < \ldots < b_r = d_B$. The two sequences

$$a_1 < a_2 < \ldots < a_r = d_C, \quad b_1 < b_2 < \ldots < b_r = d_B$$

therefore determine the two curves completely. For the parametrizations to be $1-1$ onto the image, we ask that the a_i have no common factor, and likewise for the b_i. The join is a threefold, so its degree coincides with the number of lines meeting the two curves in distinct points that also meet a given codimension 3 linear space L in $\mathbb{P}^r$. But this number equals the number of new intersection points obtained by projecting the union of the two curves from L to a plane. Denote by π_L the projection from L, and let $\overline{C} = \pi_L(C)$ and $\overline{D} = \pi_L(D)$ be the images of C and D respectively. The total intersection number

$$\overline{C} \cdot \overline{D} = d_C \cdot d_D$$

by Bezout's theorem, so to get the degree we have to subtract the intersection multiplicity at the points of $\pi_L(C \cap D)$. In our special situation there certainly are points in $C \cap D$:

$$\{p = (1 : 0 \ldots : 0), q = (0 : \cdots : 0 : 1), u = (1 : \ldots : 1)\} \subset C \cap D.$$

Furthermore, if

$$\beta = gcd(b_1 - a_1, b_2 - a_2, ..., b_r - a_r),$$

then all roots of $t^\beta - 1$ define intersection points. This is, however, all as is easily checked. Namely, we may assume that the first coordinate is 1 at an intersection point and that $t_1^{a_i} = t_2^{b_i}$ for $i = 1, .., r$, for some t_1, t_2 in the ground field. Then we may assume (over $\mathbb{C}$) that there is an α such that $t_1^\alpha = t_2$. Then we get $t_1^{a_i} = t_1^{\alpha \cdot b_i}$, i.e. $t_1 = 1$ or $a_i = \alpha \cdot b_i$ for $i = 1, ..., r$, or there is some integer β such that $\beta \cdot a_i = \alpha \cdot b_i$. Therefore α is a rational number. Furthermore the b_i have no common factor, so α must be an integer. Since the a_i also have no common factor, we may assume that $\alpha = 1$, and we have precisely the intersection points described above.

The intersections at the roots of $t^\beta - 1$ are always transversal, i.e. with distinct tangents: The tangent direction is given by the derivatives $(a_1 t^{a_1 - 1}, ..., a_r t^{a_r - 1})$ and $(b_1 t^{b_1 - 1}, ..., b_r t^{b_r - 1})$. Here the powers of t coincide, but the coefficients are not proportional, so the tangent directions are distinct. Therefore the intersection multiplicity is 1 at the image of these points by π_L.

For the points $\pi_L(p)$ and $\pi_L(q)$ the intersection multiplicity is at least two, since the two curves have the same tangent(cone) at those points. In fact, since the curves are unibranched, there is a unique tangent direction at the point, i.e. if they are singular they have a cusp there. The intersection multiplicity at these points is determined by a procedure similar to the one given in the previous section. More precisely consider say the point $\pi_L(p)$. Blow it up and let p_1 be the common intersection point of the strict transforms of the two curves on the exceptional divisor. There is a unique such intersection point, since the two curves are unibranched and the tangents to $\overline{C}$ and $\overline{D}$ at $\pi_L(p)$ coincide. Now blow up in the point p_1. If the strict transforms meet on the new exceptional divisor, then denote it by p_2 and blow up in this point. Continue, until the strict transforms do not intersect on the exceptional divisor. Thus we get a finite sequence $p_0 = \pi_L(p), p_1, \ldots, p_k$, and together with it the multiplicities of the strict transforms of the two curves at each p_i. We denote these multiplicity sequences by $m_0(C), \ldots, m_k(C)$ and $m_0(D), \ldots, m_k(D)$. The intersection multiplicity between the two curves at the point $\pi_L(p)$ is:

$$\sum_{i=0}^{k} m_i(C) m_i(D).$$

The multiplicity sequences $m_0(C), \ldots, m_k(C)$ and $m_0(D), \ldots, m_k(D)$ are decreasing and similar to the multiplicity sequences constructed in the previous section. There are however a main difference in that the new ones may contain 1's. Because of the unibranch property these 1's could only be added to the end of the sequence though. Thus the new sequences coincides with part of the old one, extended possibly with 1's only in case it contains all of the old one.

The problem is how to compute these sequences from sequences $a_1, ..., a_r$ and $b_1, ..., b_r$ of the curves C and D. In this case the non-characteristic terms are as important as the characteristic ones, since the intersection point of the strict transforms with the exceptional divisor is crucial. Some special cases may illustrate the issue:

Example 4.1. Consider monomial curves $C : (1, 2, 3, 4)$ and $D : (1, 2, 3, 5)$. Then the two curves separate after four blowups and the multiplicity sequences are $m_i(C) : 1, 1, 1, 1$ and $m_i(D) : 1, 1, 1, 1$. The intersection multiplicity at $\pi_L(p)$ is $1 + 1 + 1 + 1 = 4$.

Example 4.2. The monomial curves $C : (1, 2, 3, 4)$ and $D : (2, 4, 6, 9)$ separate after 4 blowups starting at p $(t = 0)$, the multiplicity sequences are $m_i(C) : 1, 1, 1, 1$ and $m_i(D) : 2, 2, 2, 2$. The intersection multiplicity at $\pi_L(p)$ is $2 + 2 + 2 + 2 = 8$. Similarly if $b_i = ea_i, i = 1, \ldots, s$, then $m_i(D) = em_i(C)$ for $i = 1, ..., k$ where k is the length of the multiplicity sequence for a curve $D' : (b_1, \ldots, b_s, b_{s+1})$.

Example 4.3. For $C : (1, 2, 3, 4)$ and $D : (b_0, b_1, b_2, b_3)$ where $b_0 > 1$ and $b_1 \neq 2b_0$, then $k = 2$ and $m_i(C) = (1, 1)$ and $m_i(D) = (b_0, min\{(b_1 - b_0), b_0\})$.

With these examples in mind we formulate the algorithm computing the degree of the join.

4.1. **Intersection multiplicity algorithm.** Given two monomial curves C and D defined by the sequences

$$a_1 < a_2 < \ldots < a_r = d_C, \quad b_1 < b_2 < \ldots < b_r = d_D$$

respectively, and assume that for some $j \geq 1$ $b_i \geq a_i$ for $i < j$ while $b_j > a_j$. The following three steps computes the intersection multiplicity of the general projection of the two curves to a plane in the image of the origin.

Step 1. Let $\alpha = \frac{b_1}{a_1}$. If α is not an integer, then set $k = 0$, otherwise let

$$k = max\{i | b_i = \alpha a_i\}$$

Let $m_1, m_2, \ldots, m_s$ be the multiplicity sequence, the outcome of the algorithm 2.1, of the sequence $(b_1, b_2, \ldots, b_k)$.

Set

$$\delta_k = \frac{1}{\alpha}(m_1^2 + \cdots + m_s^2),$$

and let $g = \gcd(b_1, ..., b_k)$.

Step 2. Apply the multiplicity algorithm 2.1 to the sequences $(\frac{g}{\alpha}, a_{k+1} - a_k)$ and $(g, b_{k+1} - b_k)$, with outcome

$$(e_1, r_1), (e_2, r_2), ..., (e_m, r_m) \quad \text{and} \quad (e'_1, r'_1), ..., (e'_n, r'_n)$$

respectively. Let $l = \min\{j | e_j = e'_j\}$ and $f = \min\{e_{l+1}, e'_{l+1}\}$ and let

$$\epsilon = \sum_j^l e_j \cdot r_j r'_j + f \cdot r_{l+1} r'_{l+1}.$$

Step 3. The intersection multiplicity between the curves C and D at the origin is

$$I(C, D) = \delta_k + \epsilon.$$

Proof. To start we project C and D into the plane and may choose coordinates in the plane such that $\pi(C)$ and $\pi(D)$ have the parametrizations

$$\pi(C) : x = t^{a_1} + c_{1,3}t^{a_3} + ... + c_{1,r}t^{a_r}, y = t^{a_2} + c_{2,3}t^{a_3} + ... + c_{2,r}t^{a_r}$$

and

$$\pi(D) : x = t^{b_1} + c_{1,3}t^{b_3} + ... + c_{1,r}t^{b_r}, y = t^{b_2} + c_{2,3}t^{b_3} + ... + c_{2,r}t^{b_r}.$$

By assumption $a_1 < a_2$ and $b_1 < b_2$, so both curves are tangent along the x-axis. Now, we blow up the plane in the origin. The strict transforms of these curves on the blowup intersect the exceptional curve in the x-chart (with coordinates (x, xy)). In this chart the strict transforms $\pi(C)'$ and $\pi(D)'$ have local parameterizations:

$$\pi(C)' : x = t^{a_1}+c_{1,3}t^{a_3}+...+c_{1,r}t^{a_r}, y = t^{a_2-a_1}+c_{2,3}t^{a_3-a_1}+...+c_{2,r}t^{a_r-a_1}-c_{1,3}t^{a_2+a_3-2a_1}+\dots$$

and

$$\pi(D)' : x = t^{b_1}+c_{1,3}t^{b_3}+...+c_{1,r}t^{b_r}, y = t^{b_2-b_1}+c_{2,3}t^{b_3-b_1}+...+c_{2,r}t^{b_r-b_1}-c_{1,3}t^{b_2+b_3-2b_1}+\dots.$$

The tangent at the origin is $y = 0$ if $a_1 < a_2 - a_1$, it is $x = 0$ if $a_1 - a_2 < a_1$ and it is $x = y$ if $a_1 = a_2 - a_1$.

Notice, that the terms of order less than $a_k - a_1$ and $b_k - b_1$ respectively, have the same coefficients and differ only by the factor t^{α_p}. Therefore, if $k > 0$ the two curves $\pi(C)'$ and $\pi(D)'$ have the same tangent direction at the origin, and their strict transform on the blow up in the origin intersect. Proceeding we need to know after how many blowups, the strict transforms does not intersect, and keep track of the multiplicities of the two strict transforms up to that point. Computing the number of blowups needed to separate the two curves, comes down to keeping track of first terms of the parametrizations of the strict transforms after successive blowups. The tangent direction decides the parametrization of the strict transform: If the tangent direction is $y = 0$ then the strict transform is parametrized by $x, \frac{y}{x}$, if the tangent direction is $x = 0$, then the strict transform is parametrized by $\frac{x}{y}, y$, and if the tangent direction is $x = y$, then the strict transform is parametrized by $x, \frac{y-x}{x}$. Now, the multiplicities of the strict transforms at the origin form the multiplicity sequence obtained by the algorithm 2.1, but keeping track of which of the tangent directions at each point, we actually also control the intersection between the two curves. The change from $y = 0$ to $x = 0$ of tangent direction correspond to going from (i, j) to $(i, j + 1)$ in the Euclidean algorithm, while the third kind of tangent corresponds to going from $(i, w(i))$ to (i+1,1) or to non-characteristic terms. In this algorithm, as long as $i \leq k$, the leading terms of the parametrizations differ only by a factor of t^{α}. So the corresponding tangent directions coincide. When $i = k + 1$ and $j = 1$ we have parametrizations $t^g + ..., t^{a_{k+1}-a_k} + ...$ and $t^{\alpha+g} + ..., t^{b_{k+1}-b_k} + ...$ To see when these two curves separate, we apply again the Euclidean algorithm. So here we compare the coefficients $e_{k,j}$ for the two curves. The curves split after $e_{k,1}+e_{k,2}+...+e_{k,s-1}+\epsilon$ blowups if $e_{k,j} = e'_{k,j}$ for $j < s$, while $\epsilon = \min\{e_{k,s}, e'_{k,s}\}$. □

Proposition 4.4. *Given two monomial curves C and D defined by the sequences*

$$a_1 < a_2 < \ldots < a_r = d_C, \quad b_1 < b_2 < \ldots < b_r = d_D$$

respectively, then the intersection multiplicity I_p of C and D at p is computed by the algorithm 4.1. Likewise the intersection multiplicity I_q of C and D is computed.

Let $\beta = gcd(b_1 - a_1, ...b_r - a_r)$, then the degree of the join of C and D is

$$\deg Join(C, D) = d_C \cdot d_D - I_p - I_q - \beta.$$

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Matematisk Institutt, Universitetet i Oslo P.O.Box 1053, Blindern, N-0316 Oslo, NORWAY

E-mail address: ranestad@math.uio.no