William P. Thurston

The Geometry and Topology of Three-Manifolds

Electronic version 1.1 - March 2002

http://www.msri.org/publications/books/gt3m/

This is an electronic edition of the 1980 notes distributed by Princeton University. The text was typed in TEX by Sheila Newbery, who also scanned the figures. Typos have been corrected (and probably others introduced), but otherwise no attempt has been made to update the contents. Genevieve Walsh compiled the index.

Numbers on the right margin correspond to the original edition's page numbers.

Thurston's *Three-Dimensional Geometry and Topology*, Vol. 1 (Princeton University Press, 1997) is a considerable expansion of the first few chapters of these notes. Later chapters have not yet appeared in book form.

Please send corrections to Silvio Levy at levy@msri.org.

NOTE

Since a new academic year is beginning, I am departing from the intended order in writing these notes. For the present, the end of chapter 9 and chapters 10, 11 and 12, which depend heavily on chapters 8 and 9, are to be omitted. The tentative plan for the omitted parts is to cover the following topics:

The end of chapter 9—a more general discussion of algebraic convergence.

Chapter 10—Geometric convergence: an analysis of the possibilities for geometric limits.

Chapter 11. The Riemann mapping theorem; parametrizing quasi-conformal deformations. Extending quasi-conformal deformations of S^2_∞ to quasi-isometric deformations of H^3. Examples; conditions for the existence of limiting Kleinian groups.

Chapter 12. Boundaries for Teichmüller space, classification of diffeomorphisms of surfaces, algorithms involving the mapping class group of a surface.

CHAPTER 11

Deforming Kleinian manifolds by homeomorphisms of the sphere at infinity

A pseudo-isometry between hyperbolic three-manifolds gives rise to a quasi-conformal map between the spheres at infinity in their universal covering spaces. This is a key point in Mostow's proof of his rigidity theorem (Chapter 5). In this chapter, we shall reverse this connection, and show that a k-quasi-conformal map of S^2_∞ to itself gives rise to a k-quasi-isometry of hyperbolic space to itself. A self-map $f : X \to X$ of a metric space is a *k-quasi-isometry* if

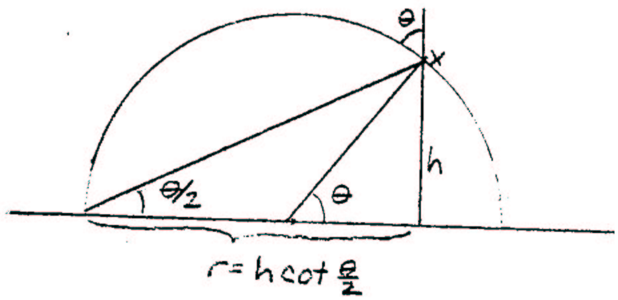
$$\frac{1}{k} d(fx, fy) \le d(x, y) \le k d(fx, fy)$$

for all x and y. By use of a version of the Riemann mapping theorem, the space of quasi-conformal maps of S^2 can be parametrized by the non-conformal part of their derivatives. In this way we obtain a remarkable global parametrization of quasi-isometric deformations of Kleinian manifolds by the Teichmüller spaces of their boundaries. 11.2

11.1. Extensions of vector fields

In §§8.4 and 8.12, we made use of the harmonic extensions of measurable functions on S_∞ to study the limit set of a Kleinian group. More generally, any tensor field on S^2_∞ extends, by a visual average, over H^3. To do this, first identify S^2_∞ with the unit sphere in $T_x(H^3)$, where x is a given point in H^3. If $y \in S^2_\infty$, this gives an identification $i : T_y(S^2_\infty) \to T_x(H^3)$. There is a reverse map $p : T_x(H^3) \to T_y(S^2_\infty)$ coming from orthogonal projection to the image of i. We can use i_* and p^* to take care of covariant tensor fields, like vector fields, and contravariant tensor fields, like differential forms and quadratic forms, as well as tensor fields of mixed type. The visual average of any tensor field T on S^2_∞ is thus a tensor field $\operatorname{av} T$, of the same type, on H^3. In general, $\operatorname{av} T$ needs to be modified by a constant to give it the right boundary behavior.

We need some formulas in order to make computations in the upper half-space model. Let x be a point in upper half-space, at Euclidean height h above the bounding plane $\mathbb{C}$. A geodesic through x at angle θ from the vertical hits $\mathbb{C}$ at a distance $r = h\cot(\theta/2)$ from the foot z_0 of the perpendicular from x to $\mathbb{C}$.



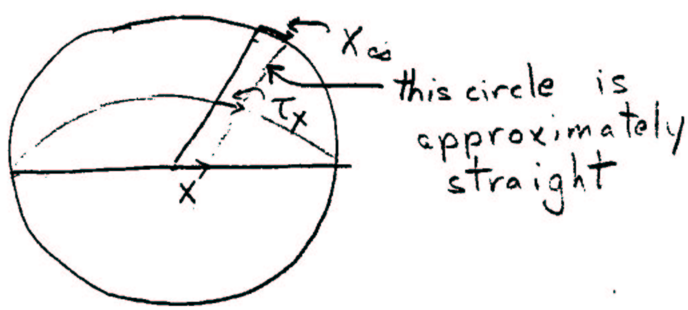
11.3

Thus, $dr = -(h/2)\csc^2(\theta/2)\,d\theta = -\frac{1}{2}(h+r^2/h)\,d\theta$. Since the map from the visual sphere at x to S^2_∞ is conformal, it follows that

$$dV_x = 4\Big(h + \frac{r^2}{h}\Big)^{-2}\,d\mu,$$

where μ is Lebesgue measure on $\mathbb{C}$ and V_x is visual measure at x.

Any tensor T at the point x pushes out to a tensor field T_∞ on $S^2_\infty = \hat{C}$ by the maps i^* and p_*. When X is a vector, then X_∞ is a holomorphic vector field, with derivative field, with derivative $\pm\|X\|$ at its zeros. To see this, let τ_X be the vector field representing the infinitesimal isometry of translation in the direction X. The claim is that $X_\infty = \tau_X|S_\infty$. This may be seen geometrically when X is at the center in the Poincaré disk model.



Alternatively if X is a vertical unit vector in the upper half-space, then we can compute that

$$X_\infty = -\sin\theta\,\frac{\partial}{\partial\theta} = \frac{h}{2}\,\frac{\sin\theta}{\sin^2\theta/2}\,\frac{\partial}{\partial r} = r\frac{\partial}{\partial r} = (z - z_0)\,\frac{\partial}{\partial z},$$

where z_0 is the foot of the perpendicular from x to $\mathbb{C}$. This clearly agrees with the corresponding infinitesimal isometry. (As a "physical" vector field, $\partial/\partial z$ is the same as the unit horizontal vector field, $\partial/\partial x$, on $\mathbb{C}$. The reason for this notation is that the differential operators $\partial/\partial x$ and $\partial/\partial z$ have the same action on holomorphic

functions: they are directional derivatives in the appropriate direction. Even though 11.4
the complex notation may at first seem obscure, it is useful because it makes it meaningful to multiply vectors by complex numbers.)

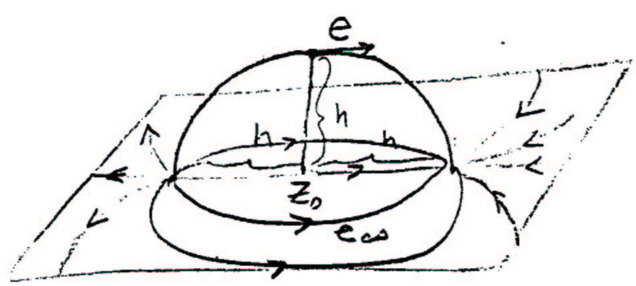
When g is the standard inner product on $T_x(H^3)$, then

$$g_\infty(Y_1, Y_2) = 4\Big(h + \frac{r^2}{h}\Big)^{-2} Y_1 \cdot Y_2,$$

where $Y_1 \cdot Y_2$ is the inner product of two vectors on $\mathbb{C}$.

Let us now compute $\mathrm{av}(\partial/\partial z)$. By symmetry considerations, it is clear that $\mathrm{av}(\partial/\partial z)$ is a horizontal vector field, parallel to $\partial/\partial z$. Let e be the vector of unit hyperbolic length, parallel to $\partial/\partial z$ at a point x in upper half-space. Then

$$e_\infty = -\frac{1}{2h}(z - z_0 - h)(z - z_0 + h)\frac{\partial}{\partial z}.$$



We have

$$\mathrm{av}\,\frac{\partial}{\partial z} = \frac{1}{4\pi}\int_{S^2} i_x\Big(\frac{\partial}{\partial z}\Big)\, dV_x,$$

so

$$\begin{aligned}\mathrm{av}\,\frac{\partial}{\partial z}\cdot e &= \frac{1}{4\pi}\int_{\mathbb{C}} g_\infty\Big(\frac{\partial}{\partial z}, e_\infty\Big)\, dV_x \\ &= \frac{1}{4\pi}\int_{\mathbb{C}} \mathrm{Re}\Big(-\frac{1}{2h}(z-z_0)^2 - h^2\Big)\, 16\Big(h + \frac{r^2}{h}\Big)^{-4} d\mu.\end{aligned}$$

Clearly, by symmetry, the term involving $\mathrm{Re}(z - z_0)^2$ integrates to zero, so we have 11.5

$$\begin{aligned}\mathrm{av}\,\frac{\partial}{\partial z}\cdot e &= \frac{1}{4\pi}\int_0^\infty \int_0^{2\pi} r\, d\theta \cdot 8h\Big(h + \frac{r^2}{h}\Big)^{-4} dr \\ &= -\frac{2h^2}{3}\Big(h + \frac{r^2}{h}\Big)^{-3}\Big|_0^\infty = \Big(\frac{2}{3}\Big)\frac{1}{h}.\end{aligned}$$

Note that the hyperbolic norm of $\mathrm{av}(\partial/\partial z)$ goes to ∞ as $h \to 0$, while the Euclidean norm is the constant $\frac{2}{3}$.

We now introduce the fudge factor by defining the extension of a vector field X on S^2_∞ to be

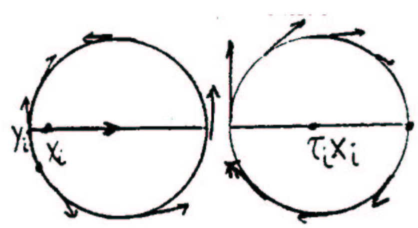
$$\operatorname{ex}(X) = \begin{cases} \frac{3}{2}\operatorname{av}(X) & \text{in } H^3, \\ X & \text{on } S^2_\infty. \end{cases}$$

Proposition 11.1.1. *If X is continuous or Lipschitz, then so is $\operatorname{ex}(X)$. If X is holomorphic, then $\operatorname{ex}(X)$ is an infinitesimal isometry.*

Proof. When X is an infinitesimal translation of $\mathbb{C}$, then $\operatorname{ex}(X)$ is the same infinitesimal translation of upper half-space. Thus every "parabolic" vector field (with a zero of order 2) on S^2_∞ extends to the correct infinitesimal isometry. A general holomorphic vector field on S^2_∞ is of the form $(az^2 + bz + c)(\partial/\partial z)$ on $\mathbb{C}$. Since such a vector field can be expressed as a linear combination of the parabolic vector fields $\partial/\partial z$, $z^2\,\partial/\partial z$ and $(z-1)^2\,\partial/\partial z$, it follows that every holomorphic vector field extends to the correct infinitesimal isometry.

Suppose X is continuous, and consider any sequence $\{x_i\}$ of points in H^3 converging to a point at infinity. Bring x_i back to the origin O by the translation τ_i along 11.6 the line $\overline{Ox_i}$. If x_i is close to S^2_∞, τ_i spreads a small neighborhood of the endpoint y_i of the geodesic from O to x_i over almost all the sphere. $\tau_{i*}X$ is large on most of the sphere, except near the antipodal point to y_i, so it is close to a parabolic vector field P_i, in the sense that for any ϵ, and sufficiently high i,

$$\|\tau_{i*}X - P_i\| \le \epsilon \cdot \lambda_i,$$



where λ_i is the norm of the derivative of τ_i at y_i. Here P_i is the parabolic vector field agreeing with $\tau_{i*}X$ at y_i, and 0 at the antipodal point of y_i. It follows that

$$\operatorname{ex} X(x_i) - X(y_i) \to 0,$$

so X is continuous along ∂B^3. Continuity in the interior is self-evident (if you see the evidence).

Suppose now that X is a vector field on $S^2_\infty \subset \mathbb{R}^3$ which has a global Lipschitz constant

$$k = \sup_{y,y' \in S^2} \frac{\|X_y - X_{y'}\|}{\|y - y'\|}.$$

Then the translates $\tau_{i*}X$ satisfy

$$\|\tau_{i*}X - P_i\| \leq B,$$

where B is some constant independent of i. This may be seen by considering stereographic projection from the antipodal point of y_i. The part of the image of $X - \tau_{i*}^{-1}P_i$ in the unit disk is Lipschitz and vanishes at the origin. When τ_{i*} is applied, the resulting vector field on $\mathbb{C}$ satisfies a linear growth condition (with a uniform growth constant). This shows that, on S^2_∞, $\|\tau_{i*}X - P_i\|$ is uniformly bounded in all but a 11.7
neighborhood of the antipodal point of Y, where boundedness is obvious. Then

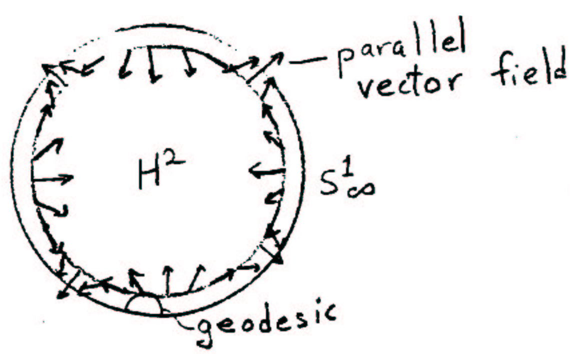
$$\|\operatorname{ex} X(x_i) - \operatorname{ex} \tau_{i*}P_i(x_i)\| \leq B \cdot \mu_i,$$

where μ_i is the norm of the derivative of τ_i^{-1} at the origin in B^3, or $1/\lambda_i$ up to a bounded factor.

Since μ_i is on the order of the (Euclidean) distance of x_i from y_i, it follows that $\operatorname{ex} X$ is Lipschitz along S^2_∞.

To see that $\operatorname{ex} X$ has a global Lipschitz constant in B^3, consider $x \in B^3$, and let τ be a translation as before taking x to O, and P a parabolic vector field approximating $\tau_* X$. The vector fields $\tau_* X - P$ obtained in this way are uniformly bounded, so it is clear that the vector fields $\operatorname{ex}(\tau_* X - P)$ have a uniform Lipschitz constant at the origin in B^3. By comparison with the upper half-space model, where τ_* can be taken to be a similarity, we obtain a uniform bound on the local Lipschitz constant for $\operatorname{ex}(X - \tau_*^{-1}P)$ at an arbitrary point x. Since the vector fields $\tau_*^{-1}P$ are uniformly Lipschitz, it follows that X is globally Lipschitz. □

Note that the stereographic image in $\mathbb{C}$ of a uniformly Lipschitz vector field on S^2_∞ is not necessarily uniformly Lipschitz—consider $z^2\,\partial/\partial z$, for example. This is explained by the large deviation of the covariant derivatives on S^2_∞ and on $\mathbb{C}$ near the point at infinity. Similarly, a uniformly Lipschitz vector field on B^3 is not generally uniformly Lipschitz on H^3. In fact, because of the curvature of H^3, a uniformly Lipschitz vector field on H^3 must be bounded; such vector fields correspond precisely to those Lipschitz vector fields on B^3 which vanish on ∂B^3. 11.8



A hyperbolic parallel vector field along a curve near S^1_∞ appears to turn rapidly.

The significance of the Lipschitz condition stems from the elementary fact that Lipschitz vector fields are uniquely integrable. Thus, any isotopy φ_t of the boundary of a Kleinian manifold $O_\Gamma = (B^3 - L_\Gamma)/\Gamma$ whose time derivative $\dot\varphi_t$ is Lipschitz as a vector field on $I \times \partial O_\Gamma$ extends canonically to an isotopy $\operatorname{ex}\varphi_t$ on O_Γ. One may see this most simply by observing that the proof that $\operatorname{ex} X$ is Lipschitz works locally.

A *k-quasi-isometric* vector field is a vector field whose flow, φ_t, distorts distances at a rate of at most k. In other words, for all x, y and t, φ_t must satisfy

$$e^{-kt} d(x,y) \le d(\varphi_t x, \varphi_t y) \le e^{kt} d(x,y).$$

A k-Lipschitz vector field on a Riemannian manifold is k-quasi-isometric. In fact, a Lipschitz vector field X on B^3 which is tangent to ∂B^3 is quasi-isometry as a vector field on $H^3 = \operatorname{int} B^3$. This is clear in a neighborhood of the origin in B^3. To see this for an arbitrary point x, approximate X near x by a parabolic vector field, as in the proof of 11.1.1, and translate x to the origin.

In particular, if φ_t is an isometry of ∂O_Γ with Lipschitz time derivative, then $\operatorname{ex}\varphi_t$ has a quasi-isometric time derivative, and φ_1 is a quasi-isometry.

Our next step is to study the derivatives of $\operatorname{ex} X$, so we can understand how a 11.9
more general isotopy such as $\operatorname{ex}\varphi_t$ distorts the hyperbolic metric. From the definition of $\operatorname{ex} X$, it is clear that ex is natural, or in other words,

$$\operatorname{ex}(T_* X) = T_*\big(\operatorname{ex}(X)\big)$$

where T is an isometry of H^3 (extended to S^2_∞ where appropriate).

If X is differentiable, we can take the derivative at $T = \mathrm{id}$, yielding

$$\operatorname{ex}[Y, X] = [Y, \operatorname{ex} X]$$

for any infinitesimal isometry Y. If Y is a pure translation and X is any point on the axis of Y, then $\nabla_X Y_x = 0$. (Here, ∇ is the hyperbolic covariant derivative, so $\nabla_Z W$ is the directional derivative of a vector field W in the direction of the vector field Z.) Using the formula

$$[Y, X] = \nabla_Y X - \nabla_X Y,$$

we obtain:

Proposition 11.1.2. *The direction derivative of* $\operatorname{ex} X$ *in the direction* Y_x*, at a point* $x \in H^3$*, is*

$$\nabla_{Y_x} \operatorname{ex} X = \operatorname{ex}[Y, X],$$

where Y *is any infinitesimal translation with axis through* x *and value* Y_x *at* x. □

The covariant derivative ∇X_x, which is a linear transformation of the tangent space $T_x(H^3)$ to itself, can be expressed as the sum of its symmetric and antisymmetric parts, 11.10

$$\nabla X = \nabla^s X + \nabla^a X,$$

where

$$\nabla^s_Y X \cdot Y' = \tfrac{1}{2}(\nabla_Y X \cdot Y' + \nabla_{Y'} X \cdot Y)$$

and

$$\nabla^a_Y X \cdot Y' = \tfrac{1}{2}(\nabla_Y X \cdot Y' - \nabla_{Y'} X \cdot Y).$$

The anti-symmetric part $\nabla^a X$ describes the infinitesimal rotational effect of the flow generated by X. It can be described by a vector field curl X pointing along the axis of the infinitesimal rotation, satisfying the equation

$$\nabla^a_Y X = \tfrac{1}{2} \operatorname{curl} X \times Y$$

where $\times$ is the cross-product. If e_0, e_1, e_2 forms a positively oriented orthonormal frame at X, the formula is

$$\operatorname{curl} X = \sum_{i \in \mathbb{Z}/3} (\nabla_{e_i} X \cdot e_{i+1} - \nabla_{e_{i+1}} X \cdot e_i)\, e_{i+2}.$$

Consider now the contribution to $\operatorname{ex} X$ from the part of X on an infinitesimal area on S^2_∞, centered at y. This part of $\operatorname{ex} X$ has constant length on each horosphere about y (since the first derivative of a parabolic transformation fixing y is the identity), and it scales as e^{-3t}, where t is a parameter measuring distance between horospheres and increasing away from y. (Linear measurements scale as e^{-t}. Hence, there is a factor of e^{-2t} describing the scaling of the apparent area from a point in H^3, and a factor of $-e^t$ representing the scaling of the lengths of vectors.) Choose positively oriented coordinates t, x_1, x_2, so that $ds^2 = dt^2 + e^{2t}(dx_1^2 + dx_2^2)$, and this infinitesimal contribution to $\operatorname{ex} X$ is in the $\partial/\partial x_1$ direction. Let e_0, e_1 and e_2 be unit vectors in the three coordinate directions. The horospheres $t =$ constant are parallel surfaces, of constant normal curvature 1 (like the unit sphere in $\mathbb{R}^3$), so you can see that 11.11

$$\nabla_{e_0} e_0 = \nabla_{e_0} e_1 = \nabla_{e_0} e_2 = 0$$

$$\nabla_{e_1} e_0 = +e_1, \nabla_{e_1} e_1 = -e_0, \nabla_{e_1} e_2 = 0$$

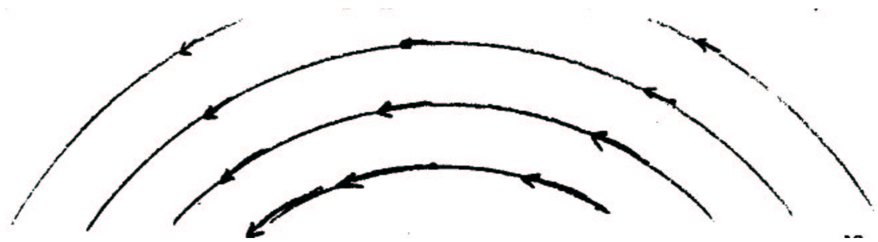
and

$$\nabla_{e_2} e_0 = e_2, \nabla_{e_2} e_2 = -e_0, \nabla_{e_2} e_1 = 0.$$

(This information is also easy to compute by using the Cartan structure equations.) The infinitesimal contribution to $\operatorname{ex} X$ is proportional to $Z = e^{-3t} e_1$, so

$$\begin{aligned} \operatorname{curl} Z &= (\nabla_{e_0} Z \cdot e_1 - \nabla_{e_1} Z \cdot e_0)\, e_2 \\ &= -2e^{-3t} e_2. \end{aligned}$$

(The curl is in the opposite sense from the curving of the flow lines because the effect of the flow speeding up on inner horospheres is stronger.)



This is proportional to the contribution of iX to $\operatorname{ex} iX$ from the same infinitesimal region, so we have

Proposition 11.1.3.

$$\operatorname{Curl}(\operatorname{ex} X) = 2\operatorname{ex}(iX),$$

and consequently

$$\operatorname{Curl}^2(\operatorname{ex} X) = -4\operatorname{ex} X$$

and

$$\operatorname{Div}(\operatorname{ex} X) = 0.$$

Proof. The first statement follows by integration of the infinitesimal contributions to $\operatorname{curl}\operatorname{ex} X$. The second statement

$$\operatorname{curl}\operatorname{curl}\operatorname{ex} X = 2\operatorname{curl}\operatorname{ex} iX = 4\operatorname{ex} i^2 X = -4\operatorname{ex} X,$$

is immediate. The third statement follows from the identity $\operatorname{div}\operatorname{curl} Y = 0$, or by considering the infinitesimal contributions to $\operatorname{ex} X$. □

The differential equation $\operatorname{curl}^2\operatorname{ex} X + \operatorname{ex} X = 0$ is the counterpart to the statement that $\operatorname{ex} f = \operatorname{av} f$ is harmonic, when f is a function. The symmetric part $\nabla^s X$ of the covariant derivative measures the infinitesimal strain, or distortion of the metric, of the flow generated by X. That is, if Y and Y' are vector fields invariant by the flow of X, so that $[X,Y] = [X,Y'] = 0$, then $\nabla_Y X = \nabla_X Y$ and $\nabla_{Y'} X = \nabla_X Y'$, so the derivative of the dot product of Y and Y' in the direction X, by the Leibniz rule is

$$\begin{aligned} X(Y\cdot Y') &= \nabla_X Y\cdot Y' + Y\cdot\nabla_X Y' \\ &= \nabla_Y X\cdot Y' + \nabla_{Y'} X\cdot Y \\ &= 2(\nabla^s_Y X\cdot Y'). \end{aligned}$$

The symmetric part of ∇ can be further decomposed into its effect on volume and a part with trace 0,

$$\nabla^s X = \tfrac{1}{3}\operatorname{Trace}(\nabla^s X)\cdot I + \nabla^{s_0} X.$$

Here, I represents the identity transformation (which has trace 3 in dimension 3). Note that $\text{trace}\,\nabla^s X = \text{trace}\,\nabla X = \text{divergence}\,X = \Sigma \nabla_{e_i} X \cdot e_i$ where $\{e_i\}$ is an orthonormal basis, so for a vector field of the form $\text{ex}\,X$, $\nabla^s \,\text{ex}\,X = \nabla^{s_0}\,\text{ex}\,X$. 11.13

Now let us consider the analagous decomposition of the covariant derivative ∇X of a vector field on the Riemann sphere (or any surface). There is a decomposition

$$\nabla X = \nabla^a X + \tfrac{1}{2}(\text{trace}\,\nabla X)I + \nabla^{s_0} X.$$

Define linear maps ∂ and $\overline{\partial}$ of the tangent space to itself by the formulas

$$\partial X(Y) = \tfrac{1}{2}\{\nabla_Y X - i\nabla_{iY} X\}$$

and

$$\overline{\partial} X(Y) = \tfrac{1}{2}\{\nabla_Y X + i\nabla_{iY} X\}$$

for any vector field Y. (On a general surface, i is interpreted as a 90° counter-clockwise rotation of the tangent space of the surface.)

Proposition 11.1.4.

$$\begin{aligned}\partial X &= \tfrac{1}{2}(\text{trace}\,\nabla X)I + \nabla^a X \\ &= \tfrac{1}{2}\{(\text{div}\,X)I + (\text{curl}\,X)iI\}\end{aligned}$$

and

$$\overline{\partial} X = \nabla^{s_0} X.$$

∂X is invariant under conformal changes of metric.

Remark (Notational remark). Any vector field on $\mathbb{C}$ be written $X = f(z)\partial/\partial z$, in local coordinates. The derivative of f can be written $df = f_x dx + f_y dy$. This can be re-expressed in terms of $dz = dx + idy$ and $d\bar{z} = dx - idy$ as

$$df = f_z dz + f_{\bar{z}} d\bar{z}$$

where

$$f_z = \tfrac{1}{2}(f_x - if_y)$$

and 11.14

$$f_{\bar{z}} = \tfrac{1}{2}(f_x + if_y).$$

Then $\partial f = f_z dz$ and $\bar{\partial} f = f_{\bar{z}} d\bar{z}$ are the complex linear and complex conjugate linear parts of the real linear map df. Similarly, $\partial X = f_z\, dz\, \partial/\partial z$ and $\bar{\partial} X = f_{\bar{z}}\, d\bar{z}\, \partial/\partial z$ are the complex linear and conjugate linear parts of the map $dX = \nabla X$.

Proof. If $L : \mathbb{C} \to \mathbb{C}$ is any real linear map, then

$$L = \tfrac{1}{2}(L - i \circ L \circ i) + \tfrac{1}{2}(L + i \circ L \circ i)$$

is clearly the decomposition into its complex linear and conjugate linear parts. A complex linear map, in matrix form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

is an expansion followed by a rotation, while a conjugate linear map in matrix form

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix},$$

is a symmetric map with trace 0.

To see that $\bar{\partial} X$ is invariant under conformal changes of metric, note that $\nabla_X iY = i\nabla_X Y$ and write $\bar{\partial} X$ without using the metric as

$$\begin{aligned} \bar{\partial} X(Y) &= \tfrac{1}{2}\{\nabla_Y X + i\nabla_{iY} X\} \\ &= \tfrac{1}{2}\{\nabla_Y X - \nabla_X Y + i\nabla_{iY} X - i\nabla_X iY\} \\ &= \tfrac{1}{2}\{[Y, X] + i[iY, X]\}. \qquad \square \end{aligned}$$

We can now derive a nice formula for $\nabla^s \operatorname{ex} X$:

Proposition 11.1.5. *For any vector $Y \in T_x(H^3)$ and any C^1 vector field X on S^2_∞,* 11.15

$$\nabla^s_Y \operatorname{ex} X = 3/4\pi \int_{S^2_\infty} i_*\big(\bar{\partial} X(Y_\infty)\big)\, dV_x.$$

Proof. Clearly both sides are symmetric linear maps applied to Y, so it suffices to show that the equation gives the right value for $\nabla_Y \operatorname{ex} X \cdot Y$. From 11.1.2, we have

$$\begin{aligned} \nabla_Y \operatorname{ex} X \cdot Y &= \operatorname{ex} [Y_\infty, X] \cdot Y \\ &= 3/8\pi \int_{S^2} [Y_\infty, X] \cdot Y_\infty \, dV_x \end{aligned}$$

and also, at the point x (where $\operatorname{ex} iY_\infty = 0$),

$$\begin{aligned} 0 &= [\operatorname{ex} iY_\infty, X] \cdot \operatorname{ex} iY_\infty \\ &= 3/8\pi \int_{S^2} [iY_\infty, X] \cdot iY_\infty \, dV_x \\ &= 3/8\pi \int_{S^2} -i\, [iY_\infty, X] \cdot Y_\infty \, dV_x. \end{aligned}$$

Therefore

$$\begin{aligned}\nabla^s_Y \operatorname{ex} X \cdot Y &= \nabla_Y \operatorname{ex} X \cdot Y \\ &= 3/8\pi \int_{S^2} [Y_\infty, X] \cdot Y_\infty + i\,[iY_\infty, X] \cdot Y_\infty \, dV_x \\ &= 3/4\pi \left(\int \bar{\partial} X(Y_\infty) \, dV_x \right) Y.\end{aligned}$$

□