Guide to the Video

Part I and II of the video presentation follow rather closely Chapter 1 /resp. Chapter 2 of the text. In the third part, however, the viewer might find it difficult to locate the details of what is discussed in the main text. Therefore we have provided this guide. It gives precise references to the main text (bold face) and repeats the formulae which were written on the black board for simple identification (underlined).

Part III

We recall the long time behavior of the driven lattice and its dependence on the driving frequency γ, which was described in detail in Part I. The goal of this presentation is to explain our result on the construction of $\frac{2\pi}{\gamma}$ – time periodic solutions of the semi-infinite lattice, (**cf Chapters 3-5**), which are in good agreement with the numerically observed time asymptotic motion of the lattice. More precisely we construct solutions of

$$\underline{\ddot{x}_n = F(x_{n-1} - x_n) - F(x_n - x_{n+1}), \quad n \geq 1}, \tag{1}$$

with

$$\underline{x_0(t) := \epsilon \sum_{m \in \mathbb{Z}} b_m e^{i\gamma m t}} \tag{2}$$

(**cf (3.1), (3.2)**). For $\underline{\epsilon = 0}$ the lattice at rest $\underline{x_n(t) = cn}$ solves (1), (2). We ask whether these solutions persist for $\underline{\epsilon \neq 0}$? Our results can be summarized as follows.

Let $\gamma_k := \frac{2}{k}\sqrt{F'(-c)}, k \in \mathbb{N}$, be the sequence of threshold values for the driving frequency.

- In the case of arbitrary force functions F (**cf Section 3.2.2, General Assumptions**) and for $\underline{\gamma > \gamma_1}$ or $\underline{\gamma_1 > \gamma > \gamma_2}$ there exist $\epsilon_0 > 0$ such that for all $0 \leq \epsilon < \epsilon_0$ we can construct $\frac{2\pi}{\gamma}$ – time periodic solutions of (1), (2), satisfying $x_n(t) - cn = O(\epsilon)$ (**cf Theorems 3.56 and 4.24**).
- In the case of the Toda lattice $\underline{(F(x) = e^x)}$, for all $\gamma \in \mathbb{R}^+ \setminus \{\gamma_k : k \in \mathbb{N}\}$ there exist $\epsilon_0 > 0$ such that for all $0 \leq \epsilon < \epsilon_0$ we can construct $\frac{2\pi}{\gamma}$ – time periodic solutions of (1), (2), satisfying $x_n(t) - cn = O(\epsilon)$ (**cf Theorem 5.13**).

It is instructive to consider the linear case $F(x) = x$. By separation of variables we find that

$$x_n(t) = \epsilon \sum_{m\in\mathbb{Z}} b_m z_m^n e^{i\gamma m t} \tag{3}$$

solves (1), (2) , if

$$z_m^2 + \delta_m z_m + 1 = 0, \text{ with } \delta_m = -2 + (\gamma m)^2. \tag{4}$$

We notice that a mode $z_m^n e^{i\gamma mt}$ corresponds to a travelling wave in the case $|\delta_m| \leq 2$, ($z_m = e^{i\beta_m}$ for $\beta_m \in \mathbb{R}$). For $|\delta_m| > 2$ one can choose $|z_m| < 1$, i.e. the corresponding mode is decaying exponentially in n. (**cf (1.25)**). The solutions we will construct in **Chapters 3-5** will be nonlinear versions of (3).

Our first result in **Chapter 3 (Lemma 3.31 and Theorem 3.38)** will reduce the construction of the desired solutions of (1), (2) to showing that there exists a sufficiently large parameter family of travelling wave solutions of the doubly infinite lattice. More precisely, in the case that the driving frequency satisfies $\gamma_k > \gamma > \gamma_{k+1}$, we have to construct a $2k$ – parameter family of k-phase waves. This reduction is proved by separating the equations for the "travelling wave part" and the "exponentially decaying part" (**cf Lemma 3.31**) and then by solving for the exponentially decaying part (**cf Theorem 3.38**) (see also **Section 3.1**).

In the case of the Toda lattice the multi-phase travelling waves can be constructed for arbitrary number of phases in terms of theta functions, using the well known g-gap solutions. Due to limited time these solutions are not discussed any further in this presentation. See **Chapter 5**, **Appendix A** and **Appendix B** for more detail.

We will spend the remaining time to show how one can obtain single – phase waves in the case of arbitrary force functions F and what the difficulties are in the construction of multi-phase waves (in particular two-phase waves), which we have not yet overcome. By a two-phase wave we denote a solution of the doubly infinite lattice of the form

$$x_n(t) = cn + \chi_2(n\beta_1 + \gamma t, n\beta_2 + 2\gamma t), \tag{5}$$

where χ_2 is 2π-periodic in both arguments. Functions of this form are $\frac{2\pi}{\gamma}$ – time periodic. The spatial frequencies β_1, β_2 turn out to be approximately given by the dispersion relation for the linearized lattice,

$$2\cos\beta_k = 2 - \frac{(\gamma k)^2}{F'(-c)}. \tag{6}$$

Remarks:

(1) This formula was given incorrectly in the video presentation.

(2) As mentioned above, two-phase waves are considered in the case $\gamma_2 > \gamma > \gamma_3$, which means that there exist real solutions β_k of (6) only in the case $|k| \leq 2$.

We first dicuss the construction of single-phase waves (**see Chapter 4**). Expand

$$x_n(t) = cn + \chi_1(n\beta + \gamma t) \tag{7}$$

in a Fourier series

$$\underline{\chi_1(\theta) = \sum_{m \in \mathbb{Z}} r(m) e^{im\theta}.} \tag{8}$$

The resulting equation for the sequence r can be written in the form

$$\underline{\Lambda(\beta) r + W(r) = 0,} \tag{9}$$

where $\Lambda(\beta)$ is a linear operator depending on β and $W(r)$ denotes the nonlinear term. The operator $\Lambda(\beta)$ is diagonal and its entries are given by

$$\begin{aligned} \Lambda(\beta)(m,m) &= 2\cos\beta m - 2 + \frac{(\gamma m)^2}{F'(-c)} \\ &= -4\sin^2\frac{\beta m}{2} + \frac{(\gamma m)^2}{F'(-c)} \end{aligned} \tag{10}$$

(See **Lemma 4.10** for a derivation of (9), (10)).

Remark:

In the video presentation we have given a different formula for $\Lambda(\beta)$ which can be obtained by dividing (9) by $-4\sin^2\frac{\beta m}{2}$, which then represents the equation for the sequence $\tilde{r}(m) := r(m)(e^{-i\beta m} - 1)$. Furthermore we have assumed $F'(-c) = 1$, which can always be achieved by rescaling time. Therefore we obtain

$$\underline{\Lambda(\beta)(m,m) = 1 - \frac{(\gamma m)^2}{4\sin^2\frac{\beta m}{2}}.} \tag{11}$$

In the video we proceed to give a brief description of the Lyapunov – Schmidt decomposition which is explained in **the introduction of Chapter 4 and after the proof of Remark 4.12**. The final result is that we construct a two-parameter family of single phase waves of the doubly infinite lattice.

We repeat this ansatz in the case of two-phase waves,

$$x_n(t) = cn + \chi_2(n\beta_1 + \gamma t, n\beta_2 + 2\gamma t), \tag{12}$$

$$\underline{\chi_2(\theta_1, \theta_2) = \sum_{m\in\mathbb{Z}^2} r(m) e^{i<m,\theta>}.} \tag{13}$$

The equation for $\tilde{r}(m) := r(m)(e^{-i<\beta,m>} - 1)$ can be written as

$$\underline{\Lambda(\beta) r + W(r) = 0,} \tag{14}$$

with

$$\underline{\Lambda(\beta)(m,m) = 1 - \frac{\gamma^2(m_1 + 2m_2)^2}{4\sin^2\frac{<\beta,m>}{2}}.} \tag{15}$$

Note that we have again set $F'(-c) = 1$ without loss of generality. Proceeding analogously to the single-phase case, we define the projection P, $(Pr)(m) := \mathbf{1}_{\{|m|\neq 1\}} r(m)$, ($|m| = |m_1| + |m_2|$), $Q := I - P$. Equation (14) is equivalent to the following two equations. Denote $\mu := Pr, \varphi := Qr$.

$$P\Lambda(\beta)\mu + PW(\mu + \varphi) = 0. \tag{16}$$

$$Q\Lambda(\beta)\varphi + QW(\mu + \varphi) = 0. \tag{17}$$

The first step in the Lyapunov-Schmidt decomposition is to satisfy the infinite dimensional equation (16) by choosing $\mu \in$ Ran (P) as a function of β and $\varphi \in$ Ran (Q) for (β, φ) in a neighborhood of $(\beta^{(0)}, 0)$. We choose $\beta^{(0)}$ such that $Q\Lambda(\beta^{(0)}) = 0$, i.e. $\beta^{(0)} = (\beta_1^{(0)}, \beta_2^{(0)})$, with

$$4\sin^2\frac{\beta_1^{(0)}}{2} = \gamma^2, \tag{18}$$

$$4\sin^2\frac{\beta_2^{(0)}}{2} = (2\gamma)^2. \tag{19}$$

Note again that (18) and (19) have real solutions $\beta_1^{(0)}, \beta_2^{(0)}$ because of the condition $\gamma_2 > \gamma > \gamma_3$.

In order to solve (16) by implicit function theory, we have to investigate the invertibility of $\Lambda(\beta)$: Ran $(P) \to$ Ran (P). In contrast to the construction of the single-phase waves it is here that we encounter a major obstacle. In fact, a small divisor problem occurs, as there exist sites $m \in \mathbf{Z}$ for which $\Lambda(\beta)(m,m)$ is arbitrarily close to zero. We call such sites *singular sites.*

We will determine the location of the singular sites of $\Lambda(\beta^{(0)})$ (cf Fig. below). First of all one realizes that $\gamma_2 > \gamma > \gamma_3$ implies that $m \in \mathbf{Z}^2$ can only be a singular site if $m_1 + 2m_2 = \pm 1, \pm 2$. We find the following cases.

- For $m \in \mathbf{Z}$, satisfying $m(-2\beta_1^{(0)} + \beta_2^{(0)}) \approx 0 \pmod{2\pi}$:
  $(\pm 1 - 2m, m)$, $(-2m, m \pm 1)$ are singular sites. They form a cross.

- For $m \in \mathbf{Z}$, satisfying $m(-2\beta_1^{(0)} + \beta_2^{(0)}) \approx 2\beta_1^{(0)} \pmod{2\pi}$:
  $(-1 - 2m, m)$ is a singular site.

- For $m \in \mathbf{Z}$, satisfying $m(-2\beta_1^{(0)} + \beta_2^{(0)}) \approx -2\beta_1^{(0)} \pmod{2\pi}$:
  $(1 - 2m, m)$ is a singular site.

- For $m \in \mathbf{Z}$, satisfying $m(-2\beta_1^{(0)} + \beta_2^{(0)}) \approx 2\beta_2^{(0)} \pmod{2\pi}$:
  $(-2m, m - 1)$ is a singular site.

- For $m \in \mathbf{Z}$, satisfying $m(-2\beta_1^{(0)} + \beta_2^{(0)}) \approx -2\beta_2^{(0)} \pmod{2\pi}$:
  $(-2m, m + 1)$ is a singular site.

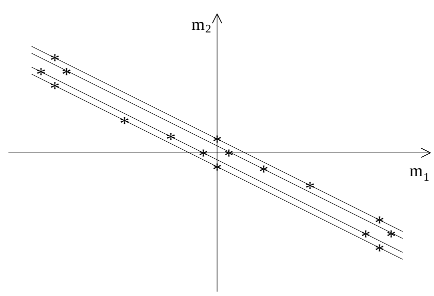

Figure 20: The location of the singular sites of $\Lambda(\beta^{(0)})$

A guide of how to construct solutions of (16) in such a situation is the work of W. Craig and C. E. Wayne on the existence of small solutions of the nonlinear wave equation [CW], where a similar small divisor problem occured. Their construction proceeds via a Newton iteration scheme. The crucial point is to control the small eigenvalues of a truncation of the linearized operator. One recalls that in a Newton scheme it is necessary to linearize around the last approximation u_n in order to obtain the better approximation u_{n+1} and therefore the linearized operator is in general not in diagonal form with respect to the standard basis. The main technical tool in [CW] is a technique introduced by J. Fröhlich and T. Spencer([FS]) for the construction of inverse operators which makes it possible to first examine one singular site at a time and then to control the interaction between them.

If one tries to apply this procedure to our problem one recognizes that the singular sites are not as well seperated as in [CW]. This causes additional difficulties which we have not resolved yet. Recent work of Surace ([S1], [S2]) contains a version of the Fröhlich-Spencer technique which might be helpful in overcoming these difficulties.

Bibliography

[CW] W. Craig and C. E. Wayne, *Newton's method and periodic solutions of non-linear wave equations*, Comm. Pure Appl. Math., **46**, (1993), 1409-1498.

[FS] J. Fröhlich and T. Spencer, *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Comm. Math. Phys., **88**, (1983), 151 –184.

[S1] S. Surace, *The Schrödinger equation with a quasi-periodic potential*, Trans. Am. Math. Soc., **320** (1), July 1990, 321 –370.

[S2] S. Surace, *A finite range operator with a quasi-periodic potential*, preprint.