SOBOLEV SPACES ON LIE MANIFOLDS AND POLYHEDRAL DOMAINS

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Abstract. We study Sobolev spaces on Lie manifolds, which we define as a class of manifolds described by vector fields (see Definition 1.2). The class of Lie manifolds includes the Euclidean spaces $\mathbb{R}^n$, asymptotically flat manifolds, conformally compact manifolds, and manifolds with cylindrical and polycylindrical ends. As in the classical case of $\mathbb{R}^n$, we define Sobolev spaces using derivatives, powers of the Laplacian, or a suitable class of partitions of unity. We extend the basic results about Sobolev spaces on Euclidean spaces to the setting of Lie manifolds. These results include the definition of the trace map, a characterization of its range, the extension theorem, the density of smooth functions, and interpolation properties. One of the main motivations is that, in the examples we have studied so far, the totally-characteristic Sobolev spaces on polyhedral domains identify with Sobolev spaces on suitable Lie manifolds with boundary. The analysis we develop may be useful for solving certain types of non-linear partial differential equations on non-compact manifolds that appear, for instance, in Einstein's constraint equations. We also sketch two applications, one to the Yamabe functional and one to the regularity of boundary value problems on polyhedral domains.

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Introduction

Function spaces play a central role in Analysis and are often used in practical applications of mathematics. In many of these applications the domains are not smooth, which has lead to work under Lipschitz-type conditions, as in [35, 54, 57, 77] etc. These papers have extended many classical results on function spaces to Lipschitz

Ammann was partially supported by MSRI's NSF grant DMS-9810361. Ionescu was supported in part by NSF Grant No. 0100021. Nistor was partially supported by NSF Grants DMS 0200808 and DMS 0209497. Manuscripts available from **http://www.math.psu.edu/nistor/** .

domains, but have also revealed the limitations of the standard Sobolev spaces. For example, the usual regularity theorems for solutions of elliptic differential equations on smooth domains [22, 73] do not hold on Lipschitz domains. See [20, 21, 25, 26] in addition to the papers quoted above.

To explain the question of regularity, denote by Δ the Laplace operator and consider the following example. Let $\mathbb{P}$ be a polygon in the plane and let $u \in H^1(\mathbb{P})$ be a solution of the Poisson problem $\Delta u = f \in \mathcal{C}^\infty(\mathbb{P})$ with Dirichlet boundary conditions (*i.e.*, $u = 0$ on $\partial\mathbb{P}$). One can show [25, 26] that there exists a constant s_0, explicitly determined in terms of the angles of $\mathbb{P}$, such that $u \in H^s(\mathbb{P})$ for any $f \in \mathcal{C}^\infty(\mathbb{P})$ and any $s < s_0$, but not better. That is, $u \notin H^{s_0}(\mathbb{P})$ for a suitable choice of $f \in \mathcal{C}^\infty(\mathbb{P})$. This is in sharp contrast with the case of smooth boundary, in which case u is smooth whenever f is smooth. A deep study of these issues in the setting of Lipschitz domains can be found in the papers of Jerison and Kenig [35] and Mitrea and Taylor [54], and in the other papers quoted there.

The loss of regularity in the Poisson problem mentioned above can be avoided, however, if one considers a different class of Sobolev spaces on the polygon $\mathbb{P}$ [12, 11, 53, 56]. These Sobolev spaces, sometimes denoted $H_b^m(\mathbb{P})$ are the so called *totally characteristic Sobolev spaces* and were used by many researchers, see [27, 10, 20, 21, 38, 45, 50, 52, 56, 65] and the references therein. The definition of the spaces $H_b^m(\mathbb{P})$ uses the distance function $\rho(x)$ from $x \in \mathbb{P}$ to the vertices of $\mathbb{P}$:

$$H_b^m(\mathbb{P}) := \{u \in L^2_{loc}(\mathbb{P}),\ \rho^{|\alpha|-1} D^\alpha u \in L^2(\mathbb{P}),\quad |\alpha| \leq m\}. \tag{1}$$

It is one of the purposes of this paper to study totally characteristic Sobolev spaces on polyhedral domains and to extend some of the main results in the theory of classical Sobolev spaces to this setting. This will help clarify the role of the totally characteristic Sobolev spaces in the study of boundary value problems on polyhedral domains, in particular. More specific results in three dimension will be included in [11].

Our approach to Sobolev spaces on polyhedral domains is to reduce their study to that of Sobolev spaces on certain non-compact manifolds with boundary. These non-compact manifolds are obtained from our polyhedral domain by conformally changing the metric with a factor that blows up at the faces of codimension ≥ 2. The resulting non-compact manifolds are "Lie manifolds with boundary," (See Definition 1.2 and Subsection 1.5 for definitions.)

Lie manifolds were first introduced informally in [49, 51]. Their definition was formalized in [5], where several simple but basic properties of these manifolds were proved in a general setting. (Lie manifolds were called "manifolds with a Lie structure at infinity" in that paper.) In addition to the non-compact manifolds that arise from polyhedral domains, other examples of Lie manifolds include the Euclidean spaces $\mathbb{R}^n$, manifolds that are Euclidean at infinity, conformally compact manifolds, manifolds with cylindrical and polycylindrical ends, and asymptotically hyperbolic manifolds. These classes of non-compact manifolds are relevant in many problems in Mathematical Physics and Computational Sciences, such as domain decomposition methods and the Finite Element Method, quasi-linear parabolic equations, Yamabe's problem, Einstein's equations, and the positive mass theorem. Classes of Sobolev spaces on non-compact manifolds have been studied in many papers, of which we mention only a few [7, 2, 34, 37, 39, 40, 43, 51, 47, 46, 63, 64, 71] in addition to the works mentioned before.

A large part of the technical material in this paper is devoted to the study of Sobolev spaces on Lie manifolds, with or without boundary. The first three sections of the paper are more elementary and we have attempted to make them essentially self-contained. We begin in Section 1 with a review of the definition of a structural Lie algebra of vector fields $\mathcal{V}$ on a manifold with corners M. This Lie algebra of vector fields will provide the derivatives appearing in the definition of the Sobolev spaces. Then we define Lie manifolds. The interior M_0 of M is by definition a *Lie manifold*. It turns out that M_0 carries a complete metric g, unique up to Lipschitz equivalence (some authors use the term "quasi-isometric" to describe two Lipschitz equivalent metrics).

In Section 1 we define the Sobolev spaces $W^{s,p}(M_0)$ on a Lie manifold M_0, where the superscripts have the following possible ranges: either $s \in \mathbb{Z}_+$ and $1 \leq p \leq \infty$ or $s \in \mathbb{R}$ and $1 < p < \infty$. The main goal of this paper is to study the spaces $W^{s,p}(M_0)$. We first define the spaces $W^{s,p}(M_0)$, $s \in \mathbb{Z}_+$ and $1 \leq p \leq \infty$, by differentiating with respect to vector fields in $\mathcal{V}$. This definition is in the spirit of the definition of Sobolev spaces on $\mathbb{R}^n$. Then we prove two alternative definitions of these Sobolev spaces, either using a suitable class of partitions of unity (as in [66, 68] for example), or as the domains of the powers of the Laplace operator (for $p = 2$). We also consider these spaces on open subsets $\Omega_0 \subset M_0$. The spaces $W^{s,p}(M_0)$, for $s \in \mathbb{R}$, $1 < p < \infty$ are defined by interpolation and duality or, alternatively, using partitions of unity. In Section 3, we discuss domains Ω_0 whose boundary $\partial\Omega_0$ is a (smooth) Lie submanifold of M_0.

We extend several of the classical results on Sobolev spaces to the setting of the spaces $W^{s,p}(M_0)$. These results include the density of smooth, compactly supported functions, the Gagliardo-Nirenberg-Sobolev inequalities, the extension theorem, the trace theorem, the characterization of the range of the trace map in Hilbert space case ($p = 2$), and the Rellich-Kondrachov compactness theorem. Some of these results follow from the analogous results for manifolds with bounded geometry. We conclude the first three, more elementary sections, with an application to the regularity of boundary value problems on polyhedral domains, Theorem 3.8.

The last three sections are slightly less elementary. In particular, we no longer attempt to make them self-contained. Some applications to geometry (the Yamabe problem) are given in Section 4. In Section 5 we discuss the geometric results needed on Lie submanifolds, most importantly, the global tubular neighborhood theorem. Finally, in Section 6 we discuss the continuity of pseudodifferential operators acting on the spaces $W^{s,p}(M_0)$. The class of pseudodifferential operators that we consider is $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$. This algebra was introduced in [6] as a quantization of the algebra of differential operators $\mathrm{Diff}^*_{\mathcal{V}}(M)$ generated by the structural Lie algebra of vector fields $\mathcal{V}$ defining the Lie structure on M_0, thus solving a conjecture from [49]. In particular, we obtain a description of the spaces $W^{s,p}(M_0)$, $s \geq 0$, $1 < p < \infty$, as the domain of any elliptic operator $P \in \Psi^{s}_{1,0,\mathcal{V}}(M_0)$.

We include few concrete examples of manifolds with a Lie structure at infinity besides those needed to treat polyhedral domains. The reader can find more examples in [41] or in [5], for example.

Acknowledgements: We would like to thank W. Dahmen, R. Lauter, R. Strichartz, and A. Vasy for useful discussions. The first named author wants to thank MSRI, Berkeley, CA for its hospitality.

1. Lie manifolds

As mentioned already in the Introduction, our approach to the study of totally-characteristic Sobolev spaces on polyhedral domains is based on their relation to Sobolev spaces on Lie manifolds with boundary. For the convenience of the reader, we recall the definition of a Lie manifold and some basic results. This Section is to a large extent based on [5]. (Note that what we are calling here "Lie manifolds" were called "manifolds with a Lie structure at infinity" in [5].)

We shall treat Lie manifolds as well as Lie submanifolds (of Lie manifolds) in Section 3.

1.1. **Definition.** We need to recall first manifolds with corners. By definition, every point p in a manifold with corners M has a coordinate neighborhood diffeomorphic to $[0,\infty)^k \times \mathbb{R}^{n-k}$ such that the transition functions are smooth up to the boundary.

We write M_0 for the interior of M, and $\partial M = M \smallsetminus M_0$ for the boundary, *i.e.,* ∂M is the union of all boundary faces of dimension 0 to $n-1$. In the sequel, by a *manifold* we shall always understand a C^∞-manifold *possibly with corners*, whereas a *smooth manifold* is a C^∞-manifold *without corners*.

As we shall see below, a Lie manifold is described by a Lie algebra of vector fields satisfying certain conditions. We now discuss some of these conditions.

Definition 1.1. A subspace $\mathcal{V} \subseteq \Gamma(M, TM)$ of the Lie algebra of all smooth vector fields on M is said to be a *structural Lie algebra of vector fields on* M provided that the following conditions are satisfied:

(i) $\mathcal{V}$ is closed under the Lie bracket of vector fields;
(ii) every $V \in \mathcal{V}$ is tangent to all hyperfaces of M;
(iii) $\mathcal{C}^\infty(M)\mathcal{V} = \mathcal{V}$; and
(iv) each point $p \in M$ has a neighborhood U_p such that
$$\mathcal{V}_{U_p} := \{X|_{\overline{U}_p} \,|\, X \in \mathcal{V}\} \simeq C^\infty(\overline{U}_p)^k,$$
that is, $\mathcal{V}_{U_p}$ is a free $C^\infty(\overline{U}_p)$-module of dimension k, for some k.

The condition (iv) in the definition above can be reformulated as follows:

(iv') For every $p \in M$ there exist a neighborhood $U_p \subset M$ of p and vector fields $X_1, X_2, \ldots, X_k \in \mathcal{V}$ with the property that for any $Y \in \mathcal{V}$, there exist functions $f_1, \ldots, f_k \in \mathcal{C}^\infty(M)$, uniquely determined on U_p, such that

$$Y = \sum_{j=1}^{k} f_j X_j \quad \text{on } U_p. \tag{2}$$

Here are some examples of structural Lie algebras of vector fields. If $F \subset TM$ is a sub-bundle of the tangent bundle of a smooth manifold (so M has no boundary) such that $\mathcal{V}_F := \Gamma(M, F)$ is closed under the Lie bracket, then $\mathcal{V}_F$ is a structural Lie algebra of vector fields. Another example arises from manifolds with boundary and is related to the totally characteristic Sobolev spaces defined on an angle, as in the introduction. More precisely, let M be a manifold with boundary and let $\mathcal{V}_b$ be the space of vector fields on M *tangent* to the boundary of M. Then $\mathcal{V}_b$ is a structural Lie algebra of vector fields. See [48, 50] and Subsection 1.5.

Definition 1.2. A *Lie structure at infinity* on a smooth manifold M_0 is a pair $(M, \mathcal{V})$, where M is a compact manifold, possibly with corners, and $\mathcal{V} \subset \Gamma(M, TM)$ is a structural Lie algebra of vector fields on M with the following properties:

(i) $M_0 = M \smallsetminus \partial M$, the interior of M, and

(ii) If $p \in M_0$, then any local basis of $\mathcal{V}$ in a neigborhood of p is also a local basis of the tangent space to M_0. (In particular, the constant k of Equation (2) equals the dimension of M_0.)

A *manifold with a Lie structure at infinity* (or, simply, a *Lie manifold*) is a manifold M_0 together with a Lie structure at infinity $(M, \mathcal{V})$ on M_0. We shall sometimes denote a Lie manifold as above by $(M_0, M, \mathcal{V})$, or, simply, by $(M, \mathcal{V})$, because M_0 is determined as the interior of M.

We include only a few examples of Lie manifolds. The reader can find more examples in [51], from where these examples were borrowed or in [41, 5].

Examples 1.3.

(a) Take $\mathcal{V}_b$ to be the set of all vector fields tangent to all faces of a manifold with corners M. Then $(M, \mathcal{V}_b)$ is a Lie manifold. We shall say following Melrose's terminology that $M_0 = M \smallsetminus \partial M$ is endowed with the b-structure at infinity.

(b) Take $\mathcal{V}_0$ to be the set of all vector fields vanishing on all faces of a manifold with corners M. Then $(M, \mathcal{V}_0)$ is a Lie manifold. We shall say following Melrose's terminology that $M_0 = M \smallsetminus \partial M$ is endowed with the zero-structure at infinity.

Remark 1.4. Let us observe, that Conditions (iii) and (iv) of Definition 1.1 are equivalent to the condition that $\mathcal{V}$ be a projective $\mathcal{C}^\infty(M)$-module. Thus, by the Serre-Swan theorem [36], there exists a vector bundle $A \to M$, unique up to isomorphism, such that $\mathcal{V} = \Gamma(M, A)$. Since $\mathcal{V}$ consists of vector fields, that is $\mathcal{V} \subset \Gamma(M, TM)$, we also obtain a natural vector bundle morphism $\varrho : A \to M$, called the *anchor map*. The Condition (ii) of Definition 1.2 is then equivalent to the fact that ϱ is an isomorphism $A|_{M_0} \simeq TM_0$ on M_0. We will take this isomorphism to be an identification, and thus we can say that A is *an extension* of TM_0 to M (that is, $TM_0 \subset A$).

1.2. **Riemannian metric.** Let $(M_0, M, \mathcal{V})$ be a Lie manifold. By definition, *a Riemannian metric on M_0 compatible with the Lie structure at infinity* $(M, \mathcal{V})$ is a metric g such that for any $p \in M$, we can choose the basis $X_1, \ldots, X_k$ in Definition 1.1, (iv') and (2) to be orthonormal with respect to this metric everywhere on U_p. (Note that this condition is a restriction only for $p \in \partial M := M \smallsetminus M_0$.) Alternatively, we will also say that (M_0, g_0) is a *Riemannian Lie manifold*. Any Lie manifold carries a compatible Riemannian metric, and any two compatible metrics are bi-Lipschitz to each others.

Remark 1.5. Using the language of Remark 1.4, g is a compatible metric on M_0 if, and only if, there exists a metric on the vector bundle $A \to M$ which restricts to g on $TM_0 \subset A$.

The geometry of a Riemannian manifold (M_0, g_0) with a Lie structure $(M, \mathcal{V})$ at infinity has been studied in [5]. For instance, (M_0, g_0) is necessarily of infinite volume and complete. Moreover, all the covariant derivatives of the Riemannian curvature tensor are bounded. Under additional mild assumptions, we also know that the injectivity radius is bounded from below by a positive constant, *i.e.,* (M_0, g_0) is of bounded geometry. (A *manifold with bounded geometry* is a Riemannian manifold with positive injectivity radius and with bounded covariant derivatives of the curvature tensor, see [66] and references therein).

On a Riemannian Lie manifold $(M_0, M, \mathcal{V})$, the exponential map $\exp_p : TM_0 \to M_0$ is well-defined for all $p \in M_0$ and extends to a differentiable map $\exp_p : A_p \to M$ depending smoothly on $p \in M$. A convenient way to introduce the exponential map is via the geodesic spray, as done in [5]. Similarly, any vector field $X \in \mathcal{V} = \Gamma(M, A)$ is integrable and will map any (connected) face of M to itself. The resulting diffeomorphism of M_0 will be denoted ψ_X.

We assume from now on that $r_{\mathrm{inj}}(M_0)$*, the injectivity radius of* (M_0, g_0)*, is positive.*

1.3. **$\mathcal{V}$-differential operators.** We are especially interested in the analysis of the differential operators generated using only derivatives in $\mathcal{V}$. Let $\mathrm{Diff}^*_{\mathcal{V}}(M)$ be the algebra of differential operators on M generated by multiplication with functions in $\mathcal{C}^\infty(M)$ and by differentiation with vector fields $X \in \mathcal{V}$. The space of order m differential operators in $\mathrm{Diff}^*_{\mathcal{V}}(M)$ will be denoted $\mathrm{Diff}^m_{\mathcal{V}}(M)$. A differential operator in $\mathrm{Diff}^*_{\mathcal{V}}(M)$ will be called a $\mathcal{V}$-differential operator.

We can define $\mathcal{V}$-differential operators acting between sections of smooth vector bundles $E, F \to M$, $E, F \subset M \times \mathbb{C}^N$ by

$$\mathrm{Diff}^*_{\mathcal{V}}(M; E, F) := e_F M_N(\mathrm{Diff}^*_{\mathcal{V}}(M)) e_E \,, \tag{3}$$

where $e_E, e_F \in M_N(\mathcal{C}^\infty(M))$ are the projections onto E and, respectively, F. It follows that $\mathrm{Diff}^*_{\mathcal{V}}(M; E, E) =: \mathrm{Diff}^*_{\mathcal{V}}(M; E)$ is an algebra. It is also closed under taking adjoints of operators in $L^2(M_0)$, where the volume form is defined using a compatible metric g on M_0.

1.4. **Lie manifolds with boundary.** One of the main motivation for this work is to study Sobolev spaces on polyhedral domains. We shall do that by reducing their study to that of Sobolev spaces on "Lie manifolds with boundary," a class of manifolds with boundary that we introduce below.

To understand the following constructions, let us take a closer look at the the local structure of the Sobolev space $H^m_b(\mathbb{P})$ associated to a polygon $\mathbb{P}$ (recall (1)). Consider $\Omega := \{(r, \theta) \,|\, 0 < r < r_0, 0 < \theta < \alpha\}$, which models an angle of $\mathbb{P}$. Then the *totally* *characteristic Sobolev spaces* associated to Ω, $H_b^m(\Omega)$, can alternatively be described as

$$H_b^m(\Omega) := \{u \in L^2_{loc}(\Omega),\ r^{-1}(r\partial_r)^i \partial_\theta^j u \in L^2(\Omega), \quad i+j \le m\}. \tag{4}$$

An important point of the above definition is that first the angle Ω was desingularized and then a different basis of vector fields on the desingularization was used instead of the standard basis in the definition of the usual Sobolev spaces. For example, in the case above of the angle Ω, the basis $r\partial_r$ and ∂_θ was used instead of the usual basis ∂_x and ∂_y. This underscores the importance of vector fields in our approach, which owes to the work of several authors. See [17, 19, 18, 49, 51, 45, 46, 56, 63, 65, 66, 67, 68, 69] and the references therein.

Let $N \subset M$ be a submanifold with corners of codimension one of M (see Section 5). Recall that this implies that N is transverse to all faces of M. We shall say that N is a *regular* submanifold of $(M, \mathcal{V})$ if we can choose a tubular neighborhood V of $N_0 := N \smallsetminus \partial N = N \cap M_0$ in M_0 and a compatible metric g on M_0 that restricts to a product-type metric on $V \simeq (\partial N_0) \times (-\varepsilon_0, \varepsilon_0)$. (In Section 5, we shall show that every tame submanifold of codimension one is regular; in turn, this will give an easy, geometric criterion to decide when a codimension one submanifold of M is regular.)

Let $\Omega \subset M$ be an open subset. We say that Ω is a *Lie domain* in M if and only if $\partial\Omega = \partial\overline{\Omega}$ and $\partial\Omega$ is a regular submanifold of M. Let $\Omega_0 = \Omega \cap M_0$. Then $\partial\Omega_0 = (\partial\Omega) \cap M_0$ is a smooth submanifold of codimension one of M_0.

Definition 1.6. A *Lie manifold with boundary* is a triple $(\Omega_0, \Omega', \mathcal{V}')$, where Ω_0 is a smooth manifold with boundary, Ω' is a compact manifold with corners containing Ω_0 as an open subset, and $\mathcal{V}'$ is a Lie algebra of vector fields on Ω' with the property that there exists a Lie manifold $(M_0, M, \mathcal{V})$, a Lie domain $\Omega \subset M$ in M and a diffeomorphism $\phi : \Omega' \to \Omega$ such that $\phi(\Omega_0) = \Omega \cap M_0$ and $\phi^*(\mathcal{V}|_\Omega) = \mathcal{V}'$.

Note that if $(\Omega_0, \Omega, \mathcal{V})$ is a Lie manifold with boundary, then Ω_0 is determined by $(\Omega, \mathcal{V})$, so we can write $(\Omega, \mathcal{V})$ instead of $(\Omega_0, \Omega, \mathcal{V})$.

1.5. **Polyhedral domains.** We now discuss in an example the relation between polyhedral domains and Lie manifolds with boundary. Let us consider

$$\Delta_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n, x_j \ge 0, \sum x_j \le 1\},$$

the unit simplex in $\mathbb{R}^n$. To Δ_n we now associate a Lie manifold with boundary $(\Sigma(\Delta_n), \mathcal{V}(\Delta_n))$, together with a "desingularization" map

$$\kappa_n : \Sigma(\Delta_n) \to \Delta_n,$$

satisfying $\kappa_n(\partial\Sigma(\Delta_n)) \subset \partial\Delta_n$. Let $S \subset \Delta_n$ be the set of points not belonging to a face of codimension ≥ 2. The map κ_n will turn out to be a bijection between $\kappa_n^{-1}(S)$ and S. We shall proceed by induction as follows.

Let $\mathbb{P}$ be the angle $\{0 \le \theta \le \alpha\}$, a closed subset of $\mathbb{R}^2$, where (r, θ) are the polar coordinates in the plane. We define its canonical desingularization by $\Sigma(\mathbb{P}) := [0, \infty) \times [0, \alpha]$, which maps surjectively to $\mathbb{P}$ by $\kappa(r, \theta) = (r\cos\theta, r\sin\theta)$. The Lie algebra of vector fields on $\Sigma(\mathbb{P})$ is given by

$$\{X \in \Gamma(T\Sigma(\mathbb{P})),\ X(0,\theta) \text{ tangent to } \{0\} \times [0,\alpha]\,\} \tag{5}$$

We can realize $\Sigma(\mathbb{P})$ as a Lie domain in the manifold with boundary $[0,\infty) \times S^1$, by realizing $[0,\alpha]$ as a subset of the unit circle S^1. On $[0,\infty) \times S^1$ we then consider all vector fields tangent to the boundary.

The desingularization $\Sigma(\Delta_2)$ of the unit triangle Δ_2 is obtained as follows. First we desingularize each angle with the opposite face removed. Then we glue these desingularizations using the desingularization map κ. This yields, up to a diffeomorphism a hexagon. Let e_1, e_2, e_3 be the three non-intersecting edges of this hexagon corresponding to the vertices of Δ_2. These faces are the ones that, under the desingularization map will go to the three vertices of the triangle. Then $\mathcal{V}(\Delta_2)$ consists of vector fields tangent to the faces e_1, e_2, e_3.

Assume now that $(\Sigma(\Delta_n), \mathcal{V}(\Delta_n), \kappa_n)$ were constructed, and let us construct $(\Sigma(\Delta_{n+1}), \mathcal{V}(\Delta_{n+1}), \kappa_{n+1})$ We first construct an analogous desingularization of

$$C\Delta_n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}, x_j \geq 0\}.$$

The space $C\Delta_n$ is the "cone" over Δ_n. We shall use next the alternative description of Δ_n as

$$\Delta_n := \{(y_0, y_1, \ldots, y_n) \in \mathbb{R}^{n+1}, y_j \geq 0, \sum y_j = 1\}.$$

The desingularization of $C\Delta_n$ is, by definition, $[0,\infty) \times \Sigma(\Delta_n)$, with desingularization map $\kappa(r,z) = r\kappa_n(z)$. The Lie algebra of vector fields on this desingularization is

$$\mathcal{V} = \{(X,Y) \in \Gamma(T[0,\infty)) \times \Gamma(T\Sigma(\Delta_n)),\ X(0) = 0 \text{ and } Y \in \mathcal{V}(\Delta_n)\}.$$

In other words, if we split the tangent space to $[0,\infty) \times \Sigma(\Delta_n)$ into the direct sum of the vector bundles tangent to the two factors, then on the first component we get vector fields tangent to the boundary ("tangent to the boundary" means "vanishing at the boundary" in this case of $[0,\infty)$) and on the second component we simply get vectors in the structural Lie algebra of vector fields $\mathcal{V}(\Sigma(\Delta_n))$ corresponding to the desingularization of Δ_n.

The desingularization $\Sigma(\Delta_{n+1})$ of Δ_{n+1} is obtained by gluing the desingularizations of the cones corresponding to each of the vertices, these cones being obtained by removing the opposite face to the given vertex, one vertex at a time.

This rather complicated construction is justified by the following simple proposition.

Proposition 1.7. *Let h be the standard euclidean metric on Δ_n. Let $\rho(x)$ be the distance from the point x to the set of points belonging to a face of codimension ≥ 2. Then $\rho^{-2}h$ is Lipschitz equivalent to any compatible metric g on the interior of $\Sigma(\Delta_n)$, that is, there exists $C > 0$ such that*

$$C^{-1}g(\xi) \leq \rho^{-2}h(\xi) \leq Cg(\xi)$$

for any tangent vector $\xi \in T\Delta_n$ tangent to an interior point of Δ_n. In particular,

$$C^{-n/2}d\,\mathrm{vol}_g \leq \rho^{-n}d\,\mathrm{vol}_h \leq C^{n/2}d\,\mathrm{vol}_g\,.$$

Proof. By induction. For $n = 2$, this follows right away from the definition of totally characteristic Sobolev spaces on a triangle. For the induction step, denote by h_n the metric on Δ_n. We can cover Δ_n with open sets V_j, such that the closure $\overline{V}_j$ contains exactly one vertex of Δ_n. It is enough to prove our statement on $\overline{V}_j$, for all j. This will change the function ρ on $\overline{V}_j$, but the metric $\rho^{-2}h_n$ will be in the same Lipschitz equivalence class, because the function ρ changes only by a factor that is bounded from above and bounded from zero (*i.e.,* it takes values in a compact interval of $(0, \infty)$).

We then consider the same construction on $C\Delta_{n-1}$, the cone over the simplex Δ_{n-1}. Note that the function ρ does not change by considering the bigger set $C\Delta_{n-1}$ instead of $\overline{V}_j$. Let r denote the distance to the vertex of the cone $C\Delta_{n-1}$. After a suitable translation, we can (and will) assume that the vertex of this cone is the origin. The metric $r^{-2}h_n$ makes the interior of $C\Delta_{n-1}$ isometric to the interior of $\Delta_{n-1} \times \mathbb{R}$ with the product metric $\tilde{h}_{n-1} \times (dt)^2$, $t \in \mathbb{R}$, where $\tilde{h}_{n-1}$ is the metric on Δ_{n-1} obtained by mapping Δ_{n-1} to the unit sphere via the map $x \to x/\|x\|$. This metric is Lipschitz equivalent to h_{n-1}.

Let $\tilde{\rho}(x)$ be the distance from $x \in C\Delta_{n-1}$ to the subset of points belonging to a face of codimension ≥ 2 in the metric $h_{n-1} \times (dt)^2$. The induction hypothesis is that $\tilde{\rho}^{-2}(h_{n-1} \times (dt)^2)$ is Lipschitz equivalent to g. Hence

$$(r\tilde{\rho})^{-2}h_n = \tilde{\rho}^{-2}\tilde{h}_{n-1} \times (dt)^2$$

is Lipschitz equivalent to g. To prove our result, it is enough then to show that $f(x) := r(x)\tilde{\rho}(x)/\rho(x)$ is bounded from above and bounded from zero. Let us observe that $\rho(tx) = t\rho(x)$, $r(tx) = tr(x)$, and $\tilde{\rho}(tx) = \tilde{\rho}(x)$, for all $t \in (0, \infty)$ for which this makes sense. Consequently, $f(tx) = f(x)$, whenever both sides are defined. To prove that f takes on values in a compact interval contained in $(0, \infty)$, it is therefore enough to do that for $\|x\|$ constant. The result hence follows from the fact that f is continuous $\neq 0$ on the set $\{\|x\| = c > 0\} \cap C\Delta_{n-1}$. □

For further reference, let us record there a consequence of the proof of the above proposition.

Corollary 1.8. *We use the notation of Proposition 1.7. There exists $f \in C(\Delta_n)$, f smooth and $\neq 0$ on the interior of Δ_n such that f/ρ takes values in a compact interval of $(0, \infty)$. Moreover, $f^{-2}h$ is a compatible metric on the interior of Δ_n for any such f.*

2. Sobolev spaces

In this section we discuss the Sobolev spaces on M_0 from an elementary point of view, that is, without using pseudodifferential operators. Our treatment is standard, following [22, 28, 29]. See also [8]. Some of these elementary results simply follow from the fact that M_0 has bounded geometry whenever its injectivity radius is positive. These results include the density of smooth, compactly supported functions, the identification of the L^2–Sobolev spaces with the domains of suitable powers of $1 + \Delta$, and the Gagliardo–Nirenberg–Sobolev Embedding theorem. We include these results here for completeness and further references.

Conventions. *Throughout the rest of this paper,* $(M_0, M, \mathcal{V})$ *will be a fixed Lie manifold. We also fix a compatible metric* g *on* M_0 *(*i.e., *, a metric compatible with the Lie structure at infinity on* M_0*, see Subsection 1.2). By* Ω *we shall denote an open subset of* M *and* $\Omega_0 = \Omega \cap M_0$*. The letters* C *and* c *will be used to denote possibly different constants that may depend only on* (M_0, g) *and its Lie structure at infinity* $(M, \mathcal{V})$*.*

We shall denote the volume form (or measure) on M_0 associated to g by $d\,\mathrm{vol}_g(x)$ or simply by dx, when there is no danger of confusion. Also, we shall denote by $L^p(\Omega_0)$ the resulting L^p-space on Ω_0 (*i.e.,* defined with respect to the volume form dx). These spaces are independent of the choice of the compatible metric g on M_0, but their norms, denoted by $\|\cdot\|_{L^p}$, do depend upon this choice, although this is not reflected in the notation. Also, we shall use the fixed metric g on M_0 to trivialize all density bundles. Then the space $\mathcal{D}'(\Omega_0)$ of distributions on Ω_0 is defined, as usual, as the dual of $\mathcal{C}^\infty_c(\Omega_0)$. The spaces $L^p(\Omega_0)$ identify with spaces of distributions on Ω_0 via the pairing

$$\langle u, \phi \rangle = \int_{\Omega_0} u(x)\phi(x)dx, \qquad \text{where } \phi \in \mathcal{C}^\infty_c(\Omega_0) \text{ and } u \in L^p(\Omega_0).$$

2.1. **Definition of Sobolev spaces using vector fields and connections.** Anticipating, let us mention that we will define the Sobolev spaces $W^{s,p}(\Omega_0)$ in the following two cases: $s \in \mathbb{Z}_+$, $1 \le p \le \infty$, and arbitrary open sets Ω_0 or $s \in \mathbb{R}$, $1 < p < \infty$, and $\Omega_0 = M_0$. In fact, we will give several definitions and then show their equivalence. The first definition is in terms of the Levi-Civita connection ∇ on TM_0. We shall denote also by ∇ the induced connections on tensors (*i.e.,* on tensor products of TM_0 and T^*M_0).

Definition 2.1 (∇-definition of Sobolev spaces)**.** Let $k \in \mathbb{Z}_+$, then the Sobolev space $W^{k,p}(\Omega_0)$ is the space of distributions u on $\Omega_0 \subset M_0$ such that

$$\|u\|^p_{\nabla, W^{k,p}} := \sum_{l=1}^{k} \int_{\Omega_0} |\nabla^l u(x)|^p dx < \infty\,, \quad 1 \le p < \infty. \tag{6}$$

For $p = \infty$ we change this definition in the obvious way, namely we require that,

$$\|u\|_{\nabla, W^{k,\infty}} := \sup |\nabla^l u(x)| < \infty\,, \quad 0 \le l \le k. \tag{7}$$

Let $p = 2$. When $\overline{\Omega} = \Sigma(\Delta_n)$, see Subsection 1.5, we can regard $\overline{\Omega}$ as a subset of its double, which is a Lie manifold. This gives then the totally characteristic Sobolev spaces on Δ_n, denoted $H^k_b(\Delta_n)$. Let $\rho(x)$ be the distance to the set of points belonging to a face of codimension ≥ 2 of Δ_n, as in Proposition 1.7. Then

$$\begin{aligned} H^k_b(\Delta_n) = W^{k,2}(\Sigma(\Delta_n)) &= \{u, \int_{\Delta_n} |\partial^\alpha u|^2 \rho^{-n+2|\alpha|} d\,\mathrm{vol}_h < \infty, \quad |\alpha| \le k\} \\ &= \{u, \rho^{|\alpha|-n/2}\partial^\alpha u \in L^2(\Delta_n) = L^2(\Delta_n, dx), \quad |\alpha| \le k\}. \end{aligned} \tag{8}$$

We introduce an alternative definition of Sobolev spaces.

Definition 2.2 (vector fields definition of Sobolev spaces)**.** Let again $k \in \mathbb{Z}_+$. Choose a finite set of vector fields $\mathcal{X}$ such that $\mathcal{C}^\infty(M)\mathcal{X} = \mathcal{V}$. This condition is equivalent to the fact that the set $\{X(p), X \in \mathcal{X}\}$ generates A_p linearly, for any $p \in M$. Then the system $\mathcal{X}$ provides us with the norm

$$\|u\|^p_{\mathcal{X},W^{k,p}} := \sum \|X_1 X_2 \dots X_l u\|^p_{L^p}\,, \quad 1 \le p < \infty, \tag{9}$$

the sum being over all possible choices of $0 \le l \le k$ and all possible choices of not necessarily distinct vector fields $X_1, X_2, \dots, X_l \in \mathcal{X}$. For $p = \infty$, we change this definition in the obvious way:

$$\|u\|_{\mathcal{X},W^{k,\infty}} := \max \|X_1 X_2 \dots X_l u\|_{L^\infty}\,, \tag{10}$$

the maximum being taken over the same family of vector fields.

In particular,

$$W^{k,p}(\Omega_0) = \{u \in L^p(\Omega_0), Pu \in L^p(\Omega_0), \text{ for all } P \in \mathrm{Diff}^k_{\mathcal{V}}(M)\} \tag{11}$$

The following proposition shows that the second definition yields equivalent norms.

Proposition 2.3. *The norms $\|\cdot\|_{\mathcal{X},W^{k,p}}$ and $\|\cdot\|_{\nabla,W^{k,p}}$ are equivalent for any choice of the compatible metric g on M_0 and any choice of a system of the finite set $\mathcal{X}$ such that $\mathcal{C}^\infty(M)\mathcal{X} = \mathcal{V}$. The spaces $W^{k,p}(\Omega_0)$ are complete Banach spaces in the resulting topology. Moreover, $W^{k,2}(\Omega_0)$ is a Hilbert space.*

Proof. As all compatible metrics g are bi-Lipschitz to each others, the equivalence classes of the $\|\cdot\|_{\mathcal{X},W^{k,p}}$-norms are independent of the choice of g. We will show that for any choice $\mathcal{X}$ and g, $\|\cdot\|_{\mathcal{X},W^{k,p}}$ and $\|\cdot\|_{\nabla,W^{k,p}}$ are equivalent. It is clear that then the equivalence class of $\|\cdot\|_{\mathcal{X},W^{k,p}}$ is independent of the choice of $\mathcal{X}$, and the equivalence class of $\|\cdot\|_{\nabla,W^{k,p}}$ is independent of the choice of g.

We argue by induction in k. The equivalence is clear for $k = 0$. We assume now that the $W^{l,p}$-norms are already equivalent for $l = 0, \dots, k-1$. Observe that if $X, Y \in \mathcal{V}$, then the Koszul formula implies $\nabla_X Y \in \mathcal{V}$ [5]. To simplify notation, we define inductively $\mathcal{X}^0 := \mathcal{X}$, and $\mathcal{X}^{i+1} = \mathcal{X}^i \cup \{\nabla_X Y \,|\, X, Y \in \mathcal{X}^i\}$.

By definition any $V \in \Gamma(T^*M^{\otimes k})$ satisfies $(\nabla\nabla V)(X,Y) = \nabla_X \nabla_Y V - \nabla_{\nabla_X Y} V$. This implies for $X_1, \dots, X_k \in \mathcal{X}$

$$(\underbrace{\nabla \dots \nabla f}_{k\text{-times}})(X_1, \dots, X_k) = X_1 \dots X_k f + \sum_{l=0}^{k-1} \sum_{Y_j \in \mathcal{X}^{k-l}} a_{Y_1,\dots,Y_l} Y_1 \dots Y_l\, f,$$

for appropriate choices of $a_{Y_1,\dots,Y_l} \in \mathbb{Z}_+$. Hence,

$$\|(\underbrace{\nabla \dots \nabla f}_{k\text{-times}})\|_{L^p} \le C \sum \|\nabla \dots \nabla f(X_1, \dots, X_k)\|_{L^p} \le C \|f\|_{\mathcal{X},W^{k,p}}.$$

By induction, we know that $\|Y_1,\dots,Y_l f\|_{L^p} \le C\|f\|_{\nabla,W^{l,p}}$ for $Y_i \in \mathcal{X}^{k-l}$, $0 \le l \le k-1$, and hence

$$\|X_1 \dots X_k f\|_{L^p} \le \underbrace{\|\nabla \dots \nabla f\|_{L^p}\|X_1\|_{L^\infty}\cdots\|X_k\|_{L^\infty}}_{\le C\|f\|_{\nabla,W^{k,p}}} + \underbrace{\sum_{l=0}^{k-1}\sum_{Y_1,\dots,Y_l\in\mathcal{X}^{k-l}} a_{Y_1,\dots,Y_l} Y_1 \dots Y_l\, f}_{\le C\|f\|_{\nabla,W^{k-1,p}}}.$$

This implies the equivalence of the norms.

The proof of completeness is standard, see for example [22, 75]. □

We shall also need the following simple observation.

Lemma 2.4. *Let $\Omega' \subset \Omega \subset M$ be open subsets, $\Omega_0 = \Omega \cap M_0$, and $\Omega_0' = \Omega' \cap M_0$. The restriction then defines continuous operators $W^{s,p}(\Omega_0) \to W^{s,p}(\Omega_0')$. If the various choices $(\mathcal{X}, g, x_j)$ are done in the same way on Ω and Ω', then the restriction operator has norm 1.*

2.2. **Definition of Sobolev spaces using partitions of unity.** Yet another description of the spaces $W^{k,p}(\Omega_0)$ can be obtained by using suitable partitions of unity as in [66, Lemma 1.3], whose definition we now recall. See also [15, 68, 58].

Lemma 2.5. *For any $0 < \epsilon < r_{\mathrm{inj}}(M_0)/6$ there is a sequence of points $\{x_j\} \subset M_0$, and a partition of unity $\phi_j \in \mathcal{C}_c^\infty(M_0)$ with the following properties:*

(i) $\operatorname{supp}(\phi_j) \subset B(x_j, 2\epsilon)$;
(ii) $\|\nabla^k \phi_j\|_{L^\infty(M_0)} \le C_{k,\epsilon}$, *with $C_{k,\epsilon}$ independent of j; and*
(iii) *the sets $B(x_j, \epsilon/2)$ are disjoint, the sets $B(x_j,\epsilon)$ form a covering of M_0, and the sets $B(x_j, 4\epsilon)$ form a covering of M_0 of finite multiplicity,* i.e.,

$$\sup_{y \in M_0} \#\{x_j \,|\, y \in B(x_j, 4\epsilon)\} < \infty.$$

Fix $\epsilon \in (0, r_{\mathrm{inj}}(M_0)/6)$. Let $\psi_j : B(x_j, 4\epsilon) \to B_{\mathbb{R}^n}(0, 4\epsilon)$ normal coordinates around x_j, *i.e.,* a composition of the exponential maps $\exp_{x_j} : T_{x_j}M_0 \to M_0$ and by some isometries $T_{x_j}M_0 \simeq \mathbb{R}^n$. The uniform bounds on the Riemann tensor R and its derivatives $\nabla^k R$ imply uniform bounds on

$$\nabla^k d\exp_{x_j} : B_{T_{x_j}M_0}(0, 4\epsilon) \to \mathbb{R}^{n(k+1)} \otimes TM,$$

which simply means that all derivatives of ψ_j are uniformly bounded.

Proposition 2.6. *Let ϕ_i and ψ_i be as in the two paragraphs above. Let $U_j = \psi_j(\Omega_0 \cap B(x_j, 2\epsilon)) \subset \mathbb{R}^n$. We define*

$$\nu_{k,\infty}(u) := \sup_j \|(\phi_j u) \circ \psi_j^{-1}\|_{W^{k,\infty}(U_j)}$$

and, for $1 \leq p < \infty$,

$$\nu_{k,p}(u)^p := \sum_j \|(\phi_j u) \circ \psi_j^{-1}\|^p_{W^{k,p}(U_j)}.$$

Then $u \in W^{k,p}(\Omega_0)$ *if, and only if,* $\nu_{k,p}(u) < \infty$. *Moreover,* $\nu_{k,p}(u)$ *defines an equivalent norm on* $W^{k,p}(\Omega_0)$.

Proof. We shall assume $p < \infty$, for simplicity of notation. The case $p = \infty$ is completely similar. Consider then $\mu(u)^p = \sum_j \|\phi_j u\|^p_{W^{k,p}(\Omega_0)}$. Then there exists $C_{k,\varepsilon} > 0$ such that

$$C_{k,\varepsilon}^{-1}\|u\|_{W^{k,p}(\Omega_0)} \leq \mu(u) \leq C_{k,\varepsilon}\|u\|_{W^{k,p}(\Omega_0)}, \tag{12}$$

for all $u \in W^{k,p}(\Omega_0)$, by Lemma 2.5 (*i.e.,* the norms are equivalent). The fact that all derivatives of $\exp_{x_j}$ are bounded uniformly in j further shows that μ and $\nu_{k,p}$ are also equivalent. □

The proposition gives rise to a third, equivalent definition of Sobolev spaces. This definition was inspired from [66, 68], and can be used to define the spaces $W^{s,p}(\Omega_0)$, for any $s \in \mathbb{R}$, $1 < p < \infty$, and $\Omega_0 = M_0$. The cases $p = 1$ or $p = \infty$ are more delicate and we shall not discuss them here.

Recall that the spaces $W^{s,p}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 < p < \infty$ are defined using the powers of $1 + \Delta$, see [67, Chapter V] or [75, Section 13.6].

Definition 2.7 (Partition of unity definition of Sobolev spaces)**.** Let $s \in \mathbb{R}$, and $1 < p < \infty$. Then we define

$$\|u\|^p_{W^{s,p}(M_0)} := \sum_j \|(\phi_j u) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)}, \quad 1 < p < \infty. \tag{13}$$

By Proposition 2.6 this norm is equivalent to our previous norm on $W^{k,p}(M_0)$ when k is a nonnegative integer.

Proposition 2.8. *The space* $\mathcal{C}^\infty_c(M_0)$ *is dense in* $W^{k,p}(M_0)$, *for* $1 < p < \infty$ *and* $s \in \mathbb{R}$, *or* $1 \leq p < \infty$ *and* $s = k \in \mathbb{Z}_+$.

Proof. For $k \in \mathbb{Z}_+$, the result is true for any manifold with bounded geometry, see [8, Theorem 2] or [28, Theorem 2.8], or [29]. For $\Omega_0 = M_0$, $s \in \mathbb{R}$, and $1 < p < \infty$, the definition of the norm on $W^{s,p}(M_0)$ allows us to reduce right away the proof to the case of $\mathbb{R}^n$, by ignoring enough terms in the sum defining the norm (13). (We also use a cut-off function $0 \leq \chi \leq 1$, $\chi \in \mathcal{C}^\infty_c(B_{\mathbb{R}^n}(0, 4\epsilon))$, $\chi = 1$ on $B_{\mathbb{R}^n}(0, 4\epsilon)$.) □

We now give a characterization of the spaces $W^{s,p}(M_0)$ using interpolation. Let $\widetilde{W}^{-k,p}(M_0)$ be the set of distributions on M_0 that extend by continuity to linear functionals on $W^{k,q}(M_0)$, $p^{-1}+q^{-1} = 1$, using Proposition 2.8. That is, $\widetilde{W}^{-k,p}(M_0)$ be the set of distributions on M_0 that define continuous linear functionals on $W^{k,q}(M_0)$, $p^{-1} + q^{-1} = 1$. We let

$$\widetilde{W}^{\theta k,k,p}(M_0) := [\widetilde{W}^{0,p}(M_0), W^{k,p}(M_0)]_\theta\,, \quad 0 \leq \theta \leq 1\,,$$

be the complex interpolation spaces. Similarly, we define

$$\widetilde{W}^{-\theta k,k,p}(M_0) = [\widetilde{W}^{0,p}(M_0), W^{-k,p}(M_0)]_\theta.$$

(See [13] or [73, Chapter 4] for the definition of the complex interpolation spaces.)

The following proposition is an analogue of Proposition 2.6. Its main role is to give an intrinsic definition of the spaces $W^{s,p}(M_0)$, a definition that is independent of choices.

Proposition 2.9. *Let $1 < p < \infty$ and $k > |s|$. Then we have a topological equality $\widetilde{W}^{s,k,p}(M_0) = W^{s,p}(M_0)$. In particular, the spaces $W^{s,p}(M_0)$, $s \in \mathbb{R}$, do not depend on the choice of the covering $B(x_j, \epsilon)$ and of the subordinated partition of unity and we have*

$$[W^{s,p}(M_0), W^{0,p}(M_0)]_\theta = W^{\theta s,p}(M_0)\,, \quad 0 \le \theta \le 1\,.$$

Moreover, the pairing between functions and distributions defines an isomorpism $W^{s,p}(M_0)^* \simeq W^{-s,q}(M_0)$, where $1/p + 1/q = 1$.

Proof. This proposition is known if $M_0 = \mathbb{R}^n$ with the usual metric [75][Equation (6.5), page 23]. In particular, $\tilde{W}^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$. As in the proof of Proposition 2.6 one shows that the quantity

$$\nu_{s,p}(u)^p := \sum_j \|(\phi_j u) \circ \psi_j^{-1}\|^p_{\tilde{W}^{s,p}(\mathbb{R}^n)}, \tag{14}$$

is equivalent to the norm on $\tilde{W}^{s,p}(M_0)$. This implies $\tilde{W}^{s,p}(M_0) = W^{s,p}(M_0)$.

Choose k large. Then we have

$$\begin{aligned}[W^{s,p}(M_0), W^{0,p}(M_0)]_\theta = [W^{s,k,p}(M_0), W^{0,k,p}(M_0)]_\theta \\ = W^{\theta s,k,p}(M_0) = W^{\theta s,p}(M_0).\end{aligned}$$

The last part follows from the compatibility of interpolation with taking duals. This completes the proof. □

The above proposition provides us with several corollaries. First, from the interpolation properties of the spaces $W^{s,p}(M_0)$, we obtain the following corollary.

Corollary 2.10. *Let $\phi \in W^{k,\infty}(M_0)$ and $p \in (1,\infty)$. Then multiplication by ϕ defines a bounded operator on $W^{s,p}(M_0)$ of norm at most $C_k\|\phi\|_{W^{k,\infty}(M_0)}$ where $k \ge |s|$, $k \in \mathbb{Z}_+$. Similarly, any differential operator $P \in \operatorname{Diff}^m_{\mathcal{V}}(M)$ defines continuous maps $P : W^{s,p}(M_0) \to W^{s-m,p}(M_0)$.*

Proof. For $s \in \mathbb{Z}_+$, this follows from the definition of the norm on $W^{k,\infty}(M_0)$ and from the definition of $\operatorname{Diff}^m_{\mathcal{V}}(M)$ as the linear span of differential operators of the form $fX_1 \ldots X_k$, ($f \in \mathcal{C}^\infty(M) \subset W^{k,\infty}$, $X_j \in \mathcal{V}$, and $0 \le k \le m$), and from the definition of the spaces $W^{k,p}(\Omega_0)$.

For $s \le m$, the follows by duality. For the other values of s, the result follows by interpolation. □

Next, recall that an isomorphism $\phi : M \to M$ of the Lie manifolds $(M_0, M, \mathcal{V})$ and $(M_0', M', \mathcal{V}')$ is a diffeomorphism such that $\phi_*(\mathcal{V}) = \mathcal{V}'$. We then have the following invariance property of the Sobolev spaces that we have introduced.

Corollary 2.11. *Let $\phi : M \to M'$ be an isomorphism of Lie manifolds, $\Omega_0 \subset M_0$ be an open subset and $\Omega' = \phi(\Omega)$. Let $p \in [1,\infty]$ if $s \in \mathbb{Z}_+$, and $p \in (1,\infty)$ if $s \notin \mathbb{Z}_+$. Then $f \to f \circ \phi$ extends to an isomorphism $\phi^* : W^{s,p}(\Omega') \to W^{s,p}(\Omega)$ of Banach spaces.*

Proof. For $k \in \mathbb{Z}_+$, this follows right away from definitions and Proposition 2.3. For $-k \in \mathbb{Z}_+$, this follows by duality, Proposition (2.9). For the other values of s, the result follows from the same proposition, by interpolation. □

Recall now that M_0 is complete [5]. Hence the Laplace operator $\Delta = \nabla^*\nabla$ is essentially self-adjoint on $\mathcal{C}_c^\infty(M_0)$ by [23, 59, 70]. We shall define then $(1+\Delta)^{s/2}$ using the spectral theorem.

Proposition 2.12. *The space $H^s(M_0) := W^{s,2}(M_0)$, $s \geq 0$, identifies with the domain of $(1+\Delta)^{s/2}$, if we endow the latter with the graph topology.*

Proof. For $s \in \mathbb{Z}_+$, the result is true for any manifold of bounded geometry, by [8, Proposition 3]. For $s \in \mathbb{R}$, the result follows from interpolation, because the interpolation spaces are compatible with powers of operators, see the chapter on Sobolev spaces in Taylor's book [73]. □

The well known Gagliardo–Nirenberg–Sobolev inequality [8, 22, 24, 28] holds also in our setting.

Proposition 2.13. *Denote by n the dimension of M_0. Assume that $1/p = 1/q - m/n$, $1 < q \leq p < \infty$, where $m \geq 0$. Then $W^{s,q}(M_0)$ is continuously embedded in $W^{s-m,p}(M_0)$.*

Proof. If s and m are integers, $s \geq m \geq 0$, the statement of the proposition is true for manifolds with bounded geometry, [8, Theorem 7] or [28, Corollary 3.1.9]. By duality (see Proposition 2.9), we obtain the same result when $s \leq 0$, $s \in \mathbb{Z}$. Then, for integer s, m, $0 < s < m$ we obtain the corresponding embedding by composition $W^{s,q}(M_0) \to W^{0,r}(M_0) \to W^{s-m,p}(M_0)$, with $1/r = 1/q - s/n$. This proves the result for integral values of s. For non-integral values of s, the result follows by interpolation using again Proposition 2.9. □

The Rellich-Kondrachov's theorem on the compactness of the embeddings of Proposition 2.13 for $1/p > 1/q - m/n$ is true if M_0 is compact [8, Theorem 9]. This happens precisely when $M = M_0$, which is a trivial case of a manifold with a Lie structure at infinity. On the other hand, it is easily seen (and well known) that these compactness cannot be true for M_0 non-compact. We will nevertheless restore this by using Sobolev spaces with weights in the next section, see Theorem 3.6.

3. Manifolds with boundary

We continue to assume that $(M_0, M, \mathcal{V})$ is Lie manifold. Let $N \subset M$ be a submanifold with corners of codimension one of M. Recall that this implies that N is transverse to all faces of M. Also, recall that N is a *regular* submanifold of $(M, \mathcal{V})$ if we can choose a tubular neighborhood V of $N_0 := N \smallsetminus \partial N = N \cap M_0$ in M_0 and a compatible metric g on M_0 that restricts to a product-type metric on

$V \simeq (\partial N_0) \times (-\varepsilon_0, \varepsilon_0)$ (see Proposition 5.9). In Section 5, we shall show that every tame manifold is regular; in turn, this will give an easy, geometric criterion to decide when a codimension one submanifold of M is regular.

Let $\Omega \subset M$ be an open subset. Recall that Ω is a Lie domain in M if, and only if, $\partial\Omega = \partial\overline{\Omega}$ and $\partial\Omega$ is a regular submanifold of M. Let $\Omega_0 = \Omega \cap M_0$. Then $\partial\Omega_0 := (\partial\Omega) \cap M_0$ is a smooth submanifold of codimension one of M_0. We have the following analogue of the classical extension theorem.

Theorem 3.1. *There exists a linear operator E mapping measurable functions on Ω_0 to measurable functions on M_0 with the properties:*

(i) *E maps $W^{k,p}(\Omega_0)$ continuously into $W^{k,p}(M_0)$ for every $p \in [1, \infty]$ and every integer $k \geq 0$, and*
(ii) *$Eu|_{\Omega_0} = u$.*

Proof. Since $\partial\Omega_0$ is a regular submanifold we can fix a compatible metric g on M_0 and a tubular neighborhood V_0 of $\partial\Omega_0$ such that $V_0 \simeq (\partial\Omega_0) \times (-\varepsilon_0, \varepsilon_0)$, $\varepsilon_0 > 0$. Let $\varepsilon = \min(\varepsilon_0, r_{\text{inj}}(M_0))/20$, where $r_{\text{inj}}(M_0) > 0$ is the injectivity radius of M_0. By Zorn's lemma and the fact that M_0 has bounded geometry we can choose a maximal, countable set of disjoint balls $B(x_i, \varepsilon)$, $i \in I$. Since this family of balls is maximal we have $M_0 = \cup_i B(x_i, 2\varepsilon)$. For each i we fix a smooth function η_i supported in $B(x_i, 3\varepsilon)$ and equal to 1 in $B(x_i, 2\varepsilon)$. This can be done easily in local coordinates around the point x_i; since the metric g is induced by a metric g on A we may also assume that all derivatives of order up to k of η_i are bounded by a constant $C_{k,\varepsilon}$ independent of i. By replacing η_i with $\eta_i/\sqrt{\sum_j \eta_j^2}$, we can further assume that $\sum_i \eta_i^2 = 1$.

Following [67, Ch. 6] we also define two smooth cutoff functions adapted to the set Ω_0. We start with a function $\psi : \mathbb{R} \to [0, 1]$ which is equal to 1 on $[-3, 3]$ and which has support in $[-6, 6]$

Let $\varphi = (\varphi_1, \varphi_2)$ denote the isomorphism between V_0 and $\partial\Omega_0 \times (-\varepsilon_0, \varepsilon_0)$, where $\varphi_1 : V_0 \to \partial\Omega_0$ and $\varphi_2 : V_0 \to (-\varepsilon_0, \varepsilon_0)$. We define

$$\Lambda_+(x) := \begin{cases} 0 & \text{if } x \in M_0 \setminus V_0 \\ \psi(\varphi_2(x)/\varepsilon) & \text{if } x \in V_0, \end{cases}$$

and $\Lambda_-(x) := 1 - \Lambda_+(x)$. Clearly Λ_+ and Λ_- are smooth functions on M_0 and $\Lambda_+(x) + \Lambda_-(x) = 1$. Obviously, Λ_+ is supported in a neighborhood of $\partial\Omega_0$ and Λ_- is supported in the complement of a neighborhood of $\partial\Omega_0$.

Let $\partial\Omega_0 = A_1 \cup A_2 \cup \ldots$ denote the decomposition of $\partial\Omega_0$ into connected components. Let $V_0 = B_1 \cup B_2 \cup \ldots$ denote the corresponding decomposition of V_0 into connected components, namely, $B_j = \varphi^{-1}(A_j \times (-\varepsilon_0, \varepsilon_0))$. Since $\partial\Omega_0 = \partial\overline{\Omega}_0$, we have $\varphi(\Omega_0 \cap B_j) = A_j \times (-\varepsilon_0, 0)$ or $\varphi(\Omega_0 \cap B_j) = A_j \times (0, \varepsilon_0)$. Thus, if necessary, we may change the sign of φ on some of the connected components of V_0 in such a way that

$$\varphi(\Omega_0 \cap V_0) = \partial\Omega_0 \times (0, \varepsilon_0).$$

Let ψ_0 denote a fixed smooth function, $\psi_0 : \mathbb{R} \to [0,1]$, $\psi_0(t) = 1$ if $t \geq -\varepsilon$ and $\psi_0(t) = 0$ if $t \leq -2\varepsilon$, and let

$$\Lambda_0(x) = \begin{cases} 1 & \text{if } x \in \Omega_0 \setminus V_0 \\ 0 & \text{if } x \in M_0 \setminus (\Omega_0 \cup V_0) \\ \psi_0(\varphi_2(x)) & \text{if } x \in V_0. \end{cases}$$

We look now at the points x_i defined in the first paragraph of the proof. Let $J_1 = \{i \in I : d(x_i, \partial\Omega_0) \leq 10\varepsilon\}$ and $J_2 = \{i \in I : d(x_i \partial\Omega_0) > 10\varepsilon\}$. For every point x_i, $i \in J_1$, there is a point $y_i \in \partial\Omega_0$ with the property that $B(x_i, 4\varepsilon) \subset B(y_i, 15\varepsilon)$. Let $B_{\partial\Omega_0}(y_i, 15\varepsilon)$ denote the ball in $\partial\Omega_0$ of center y_i and radius 15ε (with respect to the induced metric on $\partial\Omega_0$). Let $h_i : B_{\partial\Omega_0}(y_i, 15\varepsilon) \to B_{\mathbb{R}^{n-1}}(0, 15\varepsilon)$ denote the normal system of coordinates around the point y_i. Finally let $g_i : B_{\mathbb{R}^{n-1}}(0, 15\varepsilon) \times (-15\varepsilon, 15\varepsilon) \to V_0$ denote the map $g_i(v,t) = \varphi^{-1}(h_i^{-1}(v), t)$.

Let $E_{\mathbb{R}^n}$ denote the extension operator that maps $W^{k,p}(\mathbb{R}^n_+)$ to $W^{k,p}(\mathbb{R}^n)$ continuously, where $\mathbb{R}^n_+$ denotes the half-space $\{x : x_n > 0\}$. Clearly, $E_{\mathbb{R}^n} u|_{\mathbb{R}^n_+} = u$. The existence of this extension operator is a classical fact, for instance, see [67, Chapter 6]. For any $u \in W^{k,p}(\Omega_0)$ and $i \in J_1$ the function $(\eta_i u) \circ g_i$ is well defined on $\mathbb{R}^n_+$ simply by setting it equal to 0 outside the set $B_{\mathbb{R}^{n-1}}(0, 15\varepsilon) \times (0, 15\varepsilon)$. Clearly, $(\eta_i u) \circ g_i \in W^{k,p}(\mathbb{R}^n_+)$. We define the extension Eu by the formula

$$Eu(x) = \Lambda_0(x)\Lambda_-(x)u(x) + \Lambda_0(x)\Lambda_+(x) \sum_{i \in J_1} \eta_i(x) E_{\mathbb{R}^n}[(\eta_i u) \circ g_i](g_i^{-1} x)\,. \tag{15}$$

Notice that

$$\sum_{i \in J_1} \eta_i^2(x) = 1 \quad \text{in } \operatorname{supp} \Lambda_+. \tag{16}$$

Indeed, since $M_0 = \cup_{i \in I} B(x_i, 2\varepsilon)$, we have $\sum_{i \in I} \eta_i^2(x) \geq 1$ for any $x \in M_0$. Also, $\eta_i(x) \equiv 0$ in $\operatorname{supp} \Lambda_+$ if $i \in J_2$, thus (16) follows. This shows that Eu in (15) is well-defined. Clearly, by the formula, $Eu|_{\Omega_0} = u$. It remains to verify that

$$\|Eu\|_{W^{k,p}(M_0)} \leq C_k \|u\|_{W^{k,p}(\Omega_0)}.$$

This follows as in [67] using (16), the fact that the extension $E_{\mathbb{R}^n}$ satisfies the same bound, and the definition of the Sobolev spaces using partitions of unity (Proposition 2.6). □

Theorem 3.2. *The space $\mathcal{C}_c^\infty(\overline{\Omega}_0)$, where the closure is defined in M_0, is dense in $W^{k,p}(\Omega_0)$, for $1 \leq p < \infty$.*

Proof. For any $u \in W^{k,p}(\Omega_0)$ let Eu denote its extension from Theorem 3.1, $Eu \in W^{k,p}(M_0)$. By Proposition 2.8, there is a sequence of functions $f_j \in C_c^\infty(M_0)$ with the property that

$$\lim_{j \to \infty} f_j = Eu \text{ in } W^{k,p}(M_0).$$

Thus $\lim_{j \to \infty} f_j|_{\Omega_0} = u$ in $W^{k,p}(\Omega_0)$, as desired. □

Theorem 3.3. *The restriction map $\mathcal{C}_c^\infty(\Omega_0) \to \mathcal{C}_c^\infty(\partial\Omega_0)$ extends to a continuous map $T : W^{k,p}(\Omega_0) \to W^{k-1,p}(\partial\Omega_0)$, for $1 \leq p \leq \infty$.*

Proof. To simplify the notation assume $1 \leq p < \infty$. We shall assume that the compatible metric on M_0 restricts to a product type metric on V_0, our distinguished tubular neighborhood of $\partial\Omega_0$.

We use the definitions of the Sobolev spaces using partitions of unity, Proposition 2.6 and Lemma 2.5 with $\varepsilon = \min(\epsilon_0, r_{\rm inj}(M_0))/10$. Let $B(x_j, 2\varepsilon)$ denote the balls in the cover of X in Lemma 2.5 and $1 = \sum_j \phi_j$ the corresponding partition of unity. Then $\widetilde{\phi}_j = \phi_j|_{\partial\Omega_0}$ form a partition of unity on $\partial\Omega_0$. Clearly,

$$r_{\rm inj}(\partial\Omega_0) \geq r_{\rm inj}(M_0) > 0.$$

Start with a function $u \in W^{k,p}(\Omega_0)$ and let $u_j = (u\phi_j) \circ \psi_j^{-1}$, $u_j \in W^{k,p}(\psi_j(\Omega_0 \cap B(x_j, 4\varepsilon)))$. In addition $u_j \equiv 0$ outside the set $\psi_j(\Omega_0 \cap B(x_j, 2\varepsilon))$. If $B(x_j, 4\varepsilon) \cap \partial\Omega_0 = \emptyset$ let $\widetilde{T}(u_j) = 0$. Otherwise notice that $B(x_j, 4\varepsilon)$ is included in V_0, the tubular neighborhood of $\partial\Omega_0$, thus the set $\psi_j(\partial\Omega_0 \cap B(x_j, 4\varepsilon))$ is the intersection of a hyperplane and the ball $B_{\mathbb{R}^n}(0, 4\varepsilon)$. We can then let $\widetilde{T}(u_j)$ denote the Euclidean restriction of u_j to $\psi_j(\partial\Omega_0 \cap B(x_j, 4\varepsilon))$ (see [22, Section 5.5]). Clearly $\widetilde{T}(u_j)$ is supported in $\psi_j(\partial\Omega_0 \cap B(x_j, 2\varepsilon))$ and

$$\|\widetilde{T}(u_j) \circ \widetilde{\psi}_j\|_{W^{k-1,p}(\partial\Omega_0)} \leq C\|u_j\|_{W^{k,p}(\psi_j(\Omega_0 \cap B(x_j, 4\varepsilon)))},$$

where $\widetilde{\psi}_j = \psi_j|_{\Omega_0}$ and the constant C is independent of j (recall that $\psi_j(\partial\Omega_0 \cap B(x_j, 4\varepsilon))$ is the intersection of a hyperplane and the ball $B_{\mathbb{R}^n}(0, 4\varepsilon)$). Let

$$Tu = \sum_j \widetilde{T}(u_j) \circ \widetilde{\psi}_j.$$

Since the sum is uniformly locally finite we have

$$\begin{aligned}\|Tu\|^p_{W^{k-1,p}(\partial\Omega_0)} &\leq C \sum_j \|\widetilde{T}(u_j) \circ \widetilde{\psi}_j\|^p_{W^{k-1,p}(\partial\Omega_0)} \\ &\leq C \sum_j \|u_j\|^p_{W^{k,p}(\psi_j(\Omega_0 \cap B(x_j, 4\varepsilon)))} \leq C\|u\|^p_{W^{k,p}(\Omega_0)},\end{aligned}$$

with constants C independent of u. The fact $Tu|_{\mathcal{C}_c^\infty(\Omega_0)}$ is indeed the restriction operator follows immediately from the definition. □

We shall see that if $p = 2$, we get a surjective map $W^{s,2}(\Omega_0) \to W^{s-1/2,2}(\partial\Omega_0)$ (Theorem 3.7).

Theorem 3.4. *The closure of $\mathcal{C}_c^\infty(\Omega_0)$ in $W^{k,p}(\Omega_0)$ is the intersection of the kernels of $T \circ \partial_\nu^j$, $0 \leq j \leq k-1$.*

Proof. The proof is reduced to the Euclidean case [1] following the same pattern of reasoning as in the previous theorem. □

The Gagliardo–Nirenberg–Sobolev theorem holds also for manifolds with boundary.

Theorem 3.5. *Denote by n the dimension of M and let $\Omega \subset M$ be a Lie domain in M. Assume that $1/p = 1/q - m/n > 0$, $1 \leq q < \infty$, where $m \leq k$ is an integer. Then $W^{k,q}(\Omega_0)$ is continuously embedded in $W^{k-m,p}(\Omega_0)$.*

Proof. This can be proved using Proposition 2.13 and Theorem 3.1. Indeed, denote by

$$j : W^{k,q}(M_0) \to W^{k-m,p}(M_0)$$

the continuous inclusion of Proposition 2.13. Also, denote by r the restriction maps $W^{k,p}(M_0) \to W^{k,p}(\Omega_0)$. Then the maps

$$W^{k,q}(\Omega_0) \xrightarrow{E} W^{k,q}(M_0) \xrightarrow{j} W^{k-m,p}(M_0) \xrightarrow{r} W^{k-m,p}(\Omega_0)$$

are well defined and continuous. Their composition is the inclusion of $W^{k,q}(\Omega_0)$ into $W^{k-m,p}(\Omega_0)$. This completes the proof. □

For the proof of a variant of Rellich–Kondrachov's compactness theorem, we shall need Sobolev spaces with weights. Let $a_H > 0$ be a parameter associated to each hyperface (*i.e.,* face of codimension one) of M. Fix for any hyperface $H \subset M$ a defining function ρ_H, that is a function $\rho_H \geq 0$ such that $H = \{\rho_H = 0\}$ and $d\rho_H \neq 0$ on H. Let

$$\rho = \prod \rho_H^{a_H}, \tag{17}$$

the product being taken over all hyperfaces of M. The function ρ will be called an *admissible weight*. (The function ρ considered in the beginning of this paper is, in fact, an admissible weight, so there is no conflict in the notation.) We define then

$$\rho^s W^{k,p}(\Omega_0) := \{\rho^s u, u \in W^{k,p}(\Omega_0)\}, \tag{18}$$

with the norm $\|\rho^s u\|_{\rho^s W^{k,p}(\Omega_0)} := \|u\|_{W^{k,p}(\Omega_0)}$.

Theorem 3.6. *Denote by n the dimension of M and let $\Omega \subset M$ be a Lie domain, $\Omega_0 = \Omega \cap M_0$. Let $s < s'$ be real parameters. Assume that $1/p > 1/q - m/n > 0$, $1 \leq q < \infty$, where $m \leq k$ is an integer. Then $\rho^s W^{k,q}(\Omega_0)$ is compactly embedded in $\rho^{s'} W^{k-m,p}(\Omega_0)$.*

Proof. The same argument as that in the proof of Theorem 3.5 allows us to assume that $\Omega_0 = M_0$. The norms are chosen such that $W^{k,p}(\Omega_0) \ni u \to \rho^s u \in \rho^s W^{k,p}(\Omega_0)$ is an isometry. Thus, it is enough to prove that $\rho^s : W^{k,q}(\Omega_0) \to W^{k-m,p}(\Omega_0)$, $s > 0$, is a compact operator.

For any defining function ρ_H and any $X \in \mathcal{V}$, we have that $X(\rho_H)$ vanishes on H, since X is tangent to H. Recall now the function ρ defined in Equation (17). We obtain that $X(\rho^s) = \rho^s f_X$, for some $f_X \in \mathcal{C}^\infty(M)$. Then, by induction, $X_1 X_2 \ldots X_k(\rho^s) = \rho^s g$, for some $g \in \mathcal{C}^\infty(M)$.

Let $\chi \in \mathcal{C}^\infty([0,\infty)$ be equal to 0 on $[0, 1/2]$, equal to 1 on $[1,\infty)$, and non-negative everywhere. Define $\phi_\epsilon = \chi(\epsilon^{-1}\rho^s)$. Then

$$\|X_1 X_2 \ldots X_k\big(\rho^s \phi_\epsilon - \rho^s\big)\|_{L^\infty} \to 0\,, \quad \text{as } \epsilon \to 0,$$

for any $X_1, X_2, \ldots, X_k \in \mathcal{V}$. Corollary 2.10 then shows that $\rho^s \phi_\epsilon \to \rho^s$ in the norm of bounded operators on $W^{s,p}(\Omega_0)$. But multiplication by $\rho^s \phi_\epsilon$ is a compact operator, by the Rellich-Kondrachov's theorem for compact manifolds with boundary [8, Theorem 9]. This completes the proof. □

We end with the following generalization of the classical restriction theorem for the Hilbertian Sobolev spaces $H^s(M_0) := W^{s,2}(M_0)$. The tame submanifolds are defined in the next section.

Theorem 3.7. *Let $N_0 \subset M_0$ be a tame submanifold of codimension k of the Lie manifold $(M_0, M, \mathcal{V})$. Restriction of smooth functions extends to a bounded, surjective map*

$$H^s(M_0) \to H^{s-k/2}(N_0),$$

for any $s > k/2$. In particular, $H^s(\Omega_0) \to H^{s-1/2}(\partial\Omega_0)$ is continuous and surjective.

Proof. Let $B \to N$ be the vector bundle defining the Lie structure at infinity (N, B) on N_0 and $A \to M$ be the vector bundle defining the Lie structure at infinity (M, A) on M_0. (See Section 5 for further explanation of this notation.) The existence of tubular neighborhoods, Theorem 5.8, and a partition of unity argument, allows us to assume that $M = N \times S^1$ and that $A = B \times TS^1$ (external product). Since the Sobolev spaces $H^s(M_0)$ and $H^{s-1/2}(N_0)$ do not depend on the metric on A and B, we can assume that the circle S^1 is given the invariant metric making it of length 2π and that M_0 is given the product metric. The rest of the proof now is independent of the way we have arrived at the product metric on M_0.

Then $\Delta_{M_0} = \Delta_{N_0} + \Delta_{S^1}$ and $\Delta_{S^1} = -\partial^2/\partial\theta^2$ has spectrum $\{4\pi^2 n^2\}$, $n \in \mathbb{Z}$. We can decompose $L^2(N_0 \times S^1)$ according to the eigenvalues $n \in \mathbb{Z}$ of $-\frac{1}{2\pi\imath}\partial_\theta$:

$$L^2(N_0 \times S^1) \simeq \oplus_{n\in\mathbb{Z}} L^2(N_0 \times S^1)_n \simeq \oplus_{n\in\mathbb{Z}} L^2(N_0),$$

where the isomorphism $L^2(N_0 \times S^1)_n \simeq \oplus_{n\in\mathbb{Z}} L^2(N_0)$ is obtained by restricting to $N_0 = N_0 \times \{1\}$, $1 \in S^1$.

To prove our theorem, it is enough to check that, if $\xi_n \in L^2(N_0)$ is a sequence such that

$$\sum_n \|(1 + n^2 + \Delta_{N_0})^{s/2}\xi_n\|^2 < \infty, \tag{19}$$

then $\sum(1 + \Delta_{N_0})^{s/2-1/4}\xi_n$ is convergent.

Let $C = 1 + \int_{\mathbb{R}}(1 + t^2)^{-s}dt$ and assume that each ξ_n is in the spectral subspace of Δ_{N_0} corresponding to $[m, m + 1) \subset \mathbb{R}_+$. Then

$$(1 + m^2)^{s-1/2}\Big(\sum_n \|\xi_n\|\Big)^2 \le C \sum_n \|(1 + n^2 + m^2)^{s/2}\xi_n\|^2.$$

Since the constant C is independent of m and the spectral spaces of Δ_{N_0} corresponding to $[m, m + 1) \subset \mathbb{R}$ give an orthogonal direct sum decomposition of $L^2(N_0)$, this checks Equation (19) and completes the proof. □

We conclude this section with an application to the regularity of boundary value problems. Applications of this result to the regularity of boundary value problems on polyhedra as well as more details will be included in [11].

Let us introduce some notation first that will be also useful in the following, especially in Section 6.

Let $\exp : TM_0 \longrightarrow M_0 \times M_0$ be given by $\exp(v) := (x, \exp_x(v)),$, $v \in T_xM_0$. If E is a real vector bundle with a metric, we shall denote by $(E)_r$ the set of all vectors v of E with $|v| < r$. Let $(M_0^2)_r := \{(x,y), x, y \in M_0,\, d(x,y) < r\}$. Then the exponential map defines a diffeomorphism $\exp : (TM_0)_r \to (M_0^2)_r$. We shall also need the function ρ defined in Equation (17) and the weighted Sobolev spaces $\rho^s W^{k,p}(\Omega_0) := \{\rho^s u, u \in W^{k,p}(\Omega_0)\}$ introduced in Equation 18.

Theorem 3.8. *Let $\Omega_0 \subset M_0$ be a Lie domain in a Lie manifold $(M_0, M, \mathcal{V})$. Let $P \in \mathrm{Diff}^2_{\mathcal{V}}(M)$ be an order 2 elliptic operator on M_0 generated by $\mathcal{V}$. Let $u \in \rho^s W^{1,p}(\Omega_0)$ be such that $Pu \in \rho^s W^{t,p}(\Omega_0)$, $s \in \mathbb{R}$, $t \in \mathbb{Z}$, $1 < p < \infty$, and $u|_{\partial\Omega_0} = 0$. Then $u \in \rho^s W^{t+2,p}(\Omega_0)$.*

Proof. Note that locally, this is a well known statement. In particular, $\phi u \in W^{t+2,p}(\Omega_0)$, for any $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(M_0)$. The result will follow then if we prove that

$$\|u\|_{\rho^s W^{t+2,p}(M_0)} \leq C(\|Pu\|_{\rho^s W^{t,p}(M_0)} + \|u\|_{\rho^s W^{1,p}(M_0)}) \tag{20}$$

for any $u \in W^{t+2,p}_{\mathrm{loc}}(\Omega_0)$.

Let $r < r_{\mathrm{inj}}(M_0)$ and let $\exp : (TM_0)_r \to (M_0^2)_r$ be the exponential map. The statement is trivially true for $t \leq -1$, so we will assume $t \geq 0$ in what follows. Also, we will assume first that $s = 0$. The general case will be reduced to this one at the end. Assume first that $\Omega_0 = M_0$.

Let P_x be the differential operators on defined on $B_{T_xM_0}(0,r)$ obtained from P by the local diffeomorphism $\exp : B_{T_xM_0}(0,r) \to M_0$. We claim that there exists a constant $C > 0$, independent of $x \in M_0$ such that

$$\|u\|^p_{W^{t+2,p}(T_xM_0)} \leq C\big(\|P_x u\|^p_{W^{t,p}(T_xM_0)} + \|u\|^p_{W^{1,p}(T_xM_0)}\big), \tag{21}$$

for any function $u \in \mathcal{C}^\infty_{\mathrm{c}}(B_{T_xM_0}(0,r))$. This is seen as follows. We can find a constant $C_x > 0$ with this property for any $x \in M_0$ by the ellipticity of P_x using [73]. Choose C_x to be the least such constant. Let $\pi : A \to M$ be the extension of the tangent bundle of M_0, see Remark 1.4 and let $A_x = \pi^{-1}(x)$. The family P_x, $x \in M_0$, extends to a family P_x, $x \in M$, that is smooth in x. The smoothness of the family P_x in $x \in M$ shows that C_x is lower semi-continuous. Since M is compact, C_x will attain its minimum, which therefore must be positive. Let C be that minimum value.

Let now ϕ_j be the partition of unity and ψ_j be the diffeomorphisms appearing in Equation (14), for some $0 < \epsilon < r/6$. In particular, the partition of unity ϕ_j satisfies the conditions of Lemma 2.5, which implies that $\mathrm{supp}(\phi_j) \subset B(x_j, 2\epsilon)$ and the sets $B(x_j, 4\epsilon)$ form a covering of M_0 of finite multiplicity. Let $\eta_j = 1$ on the support of

ϕ_j, $\text{supp}(\eta_j) \subset B(x_j, 4\epsilon)$. We then have

$$\nu_{t+2,p}(u)^p := \sum_j \|(\phi_j u) \circ \psi_j^{-1}\|^p_{W^{t+2,p}(\mathbb{R}^n)} \leq C \sum_j \Big(\|P_x(\phi_j u)\|^p_{W^{t,p}(T_x M_0)} + \|\phi_j u\|^p_{W^{1,p}(T_x M_0)} \Big)$$
$$\leq C \sum_j \Big(\|\phi_j P_x u\|^p_{W^{t,p}(T_x M_0)} + \|[P_x, \phi_j] u\|^p_{W^{t,p}(T_x M_0)} + \|\phi_j u\|^p_{W^{1,p}(T_x M_0)} \Big)$$
$$\leq C \sum_j \Big(\|\phi_j P_x u\|^p_{W^{t,p}(T_x M_0)} + \|\eta_j u\|^p_{W^{t+1,p}(T_x M_0)} + \|\phi_j u\|^p_{W^{1,p}(T_x M_0)} \Big)$$
$$\leq C\big(\nu_{t,p}(Pu)^p + \nu_{t+1}(u)^p\big).$$

The equivalence of the norm $\nu_{s,p}$ with the standard norm on $W^{s,p}(M_0)$ (Propositions 2.6 and 2.9) shows that $\|u\|_{W^{t+2,p}(M_0)} \leq C(\|Pu\|_{W^{t,p}(M_0)} + \|u\|_{W^{t+1,p}(M_0)})$, for any t. This is known to imply

$$\|u\|_{W^{t+2,p}(M_0)} \leq C(\|Pu\|_{W^{t,p}(M_0)} + \|u\|_{W^{1,p}(M_0)}) \tag{22}$$

by a boot-strap procedure, for any $t \geq -1$. This proves our statement if $s = 0$ and $\Omega_0 = M_0$.

The case Ω_0 arbitrary follows in exactly the same way, but using a product type metric in a neighborhood of $\partial\Omega_0$ and the analogue of Equation (21) for a half-space, which shows that Equation (20) continues to hold for M_0 replaced with Ω_0.

The case s arbitrary is obtained by applying Equation (22) to the elliptic operator $\rho^{-s} P \rho^s \in \text{Diff}^2_{\mathcal{V}}(M)$ and to the function $\rho^{-s} u \in W^{k,p}(\Omega_0)$, which then gives Equation (20) righaway. □

For $p = 2$, by combining the above theorem (including its proof, especially Equation (20)) with Theorem 3.7, we obtain the following corollary.

Corollary 3.9. *We keep the assumptions of Theorem 3.8. Let $u \in \rho^s H^1(\Omega_0)$ be such that $Pu \in \rho^s H^t(\Omega_0)$ and $u|_{\partial\Omega_0} \in \rho^s H^{t+3/2}(\Omega_0)$, $s \in \mathbb{R}$, $t \in \mathbb{Z}$. Then $u \in \rho^s H^{t+2}(\Omega_0)$ and*

$$\|u\|_{\rho^s H^{t+2}(\Omega_0)} \leq C(\|Pu\|_{\rho^s H^t(\Omega_0)} + \|u\|_{\rho^s H^1(\Omega_0)} + \|u|_{\partial\Omega_0}\|_{\rho^s H^1(\Omega_0)}). \tag{23}$$

Proof. For $u|_{\partial\Omega_0} = 0$, the result follows from Equation (20), with M_0 replaced by Ω_0, which is proved as explained in the proof of Theorem 3.8. In general, choose $v \in H^{t+2}(\Omega_0)$ such that $v|_{\partial\Omega_0} = u|_{\partial\Omega_0}$, which is possible by Theorem 3.7. Then we use our result for $u - v$. □

4. The Yamabe functional on Lie manifolds

As an example of the consequences and applications of the analysis developped in the preceeding sections, we will show that the Yamabe functional on a Lie manifold is bounded from below.

As before, let (M_0, g) be a Riemannian Lie manifold, with $(M, \mathcal{V})$ the Lie structure at infinity. Recall that the scalar curvature $\mathrm{scal}_g(p)$ at $p \in M_0$ is defined as $\mathrm{scal}_g(p) = \sum_{i,j=1}^n \langle R(e_i, e_j)e_j, e_i\rangle$, where $e_1, e_2, \ldots, e_n$ denotes an orthonormal basis of T_pM_0.

We define the Yamabe functional for functions $u \in \mathcal{C}_c^\infty(M_0)$

$$Y(u) = \inf \frac{\int_M (4\,\frac{n-1}{n-2}|du|^2 + u^2\,\mathrm{scal}_g)\ \mathrm{dvol}_g}{\|u\|^2_{L^p(M_0,g)}},$$

where $p = 2n/(n-2)$. The infimum is a conformal invariant. It is called the *Yamabe constant* of (M_0, g).

The geometric problem of finding a minimizer of Y is called the Yamabe problem. The interest in this problem is due to the following observation. If a minimizer u exists, then $|u|$ is an everywhere strictly positive minimizer of Y. Then, the Euler Lagrange equation in this case is equivalent to the statement that the metric $|u|^{4/(n-2)}g$ is a metric of constant scalar curvature.

Intensive investigations from 1960 to 1987 (see [42] for a nice overview, and [76], [9], [60, 61, 62] for original literature) lead to the celebrated result that a minimizer of Y always exists on compact manifolds.

Remark 4.1. The original motivation for studying the Yamabe functional comes from the following observation (essentially due to Einstein and Hilbert): We view Y as a functional on the space of all metrics g on M, and positive functions u on M; then (g, u) a stationary point of Y (with respect to compactly supported perturbations of g and u) if and only if $u^{4/(n-2)}g$ is an Einstein manifold (see e.g. [14]). Yamabe's idea [78] was to find such stationary points via a minimax procedure. As a first step one mimimzes u for fixed g, as described above. Then, as a second step one takes the maximum over all metrics g. However, Yamabe's program did not succeed as Yamabe was not aware of analytical difficulties in his approach. As indicated above, on compact manifolds, the first step in his program (*i.e.,* the solution of the Yamabe problem) could be repaired. However, the second step cannot be repaired, there are obstructions to the existence of an Einstein metric.

For various reasons, one is interested in finding minimizers of Y on a large class of non-compact, but complete manifolds. E.g. to derive glueing formulas that describe the behavior of the Yamabe constant under taking a connected sum, it is helpful to find minimizers of Y on manifolds with cylindrical ends [2].

The following proposition is a preliminary step in the solution of the Yamabe problem on Lie manifolds.

Proposition 4.2. *The Yamabe functional extends to a continuous functional on $W^{1,2}(M_0)$, and it is bounded from below.*

Proof. Because of bounded geometry

$$\int_{M_0} 4\,\frac{n-1}{n-2}|du|^2 + \mathrm{scal}_g u^2 \le C\|u\|^2_{W^{1,2}(M_0)}.$$

Theorem 3.5 tells us that $W^{1,2}(M_0)$ embeds into $L^{\frac{2n}{n-2}}(M_0)$, hence the functional extends to $W^{1,2}(M_0)$, and the constant from the embedding yields a lower bound for the Yamabe functional. □

It would be interesting to obtain a criterion under which the infimum of he functional is attained on general non-compact Lie manifolds.

Another important case, in which the Yamabe constant on a Lie manifold is of central importance, is general relativity theory. A preliminary step to obtaining solutions to the Einstein equations on a space-time is to obtain solutions to the Einstein constraint equations on a asymptotically euclidean or hyperbolic Riemannian manifold M of dimension 3, embedded as a space-like hypersurface into a space-time. The constraint equation is the compatibility condition at the metric and at the second fundamental form on M in order that it can be extended to a solution of an Einstein metric on a space-time.

The "conformal-cmc method" is a method to obtain solutions to these Einstein constraint equations. After having chosen a divergence free symmetric $(0, 2)$-tensor σ on M and an additional real constant τ in the space-time, each solution the *Lichnerowicz* equation

$$8\Delta u + \mathrm{scal} \cdot u - |\sigma|^2 u^{-7} = 0$$

gives rise to a solution of the Einstein constraint equations with mean curvature τ. Whether the Lichnerowicz equation has solutions depends on the Yamabe constant of M [16, 32, 33, 44].

String theory suggests to search for solutions to the Einstein constraint equations on 9, 10 and 25 dimensional manifolds with other kind of asymptotic at infinity.

Remark 4.3. Most of the results of this paper, in particular Corollary 2.10, Proposition 2.13, and Theorem 3.6, still hold if one replaces functions by sections of a vector bundle, if this vector bundle extends to M in the sense of vector bundles with metric and connection.

For the special case that $\partial M = \emptyset$ (*i.e.*, M_0 compact) these statements are applied to sections of the spinor bundle in [4, 3]. This proves the existence of a maximizer of the functional

$$\mathcal{F}(\phi) = \frac{\int_M \langle D\phi, \phi\rangle}{\|D\phi\|^2_{L^q(M)}}, \quad q = 2n/(n+1),$$

where ϕ is a section of the spinor bundle, and D is the Dirac operator. The functional is bounded from above because of the boundedness of the Sobolev embedding

$$W^{1,2}(M_0) \to W^{1/2,q}(M_0).$$

Maximizers of $\mathcal{F}$ satisfy $D\phi = c|\phi|^{2/(n+1)}\phi$. If dim $M = 2$, then the spinorial Weierstrass representation tells us that these solutions represent constant mean curvature surfaces in $\mathbb{R}^3$ and S^3.

Extensions of this functional to non-compact Lie manifolds are the object of current research.

5. Submanifolds

In this section we introduce various classes of submanifolds of a Lie manifold. Some of these classes were already used in the previous sections.

5.1. **General submanifolds.** We first introduce the most general class of submanifolds of a Lie manifold.

We first fix some notation. Let (M_0, M, A) and (N_0, N, B) be Lie manifolds. We know that there exist vector bundles $A \to M$ and $B \to N$ such that $\mathcal{V} \simeq \Gamma(M, A)$ and $\mathcal{W} \simeq \Gamma(N, B)$, see Remark 1.4. We can assume that $\mathcal{V} = \Gamma(M, A)$ and $\mathcal{W} = \Gamma(N, B)$.

Definition 5.1. Let (M_0, M, A) be a Lie manifold. Then (N_0, N, B) is called a *submanifold* of (M_0, M, A) if

(i) N is a closed submanifold of M (possibly with corners, no transversality at the boundary required),
(ii) $N_0 \subset M_0$, $\partial N \subset \partial M$, and $B \subset A|_N$.
(iii) (N_0, N, B) is a Lie manifold,
(iv) the Lie structures at infinity satisfy the compatibility condition

$$\Gamma(N; B) = \{X \in \Gamma(N; A|_N) \mid \rho \circ X \in \Gamma(N; TN)\}$$

We now make three simple observations.

Remark 5.2. An alternative form of Condition (iv) of the above definition is

$$\Gamma(N; B) = \{X|_N \mid X \in \Gamma(M, A) \text{ and } X|_N \text{ tangent to } N\}. \tag{24}$$

Remark 5.3. Equation 24 shows that there exists a natural vector bundle morphism $f : B \to A$. Since $B_x = T_x N_0 \subset T_x M_0 = A_x$ for $x \in N_0$, the map f is injective above N_0. The assumption ii of our definition implies that f is injective everywhere.

We have the following simple corollary that justifies Condition (iv) of Definition 5.1.

Corollary 5.4. *Let g_0 be a metric on M_0 compatible with the Lie structure at infinity on M_0. Then the restriction of g to N_0 is compatible with the Lie structure at infinity on N_0.*

Proof. Let g be a metric on A whose restriction to TM_0 defines the metric g. Then g restricts to a metric h on B, which in turn defines a metric h_0 on N_0. By definition, h_0 is the restriction of g to N_0. □

We thus see that any submanifold (in the sense of the above definition) of a Riemannian Lie manifold is itself a Riemannian Lie manifold.

5.2. **Second fundamental form.** We define the A-*normal bundle* of the sub-manifold (N_0, N, B) of (M_0, M, A) as $\nu^A = (A|_N)/B$ which is a bundle over N. Then the anchor map ϱ defines a map $\rho^\nu : \nu^A \to (TM|_N)/TN$ which is an isomorphism over N_0.

We denote the Levi-Civita-connection on A by ∇^A and the Levi-Civita connection on B by ∇^B [5]. Let $X, Y \in \mathcal{V} = \Gamma(M, A)$ such that $X|_{N_0}, Y|_{N_0} \in TN_0$. Then $\nabla^A_X Y|_N$

depends only on $X|_N, Y|_N \in \mathcal{W} = \Gamma(N,B)$. Furthermore the Koszul formula implies that $\nabla^B_X Y$ is the tangential part of $\nabla^A_X Y|_N$. The normal part gives rise to the *second fundamental form*

$$\mathrm{II} : B \times B \to \nu^A, \quad \mathrm{II}(X,Y) = \nabla^A_X Y - \nabla^B_X Y.$$

The Levi-Civita connections ∇^A and ∇^B are torsion free, and hence $\mathrm{II}(X,Y) - \mathrm{II}(Y,X) = [X,Y] - [Y,X] = 0$ is symmetric. A direct computation reveals also that $\mathrm{II}(X,Y)$ is tensorial in X, and hence, because of the symmetry, it is also tensorial in Y. ("Tensorial" here means $\mathrm{II}(fX,Y) = f\mathrm{II}(X,Y)$.) It then follows from the compactness of N that

$$\|\mathrm{II}_p(X_p,Y_p)\| \leq C\|X_p\|\,\|Y_p\|,$$

with a constant C independent of $p \in N$. Clearly, on the interior $N_0 \subset M_0$ the second fundamental form coincides with the classical second fundamental form.

Corollary 5.5. *Let (N_0, N, B) be a submanifold of (M_0, M, A) with a compatible metric. Then the (classical) second fundamental form of N_0 in M_0 is uniformly bounded.*

5.3. **Tame submanifolds.** We now introduce tame manifolds. Our main interest in tame manifolds is the tubular neighborhood theorem, Theorem 5.8, which asserts that a tame submanifold of a Lie manifold has a tubular neighborhood in a strong sense. In particular, we will obtain that a tame submanifold of codimension one is regular. This is interesting because being tame is an algebraic condition that can be easily verified by looking at the structural Lie algebras of vector fields. On the other hand, being a regular submanifold is an analytic condition on the metric that is very difficult to check directly.

Definition 5.6. A submanifold (N_0, N, B) of a Lie manifold (M_0, M, A) is called *tame* if T_pN and $\varrho(A_p)$ span T_pM for all $p \in \partial N$.

As a consequence of these properties the anchor map ϱ defines an isomorphism from A_p/B_p to T_pM/T_pN for any $p \in N$. In particular, the anchor map ρ maps $B^\perp$, the orthogonal complement of B in A, injectively into $\varrho(A) \subset TM$. For any boundary face F and $p \in F$ we have $\rho(A_p) \subset T_pF$. Hence, N is transversal to F, *i.e.,* for any $p \in N \cap F$, the space T_pM is spanned by T_pN and T_pF. As a consequence, $N \cap F$ is a submanifold of F of codimension $\dim M - \dim N$. The codimension of $N \cap F$ in F is therefore independent of F, in particular independent of the dimension of F.

Examples 5.7.

(1) Let M be any compact manifold (without boundary). Fix a $p \in M$. Let (N_0, N, B) be a manifold with a Lie structure at infinity. Then $(N_0 \times \{p\}, N \times \{p\}, B)$ is a tame submanifold of $(N_0 \times M, N \times M, B \times TM)$.
(2) If $\partial N \neq \emptyset$, the diagonal N is a submanifold of $N \times N$, but not a tame submanifold.
(3) Let N be a submanifold with corners of M (so N is transverse to all faces of M). We endow these manifolds with the b-structure at infinity $\mathcal{V}_b$ (see Example 1.3 (i)). Then $(N, \mathcal{V}_b)$ is a tame Lie submanifold of $(M, \mathcal{V}_b)$.

We now prove the main theorem of this section.

Theorem 5.8 (Global tubular neighborhood theorem)**.** *Let (N_0, N, B) be a tame submanifold of the Lie manifold (M_0, M, A). For $\epsilon > 0$, let $(\nu)_\epsilon$ be the set of all vectors normal to N of length smaller than ϵ. If $\epsilon > 0$ is sufficiently small, then the normal exponential map $\exp^\nu$ defines a diffeomorphism from $(\nu)_\epsilon$ to an open neighborhood V_ϵ of N in M. Moreover, $dist(\exp^\nu(X), N) = |X|$ for $|X| < \epsilon$.*

Proof. Recall from [5] that the exponential map $\exp : TM_0 \to M_0$ extends to a map $\exp : A \to M$. The definition of the normal exponential function $\exp^\nu$ is obtained by identifying the quotient bundle ν^A with $B^\perp$, as discussed earlier. This gives

$$\exp^\nu : (\nu)_\epsilon \to M.$$

The differential $d\exp^\nu$ at $0_p \in \nu_p$, $p \in N$ is the identity, hence any point $p \in N$ has a neighborhood $U(p)$ and $\tau_p > 0$ such that

$$\exp^\nu : (\nu)_{\tau_p}|_{U_p} \to M \tag{25}$$

is a diffeomorphism onto its image. By compactness $\tau_p \geq \tau > 0$. Hence, $\exp^\nu$ is a local diffeomorphism of $(\nu)_\tau$ to a neighborhood of N in M. It remains to show that it is injective for small $\epsilon \in (0, \tau)$.

Let us assume now that there is no $\epsilon > 0$ such that the proposition holds. Then there are sequences $X_i, Y_i \in \nu$, $i \in \mathbb{N}$, $X_i \neq Y_i$ such that $\exp^\nu X_i = \exp^\nu Y_i$ with $|X_i|, |Y_i| \to 0$ for $i \to \infty$. After taking a subsequence we can assume that the basepoints p_i of X_i converge to p_∞ and the basepoints q_i of Y_i converge to q_∞. As the distance in M of p_i and q_i converges to 0, we conclude that $p_\infty = q_\infty$. However, $\exp^\nu$ is a diffeomorphism from $(\nu)_\tau|_{U(p_\infty)}$ into a neighborhood of $U(p_\infty)$. Hence, we see that $X_i = Y_i$ for large i, which contradicts the assumptions. □

We now prove that every tame, codimension one Lie submanifold is regular.

Proposition 5.9. *Let (N_0, N, B) be a tame submanifold of codimension one of (M_0, M, A). Fix a diffeomorphism*

$$\exp^\nu : (\nu)_\epsilon \cong N \times (-\epsilon, \epsilon) \to \{x \,|\, d(x, N) < \epsilon\} := U_\epsilon$$

as in Theorem 5.8. Then M_0 carries a compatible metric g such that $(\exp^\nu)^* g$ is a product metric, i.e., $(\exp^\nu)^* g = g_N + dt^2$ *on $N \times (-\epsilon/2, \epsilon/2)$.*

Proof. Choose any compatible metric g_1 on M_0. Let g_2 be a metric on U_ϵ such that $(\exp^\nu)^* g_2 = g_1|_N + dt^2$ on $N \times (-\epsilon, \epsilon)$. Let $d(x) := dist(x, N)$. Then

$$g = \chi \circ d g_1 + (1 - \chi \circ d) g_2,$$

has the desired properties, where the cut-off function $\chi : \mathbb{R} \to [0, 1]$ is 1 on $(-\epsilon/2, \epsilon/2)$ and has support in $(-\epsilon, \epsilon)$, and satisfies $\chi(-t) = \chi(t)$. □

6. Pseudodifferential operators

We now recall the definition of pseudodifferential operators on M_0 generated by a Lie structure at infinity $(M, \mathcal{V})$ on M_0.

6.1. **Definition.** We fix in what follows a compatible Riemannian metric g on M_0 (that is, a metric coming by restriction from a metric on the bundle $A \to M$ extending TM_0), see Section 1. In order to simplify our discussion below, we shall use the metric g to trivialize all density bundles on M. Recall that M_0 with the induced metric is complete [5].

Let $\exp_x : T_xM_0 \to M_0$ be the exponential map, which is everywhere defined because M_0 is complete. We let

$$\Phi : TM_0 \longrightarrow M_0 \times M_0, \quad \Phi(v) := (x, \exp_x(-v)),\ v \in T_xM_0, \tag{26}$$

If E is a real vector bundle with a metric, we shall denote by $(E)_r$ the set of all vectors v of E with $|v| < r$. Let $(M_0^2)_r := \{(x, y), x, y \in M_0,\ d(x, y) < r\}$. Then the map Φ of Equation (26) restricts to a diffeomorphism $\Phi : (TM_0)_r \to (M_0^2)_r$, for any $0 < r < r_{\mathrm{inj}}(M_0)$, where $r_{\mathrm{inj}}(M_0)$ is the injectivity radius of M_0, which was assumed to be positive. The inverse of Φ is of the form

$$(M_0^2)_r \ni (x, y) \longmapsto (x, \tau(x, y)) \in (TM_0)_r\,.$$

We shall denote by $S^m_{1,0}(E)$ the space of symbols of order m and type $(1, 0)$ on E (in Hörmander's sense) and by $S^m_{cl}(E)$ the space of classical symbols of order m on E [30, 55, 72, 74]. See [6] for a review of these spaces of symbols in our framework.

Let $\chi \in \mathcal{C}^\infty(A^*)$ be a smooth function that is equal to 1 on $(A^*)_r$ and is equal to 0 outside $(A^*)_{2r}$, for some $r < r_{\mathrm{inj}}(M_0)/3$. Then, following [6], we define

$$q(a)u(x) = (2\pi)^{-n} \int_{T^*M_0} e^{i\tau(x,y)\cdot\eta} \chi(x, \tau(x, y)) a(x, \eta) u(y)\, d\eta\, dy\,.$$

This integral is an oscillatory integral with respect to the symplectic measure on T^*M_0 [31]. Alternatively, we consider the measures on M_0 and on $T^*_xM_0$ defined by some choice of a metric on A and we integrate first in the fibers $T^*_xM_0$ and then on M_0. The map $\sigma_{tot} : S^m_{1,0}(A^*) \to \Psi^m(M_0)/\Psi^{-\infty}(M_0)$,

$$\sigma_{tot}(a) := q(a) + \Psi^{-\infty}(M_0)$$

is independent of the choice of the function $\chi \in \mathcal{C}^\infty_{\mathrm{c}}((A)_r)$ [6].

We now enlarge the class of order $-\infty$ operators that we consider. Any $X \in \Gamma(A)$ generates a global flow $\Psi_X : \mathbb{R} \times M \to M$ because X is tangent to all boundary faces of M and M is compact. Evaluation at $t = 1$ yields a diffeomorphism

$$\psi_X := \Psi_X(1, \cdot) : M \to M. \tag{27}$$

We now define the pseudodifferential calculus on M_0 that we will consider following [6].

Definition 6.1. Fix $0 < r < r_{\mathrm{inj}}(M_0)$ and $\chi \in \mathcal{C}^\infty_{\mathrm{c}}((A)_r)$ such that $\chi = 1$ in a neighborhood of $M \subseteq A$. For $m \in \mathbb{R}$, the space $\Psi^m_{1,0,\mathcal{V}}(M_0)$ of *pseudodifferential operators generated by the Lie structure at infinity* $(M, \mathcal{V})$ is defined to be the linear space of operators $\mathcal{C}^\infty_{\mathrm{c}}(M_0) \to \mathcal{C}^\infty_{\mathrm{c}}(M_0)$ generated by $q(a)$, $a \in S^m_{1,0}(A^*)$, and $q(b)\psi_{X_1} \dots \psi_{X_k}$, $b \in S^{-\infty}(A^*)$ and $X_j \in \Gamma(A)$, $\forall j$.

Similarly, the space $\Psi^m_{cl,\mathcal{V}}(M_0)$ of *classical pseudodifferential operators generated by the Lie structure at infinity* $(M,\mathcal{V})$ is obtained by using classical symbols a in the construction above.

We have that $\Psi^{-\infty}_{cl,\mathcal{V}}(M_0) = \Psi^{-\infty}_{1,0,\mathcal{V}}(M_0) =: \Psi^{-\infty}_{\mathcal{V}}(M_0)$ (we dropped some subscripts).

6.2. **Properties.** We now review some properties of the operators in $\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^m_{cl,\mathcal{V}}(M_0)$ from [6]. These properties will be used below. Let $\Psi^\infty_{1,0,\mathcal{V}}(M_0) = \cup_{m\in\mathbb{Z}}\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^\infty_{cl,\mathcal{V}}(M_0) = \cup_{m\in\mathbb{Z}}\Psi^m_{cl,\mathcal{V}}(M_0)$.

First of all, each operator $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ defines continuous maps $\mathcal{C}^\infty_{\rm c}(M_0) \to \mathcal{C}^\infty(M_0)$, and $\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$, still denoted by P. An operator $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ has a distribution kernel k_P in the space $I^m(M_0\times M_0, M_0)$ of distributions on $M_0\times M_0$ that are conormal of order m to the diagonal, by [31]. If $P = q(a)$, then k_P has support in $(M_0\times M_0)_r$. If we extend the exponential map $(TM_0)_r \to M_0\times M_0$ to a map $A \to M$, then the distribution kernel of $P = q(a)$ is the restriction of a distribution, also denoted k_P in $I^m(A,M)$.

If $\mathcal{P}$ denotes the space of polynomial symbols on A^* and $\mathrm{Diff}(M_0)$ denotes the algebra of differential operators on M_0, then

$$\Psi^\infty_{1,0,\mathcal{V}}(M_0)\cap \mathrm{Diff}(M_0) = \mathrm{Diff}^\infty_{\mathcal{V}}(M) = q(\mathcal{P}). \tag{28}$$

The spaces $\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^m_{1,0,\mathcal{V}}(M_0)$ are independent of the choice of the metric on A and the function χ used to define it, but depend, in general, on the Lie structure at infinity (M,A) on M_0. They are also closed under multiplication, which is a quite non-trivial fact.

Theorem 6.2. *The spaces $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ and $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ are filtered algebras that are closed under adjoints.*

For $\Psi^m_{1,0,\mathcal{V}}(M_0)$, the meaning of the above theorem is that

$$\Psi^m_{1,0,\mathcal{V}}(M_0)\Psi^{m'}_{1,0,\mathcal{V}}(M_{\mathcal{V}}) \subseteq \Psi^{m+m'}_{1,0,\mathcal{V}}(M_0) \text{ and } \big(\Psi^m_{1,0,\mathcal{V}}(M_0)\big)^* = \Psi^m_{1,0,\mathcal{V}}(M_0)$$

for all $m, m' \in \mathbb{C}\cup\{-\infty\}$.

The usual properties of the principal symbol remain true.

Proposition 6.3. *The principal symbol establishes isomorphisms*

$$\sigma^{(m)} : \Psi^m_{1,0,\mathcal{V}}(M_0)/\Psi^{m-1}_{1,0,\mathcal{V}}(M_0) \to S^m_{1,0}(A^*)/S^{m-1}_{1,0}(A^*) \tag{29}$$

and

$$\sigma^{(m)} : \Psi^m_{cl,\mathcal{V}}(M_0)/\Psi^{m-1}_{cl,\mathcal{V}}(M_0) \to S^m_{cl}(A^*)/S^{m-1}_{cl}(A^*). \tag{30}$$

Moreover, $\sigma^{(m)}(q(a)) = a + S^{m-1}_{1,0}(A^*)$ for any $a \in S^m_{1,0}(A^*)$ and $\sigma^{(m+m')}(PQ) = \sigma^{(m)}(P)\sigma^{(m')}(Q)$, for any $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ and $Q \in \Psi^{m'}_{1,0,\mathcal{V}}(M_0)$.

We shall need also the following result.

Proposition 6.4. *Let x be a defining function of some hyperface of M. Then $\rho^s\Psi^m_{1,0,\mathcal{V}}(M_0)\rho^{-s} = \Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\rho^s\Psi^m_{cl,\mathcal{V}}(M_0)\rho^{-s} = \Psi^m_{cl,\mathcal{V}}(M_0)$ for any $s\in\mathbb{C}$.*

6.3. **Continuity on $W^{s,p}(M_0)$.** The preparations above will allow us to prove the continuity of the operators $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ between suitable Sobolev spaces. This is the main result of this section. Some of the ideas and constructions in the proof below have already been used in 3.8, which the reader may find convenient to review first.

Theorem 6.5. *Let $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ and $p \in (0,\infty)$. Then P maps $\rho^r W^{s,p}(M_0)$ continuously to $\rho^r W^{s-m,p}(M_0)$ for any $r, s \in \mathbb{R}$.*

Proof. We have that P maps $\rho^r W^{s,p}(M_0)$ continuously to $\rho^r W^{s-m,p}(M_0)$ if, and only if, $\rho^{-r} P \rho^r$ maps $W^{s,p}(M_0)$ continuously to $W^{s-m,p}(M_0)$. By Proposition 6.4 it is therefore enough to check our result for $r = 0$.

We shall first prove our result if the Schwartz kernel of P has support close enough to the diagonal. To this end, let us choose $\epsilon < r_{\text{inj}}(M_0)/9$ and assume that the distribution kernel of P is supported in the set $(M_0^2)_\epsilon := \{(x,y), d(x,y) < \epsilon\} \subset M_0^2$. This is possible by choosing the function χ used to define the spaces $\Psi^m_{1,0,\mathcal{V}}(M_0)$ to have support in the set $(M_0^2)_\epsilon$. There will be no loss of generality then to assume that $P = q(a)$.

Then choose a smooth function $\eta : [0,\infty) \to [0,1]$, $\eta(t) = 1$ if $t \le 6\epsilon$, $\eta(t) = 0$ if $t \le 7\epsilon$. Let $\psi_x : B(x, 8\epsilon) \to B_{T_x M_0}(0, 8\epsilon)$ denote the normal system of coordinates induced by the exponential maps $\exp_x : T_x M_0 \to M_0$. Denote $\pi : A \to M$ be the natural (vector bundle) projection and

$$B := A \times_M A := \{(\xi_1, \xi_2) \in A \times A, \pi(\xi_1) = \pi(\xi_2)\}, \tag{31}$$

which defines a vector bundle $B \to M$. In the language of vector bundles, $B := A \oplus A$. For any $x \in M_0$, let η_x denote the function $\eta \circ \exp_x$, and consider the operator $\eta_x P \eta_x$ on $B(x, 13\epsilon)$. The diffeomorphism ψ_x then will map this operator to an operator P_x on $B_{T_x M_0}(0, 8\epsilon)$. Then P_x maps continuously $W^{s,p}(T_x M_0) \to W^{s-m,p}(T_x M_0)$, by the continuity of pseudodifferential operators on $\mathbb{R}^n$ [75, XIII, §5] or [72, 69].

The distribution kernel k_x of P_x is a distribution with compact support on

$$T_x M_0 \times T_x M_0 = A_x \times A_x = B_x$$

If $P = q(a) \in \Psi^m_{1,0,\mathcal{V}}(M_0)$, then the distributions k_x can be determined in terms of the distribution $k_P \in I^m(A, M)$ associated to P. This shows that the distributions k_x extend to a smooth family of distributions on the fibers of $B \to M$. From this, it follows that the family of operators $P_x : W^{s,p}(A_x) \to W^{s-m,p}(A_x)$, $x \in M_0$, extends to a family of operators defined for $x \in M$ (recall that $A_x = T_x M_0$ if $x \in M_0$). This extension is obtained by extending the distribution kernels. In particular, the resulting family P_x will depend smoothly on $x \in M$. Since M is compact, we obtain, in particular, that the norms of the operators P_x are uniformly bounded for $x \in M_0$.

By abuse of notation, we shall denote by $P_x : W^{s,p}(M_0) \to W^{s-m,p}(M_0)$ the induced family of pseudodifferential operators, and we note that it will still be a smooth family that is uniformly bounded in norm. Note that it is possible to extend P_x to an operator on M_0 because its distribution kernel has compact support.

Then choose the sequence of points $\{x_j\} \subset M_0$ and a partition of unity $\phi_j \in \mathcal{C}^\infty_c(M_0)$ as in Lemma 2.5. In particular, ϕ_j will have support in $B(x_j, 2\epsilon)$. Also, let

$\psi_j : B(x_j, 4\epsilon) \to B_{\mathbb{R}^n}(0, 4\epsilon)$ denote the normal system of coordinates induced by the exponential maps $\exp_x : T_xM_0 \to M_0$ and some fixed isometries $T_xM_0 \simeq \mathbb{R}^n$. Then all derivatives of $\psi_j \circ \psi_k^{-1}$ are bounded on their domain of definition, with a bound that may depend on ϵ but does not depend on j and k [15, 66].

Let

$$\nu_{s,p}(u)^p := \sum_j \|(\phi_j u) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)}.$$

be one of the several equivalent norms defining the topology on $W^{s,p}(M_0)$ (see Proposition 2.9 and Equation (13). It is enough to prove that

$$\begin{aligned}(32)\quad \nu_{s,p}(Pu)^p := \sum_j \|(\phi_j Pu) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)} \\ \le C \sum_j \|(\phi_j u) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)} =: C\nu_{s,p}(u)^p,\end{aligned}$$

for some constant C independent of u.

We now prove this statement. Indeed, for the reasons explained below, we have the following inequalities.

$$\begin{aligned}\sum_j \|(\phi_j Pu) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)} &\le C \sum_{j,k} \|(\phi_j P \phi_k u) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)} \\ &= C \sum_{j,k} \|(\phi_j P_{x_j} \phi_k u) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)} \le C \sum_{j,k} \|(\phi_j \phi_k u) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)} \\ &\le C \sum_j \|(\phi_j u) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)} = C\nu_{s,p}(u)^p.\end{aligned}$$

Above, the first and last inequalities are due to the fact that the family ϕ_j is uniformly locally finite, that is, there exists a constant N such that at any given point x, at most N of the functions $\phi_j(x)$ are different from zero. The first equality is due to the support assumptions on ϕ_j, ϕ_k, and P_{x_j}. Finally, the second inequality is due to the fact that the operators P_{x_j} are continuous, with norms bounded by a constant independent of j, as explained above. We have therefore proved that $P = q(a) \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ defines a bounded operator $W^{s,p}(M_0) \to W^{s-m,p}(M_0)$, provided that the Schwartz kernel of P has support in a set of the $(M_0^2)_\epsilon$, for $\epsilon < r_{\text{inj}}(M_0)/9$.

Assume now that $P \in \Psi^{-\infty}_{\mathcal{V}}(M_0)$. We shall check that P is bounded as a map $W^{2k,p}(M_0) \to W^{-2k,p}(M_0)$. For $k = 0$, this follows from the fact that the Schwartz kernel of P is given by a smooth function $k(x, y)$ such that $\int_{M_0} |k(x, y)| d\operatorname{vol}_g(x)$ and $\int_{M_0} |k(x, y)| d\operatorname{vol}_g(y)$ are uniformly bounded in x and y. For the other values of k, it is enough to prove that the bilinear form

$$W^{2k,p}(M_0) \times W^{2k,p}(M_0) \ni (u, v) \to \langle Pu, v\rangle \in \mathbb{C}$$

is continuous. Choose Q a parametrix of Δ^k and let $R = 1 - Q\Delta^k$ be as above. Let $R' = 1 - \Delta^k Q \in \Psi^{-\infty}_{\mathcal{V}}(M_0)$. Then

$$\langle Pu, v\rangle = \langle (QPQ)\Delta^k u, \Delta^k v\rangle + \langle (QPR)u, \Delta^k v\rangle + \langle (R'PQ)\Delta^k u, v\rangle + \langle (R'PR)u, v\rangle,$$

which is continuous since $QPQ, QPR, R'PQ$, and $R'PR$ are in $\Psi_{\mathcal{V}}^{-\infty}(M_0)$ and hence they are continuous on on $L^p(M_0)$ and because $\Delta^k : W^{2k,p}(M_0) \to L^p(M_0)$ is continuous.

Since any $P \in \Psi_{1,0,\mathcal{V}}^m(M_0)$ can be written $P = P_1 + P_2$ with $P_2 \in \Psi_{\mathcal{V}}^{-\infty}(M_0)$ and $P_1 = q(a) \in \Psi_{1,0,\mathcal{V}}^m(M_0)$ with support arbitrarily close to the diagonal in M_0, the result follows. □

As in [68][Proposition 1.8], we obtain the following characterization of Sobolev spaces.

Theorem 6.6. *Let $s \in \mathbb{R}_+$ and $p \in (1, \infty)$. We have that $u \in W^{s,p}(M_0)$ if, and only if, $u \in L^p(M_0)$ and $Pu \in L^p(M_0)$ for any $P \in \Psi_{1,0,\mathcal{V}}^s(M_0)$. The norm $u \to \|u\|_{L^p(M_0)} + \|Pu\|_{L^p(M_0)}$ is equivalent to the original norm on $W^{s,p}(M_0)$ for any elliptic $P \in \Psi_{1,0,\mathcal{V}}^s(M_0)$.*

Similarly, the space $W^{-s,p}(M_0)$ is the quotient of $L^p(M_0) \oplus L^p(M_0)$ with respect to the map $(u, v) \to u + Pv$.

Proof. Clearly, if $u \in W^{s,p}(M_0)$, then $Pu, u \in L^p(M_0)$. Let us prove the converse.

Assume $Pu, u \in L^p(M_0)$. Let $Q \in \Psi_{1,0,\mathcal{V}}^{-s}(M_0)$ be a parametrix of P and let $R, R' \in \Psi_{\mathcal{V}}^{-\infty}(M_0)$ be defined by $R := 1 - QP$ and $R' = 1 - PQ$. Then $u = QPu + Ru$. Since both $Q, R : L^p(M_0) \to W^{s,p}(M_0)$ are defined and bounded, $u \in W^{s,p}(M_0)$ and $\|u\|_{W^{s,p}(M_0)} \leq C\big(\|u\|_{L^p(M_0)} + \|Pu\|_{L^p(M_0)}\big)$. This proves the first part.

To prove the second part, we observe that the mapping

$$W^{s,q}(M_0) \ni u \to (u, Pu) \in L^q(M_0) \oplus L^q(M_0), \qquad q^{-1} + p^{-1} = 1,$$

is an isomorphism onto its image. The result then follows by duality using the Hahn-Banach theorem. □

We conclude our paper with the sketch of two regularity results on solutions of elliptic equations. We formulate the first result only for order two operators with Dirichlet boundary conditions, in order to avoid a discussion of regular boundary conditions [73] in our setting.

The proof of the following result is a standard application of the previous ideas. Recall the Sobolev spaces with weights $\rho^s W^{s,p}(\Omega_0)$ introduced in Equation (18).

Theorem 6.7. *Let $P \in \mathrm{Diff}_{\mathcal{V}}^m(M)$ be an order m elliptic operator on M_0 generated by $\mathcal{V}$. Let $u \in \rho^s W^{r,p}(M_0)$ be such that $Pu \in \rho^s W^{t,p}(M_0)$, $s, r, t \in \mathbb{R}$, $1 < p < \infty$. Then $u \in \rho^s W^{t+m,p}(M_0)$.*

Proof. Let $Q \in \Psi_{\mathcal{V}}^{-\infty}(M_0)$ be a parametrix of P. Then $R = I - QP \in \Psi_{\mathcal{V}}^{-\infty}(M_0)$. This gives $u = Q(Pu) + Ru$. But $Q(Pu) \in \rho^s W^{t+m,p}(M_0)$, by Theorem 6.5, because $Pu \in \rho^s W^{t,p}(M_0)$. Similarly, $Ru \in \rho^s W^{t+m,p}(M_0)$. This completes the proof. □

Note that the above theorem was already proved in the case $t \in \mathbb{Z}$ and $m = 2$, using more elementary methods, as part of Theorem 3.8. The proof here is much shorter, however, which attests to the power of pseudodifferential operator algebra techniques.

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