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NAVIER-STOKES DYNAMICS

ON A

DIFFERENTIAL ONE-FORM

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Abstract

The Navier-Stokes equations have been solved by transforming the dynamic Navier-Stokes equation into a differential one-form on an odd-dimensional differentiable manifold and then using the principle that this one-form predicts, by analysis with exterior calculus, a set of characteristic differential equations and vortex vector characteristic of Hamiltonian geometry. The solution was shown to be divergence-free by contracting the differential 3-form corresponding to the divergence of the gradient of the velocity with a triple of tangent vectors, implying constraints on two of the tangent vectors for the system. Analysis of the solution showed that it is bounded since it remains finite as $\left|x^k\right| \to \infty$, and is physically reasonable since the square of the gradient of the principal function is bounded. By contracting this differential one-form with the vortex vector, the Lagrangian was obtained.

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1. INTRODUCTION

In fluid dynamics, the Euler and Navier-Stokes equations model the dynamics of a fluid in $\mathbb{R}^n$ ($n = 2$ or 3) for times $t \geq 0$**.** For incompressible fluids the Navier-Stokes equations are given by

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \bullet \nabla)\mathbf{v} + \left[-\nabla P + \nu \sum_{j=1}^{n} \frac{\partial}{\partial x^j}\left(\frac{\partial \mathbf{v}}{\partial x^j}\right) + \mathbf{f} \right] \quad (1)$$

$$div\, \mathbf{v} = 0 \quad (2)$$

with initial conditions

$$\mathbf{v}(x^1, ..., x^n, t_0) = \mathbf{v}^0(x^1, ..., x^n) \quad (3)$$

For the case of zero viscosity ν, these equations are the Euler equations. Eqn.(3) is the initial condition for position coordinates x^k and time $t = t_0$, eqn.(2) is the condition for incompressibility and eqn. (1) is the equation describing the dynamics, with externally applied force $\mathbf{f}(x^1, \ldots, x^n, t)$, velocity $\mathbf{v}(x^1, \ldots, x^n, t)$, pressure $P(x^1, \ldots, x^n, t)$, and with forces due to pressure gradient ∇P and viscous friction $\nu \sum_{j=1}^{n} \frac{\partial}{\partial x^j}\left(\frac{\partial \mathbf{v}}{\partial x^j}\right)$. Many investigations have focused on finding $\mathbf{v}$ satisfying eqns.(1, 2 and 3) or on proving or disproving the global existence, smoothness and breakdown of Navier-Stokes solutions on $\mathbb{R}^3$ or on $\mathbb{R}^3/\mathbb{Z}^3$; a critical discussion as to what constitutes a solution and the state of such investigations in 2000 and 1998 have been given by Fefferman[1] and by Arnold and Khesin[2].

In the present investigation the Navier-Stokes dynamic equation (eqn.(1)) is rearranged to an expression which can be written as a differential one-form, then an extension of a principle proposed in a previous manuscript [3] is applied. This principle states that the description of a dynamic system with a characteristic differential one-form on an odd-dimensional differentiable manifold leads, by analysis with exterior calculus, to a set of characteristic differential equations and a characteristic tangent vector which define transformations of the system. The extension of this principle arises because the differential one-form used in reference 2 is the exterior derivative of a scalar function; however, the present application to the Navier-Stokes equation involves the exterior derivative of a vector field. Solution to the Navier-Stokes equation reduces to synthesis and solution of a set of differential equations analogous to Hamilton's equations and the synthesis of a characteristic tangent vector which describes the direction of the system motion.

The appropriate differentiable manifold for describing the system is a cotangent bundle rather than a tangent bundle; hence the solution is the position and the conjugate to the position $(x^j, \mathbf{b}_j)$, rather than the position and the velocity. This solution is shown to be divergence-free by contracting the differential 3-form corresponding to the divergence of the gradient of the velocity on a triple of tangent vectors, and setting the result to zero; this implies constraints on two of the tangent vectors for the system. Analysis of the solution showed it is bounded since it remains finite as $\left|x^k\right| \to \infty$, and is physically reasonable since the square of the gradient of the principal function is bounded. The characteristic tangent vector (vortex vector) indicating the direction of the system change was determined; the Lagrangian was calculated by contracting the characteristic differential one-form with the vortex vector.

2. Differential One-Form for the Navier-Stokes Equation

Since $\mathbf{v} = \mathbf{v}(x^1, \ldots, x^n, t)$ and $\frac{\partial \mathbf{v}}{\partial t} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} - \left(\mathbf{v} \bullet \nabla\right)\mathbf{v}$, then eqn.(1) becomes, upon substitution, the following equation:

$$\frac{d\,\mathbf{v}}{dt} - (\mathbf{v} \bullet \nabla)\,\mathbf{v} = -(\mathbf{v} \bullet \nabla)\,\mathbf{v} + \left[-\nabla P + \nu \sum_{j=1}^{n} \frac{\partial}{\partial x^j}\left(\frac{\partial \mathbf{v}}{\partial x^j}\right) + \mathbf{f} \right] \tag{4}$$

Multiplying this equation by $-dt$ gives

$$\mathbf{B}_j\, dx^j - d\,\mathbf{v} \;= \mathbf{B}_j\, dx^j \;-\; \mathbf{\Omega}\, dt \tag{5}$$

where

$$\mathbf{B}_j \equiv \frac{\partial \mathbf{v}}{\partial x^j}$$

$$= \mathbf{B}\left(x^j, t\right) \qquad (5\text{-}1)$$

$$\mathbf{B}_j \, dx^j = (\mathbf{v} \bullet \nabla) \mathbf{v} \, dt \qquad (5\text{-}2)$$

$$\mathbf{\Omega} \equiv [-\nabla P + \nu \sum_{j=1}^{n} \left(\frac{\partial \mathbf{B}_j}{\partial x^j} \right)_t + \mathbf{f}] \qquad (5\text{-}3)$$

Upon equating each side of eqn.(5) to the exact differential $d\mathbf{S}$, eqn.(5) becomes for the right-hand side

$$d\mathbf{S} = \mathbf{B}_j \, dx^j - \mathbf{\Omega} \, dt \qquad (6)$$

where $\mathbf{S}$ will be referred to as the principal function. It is important to note that eqn.(6) is the same as eqn.(1) after substituting $\frac{\partial \mathbf{v}}{\partial t} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} - \left(\mathbf{v} \bullet \nabla\right)\mathbf{v}$ and then multiplying each side of the resulting equation by $-dt$. An equivalent route to eqn.(6) is the multiplication of eqn.(1) by $-dt$ and setting $d\mathbf{S} = -\left(\frac{\partial \mathbf{v}}{\partial t}\right) dt$.

To analyze $\mathbf{B}_j \equiv \frac{\partial \mathbf{v}}{\partial x^j}$, recall that the general concept of the gradient of a tensor field requires $\nabla \mathbf{v}$ to be given by:

$\nabla \mathbf{v}\left(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \xi\right)$ = [Value of contraction $\mathbf{v}\left(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\right)$ at the tip of vector ξ]

– [Value of contraction $\mathbf{v}\left(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\right)$ at the tail of vector ξ]

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are unit vectors and ξ is an arbitrary tangent vector all belonging to $T(T * M_x)$, the tangent space to the cotangent space at the position x^j of manifold M.

For a Lorentz frame, $\nabla \mathbf{v}\,(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \xi) = \dfrac{\partial \mathbf{v}}{\partial x^j}\,\xi^j = \mathbf{B}_j\,\xi^j$, hence $\mathbf{B}_j$ is the gradient of the velocity contracted with unit vectors and divided by the *j*–th coordinate for the tangent vector ξ. $\mathbf{B}_j$ will be simply referred to as the gradient of **v**.

To analyze the quantity $\left(\dfrac{\partial \mathbf{B}_j}{\partial x^j}\right)_t$ in $\mathbf{\Omega}$ and develop the expression $\mathbf{\Omega} = \mathbf{\Omega}(\mathbf{B}_j, x^j, t)$, note that in the neighborhood of initial position $(\mathbf{B}_j(0), x_0^j, t_0)$, Taylor's expansion gives $\mathbf{B}_j$ as

$$\begin{aligned}\mathbf{B}_j(x^j, t) &= \mathbf{B}_j(0) + (x^j - x_0^j)\,\partial_{x^j}\mathbf{B}_j(0) + (t - t_0)\,\partial_t \mathbf{B}_j(0) \\ &\quad + \sum_{N=2}^{\infty} \sum_{r=0}^{N} \frac{1}{N!}\binom{N}{r}(x^j - x_0^j)^{N-r}(t - t_0)^r\ \partial_{x^j}^{N-r}\,\partial_t^r\,\mathbf{B}_j(0)\end{aligned} \tag{7}$$

where N and r are integers such that $2 \le N \le \infty$ and $0 \le r \le N$, with notation

$$\mathbf{B}_j(0) = \mathbf{B}_j(x_0^j, t_0), \qquad \partial_{x^j}^{N-r}\partial_t^r \mathbf{B}_j = \frac{\partial^N \mathbf{B}_j}{\partial x^{(j)\,N-r}\partial t^r} \tag{8}$$

Upon taking the partial derivative of eqn.(7) with respect to x^j it results that

$$\partial_{x^j}\mathbf{B}_j = \partial_{x^j}\mathbf{B}_j(0) + \sum_{N=2}^{\infty}\sum_{r=0}^{N} \left(\frac{N-r}{N!}\right)\binom{N}{r}(x^j - x_0^j)^{N-r-1}(t-t_0)^r \, \partial_{x^j}^{N-r}\,\partial_t^r\,\mathbf{B}_j(0) \qquad (9)$$

Upon substituting $\partial_{x^j}\mathbf{B}_j(0)$ from Taylor's expansion (eqn.(7)) into eqn.(9), it results that

$$\partial_{x^j}\mathbf{B}_j = \left[\mathbf{B}_j - \mathbf{B}_j(0) - (t-t_0)\,\partial_t\mathbf{B}_j(0)\right](x^j - x_0^j)^{-1}$$
$$+ \sum_{N=2}^{\infty}\sum_{r=0}^{N} \left(\frac{N-r-1}{N!}\right)\binom{N}{r}(x^j - x_0^j)^{N-r-1}(t-t_0)^r \, \partial_{x^j}^{N-r}\,\partial_t^r\,\mathbf{B}_j(0) \qquad (10)$$

Replacing the quantity $\partial_{x^j}\mathbf{B}_j$ in $\mathbf{\Omega}$ (eqn.(5-3)) by this quantity in eqn.(10) gives $\mathbf{\Omega}$ as

$$\mathbf{\Omega} = -\nabla P + \mathbf{f}$$
$$+\,\nu\left[\sum_{j=1}^{n}\left[\mathbf{B}_j - \mathbf{B}_j(0) - (t-t_0)\,\partial_t\mathbf{B}_j(0)\right](x^j - x_0^j)^{-1} + \sum_{j=1}^{n}\sum_{N=2}^{\infty}\sum_{r=0}^{N}\left(\frac{N-r-1}{N!}\right)\binom{N}{r}(x^j - x_0^j)^{N-r-1}(t-t_0)^r\,\partial_{x^j}^{N-r}\,\partial_t^r\,\mathbf{B}_j(0)\right] \qquad (11)$$

The differential one-form corresponding to eqn.(6) is

$$\mathbf{dS} = \mathbf{B}_j\, \mathbf{d}x^j - \Omega\, \mathbf{d}t \tag{12-1}$$

where symbol "**d**" is the exterior derivative operator and **dS** is the exterior derivative of a vector field **S.** Let the set of x^j now represent a configuration space. In order for x^j and $\mathbf{B}_j$ to be a conjugate pair, three conditions must be satisfied; namely, (1) $\mathbf{B}_j$ must be the gradient of the function **S,** (2) x^j and $\mathbf{B}_j$ must be functions of temporal coordinate t alone and (3) $\Omega = \Omega\left(\mathbf{B}_k, x^j, t\right)$ The first condition is automatically satisfied by reference to eqn.(12-1), i.e., $\mathbf{B}_j$ is the gradient of **S**. Since the existence of **v** implies $x^j = x^j(t)$ and since $\mathbf{B}_j = \mathbf{B}_j\left(x^j, t\right) = \mathbf{B}_j\left(x^j(t), t\right) = \mathbf{b}_j(t)$, then the second condition is satisfied. Condition three is satisfied by the definition of Ω in eqn.(11). Hence eqn.(12-1) becomes,

$$\mathbf{dS} = \mathbf{b}_j\, \mathbf{d}x^j - \Omega\, \mathbf{d}t \tag{12-2}$$

which is analogous to the expression for the differential one-form for the action in Hamiltonian mechanics.

The geometric object **dS** is called a vector-valued differential one-form on extended cotangent space $T^*M_{x^j}$ with coordinates ($\mathbf{b}_j$, x^j, t), with basic differential one-forms $\mathbf{d}\mathbf{b}_j$, $\mathbf{d}x^j$ and $\mathbf{d}t$ and characteristic function $\Omega(\mathbf{b}_j, x^j, t)$. With this development, the Navier-Stokes equation is expressed as a differential form (eqn.(12-2)) useful for applying exterior calculus to analyze Navier-Stokes dynamics.

3. Navier-Stokes Dynamics on a Differential One-Form

3.1. Differential equations. Using the symbol $\boldsymbol{\omega} \equiv \mathbf{dS}$, the exterior derivative of eqn.(12-2) is

$$\mathbf{d}\boldsymbol{\omega} = \mathbf{db}_j \wedge \mathbf{d}x^j - \left[\left(\frac{\partial \Omega}{\partial x^j} \right) \mathbf{d}x^j + \left(\frac{\partial \Omega}{\partial \mathbf{b}_j} \right) \mathbf{db}_j + \left(\frac{\partial \Omega}{\partial t} \right) \mathbf{d}t \right] \wedge \mathbf{d}t \qquad (13)$$

Following the procedure of reference 3, note that if x^j and $\mathbf{b}_j$ are to describe mappings of the temporal coordinate t onto the direction of the system phase flow, then x^j and $\mathbf{b}_j$ must be functions of t alone, and vector $\boldsymbol{\xi}$ which belongs to the tangent space $T\left(T^*M_{x^j}\right)$ at a point x^j of manifold M, where

$$\boldsymbol{\xi} = \left(\frac{d\,\mathbf{b}_j}{dt} \right) \partial_{\mathbf{b}_j} + \left(\frac{dx^j}{dt} \right) \partial_{x^j} + \partial_t \qquad (14)$$

must satisfy at each point $(\mathbf{b}_j, x^j, t)$ of the transformation, the equation

$$\mathbf{d}\boldsymbol{\omega}(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \qquad (15)$$

for arbitrary tangent vector $\boldsymbol{\eta}$ at each point. This contraction of differential one-form $\mathbf{d\omega}$ is a mapping of a pair of tangent vectors onto an oriented surface, a mapping defined only if the coordinates of $\frac{dx^j}{dt}$ and $\frac{d\,\mathbf{b}_j}{dt}$ of $\boldsymbol{\xi}$ have the values

$$\frac{dx^j}{dt} = \frac{\partial\Omega}{\partial\mathbf{b}_j} \quad \text{and} \quad \frac{d\,\mathbf{b}_j}{dt} = -\frac{\partial\Omega}{\partial x^j} \tag{16}$$

Using the definition of $\boldsymbol{\Omega}$(eqn.(11) , eqns.(16) become

$$\frac{dx^k}{dt} = \frac{\nu}{x^k - x_0^k} \tag{17-1}$$

and

$$\frac{d\,\mathbf{b}_k}{dt} = \partial_{x^k}\left(\nabla P\right) - \partial_{x^k}\mathbf{f}$$

$$- \nu\partial_{x^k}\left[\begin{array}{l}\sum\limits_{j=1}^{n}\left[\mathbf{b}_j - \mathbf{B}_j(0) - (t-t_0)\,\partial_t\mathbf{B}_j(0)\right](x^j - x_0^j)^{-1} \\ + \sum\limits_{j=1}^{n}\sum\limits_{N=2}^{\infty}\sum\limits_{r=0}^{N}\left(\frac{N-r-1}{N!}\right)\binom{N}{r}(x^j - x_0^j)^{N-r-1}(t-t_0)^r\,\partial_{x^j}^{N-r}\partial_t^r\,\mathbf{B}_j(0)\end{array}\right] \tag{17-2}$$

Note that this is a differential equation whose solution is $\mathbf{b}_j$; only constant coefficients of the type $\partial_{x^j}^{N-r}\partial_t^r\,\mathbf{B}_j(0)$ appear, not $\mathbf{B}_j$.

3.2. The solution

The solution to eqn.(17-1) is

$$x^k = x_0^k \pm \sqrt{2\nu(t-t_0)} \tag{18}$$

To change eqn.(17-2) so that a series expansion method can be used for its solution, first P and $\mathbf{f}$ are approximated by a Taylor's series to second order and ∇P is taken, then partial derivatives $\partial_{x^k}\mathbf{f}$ and $\partial_{x^k}\nabla P$ are taken. When comparing the terms $\left[\partial_{x^k}\partial_{x^{k+1}}\mathbf{f}(0)\right]\left(x^{k+1}-x_0^{k+1}\right)$ and $\left[\partial_{x^{k+2}}\partial_{x^k}\mathbf{f}(0)\right]\left(x^{k+2}-x_0^{k+2}\right)$ with $\left[\partial_{x^k}^2\mathbf{f}(0)\right](x^k-x_0^k)$, all in $\partial_{x^k}\mathbf{f}$, it is assumed $\left[\partial_{x^k}\partial_{x^{k+1}}\mathbf{f}(0)\right] \ll \left[\partial_{x^k}^2\mathbf{f}(0)\right]$ and $\left[\partial_{x^{k+2}}\partial_{x^k}\mathbf{f}(0)\right] \ll \left[\partial_{x^k}^2\mathbf{f}(0)\right]$; these terms are excluded as an approximation. The notation $k, k+1, k+2$ is intended to imply cyclic order in x, y, z.

Following the above indicated procedure and noting once again that $\mathbf{b}_j = \mathbf{b}_j(t)$, eqn.(17-2) becomes

$$\begin{aligned}
\frac{d\,\mathbf{b}_k}{dt} = &-\sum_{N=2}^{\infty}\sum_{r=0}^{N}\left[\frac{(N-r-1)^2}{2(2\nu)^{-1}N!}\binom{N}{r}\partial_{x^k}^{N-r}\partial_t^r\,\mathbf{B}_k(0)\right](x^k-x_0^k)^{N-r-2}(t-t_0)^r \\
&+\left[-\partial_{x^k}\partial_t\mathbf{f}(0)\right](t-t_0) \\
&+\left[-\partial_{x^k}^2\mathbf{f}(0)\right](x^k-x_0^k) \\
&+\left[\mathbf{e}_k\left(\partial_{x^k}^2 P(0)\right)+\mathbf{e}_{x^{k+1}}\left(\partial_{x^k}\partial_{x^{k+1}}P(0)\right)+\mathbf{e}_{x^{k+2}}\left(\partial_{x^{k+2}}\partial_{x^k}P(0)\right)-\partial_{x^k}\mathbf{f}(0)\right] \\
&+\left[\nu\left(\mathbf{b}_k-\mathbf{B}_k(0)\right)\right](x^k-x_0^k)^{-2}+\left[-\nu\,\partial_t\mathbf{B}_k(0)\right](x^k-x_0^k)^{-2}(t-t_0)
\end{aligned} \tag{19}$$

where $\mathbf{e}_k$ are unit vectors arising from use of the gradient. After multiplying eqn.(19) by $(x^k - x_0^k)^2$ and using eqn.(18) to remove the remaining $(t - t_0)$ dependence, then eqn.19) becomes

$$(x^k - x_0^k)^2 \frac{d\,\mathbf{b}_k}{dt} = -\sum_{N=2}^{\infty}\sum_{r=0}^{N}\left[\frac{(N-r-1)^2}{2(2\nu)^{r-1}N!}\binom{N}{r}\partial_{x^k}^{N-r}\partial_t^r\,\mathbf{B}_k(0)\right](x^k - x_0^k)^{N+r}$$

$$+\left[-(2\nu)^{-1}\partial_{x^k}\partial_t\,\mathbf{f}(0)\right](x^k - x_0^k)^4 + \left[-\partial_{x^k}^2\mathbf{f}(0)\right](x^k - x_0^k)^3$$

$$+\begin{bmatrix} -\frac{1}{2}\partial_t\mathbf{B}_k(0) - \partial_{x^k}\mathbf{f}(0) \\ \mathbf{e}_k\left(\partial_{x^k}^2 P(0)\right) + \mathbf{e}_{x^{k+1}}\left(\partial_{x^k}\partial_{x^{k+1}}P(0)\right) + \mathbf{e}_{x^{k+2}}\left(\partial_{x^{k+2}}\partial_{x^k}P(0)\right) \end{bmatrix}(x^k - x_0^k)^2 \qquad (20)$$

$$+\left[\nu\left(\mathbf{b}_k - \mathbf{B}_k(0)\right)\right]$$

The series solution to eqn.(20) proceeds by assuming $\mathbf{b}_k$ is given by

$$\mathbf{b}_k = \mathbf{b}_k(t) = \sum_{N=1}^{\infty}\mathbf{C}_N\,(t - t_0)^{\frac{N}{2}}\exp\left[-Na(t - t_0)^{\frac{1}{2}}\right] \qquad (21\text{-}1)$$

Because of eqn.(18), it is possible to express $\mathbf{b}_k$ by

$$\mathbf{b}_k = \mathbf{b}_k(x^k) = \sum_{N=1}^{\infty} \frac{\mathbf{C}_N}{(2\nu)^{\frac{N}{2}}}(x^k - x_0^k)^N \exp\left[-\frac{Na}{\sqrt{2\nu}}(x^k - x_0^k)\right] \qquad (21\text{-}2)$$

where the $\mathbf{C}_N$ and "a" are constants. Computation of $\dfrac{d\,\mathbf{b}_k}{dt}$ with the use of eqn.(21-1), followed by use of eqn.(18) to express it as a function of $\left(x^k - x_0^k\right)$ and use of eqn.(21-2), gives the following result when substituted into eqn.(20):

$$\sum_{N=1}^{\infty}\left[\frac{N\,\mathbf{C}_N}{2(2\nu)^{\frac{N}{2}-1}}\right]\left(x^k-x_0^k\right)^N \exp\left[-\frac{N\,a}{\sqrt{2\nu}}\left(x^k-x_0^k\right)\right]$$

$$-\sum_{N=1}^{\infty}\left[\frac{N\,a\,\mathbf{C}_N}{2(2\nu)^{\frac{N}{2}-\frac{1}{2}}}\right]\left(x^k-x_0^k\right)^{N+1} \exp\left[-\frac{N\,a}{\sqrt{2\nu}}\left(x^k-x_0^k\right)\right]$$

$$-\sum_{N=1}^{\infty}\left[\frac{\mathbf{C}_N}{2\left(2\nu\right)^{\frac{N}{2}-1}}\right](x^k-x_0^k)^N \exp\left[-\frac{Na}{\sqrt{2\nu}}(x^k-x_0^k)\right]$$

$$+\sum_{N=2}^{\infty}\sum_{r=0}^{N}\left[\frac{\left(N-r-1\right)^2}{2\left(2\nu\right)^{r-1}N!}\binom{N}{r}\partial_{x^k}^{N-r}\,\partial_t^r\,\mathbf{B}_k(0)\right](x^k-x_0^k)^{N+r}$$

$$+\left[(2\nu)^{-1}\partial_{x^k}\partial_t\,\mathbf{f}(0)\right](x^k-x_0^k)^4+\left[\partial_{x^k}^2\mathbf{f}(0)\right](x^k-x_0^k)^3$$

$$+\left[\begin{array}{l}\frac{1}{2}\partial_t\mathbf{B}_k(0)+\partial_{x^k}\mathbf{f}(0)\\ \\ -\mathbf{e}_k\left(\partial_{x^k}^2\,P(0)\right)-\mathbf{e}_{x^{k+1}}\left(\partial_{x^k}\partial_{x^{k+1}}P(0)\right)-\mathbf{e}_{x^{k+2}}\left(\partial_{x^{k+2}}\partial_{x^k}P(0)\right)\end{array}\right](x^k-x_0^k)^2 \qquad (22\text{-}1)$$

$$+\left[\nu\,\mathbf{B}_k(0)\right]$$

$$=\ 0$$

Replacing the exponential function by a Taylor’s series to second order and rearranging gives,

$$-\sum_{N=1}^{\infty}\left[\frac{a^3N^3\mathbf{C}_N}{4(2\nu)^{\frac{N}{2}+\frac{1}{2}}}\right]\left(x^k-x_0^k\right)^{N+3}+\sum_{N=1}^{\infty}\left[\frac{a^2N^2(N+1)\mathbf{C}_N}{4(2\nu)^{\frac{N}{2}}}\right]\left(x^k-x_0^k\right)^{N+2}$$

$$-\sum_{N=1}^{\infty}\left[\frac{a\,N^2\,\mathbf{C}_N}{2(2\nu)^{\frac{N}{2}-\frac{1}{2}}}\right]\left(x^k-x_0^k\right)^{N+1}+\sum_{N=1}^{\infty}\left[\frac{\left(N-1\right)\mathbf{C}_N}{2(2\nu)^{\frac{N}{2}-1}}\right]\left(x^k-x_0^k\right)^{N}$$

$$+\sum_{N=2}^{\infty}\sum_{r=0}^{N}\left[\frac{\left(N-r-1\right)^2}{2\left(2\nu\right)^{r-1}N!}\binom{N}{r}\partial_{x^k}^{N-r}\,\partial_t^r\,\mathbf{B}_k(0)\right](x^k-x_0^k)^{N+r}$$

$$+\left[(2\nu)^{-1}\partial_{x^k}\partial_t\,\mathbf{f}(0)\right](x^k-x_0^k)^4+\left[\partial_{x^k}^2\mathbf{f}(0)\right](x^k-x_0^k)^3$$

$$+\left[\begin{array}{l}\frac{1}{2}\partial_t\mathbf{B}_k(0)+\partial_{x^k}\mathbf{f}(0)\\ \\ -\mathbf{e}_k\left(\partial_{x^k}^2\,P(0)\right)-\mathbf{e}_{x^{k+1}}\left(\partial_{x^k}\partial_{x^{k+1}}P(0)\right)-\mathbf{e}_{x^{k+2}}\left(\partial_{x^{k+2}}\partial_{x^k}P(0)\right)\end{array}\right](x^k-x_0^k)^2 \qquad (22\text{-}2)$$

$$+\left[\nu\,\mathbf{B}_k(0)\right]$$

$$=0$$

Combining terms gives

$$\sum_{N=2}^{\infty}\sum_{r=0}^{N}\left[\frac{\left(N-r-1\right)^2}{2\left(2\nu\right)^{r-1}N!}\binom{N}{r}\partial_{x^k}^{N-r}\partial_t^r\,\mathbf{B}_k(0)\right](x^k - x_0^k)^{N+r}$$

$$+\sum_{N=4}^{\infty}\left(\frac{1}{4(2\nu)^{\frac{N}{2}-1}}\right)\left[\begin{matrix}-a^3\left(N-3\right)^3\mathbf{C}_{N-3}+a^2\left(N-2\right)^2\left(N-1\right)\mathbf{C}_{N-2}\\ \\ -2a\left(N-1\right)^2\mathbf{C}_{N-1}+2\left(N-1\right)\mathbf{C}_N\end{matrix}\right]\left(x^k-x_0^k\right)^N$$

$$+\left[(2\nu)^{-1}\partial_{x^k}\partial_t\,\mathbf{f}(0)\right](x^k-x_0^k)^4$$

$$+\left[\partial_{x^k}^2\mathbf{f}(0)+\left(\frac{a^2}{2\sqrt{2\nu}}\right)\mathbf{C}_1-\left(\frac{2a}{\sqrt{2\nu}}\right)C_2+\left(\frac{1}{\sqrt{2\nu}}\right)\mathbf{C}_3\right](x^k-x_0^k)^3$$

$$+\left[\begin{matrix}-\left(\frac{a}{2}\right)\mathbf{C}_1+\left(\frac{1}{2}\right)\mathbf{C}_2+\frac{1}{2}\partial_t\mathbf{B}_k(0)+\partial_{x^k}\mathbf{f}(0)\\ \\ -\mathbf{e}_k\left(\partial_{x^k}^2P(0)\right)-\mathbf{e}_{x^{k+1}}\left(\partial_{x^k}\partial_{x^{k+1}}P(0)\right)-\mathbf{e}_{x^{k+2}}\left(\partial_{x^{k+2}}\partial_{x^k}P(0)\right)\end{matrix}\right](x^k-x_0^k)^2$$

(22-3)

$$+\left[\nu\,\mathbf{B}_k(0)\right]$$

$$=0$$

Changing $\left(x^k-x_0^k\right)^{N+r}$ to $\left(x^k-x_0^k\right)^N$ gives

$$\sum_{N=2}^{\infty}\sum_{r=0}^{r\le N/2}\mathbf{A}_{N-r,r}\left(x^k-x_0^k\right)^N-\left[\frac{\partial_t\,\mathbf{B}_k(0)}{2}\right]\left(x^k-x_0^k\right)^2$$

(23-1)

$$=\sum_{N=2}^{\infty}\sum_{r=0}^{N}\left[\frac{\left(N-r-1\right)^2}{2\left(2\nu\right)^{r-1}r!\left(N-r\right)!}\partial_{x^k}^{N-r}\partial_t^r\,\mathbf{B}_k(0)\right](x^k-x_0^k)^{N+r}$$

where $\mathbf{A}_{N-r,\,r} = \left[\dfrac{\left(N-2r-1\right)^2}{2\left(2\nu\right)^{r-1} r!\left(N-2r\right)!} \partial_{x^k}^{N-2r}\, \partial_t^r\, \mathbf{B}_k(0) \right]$, for $N-2r \geq 0$ (23-2)

and where the term $\left[\dfrac{\partial_t \mathbf{B}_k(0)}{2}\right]\left(x^k - x_0^k\right)^2$ is generated by the sums $\sum_{N=2}^{\infty}\sum_{r=0}^{r\leq N/2}$ but is not generated by the sums $\sum_{N=2}^{\infty}\sum_{r=0}^{N}$, hence it is subtracted; there are no other terms of this type. The meaning of the limit $r_{\max} \leq \dfrac{N}{2}$ is illustrated as follows: if $N = \text{odd number, say } N=3,$ then $r_{\max} = 1$ since $r_{\max} = 2$ would contradict $r_{\max} \leq \dfrac{N}{2}$. If $N = \text{even number, say } N=4$, then $r_{\max} = 2$ since $r_{\max} = 3$ would contradict $r_{\max} \leq \dfrac{N}{2}$.

This new sum is expanded to a sum starting at $N=4$ giving

$$\sum_{N=4}^{\infty}\sum_{r=0}^{r\leq N/2} \mathbf{A}_{N-r,r}\left(x^k - x_0^k\right)^N - \left(\frac{\partial_t \mathbf{B}_k(0)}{2}\right)\left(x^k - x_0^k\right)^2$$

$$+\left[\mathbf{A}_{20} + \left(\frac{\partial_t \mathbf{B}_k(0)}{2}\right)\right]\left(x^k - x_0^k\right)^2 + \left[\mathbf{A}_{30} + \mathbf{A}_{21}\right]\left(x^k - x_0^k\right)^3 \tag{23-3}$$

$$= \sum_{N=2}^{\infty}\sum_{r=0}^{N}\left[\frac{\left(N-r-1\right)^2}{2\left(2\nu\right)^{r-1} r!\left(N-r\right)!}\partial_{x^k}^{N-r}\,\partial_t^r\,\mathbf{B}_k(0)\right](x^k - x_0^k)^{N+r}$$

The terms with like coefficients can now be combined, giving

$$\sum_{N=4}^{\infty}\left(\frac{1}{4(2\nu)^{\frac{N}{2}-1}}\right)\left[\begin{array}{l}\displaystyle\sum_{r=0}^{r\le N/2}\frac{2\left(N-2r-1\right)^{2}}{\left(2\nu\right)^{r-\frac{N}{2}}r!\left(N-2r\right)!}\partial_{x^k}^{N-2r}\partial_t^{r}\mathbf{B}_k(0)\\ -a^{3}\left(N-3\right)^{3}\mathbf{C}_{N-3}+a^{2}\left(N-2\right)^{2}\left(N-1\right)\mathbf{C}_{N-2}\\ -2a\left(N-1\right)^{2}\mathbf{C}_{N-1}+2\left(N-1\right)\mathbf{C}_{N}\end{array}\right]\left(x^k-x_0^k\right)^N$$

$$+\left[(2\nu)^{-1}\partial_{x^k}\partial_t\mathbf{f}(0)\right](x^k-x_0^k)^4$$

$$+\left[\left(\frac{a^2}{2\sqrt{2\nu}}\right)\mathbf{C}_1-\left(\frac{2a}{\sqrt{2\nu}}\right)C_2+\left(\frac{1}{\sqrt{2\nu}}\right)\mathbf{C}_3+\mathbf{A}_{30}+\mathbf{A}_{21}+\partial_{x^k}^2\mathbf{f}(0)+\right](x^k-x_0^k)^3$$

$$+\left[\begin{array}{l}-\left(\frac{a}{2}\right)\mathbf{C}_1+\left(\frac{1}{2}\right)\mathbf{C}_2+\mathbf{A}_{20}+\frac{1}{2}\partial_t\mathbf{B}_k(0)+\partial_{x^k}\mathbf{f}(0)\\ -\mathbf{e}_k\left(\partial_{x^k}^2P(0)\right)-\mathbf{e}_{x^{k+1}}\left(\partial_{x^k}\partial_{x^{k+1}}P(0)\right)-\mathbf{e}_{x^{k+2}}\left(\partial_{x^{k+2}}\partial_{x^k}P(0)\right)\end{array}\right](x^k-x_0^k)^2 \qquad (24\text{-}1)$$

$$+\left[\nu\,\mathbf{B}_k(0)\right]$$

$$=0$$

The sum is expanded to begin at $N = 5$ in order to generate and group coefficients of $(x^k - x_0^k)^4$, hence eqn.(24-1) becomes

$$\sum_{N=5}^{\infty}\left(\frac{1}{4(2\nu)^{\frac{N}{2}-1}}\right)\left[\begin{array}{l}\displaystyle\sum_{r=0}^{r\le N/2}\frac{2\left(N-2r-1\right)^2}{\left(2\nu\right)^{r-\frac{N}{2}}r!\left(N-2r\right)!}\partial_{x^k}^{N-2r}\partial_t^r\mathbf{B}_k(0)\\ -a^3\left(N-3\right)^3\mathbf{C}_{N-3}+a^2\left(N-2\right)^2(N-1)\mathbf{C}_{N-2}\\ -2a\left(N-1\right)^2\mathbf{C}_{N-1}+2\left(N-1\right)\mathbf{C}_N\end{array}\right]\left(x^k-x_0^k\right)^N$$

$$+\left[\begin{array}{l}\left(\frac{-a^3\nu^{-1}}{8}\right)\mathbf{C}_1+\left(\frac{3a^2\nu^{-1}}{2}\right)\mathbf{C}_2+\left(\frac{-9a\nu^{-1}}{4}\right)\mathbf{C}_3+\left(\frac{3\nu^{-1}}{4}\right)\mathbf{C}_4\\ +(2\nu)^{-1}\partial_{x^k}\partial_t\mathbf{f}(0)+\frac{3}{8}\nu\partial_{x^k}^4\mathbf{B}_k(0)\\ +\frac{1}{4}\partial_{x^k}^2\partial_t\mathbf{B}_k(0)+\frac{1}{4}(2\nu)^{-1}\partial_t^2\mathbf{B}_k(0)\end{array}\right](x^k-x_0^k)^4$$

$$+\left[\left(\frac{a^2}{2\sqrt{2\nu}}\right)\mathbf{C}_1-\left(\frac{2a}{\sqrt{2\nu}}\right)C_2+\left(\frac{1}{\sqrt{2\nu}}\right)\mathbf{C}_3+\mathbf{A}_{30}+\mathbf{A}_{21}+\partial_{x^k}^2\mathbf{f}(0)+\right](x^k-x_0^k)^3$$

$$+\left[\begin{array}{l}-\left(\frac{a}{2}\right)\mathbf{C}_1+\left(\frac{1}{2}\right)\mathbf{C}_2+\mathbf{A}_{20}+\frac{1}{2}\partial_t\mathbf{B}_k(0)+\partial_{x^k}\mathbf{f}(0)\\ -\mathbf{e}_k\left(\partial_{x^k}^2P(0)\right)-\mathbf{e}_{x^{k+1}}\left(\partial_{x^k}\partial_{x^{k+1}}P(0)\right)-\mathbf{e}_{x^{k+2}}\left(\partial_{x^{k+2}}\partial_{x^k}P(0)\right)\end{array}\right](x^k-x_0^k)^2$$

$$+\left[\nu\mathbf{B}_k(0)\right]$$

$$=0 \qquad (24\text{-}2)$$

Evaluating the $\mathcal{A}_{N-r,r}$ values in eqn.(24-2) gives

$$\sum_{N=5}^{\infty}\left(\frac{1}{4(2\nu)^{\frac{N}{2}-1}}\right)\begin{bmatrix}\displaystyle\sum_{r=0}^{r\le N/2}\frac{2\left(N-2r-1\right)^2}{\left(2\nu\right)^{r-\frac{N}{2}}r!\left(N-2r\right)!}\partial_{x^k}^{N-2r}\partial_t^r\,\mathbf{B}_k(0)\\ -a^3\left(N-3\right)^3\mathbf{C}_{N-3}+a^2\left(N-2\right)^2(N-1)\mathbf{C}_{N-2}\\ -2a\left(N-1\right)^2\mathbf{C}_{N-1}+2\left(N-1\right)\mathbf{C}_N\end{bmatrix}\left(x^k-x_0^k\right)^N$$

$$+\begin{bmatrix}\left(\frac{-a^3\nu^{-1}}{8}\right)\mathbf{C}_1+\left(\frac{3a^2\nu^{-1}}{2}\right)\mathbf{C}_2+\left(\frac{-9a\,\nu^{-1}}{4}\right)\mathbf{C}_3+\left(\frac{3\nu^{-1}}{4}\right)\mathbf{C}_4\\ +(2\nu)^{-1}\partial_{x^k}\partial_t\,\mathbf{f}(0)+\frac{3}{8}\nu\,\partial_{x^k}^4\,\mathbf{B}_k(0)\\ +\frac{1}{4}\partial_{x^k}^2\partial_t\,\mathbf{B}_k(0)+\frac{1}{4}(2\nu)^{-1}\partial_t^2\,\mathbf{B}_k(0)\end{bmatrix}(x^k-x_0^k)^4$$

$$+\begin{bmatrix}\left(\frac{a^2}{2\sqrt{2\nu}}\right)\mathbf{C}_1-\left(\frac{2a}{\sqrt{2\nu}}\right)C_2+\left(\frac{1}{\sqrt{2\nu}}\right)\mathbf{C}_3\\ +\frac{2}{3}\nu\,\partial_{x^k}^3\,\mathbf{B}_k(0)+0+\partial_{x^k}^2\mathbf{f}(0)\end{bmatrix}(x^k-x_0^k)^3$$

$$+\begin{bmatrix}-\left(\frac{a}{2}\right)\mathbf{C}_1+\left(\frac{1}{2}\right)\mathbf{C}_2+\frac{1}{2}\nu\,\partial_{x^k}^2\mathbf{B}_k(0)+\frac{1}{2}\partial_t\,\mathbf{B}_k(0)+\partial_{x^k}\mathbf{f}(0)\\ -\mathbf{e}_k\left(\partial_{x^k}^2P(0)\right)-\mathbf{e}_{x^{k+1}}\left(\partial_{x^k}\partial_{x^{k+1}}P(0)\right)-\mathbf{e}_{x^{k+2}}\left(\partial_{x^{k+2}}\partial_{x^k}P(0)\right)\end{bmatrix}(x^k-x_0^k)^2$$

$$+\left[\nu\,\mathbf{B}_k(0)\right] \tag{24-3}$$

$$=0$$

The right side of eqn.(24-3) is zero only if the coefficients of the individual powers of $(x^k - x_0^k)$ are zero; hence,

$$\mathbf{B}_k(0) = 0 \tag{25-1}$$

$$\mathbf{C}_2 = \begin{bmatrix} a\,\mathbf{C}_1 - \left[\nu\,\partial^2_{x^k}\mathbf{B}_k(0) + \partial_t \mathbf{B}_k(0)\right] \\ -\left[2\,\partial_{x^k}\mathbf{f}(0)\right] \\ +2\left[\mathbf{e}_k\left(\partial^2_{x^k} P(0)\right) + \mathbf{e}_{x^{k+1}}\left(\partial_{x^k}\partial_{x^{k+1}}P(0)\right) + \mathbf{e}_{x^{k+2}}\left(\partial_{x^{k+2}}\partial_{x^k}P(0)\right)\right] \end{bmatrix} \tag{25-2}$$

$$\mathbf{C}_3 = \left(-\frac{a^2}{2}\right)\mathbf{C}_1 + (2a)\,\mathbf{C}_2 - \frac{(2\nu)^{\frac{3}{2}}}{3}\partial^3_{x^k}\mathbf{B}_k(0) - \sqrt{2\nu}\,\partial^2_{x^k}\mathbf{f}(0)$$

$$= \begin{bmatrix} \left(\frac{3a^2}{2}\right)\mathbf{C}_1 - \left[2a\nu\,\partial^2_{x^k}\mathbf{B}(0) + \frac{(2\nu)^{\frac{3}{2}}}{3}\partial^3_{x^k}\mathbf{B}_k(0) + 2a\,\partial_t\mathbf{B}(\mathbf{0})\right] \\ -\left[4a\,\partial_{x^k}\mathbf{f}(0) + \sqrt{2\nu}\,\partial^2_{x^k}\mathbf{f}(0)\right] \\ +4a\left[\mathbf{e}_k\left(\partial^2_{x^k} P(0)\right) + \mathbf{e}_{x^{k+1}}\left(\partial_{x^k}\partial_{x^{k+1}}P(0)\right) + \mathbf{e}_{x^{k+2}}\left(\partial_{x^{k+2}}\partial_{x^k}P(0)\right)\right] \end{bmatrix} \tag{25-3}$$

$$\mathbf{C}_4=\left(\frac{a^3}{6}\right)\mathbf{C}_1-\left(2a^2\right)\mathbf{C}_2+\left(3a\right)\mathbf{C}_3$$

$$+\left[-\frac{1}{2}\nu^2\,\partial^4_{x^k}\mathbf{B}_k(0)-\frac{1}{3}\nu\,\partial^2_{x^k}\partial_t\mathbf{B}_k(0)-\frac{1}{6}\partial^2_t\,\mathbf{B}_k(0)-\frac{2}{3}\partial_{x^k}\partial_t\,\mathbf{f}\left(0\right)\right]$$

$$=\left[\begin{array}{l}\left(\frac{8a^3}{3}\right)\mathbf{C}_1-\left[\begin{array}{l}2a^2\left(2\nu\right)\partial^2_{x^k}\,\mathbf{B}(0)+a\left(2\nu\right)^{\frac{3}{2}}\partial^3_{x^k}\,\mathbf{B}_k(0)+\frac{1}{8}\left(2\nu\right)^2\partial^4_{x^k}\mathbf{B}_k(0)\\+\,4a^2\,\partial_t\,\mathbf{B}(\mathbf{0})+\frac{1}{6}\partial^2_t\,\mathbf{B}_k(0)+\frac{1}{3}\nu\,\partial^2_{x^k}\partial_t\mathbf{B}_k(0)\end{array}\right]\\-\left[8a^2\,\partial_{x^k}\mathbf{f}(0)+3a\left(2\nu\right)^{\frac{1}{2}}\partial^2_{x^k}\mathbf{f}(0)+\frac{2}{3}\partial_{x^k}\partial_t\,\mathbf{f}\left(0\right)\right]\\+\,8a^2\left[\mathbf{e}_k\left(\partial^2_{x^k}\,P(0)\right)+\mathbf{e}_{x^{k+1}}\left(\partial_{x^k}\partial_{x^{k+1}}P(0)\right)+\mathbf{e}_{x^{k+2}}\left(\partial_{x^{k+2}}\partial_{x^k}P(0)\right)\right]\end{array}\right] \qquad (25\text{-}4)$$

$$\mathbf{C}_N=a^3\,\frac{\left(N-3\right)^3}{2\left(N-1\right)}\mathbf{C}_{N-3}-a^2\,\frac{\left(N-2\right)^2}{2}\mathbf{C}_{N-2}+a\left(N-1\right)\mathbf{C}_{N-1}$$

$$-\sum_{r=0}^{r\le N/2}\frac{\left(N-2r-1\right)^2}{\left(N-1\right)\left(2\nu\right)^{r-\frac{N}{2}}\,r!\left(N-2r\right)!}\partial^{N-2r}_{x^k}\,\partial^r_t\,\mathbf{B}_k(0) \qquad ;N\ge 5 \qquad (25\text{-}5)$$

Eqns.(25-1) through (25-5) (recursion formula for $N\geq 5$) can be used to compute all constants relative to the value of $\mathbf{C}_1$, but do not provide an explicit calculation of $\mathbf{C}_1$. These constants are functions of constant coefficients of the type $\partial^{\alpha}_{x^k}\,\partial^{\beta}_t\,\mathbf{B}_k(0)$, $\partial^{\alpha}_{x^k}\,\partial^{\beta}_t\,\mathbf{f}(0)$ and $\partial^{\alpha}_{x^k}\,\partial^{\beta}_t\,P(0)$. Referring back to the analysis in the paragraph before eqn.(12-2), namely, $\mathbf{B}_k = \mathbf{B}_k(x^k, t) = \mathbf{B}_k(x^k(t), t) = \mathbf{b}_k(t)$, it is seen that eqn.(25-1) states that at the initial conditions defined by $\mathbf{b}_k(t_0) = 0$, $x^k = x_0^k$, $t = t_0$, the gradient of the velocity is zero.

3.3 Analysis of Solution

The present solution to the Navier-Stokes equations is based upon transformation of the Navier-Stokes dynamic equation(eqn.(1)) to Hamiltonian form. Hence, the appropriate space for characterizing the solution to this equation is the extended cotangent space with coordinates $\left(\mathbf{b}_k, x^k, t\right)$ rather than the extended tangent space with coordinates$\left(\mathbf{v}, x^k, t\right)$. This procedure leads to a description of Navier-Stokes dynamics by a pair of equations analogous to Hamiltonian's equations and a characteristic tangent vector(the vortex vector) to define the direction of system change. The solutions to these equations are x^k and $\mathbf{b}_k$, as given in eqns.(18 and 21-1).

This analysis is an analysis of $\mathbf{b}_k$ when the gradient of the velocity $\mathbf{B}_k$, the externally applied force $\mathbf{f}$ and the pressure P are expanded according to Taylor's theorem. $\mathbf{B}_k$, $\mathbf{f}$ and P are assumed to be smooth C^∞ functions on $\mathbb{R}^n \mathrm{X}\ (0,\infty]$, although $\mathbf{f}$ and P are approximated by an expansion only to second order. The constants in the series expansion of $\mathbf{b}_k$ are considered as parameters which can be determined by experiment, thereby giving a technique for obtaining $\mathbf{C}_N$ and some of the Taylor expansion coefficients for $\mathbf{B}_k$, $\mathbf{f}$ and P. This technique is employed for measuring characteristic interaction constants of atoms by some of the most precise quantum mechanical methods, e.g., the use of coupling constants for interaction energy terms as parameters in the analysis of frequency standard experiments using atomic beam magnetic resonance spectroscopy.

3.3.1 Initial conditions

One part of the solution (eqn.(18)) to the Navier-Stokes dynamic equation(1) gave the explicit functional dependence of x^k on t. This dependence, $x^k = x_0^k \pm \sqrt{2\nu(t-t_0)}$, shows that if $t = t_0$ then $x^k = x_0^k$ and so eqns.(21-1 and 21-2) give $\mathbf{b}_k = 0$. It is noted that since the appropriate variables for cotangent space are $\left(\mathbf{b}_k, x^k, t\right)$ rather than $\left(\mathbf{v}, x^k, t\right)$, the initial conditions are $\mathbf{b}_k = 0$, $x^k = x_0^k$, $t = t_0$.

3.3.2 The solution

(a) $\mathbf{B}_k$, f and P as smooth functions on $\mathbb{R}^n$ X [0, ∞)

The solution to eqn.(1) depends on the existence of smooth functions $\mathbf{B}_k$, **f,** and P such that Taylor's expansion theorem can be used; hence the solution depends on these functions being C^∞, although **f** and P are expanded only to second - order. It will be shown in **3.3.2(b)** that when eqn.(21-2) is used, the solution $\mathbf{b}_k$ is also C^∞. All these functions are real and belong to $\mathbb{R}^n$ X [0, ∞).

(b) Behavior of $\mathbf{b}_k$ as $\left|x^k\right| \to \infty$.

Eqn.(18), which is the solution to one of the set of differential equations (eqn.(17-1)), was employed in subsequent equations whenever it was useful to express particular equations as functions of $x^k - x_0^k$ rather than $t - t_0$. This knowledge was used to obtain

$\mathbf{b}_k = \mathbf{b}_k\left(x^k\right)$. Using this form of $\mathbf{b}_k$ and for arbitrary α, the α-th derivative of $\mathbf{b}_k$ is

$$\partial^{\alpha}_{x^k}\mathbf{b}_k = \sum_{N=1}^{\infty}\sum_{m=0}^{\alpha} \beta_{N m \alpha}\mathbf{C}_N\left(x^k - x^k_0\right)^{N-(\alpha-m)}\exp\left[\frac{-aN}{\sqrt{2\nu}}\left(x^k - x^k_0\right)\right] \tag{26}$$

where $$\beta_{N m \alpha} = \frac{1}{\sqrt{2\nu}}\left(\frac{-aN}{\sqrt{2\nu}}\right)^m \binom{\alpha}{m}\left(\frac{N!}{\left[N-(\alpha-m)\right]!}\right) \tag{27}$$

and where $N \geq \alpha - m$. At $t = t_0$, note that $\partial^{\alpha}_{x^k}\mathbf{b}_k\left(x^k_0\right) = 0$ since, eqn.(18) implies $x^k = x^k_0$ at $t = t_0$. This analysis shows $\lim_{\left|x^k\right|\to\infty}\left|\partial^{\alpha}_{x^k}\mathbf{b}_k\left(x^k_0\right)\right| = 0$ for any α. Hence $\mathbf{b}_k$ will not grow large as $\left|x^k\right| \to \infty$.

The behavior of $\mathbf{b}_k$ as $\left|x^k\right| \to \infty$ can also be examined directly from eqn.(21-2). As $\left|x^k\right| \to \infty$, it is noted that $\dfrac{1}{\exp\left[\dfrac{Na}{\sqrt{2\nu}}(x^k - x^k_0)\right]} \to 0$ faster than $(x^k - x^k_0)^N \to \infty$, therefore $\mathbf{b}_k \to 0$; the solution does not blow-up.

(c) Bounded energy.

Since the motion of the system occurs in cotangent space rather than tangent space, evaluation of the following integral will show that $\left|\mathbf{b}_k\right|^2$ is bounded:

$$\int_{\mathbb{R}^n} \left|\mathbf{b}_k\right|^2 dx^k < C \quad \text{for all } t \geq t_0 \text{ and } C < \infty. \tag{28-1}$$

Evaluation of this integral gives

$$\int_0^\infty |\mathbf{b}_k|^2 dx^k = \sum_{M=1}^{\infty}\sum_{N=1}^{\infty} \left|\frac{\sqrt{2\nu}\,\mathbf{C}_M\mathbf{C}_N}{a^{M+N+1}}\right| \left|\left[\frac{(M+N)!}{(M+N)^{M+N+1}}\right]\right| \quad \text{for all } t \ge t_0$$

(28-2)

$$= K \quad \text{, where} \quad 0 \le K \le \frac{\sqrt{2\nu}\,\mathbf{C}_1^2}{4a^3}$$

Hence eqn.(28-2) implies the function $|\mathbf{b}_k|^2$ is bounded.

A physically reasonable solution has a bounded energy in field-free space when

$$\int_{\mathbb{R}^n} |\mathbf{p}|^2 dx < \text{Constant, for all } t \ge 0 \tag{28-3}$$

since in this case, the energy is proportional to the square of the momentum $|\mathbf{p}|^2$. The solution $\mathbf{b}_k$ (the gradient of **S)** can be used as the integrand in eqn.(28-3) in place of the momentum (the gradient of the action) for proof of a physically reasonable solution. This is based on the fact that both principal functions(**S** and the action) can be represented by a family of surfaces and the gradient of the principal function is always perpendicular to any surface at a point; the larger the gradient, the slower the front. When the gradient of the principal function is a function of time $(\mathbf{b}_k(t) \text{ or } \mathbf{p}_k(t))$ it characterizes the motion in field-free space; hence, the square of the gradient of the principle function is proportional to the kinetic energy. Therefore, eqn.(28-2) shows the solution is physically reasonable.

(d) Graphing the solution

The solution $\mathbf{b}_k$ contains constants $\mathbf{C}_N$ and "a" and hence cannot be graphed without knowledge of these constants. Quantity "a" is merely a unit constant present to make the argument of the exponential dimensionless; hence, its value $1\sec^{-1/2}$. The constants $\mathbf{C}_N$ are functions of the constant coefficients $\partial_{x^k}^{\alpha}\,\partial_t^{\beta}\,\mathbf{B}_k(0)$, $\partial_{x^k}^{\alpha}\,\partial_t^{\beta}\,\mathbf{f}(0)$ and $\partial_{x^k}^{\alpha}\,\partial_t^{\beta}\,P(0)$. The procedure to obtain the expansion coefficients is to treat them as parameters and determine them experimentally. This involves fitting the experimental data with the use of these parameters, then designating these evaluated parameters as the characteristic constants for the system. This was suggested earlier as a commonly used technique for precise quantum mechanical measurements, for example the frequency standard work on cesium by means of atomic beam magnetic resonance spectroscopy, where hyperfine structure constants are treated as parameters.

(e) Solution for $\mathbf{b}_k$ when f = 0

By setting the external force $\mathbf{f} = 0$, $\mathbf{b}_k$ then depends on the expansion coefficients $\partial_{x^k}^{\alpha}\,\partial_t^{\beta}\,\mathbf{B}_k(0)$ and $\partial_{x^k}^{\alpha}\,\partial_t^{\beta}\,P(0)$. By this procedure it is possible to eliminate some of parameters required to fit experimental data and hence allow a first approximation for determination of some of the required coefficients.

3.3.3 Incompressibility.

Eqn.(2) is the condition for the velocity vector field $\mathbf{v}$ to be divergence-free. If ∂_{x^k} is taken on each side of eqn.(2), this equation becomes $div\ \mathbf{B}_k = 0$. In differential geometry the divergence of a vector field on an oriented Euclidean space is the density in the expression for the 3-form on that space. Extending this definition to higher dimensions, the divergence of vector field $\mathbf{B}_k$ on the oriented cotangent space T^*M_x is the density in the expression for the 3-form on T^*M_x, given by

$$\boldsymbol{\omega}^3 = \left(div \;\; \mathbf{B}_k\right) \mathbf{db}_k \wedge \mathbf{d}x^k \wedge \mathbf{d}t \tag{29}$$

where $\boldsymbol{\omega}^3$ characterizes the sources in an elementary parallelepiped with edges $\left(\varepsilon\xi_\alpha, \varepsilon\xi_\beta, \varepsilon\xi_\kappa\right)$ and tangent vectors $\xi \in T\left(T*M\right)$, and where $\mathbf{db}_k$, $\mathbf{d}x^k$ and $\mathbf{d}t$ are basis differential one-forms for cotangent space $T*M$ at point $\left(x^1,\ldots,x^n\right)$ of $M \subset \mathbb{R}^n$. In order for $div \;\; \mathbf{B}_k = 0$, then $\boldsymbol{\omega}^3\left(\xi_\alpha, \xi_\beta, \xi_\kappa\right) = 0$. For tangent vector ξ_α

$$\xi_\alpha = \dot{\mathbf{b}}_k \; \partial_{\mathbf{b}_k} + \dot{x}^k \partial_{x^k} + \partial_t \tag{30-1}$$

and arbitrary tangent vectors (also belonging to $T\left(T*M\right)$)

$$\xi_\beta = \beta_{\mathbf{b}_k} \dot{\mathbf{b}}_k \; \partial_{\mathbf{b}_k} + \beta_{x^k} \dot{x}^k \partial_{x^k} + \partial_t \tag{30-2}$$

and

$$\xi_\kappa = \kappa_{\mathbf{b}_k} \dot{\mathbf{b}}_k \; \partial_{\mathbf{b}_k} + \kappa_{x^k} \dot{x}^k \partial_{x^k} + \partial_t \tag{30-3}$$

it results that $\boldsymbol{\omega}^3\left(\xi_\alpha, \xi_\beta, \xi_\kappa\right) = 0$ only if

$$\beta_{x^{(1)}}\left(\kappa_{x^{(2)}} - \kappa_{x^{(3)}}\right) + \beta_{x^{(2)}}\left(\kappa_{x^{(3)}} - \kappa_{x^{(1)}}\right) + \beta_{x^{(3)}}\left(\kappa_{x^{(1)}} - \kappa_{x^{(2)}}\right) = 0 \tag{30-4 }$$

where $\left(\partial_{\mathbf{b}_k}, \partial_{x^k}, \partial_t\right)$ are basis tangent vectors for tangent space $T(T*M_x)$. The condition on β_{x^k} and κ_{x^k} given in eqn.(30-4) implies that the vectors ξ_β and ξ_κ are not entirely arbitrary; the condition distorts the parallelepiped $\left(\xi_{\alpha},\xi_{\beta},\xi_{\kappa}\right)$ to allow the gradient of **v** to be divergence-free.

This condition on β_{x^k} and κ_{x^k} is strictly true from a mathematical point of view, but involves assumptions which have not been adequately studied in terms of physical reasonableness. However, if the volume of this parallelepiped is in the same region of space in which the motion of the system occurs, then the requirements of eqn.(2) are fulfilled.

3.3.4 Vortex vector

The vortex vector **R ,** the vector which gives the direction of the system change, is obtained by noting that the coordinate values for the coordinates of the tangent vector ξ_{α} are given by eqns.(16). By substituting these values into eqn.(30-1), it results that

$$\begin{aligned} \mathbf{R} &= -\left(\frac{\partial\Omega}{\partial x^k}\right)\partial_{\mathbf{b}_k} + \left(\frac{\partial\Omega}{\partial \mathbf{b}_k}\right)\partial_{x^k} + \partial_t \\ &= -\left(\frac{\partial\Omega}{\partial x^k}\right)\partial_{\mathbf{b}_k} + \left(\frac{\nu}{x^k - x_0^k}\right)\partial_{x^k} + \partial_t \end{aligned} \qquad (31)$$

This form of the vortex vector can be made more detailed by means of eqn.(17-2), since

$$\frac{-\partial\Omega}{\partial x^k} = \frac{d\,\mathbf{b}_k}{dt} \quad .$$

To obtain the Lagrangian for the system the fundamental differential one-form $\mathbf{dS}$ is contracted with the vortex vector giving

$$\mathbf{dS}(\mathbf{R}) = \mathbf{b}_k \left(\frac{\partial \Omega}{\partial \mathbf{b}_k} \right) - \Omega$$

$$= \mathbf{b}_k \left(\frac{\nu}{x^k - x_0^k} \right) - \Omega \qquad (32)$$

This equation can be made more detailed by substitution for $\mathbf{b}_k$ (eqn.(21-1) and Ω (eqn.(11)). This technique for obtaining the Lagrangian has been demonstrated in reference 2 in Hamiltonian mechanics, geometric optics, irreversible thermodynamics, black hole mechanics, and electromagnetic and string field theory [2].

4. Conclusion

The technique employed in this paper for solving the Navier-Stokes model for fluid dynamics in the case of incompressible fluids was to transform the dynamic equation into a differential one-form, and then use methods from exterior calculus to generate a pair of differential equations and a vortex vector satisfying Hamiltonian geometry. This pair of equations was solved for the position x^k as a function of time and for $\mathbf{b}_k$ (the conjugate to the position) as a function of time.

The value of the solution $\mathbf{b}_k$ as $|x^k| \to \infty$ was shown to be finite, hence the solution is bounded; blow-up does not occur. The solution was shown to be physically reasonable since the gradient of the principal function is bounded. It is not possible to plot the solution without knowledge of some of the constants contained in the solution, but these constants can be treated as parameters and evaluated experimentally. One example of this procedure is the frequency standard work on cesium atom using atomic beam magnetic resonance spectroscopy, where the hyperfine interaction constants are treated as parameters and determined with experimental data. This procedure has led to results which are accurate to better than one part in 10^6.

The gradient was taken on each side of the equation for the divergence of the velocity, resulting in an expression for the divergence-free gradient of the velocity. Then the differential 3-form corresponding to the divergence of the gradient of the velocity was contracted on a triple of tangent vectors and set to zero. As a result, a condition was placed on arbitrary tangent vectors in $T(T^*M_x)$, distorting the volume where the motion of the system occurs in a manner which restricts the gradient of the velocity to be incompressible.

The vortex vector (characteristic tangent vector) giving the direction of the system change was constructed by substituting coordinate values for coordinates of a basic tangent vector in $T(T^*M_x)$. By contracting the characteristic differential one-form defining the system with the vortex vector, the Lagrangian was obtained.

The present solution to the Navier-Stokes equations is based on several assumptions, namely, (1) assuming the gradient of the velocity **v**, the pressure *P*, the force **f**, and the exponential part of the series solution for $\mathbf{b}_k$ are all smooth functions which can be represented by Taylor's expansion theorem, with all but $\mathbf{B}_k$ (infinite order expansion) expanded to second order, (2) assuming the cross terms in $\partial_{x^k}\mathbf{f}$ (see paragraph after eqn.(19-1) can be neglected and (3) assuming a certain condition on the coordinates of two otherwise arbitrary tangent vectors in the tangent space to the cotangent space where the motion of the system occurs.

The proposed series solution satisfies all the requirements for the Navier-Stokes equations for a physically reasonable bounded solution which is divergence-free and which predicts that the gradient of the principal function is bounded. The solution was obtained by (1) representing the dynamic Navier-Stokes equation as a characteristic differential one-form; this form generates a pair of characteristic differential equations for the dynamics and a characteristic tangent vector for the direction of system change, and (2) using the definition of divergence form differential geometry to represent the divergence equation by a volume 3-form, whose contraction on a triple of tangent vectors implies incompressibility when a certain condition is placed on two of the tangent vectors for the system.

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References

1. C. L. Fefferman, *Existence and Smoothness of the Navier-Stokes Equations*, http://www.claymath.org/Millennium_Prize_Problems/Navier-Stokes_Equations/_objects/Official_Problem_Description.pdf

2. V. I. Arnold and B. A. Khesin, *Topological Methods in Hydrodynamics* (Springer, 1998).

3. T. Story, *Dynamics on Differential One-Forms*, J. Math. Chem. **29** (2001) 85-96.