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The Geometry and Topology of Three-Manifolds

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This is an electronic edition of the 1980 notes distributed by Princeton University. The text was typed in TEX by Sheila Newbery, who also scanned the figures. Typos have been corrected (and probably others introduced), but otherwise no attempt has been made to update the contents. Genevieve Walsh compiled the index.

Numbers on the right margin correspond to the original edition's page numbers.

Thurston's *Three-Dimensional Geometry and Topology*, Vol. 1 (Princeton University Press, 1997) is a considerable expansion of the first few chapters of these notes. Later chapters have not yet appeared in book form.

Please send corrections to Silvio Levy at levy@msri.org.

CHAPTER 6

Gromov's invariant and the volume of a hyperbolic manifold

6.1. Gromov's invariant 6.1

Let X be any topological space. Denote the real singular chain complex of X by $C_*(k)$. (Recall that $C_k(X)$ is the vector space with a basis consisting of all continuous maps of the standard simplex Δ^k into X.) Any k-chain c can be written uniquely as a linear combination of the basis elements. Define the norm $\|c\|$ of c to be the sum of the absolute values of its coefficients,

6.1.1. $$\|c\| = \sum |a_i| \qquad \text{where } c = \sum a_i \sigma_i, \quad \sigma_i : \Delta^k \to X.$$

Gromov's norm on the real singular homology (really it is only a pseudo-norm) is obtained from this norm on cycles by passing to homology: if $a \in H_k(X; \mathbb{R})$ is any homology class, then the norm of α is defined to be the infimum of the norms of cycles representing α, Labelled this 6.1.2.def

Definition 6.1.2 (First definition).

$$\|\alpha\| = \inf \{\|z\| \mid z \text{ is a singular cycle representing } \alpha\}.$$

It is immediate that

$$\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$$

and for $\lambda \in \mathbb{R}$,

$$\|\lambda\alpha\| \leq |\lambda| \, \|\alpha\|.$$

If $f : X \to Y$ is any continuous map, it is also immediate that 6.2

6.1.2. $$\|f_*\alpha\| \leq \|\alpha\|.$$

In fact, for any cycle $\sum a_i \sigma_i$ representing α, the cycle $\sum a_i f \circ \sigma_i$ represents $f_*\alpha$, and $\| \sum a_i f \circ \sigma_i\| = \sum |a_i| \leq \| \sum a_i \sigma_i\|$. (It may happen that $f \circ \sigma_i = f \circ \sigma_j$; even when $\sigma_i \neq \sigma_j$.) Thus $\|f_*\alpha\| \leq \inf \|a_i f \circ \sigma_i\| \leq \|\alpha\|$. In particular, the norm of the fundamental class of a closed oriented manifold M gives a characteristic number of M, Gromov's invariant of M, satisfying the inequality that for any map $f : M_1 \to M_2$,

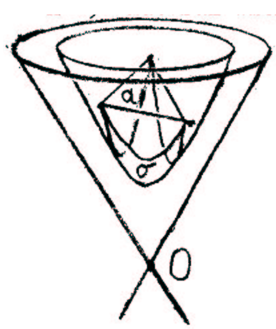
6.1.3. $$\| [M_1] \| \geq | \deg f| \, \| [M_2] \|.$$

What is not immediate from the definition is the existence of any non-trivial examples where $\| [M] \| \neq 0$.

Example. The n-sphere $n \geq 1$ admits maps $f : S^n \to S^n$ of degree 2 (and higher). As a consequence of 6.1.2 $\| [S^n] \| = 0$. More explicitly, one may picture a sequence $\{z_i\}$ representing the fundamental class of S^1, where z_i is $(\frac{l}{i})\sigma_i$ and σ_i wraps a 1-simplex i times around S^1. Since $\|z_i\| = \frac{1}{i}$, $\| [S^1] \| = 0$.

As a trivial example, $\| [S^0] \| = 2$.

Consider now the case of a complete hyperbolic manifold M^n. Any $k+1$ points $v_0, \ldots, v_k$ in $\tilde{M}^n = H^n$ determine a *straight* k-simplex $\sigma_{v_0,\ldots,v_k} : \Delta^k \to H^n$, whose image is the convex hull of $v_0, \ldots, v_k$. There are various ways to define canonical parametrizations for $\sigma_{v_0,\ldots,v_k}$; here is an explicit one. Consider the quadratic form 6.3
model for H^n (§2.5). In this model, $v_0, \ldots, v_k$ become points in $\mathbb{R}^{n+1}$, so they determine an affine simplex α. [In barycentric coordinates, $\alpha(t_0, \ldots, t_k) = \sum t_i v_i$. This parametrization is natural with respect to affine maps of $\mathbb{R}^{n+1}$]. The central projection from O of α back to one sheet of hyperboloid $Q = x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1$ gives a parametrized straight simplex $\sigma_{v_0,\ldots,v_k}$ in H^n, natural with respect to isometries of H^n.



Any singular simplex $\tau : \Delta^k \to M$ can be lifted to a singular simplex $\tilde{\tau}$ in $\tilde{M} = H^n$, since Δ^k is simply connected. Let straight $(\tilde{\tau})$ be the straight simplex with the same vertices as $\tilde{\tau}$ and let straight(τ) be the projection of $\tilde{\tau}$ back to M. Since the straightening operation is natural, straight(τ) does not depend on the lift $\tilde{\tau}$. Straight extends linearly to a chain map

$$\text{straight } : C_*(M) \to C_*(M),$$

chain homotopic to the identity. (The chain homotopy is constructed from a canonical homotopy of each simplex τ to straight(τ).) It is clear that for any chain c, $\|$ straight $(c)\| \leq \|c\|$. Hence, in the computation of the norm of a homology class in M, it suffices to consider only straight simplices. 6.3

Proposition 6.1.4. *There is a finite supremum v_k to the k-dimensional volume of a straight k-simplex in hyperbolic space H^n provided $k \neq 1$.*

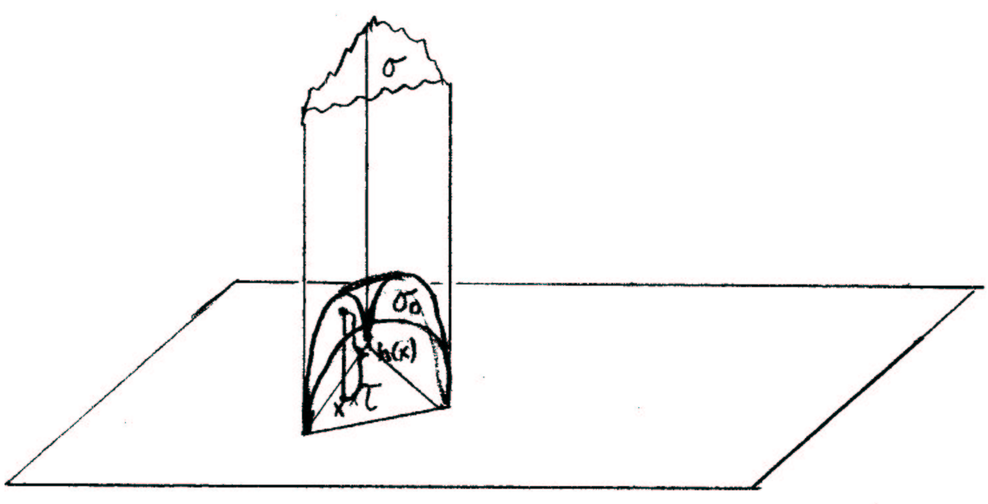
Proof. It suffices to consider ideal simplices with all vertices on S_∞, since any finite simplex fits inside one of these. For $k = 2$, there is only one ideal simplex up to isometry. We have seen that 2 copies of the ideal triangle fit inside a compact surface (§3.9). Thus it has finite volume, which equals π by the Gauss-Bonnet theorem. When $k = 3$, there is an efficient formula for the computation of the volume of an ideal 3-simplex; see Milnor's discussion of volumes in chapter 7. The volume of such simplices attains its unique maximum at the regular ideal simplex, which has all angles equal to 60°. Thus we have the values

6.1.5.
$$\begin{aligned} v_2 &= 3.1415926\ldots = \pi \\ v_3 &= 1.0149416\ldots \end{aligned}$$

It is conjectured that in general, v_k is the volume of the regular ideal k-simplex; if so, Milnor has computations for more values, and a good asymptotic formula as $k \to \infty$. In lieu of a proof of this conjecture, an upper bound can be obtained for v_k from the inductive estimate

6.1.6.
$$v_k < \frac{v_{k-1}}{k-1}.$$

To prove this, consider any ideal k-simplex σ in H^k. Arrange σ so that one of its vertices is the point at ∞ in the upper half-space model, so that σ looks like a triangular chimney lying above a $k-1$ face σ_0 of σ. 6.5



Let dW^k be the Euclidean volume element, so hyperbolic volume is $dV^k = (\frac{1}{x_k})^k dW^k$. Let τ denote the projection of σ_0 to E^{n-1}, and let $h(x)$ denote the Euclidean height of σ_0 above the point $x \in \tau$. The volume of σ is

$$v(\sigma) = \int_\tau \int_h^\infty t^{-k}\, dt\, dW^{k-1}$$

(where dW^{k-1} is the Euclidean $k-1$ volume element for τ). Integrating, we obtain

$$(k-1)\, v(\sigma) = \int_\tau h^{-(k-1)}\, dW^{k-1}.$$

The volume of σ_0 is obtained by a similar integral, where dW^{k-1} is replaced by the Euclidean volume element for σ, which is never smaller than dW^{k-1}. We have $(k-1)v(\sigma) < v(\sigma_0) \le v_{k-1}$. $\square$

We are now ready to find non-trivial examples for Gromov's invariant:

COROLLARY 6.1.7. *Every closed oriented hyperbolic manifold M^n of dimension $n > 1$ satisfies the inequality*

$$\|\,[M]\,\| \ge \frac{v(M)}{v_n}.$$

PROOF. Let Ω be the hyperbolic volume form for M, so that $\int_M \Omega = v(M)$. If $z = \sum z_i \sigma_i$ is any straight cycle representing $[M]$, then 6.6

$$v(M) = \int_M \Omega = \sum z_i \int_{\Delta^n} \sigma_i^* \Omega \le \sum |z_i|\, v_n.$$

Dividing by v_n, we obtain $\|z\| \ge v(M)/v_n$. The infimum over all such z gives 6.1.7 $\square$

A similar proof shows that the norm of element $0 \ne \alpha \in H_k(M, \mathbb{R})$ where $k \ne 1$ is non-zero. Instead of Ω, use an k-form ω representing some multiple $\lambda\alpha$ such that ω has Riemannian norm ≤ 1 at each point of M. (In fact, ω need only satisfy the inequality $\omega(V) \le 1$ where V is a *simple* k-vector of Riemannian norm 1.) Then the inequality $\|\alpha\| \ge \lambda/v_k$ is obtained.

Intuitively, Gromov's norm measures the efficiency with which multiples of a homology class can be represented by simplices. A complicated homology class needs many simplices.

Gromov proved the remarkable theorem that the inequality of 6.1.7 is actually equality. Instead of proving this, we will take the alternate approach to Gromov's theorem developed in [Milnor and Thurston, "Characteristic numbers for three-manifolds"], of changing the definition of $\|\ \|$ to one which is technically easier to work with. It can be shown that past and future definitions are equivalent. However, we have no further use for the first definition, 6.1.2, so henceforth we shall simply abandon it.

For any manifold M, let $C^1(\Delta^k, M)$ denote the space of maps of Δ^k to M, with the C^1 topology. We define a new notion of chains, where a k-chain is a Borel measure μ on $C^1(\Delta^K, M)$ with compact support and bounded total variation. [The total variation of a measure μ is $\|\mu\| = \sup\{\int f\, d\mu \,|\, |f| \le 1\}$. Alternately, μ can be decomposed into a positive and negative part, $\mu = \mu_+ - \mu_-$ where μ_+ and μ_- are 6.7

positive. Then $\|\mu\| = \int d\mu_+ + \int d\mu_-$]. Let the group of k-chains be denoted $\mathcal{C}_k(M)$. There is a map $\partial : \mathcal{C}_k(M) \to \mathcal{C}_{k-1}(M)$, defined in an obvious way. It is not difficult to prove that the homology obtained by using these chains is the standard homology for M; see [Milnor and Thurston, "Characteristic numbers for three-manifolds"] for more details. (Note that integration of a k-form over an element of $\mathcal{C}_k(M)$ is defined; this gives a map from $\mathcal{C}_*(M)$ to currents on M. Some condition such as compact support for μ is necessary; otherwise one would have pathological cycles such as $\sum(\frac{1}{i})^2\sigma_i$, where σ_i wraps Δ^1 i times around S^1. The measure has total variation $\sum(\frac{1}{i})^2 < \infty$, yet the cycle would seem to represent the infinite multiple $\sum(\frac{1}{i})[S^1]$ of $[S^1]$.)

Definition 6.1.8 (Second definition). i Let $\alpha \in H^k(M;\mathbb{R})$, where M is a manifold. Gromov's norm $\|\alpha\|$ is defined to be

$$\|\alpha\| = \inf\{\|u\| \,|\mu \in \mathcal{C}^k(M) \text{ represents } \alpha\}.$$

Theorem 6.2 (Gromov). *Let M^n be any closed oriented hyperbolic manifold. Then*

$$\|\,[M]\,\| = \frac{v(M^n)}{v_n}.$$

Proof. The proof of corollary 6.1.7 works equally well with the new definition as with the old. The point is that the straightening operation is completely uniform, so it works with measure-cycles. What remains is to prove that $\|\,[M]\,\| \le v(M)/v_n$, or in other words, the fundamental cycle of M can be represented efficiently by a cycle using simplices which have (on the average) nearly maximal volume. 6.8

Let σ be any singular k-simplex in H^n. A chain $\mathrm{smear}_M(\sigma) \in \mathcal{C}_k(M)$ can be constructed, which is a measure supported on all isometric maps of σ into M, weighted uniformly. With more notation, let h denote Haar measure on the group of orientation-preserving isometries of H^n, $\mathrm{Isom}_+(H^n)$. Let h be normalized so that the measure of the set of isometries taking a point $x \in H^n$ to a region $R \subset H^n$ is the volume of R. Haar measure on $\mathrm{Isom}_+(H^n)$ is invariant under both right and left multiplication, so it descends to a measure (also denoted h) on the quotient space $P(M) = \pi_1 M \backslash \mathrm{Isom}_+(H^n)$.

There is a map from $P(M)$ to $\mathcal{C}^1(\Delta^k, M)$, which associates to a coset $\pi_1 M\varphi$ the singular simplex $p\circ\varphi\circ\sigma$, where $p : H^n \to M$ is the covering projection. The measure h pushes forward to give a chain $\mathrm{smear}_M(\sigma) \in \mathcal{C}_k(M)$. Since h is invariant on both sides, $\mathrm{smear}_M(\sigma)$ depends only on the isometry class of σ. Smearing extends linearly to $\mathcal{C}_k(H^n)$. Furthermore, $\mathrm{smear}_M \partial c = \partial\, \mathrm{smear}_M c$.

Let σ now be any straight simplex in H^n, and σ_- a reflected copy of σ. Then $\frac{1}{2}\,\mathrm{smear}_M(\sigma - \sigma_-))$ is a cycle, since the faces of σ and σ_- cancel out in pairs, up to isometries. We have

$$\|\tfrac{1}{2}\ \mathrm{smear}_M\,(\sigma - \sigma_-)\| = v(M).$$

The homology class of this cycle can be computed by integration of the hyperbolic form Ω from M. The integral over each copy of σ is $v(\sigma)$, so the total integral is $v(M)v(\sigma)$. Thus, the cycle represents

$$[\tfrac{1}{2}\,\mathrm{smear}\,(\sigma - \sigma_-)] = v(\sigma)[M]$$

so that 6.9

$$\|v(\sigma)[M]\,\| \le v(M).$$

Dividing by $v(\sigma)$ and taking the infimum over σ, we obtain 6.2. □

Corollary 6.2.1. *If $f : M_1 \to M_2$ is any map between closed oriented hyperbolic n-manifolds, then*

$$v(M_1) \ge |\deg f| v(M_2).$$

Gromov's theorem can be generalized to any (G, X)-manifold, where G acts transitively on X with compact isotropy groups.

To do this, choose an invariant Riemannian metric for X and normalize Haar measure on G as before. The smearing operation works equally well, so that one has a chain map

$$\mathrm{smear}_M : \mathcal{C}_k(X) \to \mathcal{C}_k(M).$$

In fact, if N is a second (G, X)-manifold, one has a chain map

$$\mathrm{smear}_{N,M} : \mathcal{C}_k(N) \to \mathcal{C}_k(M),$$

defined first on simplices in N via a lift to X, and then extended linearly to all of $\mathcal{C}_k(N)$. If z is any cycle representing $[N]$, then $\mathrm{smear}_{N,M}(z)$ represents

$$(v(N)/v(M))[M].$$

This gives the inequality

$$\frac{\|\,[N]\,\|}{v(N)} \ge \frac{\|\,[M]\,\|}{v(M)}.$$

Interchanging M and N, we obtain the reverse inequality, so we have proved the following result:

Theorem 6.2.2. *For any pair (G, X), where G acts transitively on X with compact isotropy groups and for any invariant volume form on X, there is a constant C* 6.10
such that every closed oriented (G, X)-manifold M satisfies

$$\|\,[M]\,\| = Cv(M),$$

(where $v(M)$ is the volume of M). □

This line may be pursued still further. In a hyperbolic manifold a smeared k-cycle is homologically trivial except in dimension $k = 0$ or $k = n$, but this is not generally true for other (G, X)-manifolds when G does not act transitively on the frame bundle of X. The *invariant cohomology* $H^*_G(X)$ is defined to be the cohomology of the cochain complex of differential forms on X invariant by G. If α is any invariant cohomology class for X, it defines a cohomology class α_M on any (G, X)-manifold M. Let $PD(\gamma)$ denote the Poincaré dual of a cohomology class γ.

THEOREM 6.2.3. *There is a norm* $\| \ \|$ *in* $H^*_G(X)$ *such that for any closed oriented* (G, X)*-manifold* M,

$$\big\|\mathrm{PD}(\alpha_m)\| = v(M)\|\alpha\|.$$

PROOF. It is an exercise to show that the map

$$\mathrm{smear}_{M,M} : H_*(M) \to H_*(M)$$

is a retraction of the homology of M to the Poincaré dual of the image in M of $H^*_G(X)$. The rest of the proof is another exercise. □

In these variations, 6.2.2 and 6.2.3, on Gromov's theorem, there does not seem to be any general relation between the proportionality constants and the maximal 6.11
volume of simplices. However, the inequality 6.1.7 readily generalizes to any case when X possesses and invariant Riemannian metric of non-positive curvature.

6.3. Gromov's proof of Mostow's Theorem

Gromov gave a very quick proof of Mostow's theorem for hyperbolic three-manifolds, based on 6.2. The proof would work for hyperbolic n-manifolds if it were known that the regular ideal n-simplex were the unique simplex of maximal volume. The proof goes as follows.

This is now known to be true.

LEMMA 6.3.1. *If* M_1 *and* M_2 *are homotopy equivalent, closed, oriented hyperbolic manifolds, then* $v(M_1) = v(M_2)$.

PROOF. This follows immediately by applying 6.2 to the homotopy equivalence $M_1 \leftrightarrow M_2$. □

Let $f_1 : M_1 \to M_2$ be a homotopy equivalence and let $\tilde{f}_1 : \tilde{M}_1 \to \tilde{M}_2$ be a lift of f_1. From 5.9.5 we know that $\tilde{f}_1$ extends continuously to the sphere S^{n-1}_∞.

LEMMA 6.3.2. *If* $n = 3$, $\tilde{f}_1$ *takes every* 4*-tuple of vertices of a positively oriented regular ideal simplex to the vertices of a positively oriented regular ideal simplex.*

PROOF. Suppose the contrary. Then there is a regular ideal simplex σ such that the volume of the simplex straight$(\tilde{f}_1\sigma)$ spanned by the image of its vertices is $v_3 - \epsilon$, with $\epsilon > 0$. There are neighborhoods of the vertices of σ in the disk such that for any simplex σ' with vertices in these neighborhoods, $v\big(\text{straight}(\tilde{f}_1\sigma')\big) \le v_3 - \epsilon/2$. Then for every finite simplex σ_0' very near to σ, this means that a definite Haar measure 6.12
of the isometric copies σ' of σ_0' near σ' have $v\big(\text{straight}(\tilde{f}_1\sigma_0')\big) < v_3 - \epsilon/2$. Such a simplex σ_0' can be found with volume arbitrarily near v_3. But then the "total volume" of the cycle $z = \frac{1}{2}\,\text{smear}(\sigma_0' - \sigma_{0-}')$ strictly exceeds the total volume of straight$(f_* z)$, contradicting 6.3.1. $\square$

To complete the proof of Mostow's theorem in dimension 3, consider any ideal regular simplex σ together with all images of σ coming from repeated reflections in the faces of σ. The set of vertices of all these images of σ is a dense subset of S^2_∞. Once $\tilde{f}_1$ is known on three of the vertices of σ, it is determined on this dense set of points by 6.3.2, so $\tilde{f}_1$ must be a fractional linear transformation of S^2_∞, conjugating the action of $\pi_1 M_1$ to the action of $\pi_1 M_2$. This completes Gromov's proof of Mostow's theorem. $\square$

In this proof, the fact that f_1 is a homotopy equivalence was used to show (a) that $v(M_1) = v(M_2)$ and (b) that $\tilde{f}_1$ extends to a map of S^2_∞. With more effort, the proof can be made to work with only assumption (a):

THEOREM 6.4 (Strict version of Gromov's theorem). *Let $f : M_1 \to M_2$ be any map of degree $\neq 0$ between closed oriented hyperbolic three-manifolds such that Gromov's inequality* 6.2.1 *is equality, i.e.,*

$$v(M_1) = |\deg f|\, v(M_2).$$

Then f is homotopic to a map which is a local isometry. If $|\deg f| = 1$, f is a homotopy equivalence and otherwise it is homotopic to a covering map.

PROOF. The first step in the proof is to show that a lift $\tilde{f}$ of f to the universal covering spaces extends to S^2_∞. Since the information in the hypothesis of 6.4 has to do with volume, not topology, we will know at first only that this extension is a measurable map of S^2_∞. Then, the proof of Section 6.3 will be adapted to the current situation.

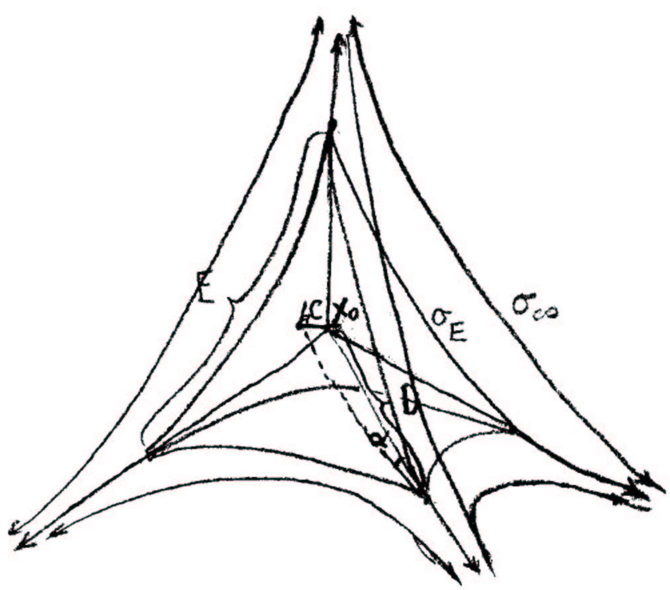
The proof works most smoothly if we have good information about the asymptotic behavior of volumes of simplices. Let σ_E be a regular simplex in H^3 all of whose edge lengths are E.

THEOREM 6.4.1. *The volume of σ_E differs from the maximal volume v_3 by a quantity which decreases exponentially with E.*

PROOF. Construct copies of simplices σ_E centered at a point $x_0 \in H^3$ by drawing the four rays from a point x_0 through the vertices of an ideal regular simplex σ_∞ centered at x_0. The simplex whose vertices are on these rays, a distance D from x_0, is isometric to σ_E for some E. Let C be the distance from x_0 to any face of this simplex. The derivative $dv(\sigma_E)/dD$ is less than the area of $\partial\sigma_E$ times the maximal normal velocity of a face of σ_E. If α is the angle between such a face and the ray through x_0, we have

$$\frac{dv(\sigma_E)}{dD} < 2\pi \sin\alpha.$$

From the hyperbolic law of sines (2.6.16) $\sin\alpha = \sinh C/\sinh D$, showing that $dv(\sigma_I)/dD$ decreases exponentially with D (since $\sinh C$ is bounded). The corresponding statement for E follows since asymptotically, $E \sim 2D + \text{constant}$. $\square$ 6.14



LEMMA 6.4.2. *Any simplex with volume close to v_3 has all dihedral angles close to $60°$.*

PROOF. Such a simplex is properly contained in an ideal simplex with any two face planes the same, so with one common dihedral angle. 6.4.2 follows form ??? $\square$

LEMMA 6.4.3. *There is some constant C such that for every simplex σ with volume near v_3 and for any angle β on a face of σ,*

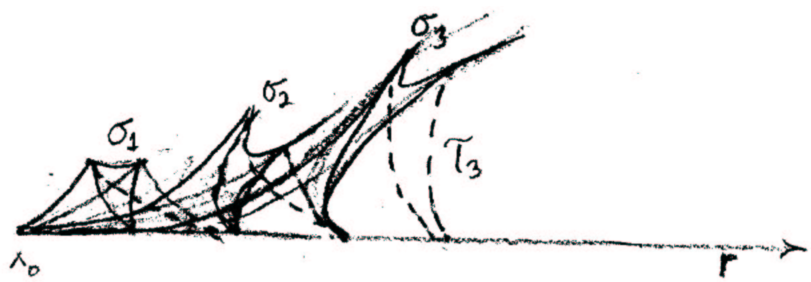
$$v_3 - v(\sigma) \geq C\beta^2.$$

PROOF. If the vertex v has a face angle of β, first enlarge σ so that the other three vertices are at ∞, without changing a neighborhood of v. Now prolong one of

the edges through v to S^2_∞, and push v out along this edge. The new spike added to σ beyond v has thickness at v estimated by a linear function of β (from 2.6.12), 6.15
so its volume is estimated by a quadratic function of β. (This uses the fact that a cross-section of the spike is approximately an equilateral triangle.) □

LEMMA 6.4.4. *For every point x_0 in M_1, and almost every ray r through x_0, $f_1(r)$ converges to a point on S^2_∞.*

PROOF. Let $x_0 \in H^3$, and let r be some ray emanating from x_0. Let the simplex σ_i (with all edges having length i) be placed with a vertex at x_0 and with one edge on r, and let τ_i be a simplex agreeing with σ_i in a neighborhood of x_0 but with the edge on r lengthened, to have length $i+1$.



The volume of σ_i and $\tau_i \supset \sigma_i$ deviate from the supremal value by an amount ϵ_i decreasing exponentially with i, so $\operatorname{smear}_{M_1} \tau_i$ and $\operatorname{smear}_{M_1} \sigma_i$ are very efficient cycles representing a multiple of $[M_1]$. Since $v(M_1) = |\deg f|\, v(M_2)$, the cycles straight $f_* \operatorname{smear}_{M_1} \sigma_i$ and straight $f_* \operatorname{smear}_{M_1} \tau_i$ must also be very efficient. In other words, for all but a set of measure at most $v(M_1)\epsilon_i/v_3$ of simplices σ in $\operatorname{smear} \sigma_i$ (or near $\operatorname{smear} \tau_i$), the simplex straight $f\sigma$ must have volume $\geq v_3 - \epsilon_i$. 6.16

Let B be a ball around x_0 which embeds in M_i. The chains $\operatorname{smear}_B \sigma_i$ and $\operatorname{smear}_B \tau_i$ correspond to the measure for $\operatorname{smear}_M \sigma_i$ and $\operatorname{smear}_M \tau_i$ restricted to those singular simplices with the first vertex in the image of B in M_1. Thus for all but a set of measure at most $\big(2v(M_1)/v_3\big) \sum_{i=i_0}^{\infty} \epsilon_i$ of isometries I with take x_0 to B, all simplices $I(\sigma_i)$ and $I(\tau_i)$ for all $i > i_0$ are mapped to simplices straight $\tilde{f} \operatorname{smear}_B \sigma$ with volume $\geq v_3 - \epsilon_i$. By 6.4.3, the sum of all face angles of the image simplices is a geometically convergent series. It follows that for all but a set of small measure of rays r emanating from points in B, $f(r)$ converges to a point on S^2_∞; in fact, by letting $i_0 \to \infty$, it follows that for almost every ray r emanating from points in B, $\tilde{f}(r)$ converges. Then there must be a point x' in B such that for almost every ray r emanating from x', $\tilde{f}(r)$ converges. Since each ray emanating from a point in H^3 is asymptotic to some ray emanating from x', this holds for rays through all points in H^3. □

REMARK. This measurable extension of $\tilde{f}$ to S^2_∞ actually exists under very general circumstances, with no assumption on the volume of M_1 and M_2. The idea is that if g is a geodesic in M_1, $\tilde{f}(g)$ behaves like a random walk on $\tilde{M}_2$. Almost every random walk in hyperbolic space converges to a point on S^{n-1}_∞. (Moral: always carry a map when you are in hyperbolic space!)

LEMMA 6.4.5. *The measurable extension of $\tilde{f}$ to S^2_∞ carries the vertices of almost every positively oriented ideal regular simplex to the vertices of another positively oriented ideal regular simplex.*

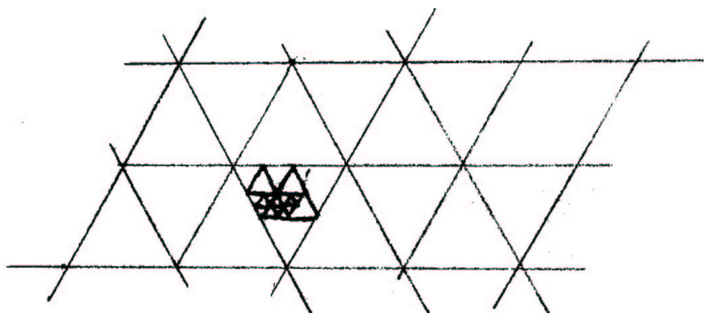
PROOF. Consider a point x_0 in H^3 and a ball B about x_0 which embeds in M, as before. Let σ_i be centered at x_0. As before, for almost all isometries I which take 6.17
x_0 to B, the sequence $\{\text{straight } \tilde{f} \circ I \circ \sigma_i\}$ has volume converging to v_3, and all four vertices converging to S^2_∞.

If for almost all I these four vertices converge to distinct points, we are done. Otherwise, there is a set of positive measure of ideal regular simplices such that the image of the vertex set of σ is degenerate: either all four vertices are mapped to the same point, or three are mapped to one point and the fourth to an arbitrary point. We will show this is absurd. If the degenerate cases occur

with positive measure, there is some pair of points v_0 and v_1 with $\tilde{f}(v_0) = \tilde{f}(v_1)$ such that for almost all regular ideal simplices spanned by v_0, v_1, v_2, v_3, either $\tilde{f}(v_2) = \tilde{f}(v_0)$ or $\tilde{f}(v_3) = \tilde{f}(v_0)$. Thus, there is a set A of positive measure with $\tilde{f}(A)$ a single point. Almost every regular ideal simplex with two vertices in A has one other vertex in A. It is easy to conclude that A must be the entire sphere. (One method is to use ergodicity as in the proof of 6.4 which will follow.) The image point $\tilde{f}(A)$ is invariant under covering transformations of M_1. This implies that the image of $\pi_1 M_1$ in $\pi_1 M_2$ has a fixed point on S_∞, which is absurd. □

We resume the proof of 6.4 here. It follows from 6.4.5 that there is a vertex v_0 such that for almost all regular ideal simplices spanned by v_0, v_1, v_2, v_3, the image vertices span a regular ideal simplex. Arrange v_0 and $\tilde{f}(v_0)$ to be the point at infinity in the 6.18
upper half-space model. Three other points v_1, v_2, v_3 span a regular ideal simplex with v_0 if and only if they span an equilateral triangle in the plane, E^2. By changing coordinates, we may assume that f maps vertices of almost all equilateral triangles parallel to the x-axis to the vertices of an equilateral triangle in the plane. In complex

notation, let $\omega = \sqrt[3]{-1}$, so that $0, 1, \omega$ span an equilateral triangle. For almost all $z \in \mathbb{C}$, the entire countable set of triangles spanned by vertices of the form $z + 2^{-k}n$, $z + 2^{-k}(n+1)$, $z + 2^{-k}(n+\omega)$, for $k, n \in \mathbb{Z}$, are mapped to equilateral triangles.



Then the map $\tilde{f}$ must take the form

$$\tilde{f}\big(z + 2^{-k}(n + m\omega)\big) = g(z) + h(z) \cdot 2^{-k}(n + m\omega), \qquad k, n, m \in \mathbb{Z},$$

for almost all z. The function h is invariant a.e. by the dense group T of translations of the form $z \mapsto z + 2^{-k}(n + m\omega)$. This group is ergodic, so h is constant a.e. Similar reasoning now shows that g is constant a.e., so that f is essentially a fractional linear transformation on the sphere S^2_∞. Since $\tilde{f} \circ T_\alpha = T_{f_*\alpha} \circ \tilde{f}$, this shows that $\pi_1 M_1$ is conjugate, in $\mathrm{Isom}(H^3)$, to a subgroup of $\pi_1 M_2$. ☐

6.19

6.5. Manifolds with Boundary

There is an obvious way to extend Gromov's invariant to manifolds with boundary, as follows. If M is a manifold and $A \subset M$ a submanifold, the relative chain group $\mathcal{C}_k(M, A)$ is defined to be the quotient $\mathcal{C}_k(M)/\mathcal{C}_k(A)$. The norm on $\mathcal{C}_k(M)$ goes over to a norm on $\mathcal{C}_k(M, A)$: the norm $\|\mu\|$ of an element of $\mathcal{C}_k(M, A)$ is the total variation of μ restricted to the set of singular simplices that do not lie in A. The norm $\|\gamma\|$ of a homology class $\gamma \in H_k(M, A)$ is defined, as before, to be the infimal norm of relative cycles representing γ. Gromov's invariant of a compact, oriented manifold with boundary $(M, \partial M)$ is $\big\|[M, \partial M]\big\|$, where $[M, \partial M]$ denotes the relative fundamental cycle.

There is a second interesting definition which makes sense in an important special case. For concreteness, we shall deal only with the case of three-manifold whose boundary consists of tori. For such a manifold M, define

$$\|\,[M, \partial M]\,\|_0 = \lim_{a \to 0} \inf\{\|z\| \,|\, z \,\text{straight}\, [M, \partial M] \text{ and } \|\partial Z\| \le a\}.$$

Observe that ∂z represents the fundamental cycle of ∂M, so that a necessary condition for this definition to make sense is that $\|\,[\partial M]\,\| = 0$. This is true in the present situation that ∂M consists of tori, since the torus admits self-maps of degree > 1.

Then $\|(M, \partial M)\|_0$ is the limit of a non-decreasing sequence, so to insure the existence of the limit we need only find an upper bound. This involves a special property of the torus.

PROPOSITION 6.5.1. *There is a constant K such that z is any homologically trivial cycle in $\mathcal{C}_2(T^2)$, then z bounds a chain c with $\|c\| \leq K\|z\|$.*

PROOF. Triangulate T^2 (say, with two "triangles" and a single vertex). Partition T^2 into disjoint contractible neighborhoods of the vertices. Consider first the case that no simplices in the support of z have large diameter. Then there is a chain homotopy of z to its simplicial approximation $a(z)$.

The chain homotopy has a norm which is a bounded multiple of the norm of z. Since simplicial singular chains form a finite dimensional vector space, $a(z)$ is homologous to zero by a homology whose norm is a bounded multiple of the norm of $a(z)$. This gives the desired result when the simplices of z are not large. In the general case, pass to a very large cover $\tilde{T}^2$ of T^2. For any finite sheeted covering space $p : \tilde{M} \to M$ there is a canonical chain map, transfer: $\mathcal{C}_*(M) \to \mathcal{C}_*(\tilde{M})$. The transfer of a singular simplex is simply the average of its lifts to $\tilde{M}$; this extends in an obvious way to measures on singular simplices. Clearly $p \circ$ transfer $=$ id, and $\|$ transfer $c\| = \|c\|$. If z is any cycle on T^2, then for a sufficiently large finite cover $\tilde{T}^2$ of T^2, the transfer of z to $\tilde{T}^2 = T^2$ has no large 2-simplices in its support. Then transfer z is the boundary of a chain c with $\|c\| \leq K\|z\|$ for some fixed K. The projection of c back to the base space completes the proof. □

6.21

We now have upper bounds for $\|\,[M, \partial M]\,\|_0$. In fact, let z be any cycle representing $[M, \partial M]$, and let ϵ be any cycle representing $[\partial M]$. By piecing together z with a homology from ∂z to ϵ given by 6.5.1, we find a cycle z' representing $[M, \partial M]$ with $\|z'\| \leq \|z\| + K(\|\partial z\| + \|\epsilon\|)$. Passing to the limit as $\|\epsilon\| \to 0$, we find that $\|\,[M, \partial M]\,\| \leq \|z\| + K\|\partial z\|$.

The usefulness of the definition of $\|\,[M, \partial M]\,\|_0$ arises from the easy

PROPOSITION 6.5.2. *Let $(M, \partial M)$ be a compact oriented three-manifold, not necessarily connected, with ∂M consisting of tori. Suppose $(N, \partial N)$ is an oriented manifold obtained by gluing together certain pairs of boundary components of M. Then*

$$\| [N, \partial N] \|_0 \leq \| [M, \partial M] \|_0.$$

COROLLARY 6.5.3. *If $(S, \partial S)$ is any Seifert fiber space, then*

$$\| [S, \partial S] \|_0 = \| [S, \partial S] \| = 0.$$

(The case $\partial S = \phi$ is included.)

PROOF OF COROLLARY. If S is a circle bundle over a connected surface M with non-empty boundary, then S (or a double cover of it, if the fibers are not oriented) is $M \times S^1$. Since it covers itself non-trivially its norm (in either sense) is 0. If S is a circle bundle over a closed surface M, it is obtained by identification of $(M - D^2) \times S^1$ with $D^2 \times S^1$, so its norm is also zero. If S is a Seifert fibration, it is obtained by identifying solid torus neighborhoods of the singular fibers with the complement which is a fibration. □

PROOF OF 6.5.2. A cycle z representing $[M, \partial M]$ with $\|\partial z\| \leq \epsilon$ goes over to a chain on $[N, \partial N]$, which can be corrected to be a cycle z' with $\|z\|' \leq \|z\| + K\epsilon$. □ 6.22

If M is a complete oriented hyperbolic manifold with finite total volume, recall that M is the interior of a compact manifold $\bar{M}$ with boundary consisting of tori. Both $\| [\bar{M}, \partial \bar{M}] \|$ and $\| [\bar{M}, \partial \bar{M}] \|_0$ can be computed in this case:

LEMMA 6.5.4 (Relative version of Gromov's Theorem). *If M is a complete oriented hyperbolic three-manifold with finite volume, then*

$$\| [\bar{M}, \partial \bar{M}] \|_0 = \| [\bar{M}, \partial \bar{M}] \| = \frac{v(M)}{v_3}.$$

PROOF. Let σ be a 3-simplex whose volume is nearly the maximal value, v_3. Then $\mathrm{smear}_M \sigma$ is a measure on singular cycles with non-compact support. Restrict this measure to simplices not contained in $M_{(0,\epsilon]}$, and project to $M_{[\epsilon,\infty)}$ by a retraction of M to $M_{[\epsilon,\infty)}$. Since the volume of $M_{(0,\epsilon]}$ is small for small ϵ, this gives a relative fundamental cycle z' for

$$(M_{[\epsilon,\infty)}, \partial M_{[\epsilon,\infty)}) = (\bar{M}, \partial \bar{M})$$

with $\|z'\| \approx \frac{v(M)}{v_3}$ and with $\|\partial z'\|$ small. This proves that

$$\frac{v(M)}{v_3} \geq \| [\bar{M}, \partial \bar{M}] \|_0.$$

There is an immediate inequality

$$\| \, [\bar{M}, \partial \bar{M}] \, \|_0 \geq \| \, [\bar{M}, \partial \bar{M}] \, \|.$$

To complete the proof, we will show that $\| \, [\bar{M}, \partial \bar{M}] \, \| \geq v(M)/v_3$. This is done by a straightening operation, as in 6.1.7. For this, note that if σ is any simplex lying in 6.23
$M_{(0,\epsilon]}$, then straight(σ) also lies in $M_{(0,\epsilon]}$, since $M_{(0,\epsilon]}$ is convex. Hence we obtain a chain map

$$\text{straight} : \mathcal{C}_*(M, M_{(0,\epsilon]}) \to \mathcal{C}_*(M, M_{(0,\epsilon]}),$$

chain homotopic to the identity, and not increasing norms. As in 6.1.7, this gives the inequality

$$\| \, [M, M_{(0,\epsilon]}] \, \| \geq \frac{v(M_{[\epsilon,\infty)})}{v_3}.$$

Since for small ϵ there is a chain isomorphism between $\mathcal{C}_k(M, M_{(0,\epsilon]})$ and $\mathcal{C}_k(\bar{M}, \partial \bar{M})$ which is a $\| \; \|$-isometry, this proves 6.5.4. □

Here is an inequality which enables one to compute Gromov's invariant for much more general three-manifolds:

THEOREM 6.5.5. *Suppose M is a closed oriented three-manifold and $H \subset M$ is a three-dimensional submanifold with a complete hyperbolic structure of finite volume. Suppose $\bar{H}$ is embedded in M and that $\partial \bar{H}$ is incompressible. Then*

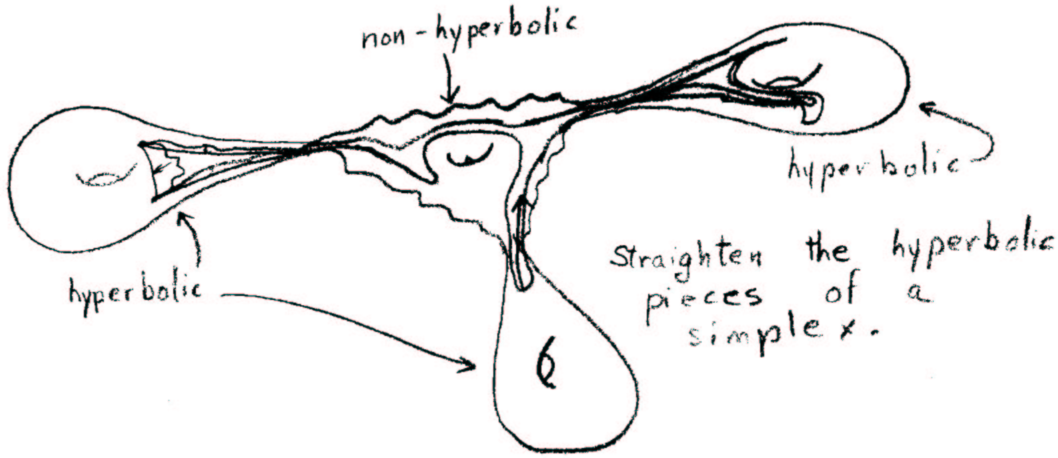
$$\| \, [M] \, \| \geq \frac{v(H)}{v_3}.$$

REMARK. Of course, the hypothesis that $\partial \bar{H}$ is incompressible is necessary; otherwise M might be S^3. If H were not hyperbolic, further hypotheses would be needed to obtain an inequality. Consider, for instance, the product $M_g \times I$ where M_g is a surface of genus $g > 1$. Then $\| \, [M_g] \, \| = 2\, v(M_g)/\pi = 4\, |\chi(M_g)|$, so

$$\| \, [M_g \times I, \partial(M_g \times I)] \, \| \geq \| \, [M_g] \, \| \geq 4\, |\chi(M_g)|.$$

On the other hand, one can identify the boundary of this manifold to obtain $M_g \times S^1$, 6.24
which has norm 0. The boundary can also be identified to obtain hyperbolic manifolds (see §4.6, or §). Since finite covers of arbitrarily high degree and with arbitrarily high norm can also be obtained by gluing the boundary of the same manifold, no useful inequality is obtained in either direction.

PROOF. Since this is a digression, we give only a sketch of a proof.



□

With 6.5.5 combined with 6.5.2, one can compute Gromov's invariant for any manifold which is obtained from Seifert fiber spaces and complete hyperbolic manifolds of finite volume by identifying along incompressible tori.

The strict and relative versions of Gromov's theorems may be combined; here is the most interesting case:

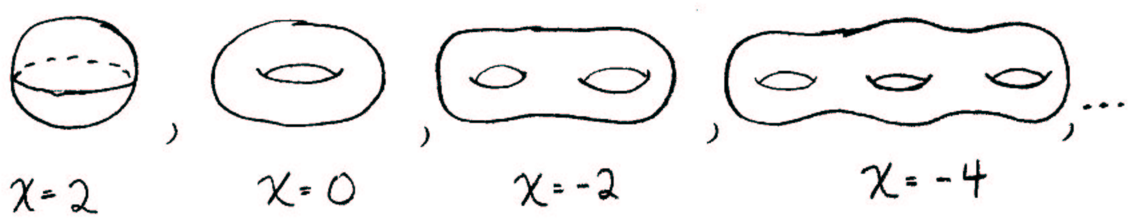
THEOREM 6.5.6. *Suppose M_1 is a complete hyperbolic manifold of finite volume and that $M_2 \neq M_1$ is a complete hyperbolic manifold obtained topologically by replacing certain cusps of M_1 by solid tori. Then $v(M_1) > v(M_2)$.*

6.25

PROOF. No new ideas are needed. Consider some map $f : M_1 \to M_2$ which collpases certain components of $M_{1_{(0,\epsilon]}}$ to short geodesics in M_2. Now apply the proof of 6.4. □

6.6. Ordinals

Closed oriented surfaces can be arranged very neatly in a single sequence,



in terms of their Euler characteristic. What happens when we arrange all hyperbolic three-manifolds in terms of their volume? From Jørgensen's theorem, 5.12 it follows that the set of volumes is a closed subset of $\mathbb{R}_+$. Furthermore, by combining Jørgensen's theorem with the relative version of Gromov's theorem, 6.5.4, we obtain

Corollary 6.6.1. *The set of volumes of hyperbolic three-manifolds is well-ordered.*

Proof. Let $v(M_1) \geq v(M_2) \geq \ldots \geq v(M_k) \geq \ldots$ be any non-ascending sequence of volumes. By Jørgensen's theorem, by passage to a subsequence we may assume that the sequence $\{M_i\}$ converges geometrically to a manifold M, with $v(M) \leq \lim v(M_i)$. By 6.5.2, eventually $\| [M_i] \|_0 \leq \| [M] \|_0$, so 6.5.4 implies that the sequence of volumes is eventually constant. $\square$

6.26

Corollary 6.6.2. *The volume is a finite-to-one function of hyperbolic manifolds.*

Proof. Use the proof of 6.6.1, but apply the strict inequality 6.5.6 in place of 6.5.2, to show that a convergent sequence of manifolds with non-increasing volume must be eventually constant. $\square$

In view of these results, the volumes of complete hyperbolic manifolds are indexed by countable ordinals. In other words, there is a smallest volume v_1, a next smallest volume v_2, and so forth. This sequence $v_1 < v_2 < v_3 < \cdots < v_k < \cdots$ has a limit point v_ω, which is the smallest volume of a complete hyperbolic manifold with one cusp. The next smallest manifold with one cusp has volume $v_{2\omega}$. It is a limit of manifolds with volumes $v_{\omega+1}, v_{\omega+2}, \ldots, v_{\omega+k}, \ldots$. The first volume of a manifold with two cusps is v_{ω^2}, and so forth. (See the discussion on pp. 5.59–5.60, as well as Theorem 6.5.6.) The set of all volumes has order type ω^ω. These volumes are indexed by the ordinals less than ω^ω, which are represented by polynomials in ω. Each volume of a manifold with k cusps is indexed by an ordinal of the form $\alpha \cdot \omega^k$, (where the product $\alpha \cdot \beta$ is the ordinal corresponding to the order type obtained by replacing each element of α with a copy of β). There are examples where α is a limit ordinal. These can be constructed from coverings of link complements. For instance, the Whitehead link complement has two distinct 2-fold covers; one has two cusps and the other has three, so the common volume corresponds to an ordinal divisible by ω^3. I do not know any examples of closed manifolds corresponding to limit ordinals.

It would be very interesting if a computer study could determine some of the low volumes, such as $v_1, v_2, v_\omega, v_{\omega^2}$. It seems plausible that some of these might come from Dehn surgery on the Borromean rings.

There is some constant C such that every manifold with k cusps has volume $\geq C \cdot k$. This follows from the analysis in 5.11.2: the number of boundary components of $M_{[\epsilon,\infty)}$ is bounded by the number of disjoint $\epsilon/2$ balls which can fit in M. It would be interesting to calculate or estimate the best constant C.

Corollary 6.6.3. *The set of values of Gromov's invariant $\| \, [\,] \, \|_0$ on the class of connected manifolds obtained from Seifert fiber spaces and complete hyperbolic manifolds of finite volume by identifying along incompressible tori is a closed well-ordered subset of $\mathbb{R}^+$, with order type ω^ω.*

We shall see later (§) that this class contains all Haken manifolds with toral boundaries.

Proof. Extend the volume function to $v(M) = v_3 \cdot \| \, [M] \, \|_0$ when M is not hyperbolic. From 6.5.5 and 6.5.2, we know that every value of v is a finite sum of volumes of hyperbolic manifolds. Suppose $\{w_i\}$ is a bounded sequence of values of v. Express each w_i as the sum of volumes of hyperbolic pieces of a manifold M_i with $v(M)_i = w_i$. The number of terms is bounded, since there is a lower bound to the volume of a hyperbolic manifold, so we may pass to an infinite subsequence where the number of terms in this expression is constant. Since every infinite sequence of ordinals has an infinite non-decreasing subsequence, we may pass to a subsequence of w_i's where all terms in these expressions are non-decreasing. This proves that 6.28
the set of values of v is well-ordered. Furthermore, our subsequence has a limit $w = v_{\alpha_1} + \cdots + v_{\alpha_k}$, which is expressed as a sum of limits of non-decreasing sequences of volumes. Each v_{α_j} is the volume of a hyperbolic manifold M_j with at least as many cusps as the limiting number of cusps of the corresponding hyperbolic piece of M_i. Therefore, the $\bar{M}_j$'s may be glued together to obtain a manifold M with $v(M) = w$. This shows the set of values of v is closed. The fact that the order type is ω^ω can be deduced easily by showing that every value of v is not in the k-th derived set, for some integer k; in fact, $k \leq v/C$, where C is the constant just discussed. □

6.7. Commensurability

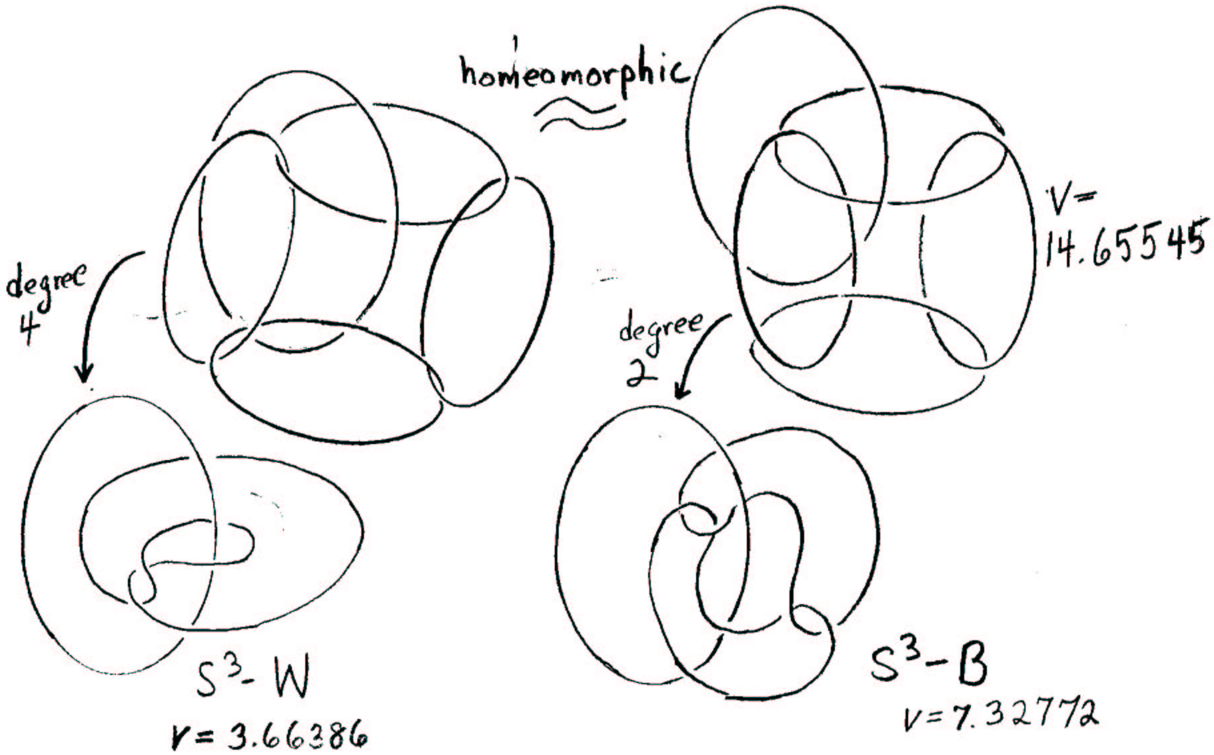
Definition 6.7.1. If Γ_1 and Γ_2 are two discrete subgroups of isometries of H^n, then Γ_1 is *commensurable* with Γ_2 if Γ_1 is conjugate (in the group of isometries of H^n) to a group Γ_1' such that $\Gamma' \cap \Gamma_2$ has finite index in Γ_1' and in Γ_2.

Definition 6.7.2. Two manifolds M_1 and M_2 are *commensurable* if they have finited sheeted covers $\tilde{M}_1$ and $\tilde{M}_2$ which are homeomorphic.

Commensurability in either sense is an equivalence relation, as the reader may easily verify. 6.29

Labelled 6.7.3.ex

Example 6.7.3. If W is the Whitehead link and B is the Borromean rings, then $S^3 - W$ has a four-sheeted cover homeomorphic with a two sheeted cover of $S^3 - B$:



The homeomorphism involves cutting along a disk, twisting 360° and gluing back. Thus $S^3 - W$ and $S^3 - B$ are commensurable. One can see that $\pi_1(S^3 - W)$ and $\pi_1(S^3 - B)$ are commensruable as discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$ by considering the tiling of H^3 by regular ideal octahedra. Both groups preserve this tiling, so they are contained in the full group of symmetries of the octahedral tiling, with finite index. Therefore, they intersect each other with finite index.

$$\pi_1(S^3 - B) \subset \text{Symmetries (octahedral tiling)} \supset \pi_1(S^3 - W)$$
$$\pi_1(S^3 - B) \supset \pi_1(S^3 - B) \cap \pi_1(S^3 - W) \subset \pi_1(S^3 - W)$$

6.30

Warning. Two groups Γ_1 and Γ_2 can be commensurable, and yet not be conjugate to subgroups of finite index in a single group.

Proposition 6.7.3. *If M_1 is a complete hyperbolic manifold with finite volume and M_2 is commensurable with M_1, then M_2 is homotopy equivalent to a complete hyperbolic manifold.*

Proof. This is a corollary of Mostow's theorem. Under the hypotheses, M_2 has a finite cover M_3 which is hyperbolic. M_3 has a finite cover M_4 which is a regular cover of M_2, so that $\pi_1(M_4)$ is a normal subgroup of $\pi_1(M_2)$. Consider the action of $\pi_1(M_2)$ on $\pi_1(M_4)$ by conjugation. $\pi_1(M_4)$ has a trivial center, so in other words the action of $\pi_1(M_4)$ on itself is effective. Then for every $\alpha \in \pi_1(M_2)$, since some power of α^k is in $\pi_1(M_4)$, α must conjugate $\pi_1(M_4)$ non-trivially. Thus $\pi_1(M_2)$ is isomorphic to a group of automorphisms of $\pi_1(M_4)$, so by Mostow's theorem it is a discrete group of isometries of H^n. □

In the three-dimensional case, it seems likely that M_1 would actually be hyperbolic. Waldhausen proved that two Haken manifolds which are homotopy equivalent are homeomorphic, so this would follow whenever M_1 is Haken. There are some sorts of properties of three-manifolds which do not change under passage to a finite-sheeted cover. For this reason (and for its own sake) it would be interesting to have a better understanding of the commensurability relation among three-manifolds. This is difficult to approach from a purely topological point of view, but there is a great deal of information about commensurability given by a hyperbolic structure. For instance, in the case of a complete non-compact 6.31

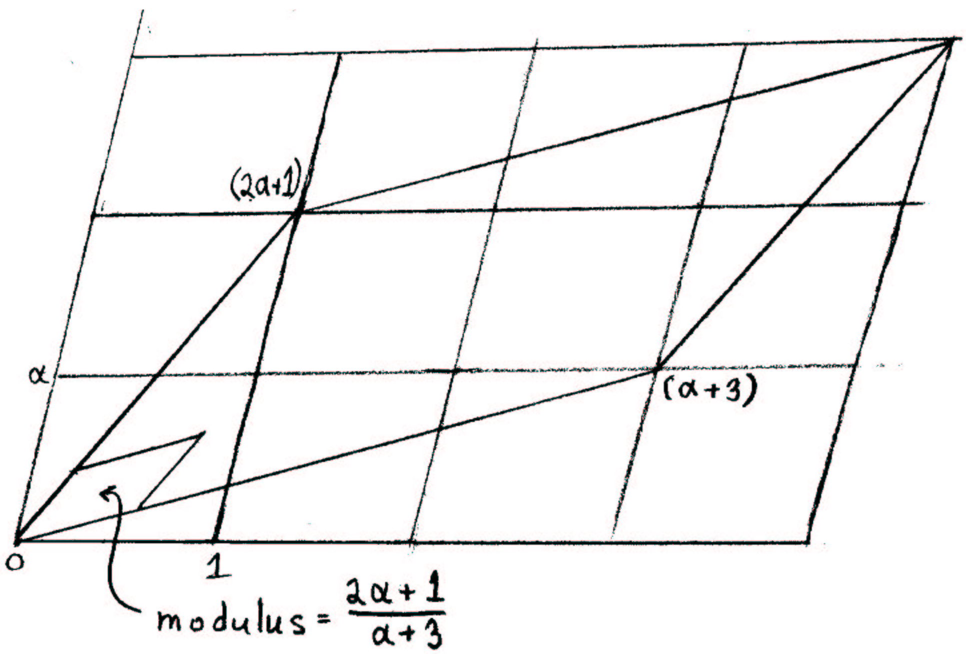
hyperbolic three-manifold M of finite volume, each cusp gives a canonical Euclidean structure on a torus, well-defined up to similarity. A convenient invariant for this structure is obtained by arranging M so that the cusp is the point at ∞ in the upper half space model and one generator of the fundamental group of the cusp is a translation $z \mapsto z + 1$. A second generator is then $z \mapsto z + \alpha$. The set of complex numbers $\alpha_1 \dots \alpha_k$ corresponding to various cusps is an invariant of the commensurability class of M well-defined up to the equivalence relation

$$\alpha_i \sim \frac{n\alpha_i + m}{p\alpha_i + q},$$

where

$$n, m, pq \in \mathbb{Z}, \qquad \begin{vmatrix} n & m \\ p & q \end{vmatrix} \neq 0.$$

(n, m, p and q depend on i).



6.32

In particular, if $\alpha \sim \beta$, then they generate the same fields $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$.

Note that these invariants α_i are always algebraic numbers, in view of

PROPOSITION 6.7.4. *If* Γ *is a discrete subgroup of* $\mathrm{PSL}(2, \mathbb{C})$ *such that* H^3/Γ *has finite volume, then* Γ *is conjugate to a group of matrices whose entries are algebraic.*

PROOF. This is another easy consequence of Mostow's theorem. Conjugate Γ so that some arbitrary element is a diagonal matrix

$$\begin{bmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{bmatrix}$$

and some other element is upper triangular,

$$\begin{bmatrix} \lambda & x \\ 0 & \lambda^{-1} \end{bmatrix}.$$

The component of Γ in the algebraic variety of representations of Γ having this form is 0-dimensional, by Mostow's theorem, so all entries are algebraic numbers. $\square$

One can ask the more subtle question, whether all entries can be made algebraic integers. Hyman Bass has proved the following remarkable result regarding this question:

THEOREM 6.7.5 (Bass). *Let* M *be a complete hyperbolic three-manifold of finite volume. Then either* $\pi_1(M)$ *is conjugate to a subgroup of* $\mathrm{PSL}(2, \mathcal{O})$, *where* $\mathcal{O}$ *is the ring of algebraic integers, or* M *contains a closed incompressible surface (not homotopic to a cusp).*

The proof is out of place here, so we omit it. See Bass. As an example, very few knot complements seem to contain non-trivial closed incompressible surfaces. The property that a finitely generated group Γ is conjugate to a subgroup of $\mathrm{PSL}(2, \mathcal{O})$ is equivalent to the property that the *additive* group of matrices generated by Γ 6.33
is finitely generated. It is also equivalent to the property that the trace of every element of Γ is an algebraic integer. It is easy to see from this that every group commensurable with a subgroup of $\mathrm{PSL}(2, \mathcal{O})$ is itself conjugate to a subgroup of $\mathrm{PSL}(2, \mathcal{O})$. (If $\operatorname{Tr} \gamma^n = a$ is an algebraic integer, then an eigenvalue λ of γ satisfies $\lambda^{2n} - a\lambda^n + 1 = 0$. Hence λ, λ^{-1} and $\operatorname{Tr} \gamma = \lambda + \lambda^{-1}$ are algebraic integers).

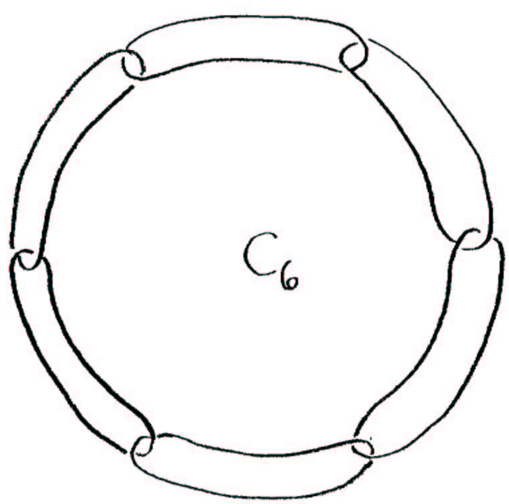
If two manifolds are commensurable, then their volumes have a rational ratio. We shall see examples in the next section of incommensurable manifolds with equal volume.

Questions 6.7.6. Does every commensurability class of discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$ have a finite collection of maximal groups (up to isomorphism)?

Is the set of volumes of three-manifolds in a given commensurability class a discrete set, consisting of multiples of some number V_0?

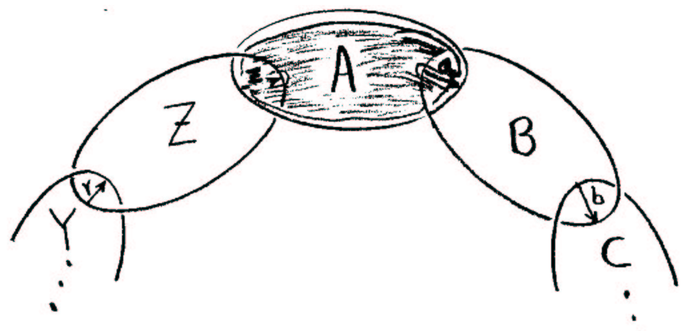
6.8. Some Examples

Example 6.8.1. Consider the k-link chain C_k pictured below:

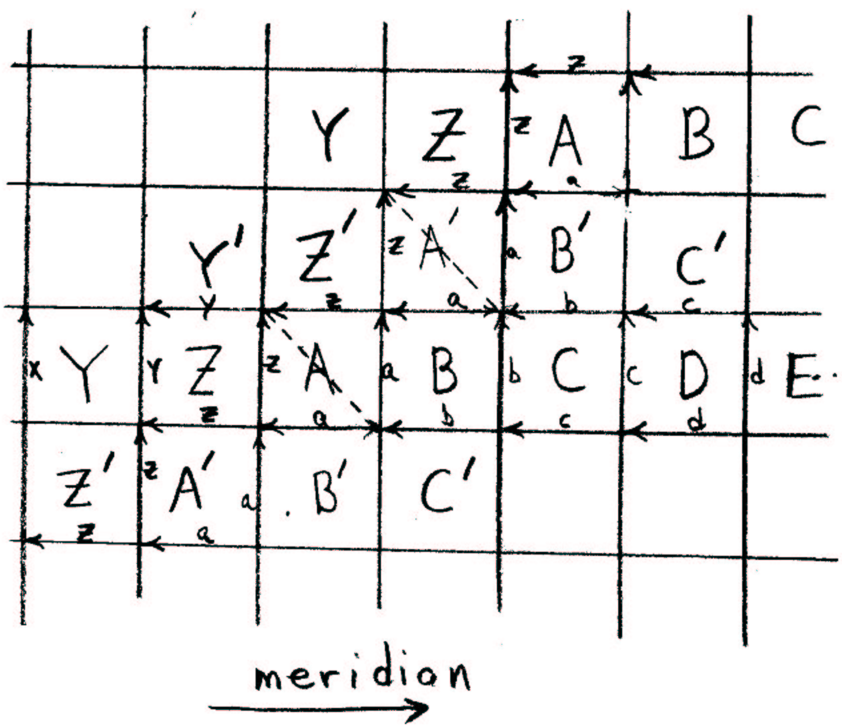


6.34

If each link of the chain is spanned by a disk in the simplest way, the complement of the resulting complex is an open solid torus.

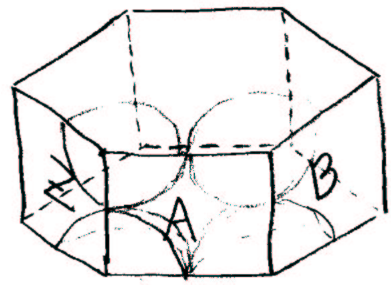


$S^3 - C_k$ is obtained from a solid torus, with the cell division below on its boundary, by deleting the vertices and identifying.

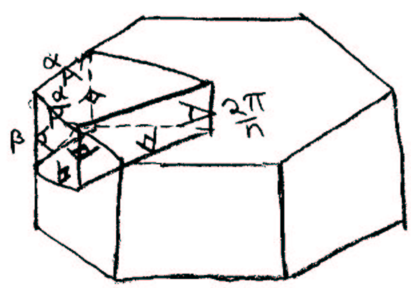


6.35

To construct a hyperbolic structure for $S^3 - C_k$, cut the solid torus into two drums.

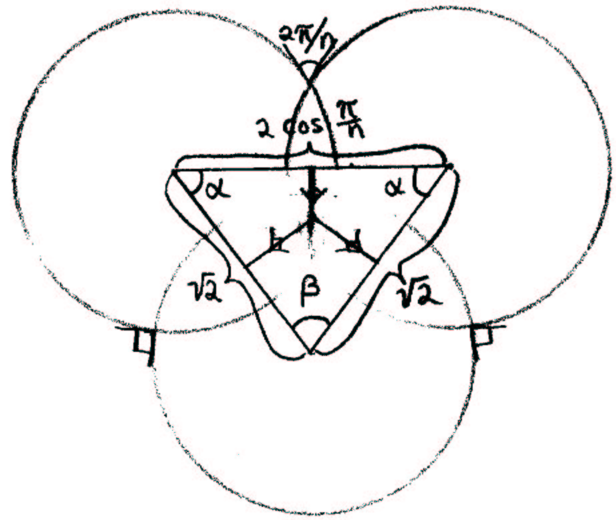


Let P be a regular k-gon in H^3 with all vertices on S^2_∞. If P' is a copy of P obtained by displacing P along the perpendicular to P through its center, then P' and P can be joined to obtain a regular hyperbolic drum. The height of P' must be adjusted so that the reflection through the diagonal of a rectangular side of the drum is an isometry of the drum. If we subdivide the drum into $2k$ pieces as shown,



6.36

the condition is that there are horospheres about the ideal vertices tangent to three faces. Placing the ideal vertex at ∞ in upper half-space, we have a figure bounded by three vertical Euclidean planes and three Euclidean hemispheres of equal radius r. Here is a view from above:



From this figure, we can compute the dihedral angles α and β of the drum to be

$$\alpha = \operatorname{arc}\cos\left(\frac{\cos \pi/k}{\sqrt{2}}\right), \quad \beta = \pi - 2\alpha.$$

Two copies of the drum with these angles can now be glued together to give a hyperbolic structure on $S^3 - C_k$. (Note that the total angle around an edge is $4\alpha + 2\beta = 2\pi$. Since the horospheres about vertices are matched up by the gluing maps, we obtain a complete hyperbolic manifold).

From Milnor's formula (6), p. 7.15, for the volume, we can compute some values.

6.37

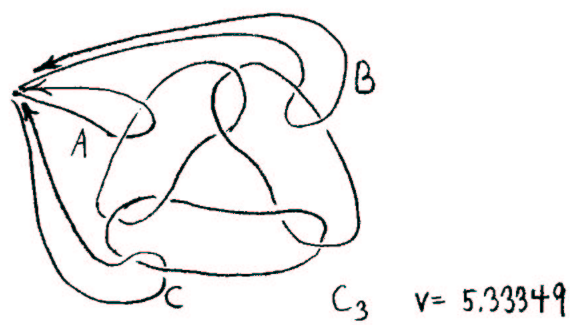
k	$v(S^3 - C_k)$	$v(S^3 - C_k)/k$	
2	0	0	(Seifert fiber space)
3	5.33349	1.77782	$\sim \mathrm{PSL}(2, \mathcal{O}_7)$
4	10.14942	2.53735	$\sim \mathrm{PSL}(2, \mathcal{O}_3)$
5	14.60306	2.92061	
6	18.83169	3.13861	
7	22.91609	3.27373	
10	34.691601	3.4691601	
50	182.579859	3.65159719	
200	732.673784	3.66336892	
1000	3663.84264	3.66384264	
8000	29310.8990	3.66386238	
∞	∞	3.66386238	Whitehead link

Note that the quotient space $(S^3 - C_k)/\mathbb{Z}_k$ by the rotational symmetry of C_k is obtained by generalized Dehn surgery on the White head link W, so the limit of 6.38
$v(C_k)/k$ as $k \to \infty$ is the volume of $S^3 - W$.

Note also that whenever k divides l, then there is a degree $\frac{l}{k}$ map from $S^3 - C_l$ to $S^3 - C_k$. This implies that $v(S^3 - C_l)/l > v(S^3 - C_k)/k$. In fact, from the table it is clear that these numbers are strictly increasing with k.

The cases $k = 3$ and 4 have particular interest.

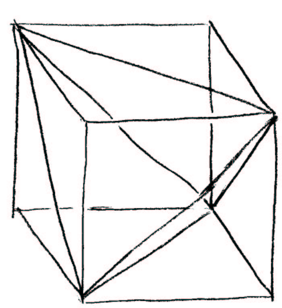
Example 6.8.2. The volume of $S^3 - C_3$ per cusp has a particularly low value (1.7778). The holonomy of the hyperbolic structure can be described by



$$H(A) = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$
$$H(B) = \begin{bmatrix} 1+\alpha & \alpha \\ -\alpha & 1-\alpha \end{bmatrix}$$
$$H(C) = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix}$$

where $\alpha = \frac{-1+\sqrt{-7}}{2}$. Thus $\pi_1(X^3 - C_3)$ is a subgroup of $\mathrm{PSL}(2, \mathcal{O}_7)$ where $\mathcal{O}_d$ is the ring of integers in $\mathbb{Q}\sqrt{-d}$. See §7.4. Referring to Humbert's formula 7.4.1, we find $v(H^3/\operatorname{PSL}(2, \mathcal{O}_7) = .8889149\ldots$, so $\pi_1(S^3 - C_3)$ has index 6 in this group.

Example 6.8.3. When $k = 4$, the rectangular-sided drum becomes a cube with all dihedral angles 60°. This cube may be subdivided into five regular ideal tetrahedra: 6.39



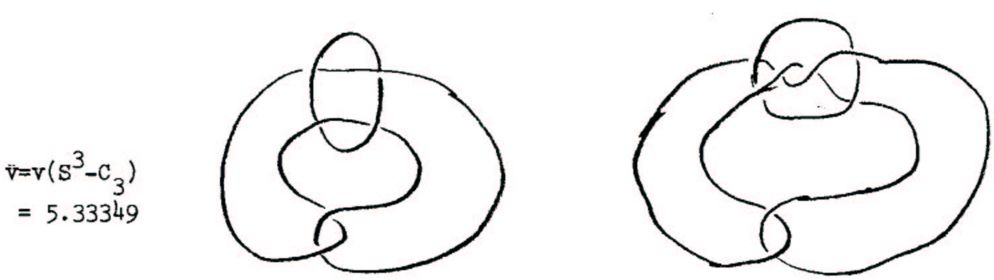
Thus $S^3 - C_4$ is commensurable with S^3- figure eight knot, since $\pi_1(S^3 - C_4)$ preserves a tiling of H^3 by regular ideal tetrahedra.

commensurable with $\mathrm{PSL}(2, \mathcal{O}_3)$

$S^3 - C_k$ is homeomorphic to many other link complements, since we can cut along any disk spanning a component of C_k, twist some integer number of times and glue back to obtain a link with a complement homeomorphic to that of C_k. Furthermore, if we glue back with a half-integer twist, we obtain a link whose complement is hyperbolic with the same volume as $S^3 - C_k$. This follows since twice-punctured spanning disks are totally geodesic thrice-punctured spheres in the hyperbolic structure of $S^3 - C_k$. The thrice-punctured sphere has a unique hyperbolic structure, and all six isotopy classes of diffeomorphisms are represented by isometries. 6.40

Using such operations, we obtain these examples for instance:

EXAMPLE 6.8.4.



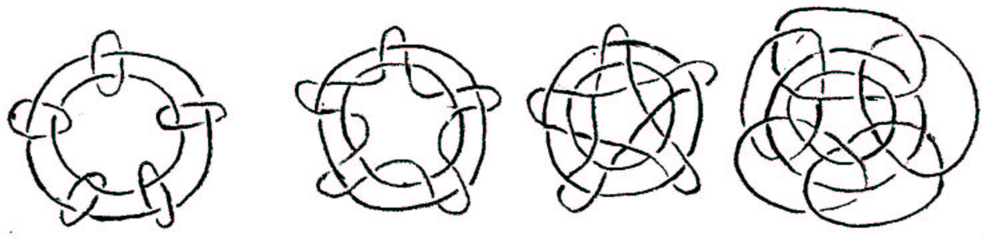
commensurable with C_3

The second link has a map to the figure-eight knot obtained by erasing a component of the link. Thus, by 6.5.6, we have

$$v(S^3 - C_3) = 5.33340\ldots > 2.02988 = v(S^3 - \text{figure eight knot}).$$

These links are commensurable with C_3, since they give rise to identical tilings of H^3 by drums. As another example, the links below are commensurable with C_{10}:

EXAMPLE 6.8.5.



$k = 5$ Commensurable with C_{10} $\quad v = 34.69616$

6.41

The last three links are obtained from the first by cutting along 5-times punctured disks, twisting, and gluing back. Since this gluing map is a diffeomorphism of the surface which extends to the three-manifold, it must come from an isometry of a 6-punctured sphere in the hyperbolic structure. (In fact, this surface comes from the top of a 10-sided drum).

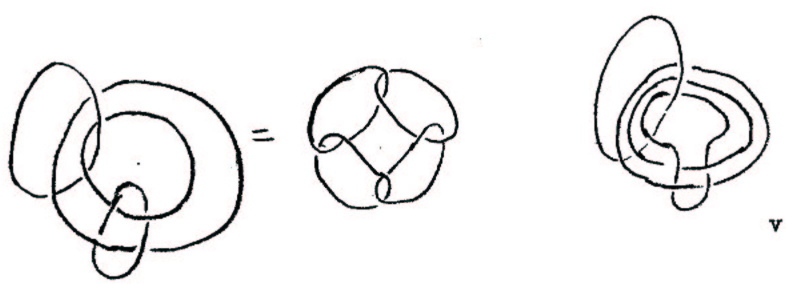
The compex modulus associated with a cusp of C_n is

$$\frac{1}{2} \quad \left(1 + \sqrt{\frac{1 + \sin^2 \frac{\pi}{n}}{\cos^2 \frac{\pi}{n}}}\ i\right).$$

Clearly we have an infinite family of incommensurable examples.

By passing to the limit $k \to \infty$ and dividing by $\mathbb{Z}$, we get these links commensurable with $S^3 - W$ and $S^3 - B$, for instance:

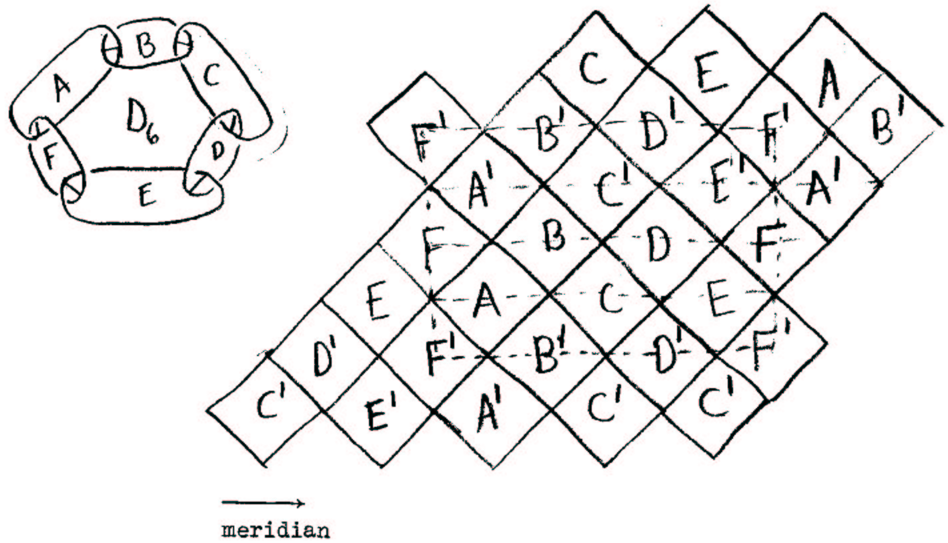
Example 6.8.6.



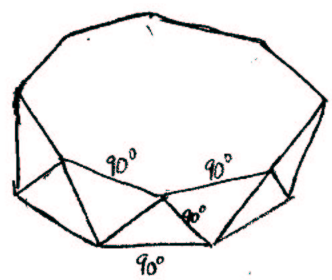
$v = v(S^3-B)$
$= 7.32772\ldots$

Many other chains, with different amounts of twist, also have hyperbolic structures. They all are obtained, topologically, by identifying faces of a tiling of the boundary of a solid torus by rectangles. Here is another infinite family $D_{2k}(\geq 3)$ which is easy to compute: 6.42

Example 6.8.7.



Hyperbolic structures can be realized by subdividing the solid torus into 4 drums with triangular sides:



6.43

Regular drums with all dihedral angles 90° can be glued together to give $S^3 - D_k$. By methods similar to Milnor's in 7.3, the formula for the volume is computed to be

$$v(S^3 - D_{2k}) = 8k\big(\mu(\tfrac{\pi}{4} + \tfrac{\pi}{2k}) + \mu(\tfrac{\pi}{4} - \tfrac{\pi}{2k})\big).$$

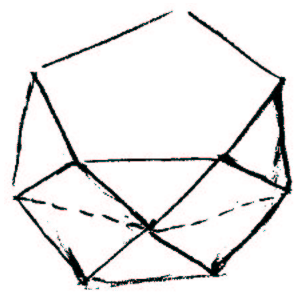
Thus we have the values

k	$v(S^3 - D_{2k})$	$v(S^3 - D_{2k})/(2k)$
3	14.655495	2.44257
4	24.09218	3.01152
5	32.55154	3.25515
6	40.59766	3.38314
100	732.750	3.66288
1000	7327.705	3.66386
∞	∞	3.66386

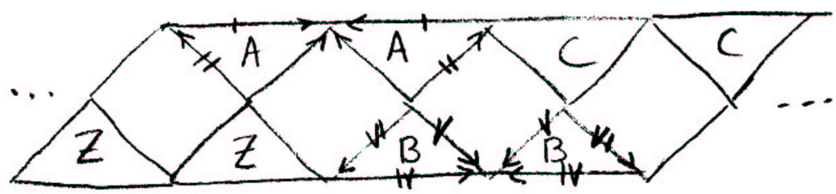
The cases $k = 3$ and $k = 4$ have algebraic significance. They are commensurable with $\mathrm{PSL}(2, \mathcal{O}_1)$ nad $\mathrm{PSL}(2, \mathcal{O}_2)$, respectively. When $k = 3$, the drum is an octahedron and $v(S^3 - D_{2k}) = 4v(S^3 - W)$.

Note that the volume of $(S^3 - D_{12})$ is 20 times the volume of the figure-eight knot complement. 6.44

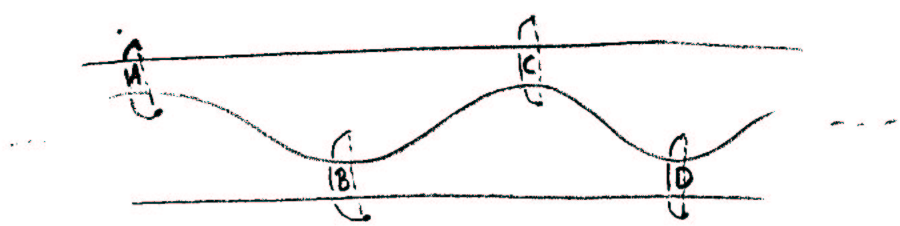
Two copies of the triangular-sided drum form this figure:



The faces may be glued in other patterns to obtain link complements. For instance, if k is even we can first identify



the triangular faces, to obtain a ball minus certain arcs and curves on the boundary.



6.45

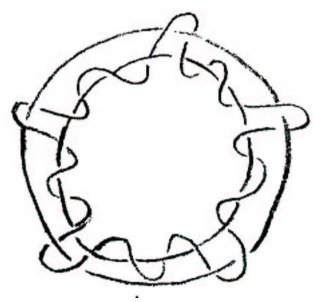
If we double this figure, we obtain a complete hyperbolic structure for the complement of this link, E_l:

Example 6.8.8.

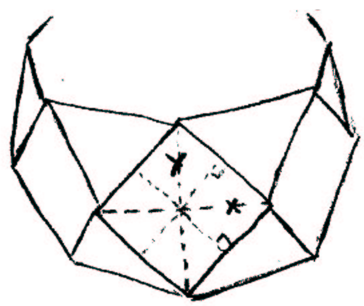
E_5

Alternatively, we can identify the boundary of the ball to obtain

EXAMPLE 6.8.9.



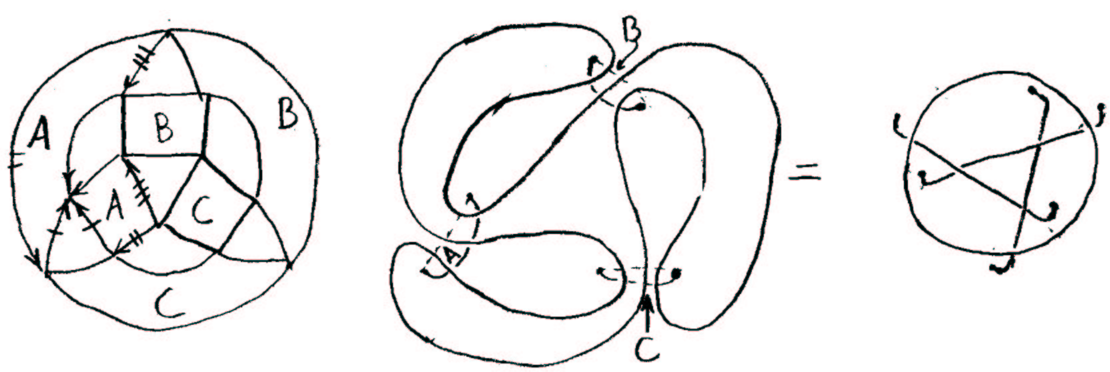
In these examples, note that the rectangular faces of the doubled drums



6.46

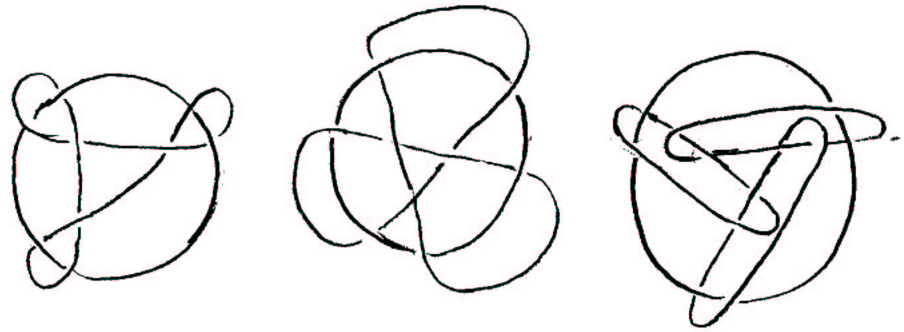
have complete symmetry, and some of the link complements are obtained by gluing maps which interchange the diagonals, while others preserve them. These links are generally commensurable even when they have the same volume; this can be proven by computing the moduli of the cusps.

There are many variations. Two copies of the drum with 8 triangular faces, glued together, give a cube with its corners chopped off. The 4-sided faces can be glued, to obtain the ball minus these arcs and curves:



The two faces of the ball may be glued together (isometrically) to give any of these link complements: 6.47

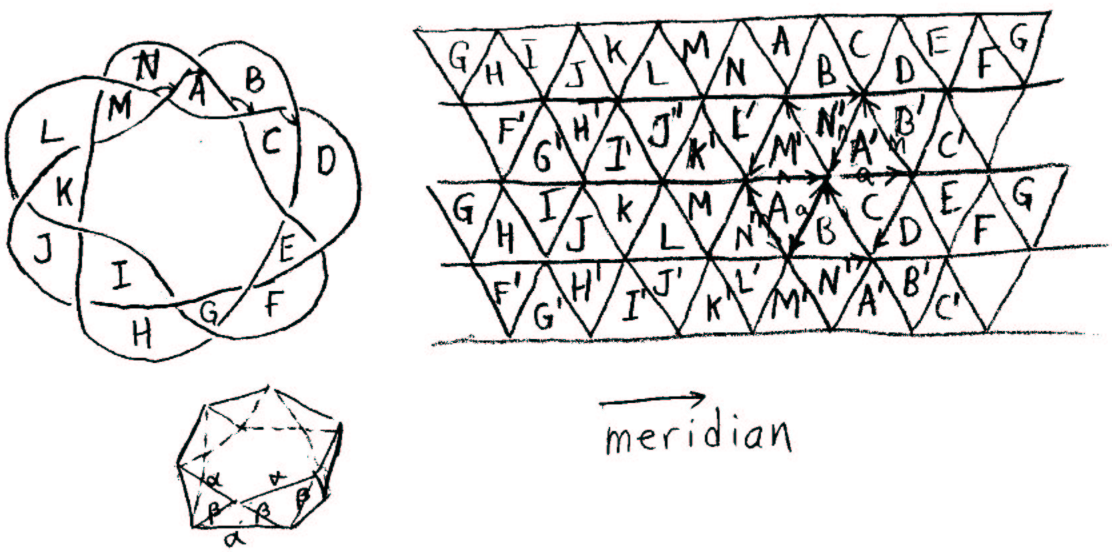
Example 6.8.10.



$v = 12.04692 = \frac{1}{2}v(S^3 - D_8) > v(C^3)$ (commensurable with $\mathrm{PSL}(2, \mathbb{Z}\sqrt{-2})$)

The sequence of link complements, F_n below can also be given hyperbolic structures obtained from a third kind of drum:

Example 6.8.11.



6.48

The regular drum is determined by its angles α and $\beta = \pi - \alpha$. Any pair of angles works to give a hyperbolic structure; one verifies that when the angle $\alpha = \arccos(\cos\frac{\pi}{2n} - \frac{1}{2})$, the hyperbolic structure is complete. The case $n = 1$ gives a trivial knot. In the case $n = 2$, the drums degenerate into simplices with 60° angles, and we obtain once more the hyperbolic structure on F_2 = figure eight knot. When $n = 3$, the angles are 90°, the drums become octahedra and we obtain $F_3 = B$. Passing to the limit $n = \infty$, and dividing by $\mathbb{Z}$, we obtain the following link, whose complement is commensurable with S^3 − figure eight knot:

EXAMPLE 6.8.12.

$v = 4 - 05977\ldots$

With these examples, many maps between link complements may be constructed. The reader should experiment for himself. One gets a feeling that volume is a very good measure of the complexity of a link *complement*, and that the ordinal structure is really inherent in three-manifolds.