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A symplectic approach to Arnold diffusion problems

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The purpose of this text is to present a symplectic approach to Arnold diffusion problems, that is, the existence of orbits of perturbed integrable systems along which the action variables experience a drift whose length is independent of the size of the perturbation. We chose to focus on the construction of orbits drifting along "chains of cylinders", taking for granted the existence of the chains. We however give a rather complete description of these chains, together with some elements on their symplectic features and some main ideas to prove their existence. We adopt the setting introduced by John Mather to prove the Arnold conjecture for perturbations of Tonelli Hamiltonians, which we see as the good one to set out the various (and numerous) problems of the construction, and give some ideas to show how the symplectic approach may enable one to enlarge its scope.

1. Introduction

In this text we denote by $\mathbb{A}^n = T^*\mathbb{T}^n$ the cotangent bundle of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, endowed with its angle-action coordinates (θ, r) and its usual exact-symplectic structure.

1. The questions addressed in this paper originate in the famous Boltzmann conjecture, rephrased in the modern mathematical language (following [54]) as:

> For (almost) all proper Hamiltonian function H on a $2n$-dimensional symplectic manifold and (almost) all real value e, the associated Hamiltonian vector field is ergodic on each connected component of $H^{-1}(e)$.

Forgetting about the real scope of this conjecture — certainly limited to m-body problems with very large m — it is well-known that the KAM theorem yields counterexamples to the previous statement as soon as $n \geq 2$. One can see the following weaker *quasiergodic* conjecture by Poincaré and Ehrenfest as an attempt to partially recover its possible dynamical applications:

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> For (almost) all proper Hamiltonian function H on a symplectic manifold and (almost) all real value e, the associated Hamiltonian vector field admits an orbit which is dense in $H^{-1}(e)$.

It turns out that the Poincaré–Ehrenfest conjecture is false too: this is a consequence of the KAM theorem if $n = 2$, while Herman proved (see [54]) that it is false for $n \geq 3$, at least on nonexact symplectic manifolds. He also asked the simpler — but still open — question of the existence of a C^∞ perturbation of $\frac{1}{2}\|r\|^2$ on $\mathbb{A}^n$ with a dense orbit on some energy level.

A possible way to state a correct but even weaker question in the spirit of the previous conjectures comes from [25], where Arnold introduced the first example of an "unstable" family of Hamiltonian systems on $\mathbb{A}^3$, namely:

$$H_\varepsilon(\theta, r) = r_1 + \tfrac{1}{2}(r_2^2 + r_3^2) + \varepsilon(\cos\theta_3 - 1) + \mu(\varepsilon)(\cos\theta_3 - 1)g(\theta), \quad (1)$$

where g is a suitably chosen trigonometric polynomial, $\varepsilon > 0$ is small enough and $\mu(\varepsilon) \ll \varepsilon$. The main result of Arnold is the existence of $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the system H_ε admits an "unstable solution" $\gamma_\varepsilon(t) = (\theta(t), r(t))$ such that

$$r_2(0) < 0, \quad r_2(T_\varepsilon) > 1, \quad (2)$$

for some (large) T_ε. Orbits experiencing this type of behavior are said to be *diffusion orbits*. In view of this result and the associated constructions, Arnold conjectured (see [25]) that for "typical" systems of the form

$$H_\varepsilon(\theta, r) = h(r) + \varepsilon f(\theta, r, \varepsilon) \quad (3)$$

on $\mathbb{A}^n$, $n \geq 3$, the projection in action of some orbits should visit any element of a prescribed collection of arbitrary open sets intersecting a connected component of a level set of h. One therefore gets an "asymptotic density" of the projection of the orbit onto the action space when the size of the perturbation tends to 0. Taking the variation of the angles into account, one can also produce examples of perturbations of $\frac{1}{2}\|r\|^2$ on the annulus $\mathbb{A}^3$ with orbits dense on subsets of Hausdorff dimension 5 inside an energy level; see [32].

2. The Arnold conjecture is directly related to the existence or nonexistence of particular invariant subsets acting as "barriers" inside an energy level. Assume that X is a complete vector field on a manifold M. Given some open connected subset O and a point x in M, consider the full orbit of O under the flow Φ of X:

$$\mathscr{O} = \Phi(\mathbb{R} \times O).$$

Hence $\mathscr{O}$ is the "accessibility domain" attached to O and its boundary $\partial\mathscr{O} = \operatorname{Adh}\mathscr{O} \setminus \mathscr{O}$ is invariant under the flow Φ. The existence of an orbit connecting O

and x is equivalent to x and O being in the same connected component of the complement of $\partial\mathscr{O}$.

Understanding the structure of the boundaries of the domains of accessibility is in general hopeless. However, in the discrete case of area-preserving twist maps of the annulus $X = \mathbb{T} \times [0, 1]$, Birkhoff's theory gives a satisfactory answer (see Appendix B). Consider a neighborhood $O = \mathbb{T} \times [0, \varepsilon[$ of the lower boundary and assume that $\mathscr{O} \subset \mathbb{T} \times [0, 1[$. Then by the standard trick of "filling the holes" (see [48]), one proves that the boundary $\partial\mathscr{O}$ admits a connected component which disconnects $\mathbb{T} \times]0, 1[$ and is the graph of a Lipschitz map $\mathbb{T} \to [0, 1]$. More generally, one proves in the same way the existence of orbits connecting any neighborhoods of the lower and upper essential circles bounding a Birkhoff zone: a first example of diffusion behavior.

In general, an area-preserving twist map of X has essential invariant circles in $\mathbb{T} \times]0, 1[$ and do not admit diffusion orbits starting arbitrarily close to $\mathbb{T} \times \{0\}$ and ending arbitrarily close to $\mathbb{T} \times \{1\}$. A crucial idea was introduced by Moeckel [48] and then by Le Calvez [41], who studied the diffusion properties of *bisystems* of maps on the annulus.[1] A bisystem is a pair of maps $(\varphi_0, \varphi_1) : X \circlearrowleft$, and one defines an orbit of (φ_0, φ_1) as a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_{n+1} = \varphi_i(x_n)$, with $i = 0$ or 1. It turns out that if φ_0 is an area-preserving twist map of $X = \mathbb{T} \times [0, 1]$ and $\varphi_1 : X \to X$ is area-preserving, then a sufficient condition for the bisystem (φ_0, φ_1) to admit an orbit connecting arbitrary neighborhoods of $\mathbb{T} \times \{0\}$ and $\mathbb{T} \times \{1\}$ is that both maps do not admit any essential invariant circle in common, apart from the boundary ones. The underlying idea, close to the setting of control theory, is that the action of φ_1 destroys the boundaries of accessibility of φ_0; see [42] for a study of diffusion bisystems of integrable Hamiltonian systems based on this type of methods.

The previous ideas have been generalized by Koropecki and Nassiri [35; 36] to the dynamics of bisystems of symplectic diffeomorphisms on compact surfaces, which are proved to be generically *transitive*. We will go back to this work in the last section of this text.

Our approach to constructing diffusion orbits for systems (3) on $\mathbb{A}^3$ is based on the embedding of bisystems on subsets of $\mathbb{A}$ into the system generated by H_ε, restricted to some energy level. More precisely, the orbits of our bisystems will only be *pseudoorbits*, which have the additional property to admit genuine shadowing orbits of the Hamiltonian system. Moreover, the bisystems satisfy the previous property of noncoincidence of invariant circles under mild nondegeneracy conditions (which can be made rather explicit), which yields the existence of diffusion pseudoorbits, and thus to diffusion orbits.

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[1] Also called IFS.

As an ubiquitous example, setting $\varepsilon = 1$ in (1) yields a class of systems for which the unperturbed part no longer depends on the actions only, but still remains completely integrable (with nondegenerate hyperbolicity). It has been a challenging question to prove the existence of unstable solutions (2) for the slightly more general class of systems

$$G_\mu(\theta, r) = r_1 + \tfrac{1}{2}(r_2^2 + r_3^2) + (\cos\theta_3 - 1) + \mu g(\theta, r), \tag{4}$$

where g belongs to a residual subset of a small enough ball in some appropriate function space (finitely or infinitely differentiable, Gevrey, analytic). This setting (with its natural generalizations) is now called the *a priori* unstable case of Arnold diffusion. In [21] we set out a geometric framework to deal with such systems, using the previous bisystem method; see also [8; 11; 12; 13; 15; 16; 19; 20; 22; 23; 24; 48; 53] amongst others for different approaches. Another different and very promising direction has been introduced in a related context by Nassiri and Pujals [49], where the notion of robust transitivity is used in place of the sole existence of diffusing orbits.

3. To some extent, the *a priori* unstable geometric and dynamical features can be recovered in the so-called *a priori* stable case (3). This problem leads one first to analyze the hyperbolic structure of such systems (under nondegeneracy conditions) in the complement of the set of Lagrangian invariant tori. Due to the many technicalities involved in this geometric part of the study, in this text we will take for granted the existence of a large family of compact 3-dimensional hyperbolic invariant submanifolds (described in the next section), with a rich homoclinic structure, which form "chains" contained in a regular energy level. Given a finite family of open subsets intersecting a connected component of a level $h^{-1}(e)$, there is an ε_0 such that these chains exist for $0 < \varepsilon < \varepsilon_0$ and connect these open sets.

The cylinders could be seen as the counterpart in the Baire category of the Lagrangian tori. The latter form subsets whose complement has relative measure tending to 0 when the size of the perturbation tends to 0, while our invariant cylinders tend to form dense subsets of a given regular level; see [32] for an example.

One main difficulty to recover the *a priori* unstable setting in *a priori* stable perturbations is the essentially singular nature of the problem: no hyperbolicity is present in the unperturbed system, so that the hyperbolicity constants of our 3-dimensional manifolds tend to 0 when $\varepsilon \to 0$, which makes their embedding properties a very delicate matter. We will limit ourselves here to give a description of the cylinders and chains and underline the various difficulties raised by their construction, we refer to [5; 7; 44] for more.

Once the chains are given, one can focus on the construction of diffusion orbits drifting along them. We will describe quite extensively two simple but relevant examples in this paper, which correspond to the two situations encountered in the $n = 3$ setting: the case of doubly resonance cylinders (the so-called *a priori* chaotic case) and the case of simple resonance cylinders (the singular *a priori* unstable case). In both cases, our method is to reduce the problem to the embedding of a bisystem of maps (or correspondences) on an annulus to which one can apply the Moeckel's method under mild nondegeneracy conditions. Then a normally hyperbolic shadowing process using the area preservation and the Poincaré recurrence theorem (as introduced and used in [15; 24]) will provide us with the diffusion orbits connecting the initially given open sets.

4. Let us briefly describe our setting, beginning with the functional spaces. Fix $n \geq 1$. For $2 \leq \kappa < +\infty$ and $f \in C^\kappa(\mathbb{A}^n) := C^\kappa(\mathbb{A}^n, \mathbb{R})$ we let

$$\|f\|_\kappa = \sum_{k \in \mathbb{N}^{2n}, 0 \leq |k| \leq \kappa} \|\partial^k f\|_{C^0(\mathbb{A}^n)} \leq +\infty$$

and we set $C^\kappa_b(\mathbb{A}^n) = \{f \in C^\kappa(\mathbb{A}^n) \mid \|f\|_\kappa < +\infty\}$, so that $C^\kappa_b(\mathbb{A}^n)$ is a Banach space. We consider systems on $\mathbb{A}^3$, of the form

$$H(\theta, r) = h(r) + f(\theta, r), \tag{5}$$

where $h : \mathbb{R}^3 \to \mathbb{R}$ is C^κ and the perturbation $f \in C^\kappa_b(\mathbb{A}^3)$ is small enough.

Even if our point of view here is essentially symplectic, we will adopt the setting introduced by Mather for proving the Arnold conjecture by variational methods. A first restriction in [47] is that the unperturbed part h is a Tonelli Hamiltonian, that is, strictly convex with superlinear growth at infinity ($\lim_{\|r\| \to +\infty} h(r)/\|r\| \to +\infty$). We will limit here to Tonelli Hamiltonians too, since convexity reveals itself to be necessary in our constructions in the neighborhood of double resonance points, in order to get well-defined classical systems as main parts of normal forms. However, the symplectic approach seems to make it possible to relax the convexity assumptions, at least to some extent.

A natural expectation, already illustrated by (1), would be the existence of diffusion orbits for all systems in "segments" in $C^\kappa_b(\mathbb{A}^3)$ originating at h, of the form

$$\{H_\varepsilon(\theta, r) = h(r) + \varepsilon f(\theta, r) \mid \varepsilon \in \,]0, \varepsilon_0[\} \tag{6}$$

where f is a fixed function, where of course the smallness threshold ε_0 may explicitly depend on f. However, it seems difficult to prove the existence of diffusion over *whole* segments such as (6). To take this observation into account, still following Mather, one uses a more global framework and introduce "anisotropic balls" in which the diffusion phenomenon can be expected to occur

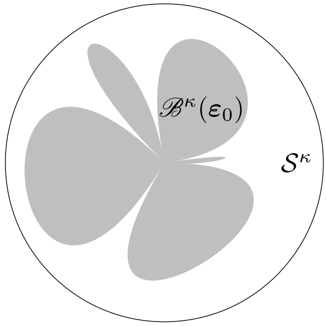

Figure 1. A generalized ball.

generically. Let $\mathcal{S}^\kappa$ be the unit sphere in $C^\kappa_b(\mathbb{A}^3)$. Given $\boldsymbol{\epsilon}_0 : \mathcal{S}^\kappa \to [0, +\infty[$ (a "threshold function"), we define the associated $\boldsymbol{\epsilon}_0$-ball:

$$\mathscr{B}^\kappa(\boldsymbol{\epsilon}_0) := \{\varepsilon \boldsymbol{f} \mid \boldsymbol{f} \in \mathcal{S}^\kappa, \varepsilon \in]0, \boldsymbol{\epsilon}_0(\boldsymbol{f})[\}. \tag{7}$$

Note also that if $\boldsymbol{\epsilon}_0$ is lower-semicontinuous, the associated ball is open in $C^\kappa_b(\mathbb{A}^3)$.

This yields the following version of the diffusion conjecture,[2] to be compared with [25].

Conjecture (diffusion conjecture in the convex setting). *There is an integer $\kappa_0 \geq 2$ such that for $\kappa \geq \kappa_0$, given a C^κ integrable Tonelli Hamiltonian h on $\mathbb{A}^3$, an $\boldsymbol{e} >$ Min h and a finite family of open sets $O_1, \ldots, O_m$ which intersect $h^{-1}(\boldsymbol{e})$, then there exists a lower semicontinuous function*

$$\boldsymbol{\epsilon}_0 : \mathcal{S}^\kappa \to \mathbb{R}^+$$

with positive values on a dense open subset of $\mathcal{S}^\kappa$ such that for f in a dense open subset of $\mathscr{B}^\kappa(\boldsymbol{\epsilon}_0)$ the system

$$H(\theta, r) = h(r) + f(\theta, r) \tag{8}$$

admits an orbit which intersects each $\mathbb{T}^3 \times O_i$.

The zeros of $\boldsymbol{\epsilon}_0$ correspond to directions along which diffusion cannot occur. Simple examples show that such directions exist in general: for instance if $h(r) = \frac{1}{2}(r_1^2 + r_2^2 + r_3^2)$, the system $H_\varepsilon = h + \varepsilon f$ with $f(\theta) = \sin\theta_3$ is completely integrable and does not admit diffusion orbits connecting open sets which are far from the $\theta_3 = 0$ plane. In view of the shape of $\mathscr{B}^\kappa(\boldsymbol{\epsilon}_0)$, a residual subset in such

[2]Mather's formulation is indeed still more precise and involved.

a ball is said to be *cusp-residual* and a property which holds on a cusp-residual subset is said to be *cusp-generic*.

From our point of view, one main interest (amongst many others) of the Mather setting comes from the possibility of proving first the existence of chains of cylinders for perturbations in a small enough generalized ball, and then prove that a new but arbitrarily small perturbation of any system in that ball yields the existence of diffusion orbits drifting along the chain, so connecting the open sets.

We wish to mention that very important advances has been achieved towards the proof of this conjecture, first by John Mather himself in his unfortunately unpublished notes, and more recently by P. Bernard, C.-Q. Cheng, V. Kaloshin, Ke Zhang and their collaborators; see [5; 7; 10; 33] and the many references therein. The methods in these works are either purely variational, or based on the weak KAM theory developed by A. Fathi; see [18]. Our methods in this text are more geometric and use in a crucial way the symplectic features of the systems.

2. The cusp-generic hyperbolic structure

This section is devoted to the geometric part of our study. We limit ourselves to a description of the main steps and refer to [44] for details and proofs.

2.1. ***Cylinders and chains.*** **1.** Let us briefly describe the various objects involved in our construction. We refer to [21] for precise definitions, which will also be recalled in the next two sections. Let X be a C^1 complete vector field on a smooth manifold M, with flow Φ. Let p be an integer ≥ 1:

- We say that $\mathscr{C} \subset M$ is a C^p *invariant cylinder with boundary* for X if $\mathscr{C}$ is a submanifold of M, C^p-diffeomorphic to $\mathbb{T}^2 \times [0, 1]$, which is invariant under the flow of X: $\Phi^t(\mathscr{C}) = \mathscr{C}$ for all $t \in \mathbb{R}$.

- We denote by $\boldsymbol{Y}$ any realization of the two-sphere S^2 minus three open discs with nonintersecting closures, so that $\partial \boldsymbol{Y}$ is the union of three circles. We say that $\mathscr{C}_\bullet \subset M$ is an *invariant singular cylinder* for X if $\mathscr{C}_\bullet$ is a C^1 submanifold of M, C^1 diffeomorphic to $\mathbb{T} \times \boldsymbol{Y}$ and invariant under Φ. The boundary of a singular cylinder is the disjoint union of three tori.

Throughout this paper we will consider vector fields generated by Hamiltonian functions $H \in C^\kappa(\mathbb{A}^3)$, $\kappa \geq 2$. The cylinders or singular cylinders will be contained in regular levels of H.

2. The notion of normal hyperbolicity for submanifolds with boundary requires some care. We refer to [9] for a general presentation, well-adapted to our setting (see also Appendix A). It suffices here to say that the normally hyperbolic invariant submanifolds we are dealing with here are invariant submanifolds

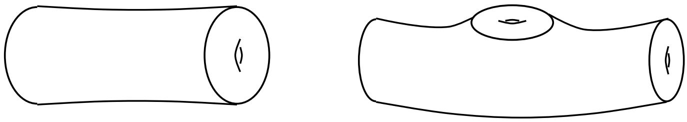

Figure 2. Cylinder and singular cylinder.

with boundary contained in usual normally hyperbolic manifolds of the same dimension (invariant for a new system slightly modified outside the submanifold at hand). In particular, our normally hyperbolic cylinders and singular cylinders admit well-defined 4-dimensional stable and unstable manifolds, contained in their energy level.

3. In addition to the normal hyperbolicity, to reduce the dynamics inside the cylinders to that of twist maps, we require that they admit global Poincaré sections, diffeomorphic to $\mathbb{T} \times [0, 1]$, whose associated Poincaré maps satisfy a twist condition. Analogous (but slightly more involved) notions are required for singular cylinders. The invariant tori contained in the cylinders which intersect these global sections along essential circles will be called *essential tori*. Moreover, in order to reduce the dynamics in the neighborhood of the cylinders to that of a suitable bisystem, we require that they satisfy specific homoclinic conditions, which yields the notion of *admissible cylinders*. Again, we refer to [21] for a complete description of the previous conditions, the necessary ones will be recalled in the following and illustrated by specific examples.

4. Finally, we will introduce various heteroclinic conditions to be satisfied by pairs of cylinders in order for them to admit orbits drifting along both of them. This yields the notion of *admissible chains*, that is, finite ordered families $(\mathscr{C}_k)_{1\leq k\leq k_*}$ of admissible cylinders or singular cylinders, in which two consecutive elements satisfy these heteroclinic conditions.

5. Our main statement regarding the existence of chains is the following one, for which we refer to [44].

Statement I (usp-generic existence of admissible chains). *There is an integer $\kappa_0 \geq 2$ such that for $\kappa \geq \kappa_0$, given a C^κ integrable Tonelli Hamiltonian h on $\mathbb{A}^3$, an $\boldsymbol{e} > \operatorname{Min} h$ and a finite family of open sets $O_1, \ldots, O_m$ which intersect $h^{-1}(\boldsymbol{e})$, then there exist a $\delta > 0$ and a lower semicontinuous function*

$$\epsilon_0 : \mathcal{S}^\kappa \to \mathbb{R}^+$$

with positive values on a dense open subset of S^κ such that for f in a dense open subset of $\mathscr{B}^\kappa(\epsilon_0)$ the system

$$H(\theta, r) = h(r) + f(\theta, r) \tag{9}$$

admits an admissible chain of cylinders and singular cylinders, such that each open set $\mathbb{T}^3 \times O_k$ contains the δ-neighborhood in $\mathbb{A}^3$ of some essential torus of the chain.

The fact that the statement is true only for f in a dense open subset of $\mathscr{B}^\kappa(\epsilon_0)$ and not for any f in $\mathscr{B}^\kappa(\epsilon_0)$ comes from the transversality conditions on the heteroclinic connections required in the definition of a chain. Less stringent conditions on a chain would be satisfied for all perturbations in $\mathscr{B}^\kappa(\epsilon_0)$.

6. One can be more precise and localize the previous chain. Since h is a Tonelli Hamiltonian, one readily checks that $\omega := \nabla h$ is a diffeomorphism from $\mathbb{R}^3$ onto $\mathbb{R}^3$, and that the level set $h^{-1}(\boldsymbol{e})$ is diffeomorphic to S^2. Given an indivisible vector $k \in \mathbb{Z}^3 \setminus \{0\}$, set

$$\Gamma_k = \omega^{-1}(k^\perp) \cap h^{-1}(\boldsymbol{e}),$$

where $k^\perp$ is the plane orthogonal to k for the Euclidean structure of $\mathbb{R}^3$. Then one checks that Γ_k is diffeomorphic to a circle, and that if $k \neq k'$ then Γ_k and $\Gamma_{k'}$ intersect at exactly two points (such intersection points are said to be *double resonance points*). By projective density, it is possible to choose a family $k_1, \dots, k_{m-1}$ of indivisible and pairwise independent vectors of $\mathbb{Z}^3$ such that

- Γ_{k_i} intersects O_i and O_{i+1} for $1 \le i \le m-1$;
- for $2 \le i \le m-1$, $\Gamma_{k_{i-1}} \cap \Gamma_{k_i}$ contains a point $a_i \in O_i$.

Fix $a_1 \in \Gamma_{k_1} \cap O_1$ and $a_m \in \Gamma_{k_{m-1}} \cap O_m$. Fix an arbitrary orientation on each circle Γ_{k_i} and let $[a_i, a_{i+1}]_{\Gamma_i}$ be the segment of Γ_i bounded by a_i and a_{i+1} according to this orientation. Set finally

$$\boldsymbol{\Gamma} = \bigcup_{1 \le i \le m-1} [a_i, a_{i+1}]_{\Gamma_i}.$$

We will prove that one can choose ϵ_0 in Theorem I so that for $f \in \mathscr{B}(\epsilon_0)$ the projection to $\mathbb{R}^3$ of the admissible chain is located in a $\rho(f)$-tubular neighborhood of $\boldsymbol{\Gamma}$, whose radius $\rho(f)$ tends to 0 when $f \to 0$ in $C^\kappa(\mathbb{A}^3)$.

2.2. *Simple resonance cylinders.* **1.** In this section we assume for simplicity and with no loss of generality that $h(r) = \frac{1}{2}(r_1^2 + r_2^2 + r_3^2)$, so that the frequency vector is just $\omega(r) = r$. We fix an energy $\boldsymbol{e} > 0$ and consider the broken line $\boldsymbol{\Gamma}$ defined in the previous section. We will focus on a single arc $\Gamma = \Gamma_{k_i}$ for which we can assume, up to a linear change, that $k_i = (0, 0, 1)$. Hence Γ is contained

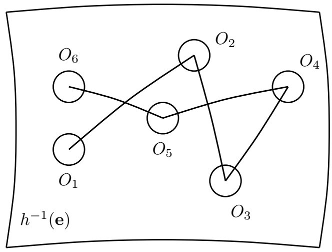

Figure 3. A "broken line" Γ of resonance arcs.

in the great circle intersection of the plane $r_3 = 0$ with the sphere $h^{-1}(\boldsymbol{e})$. The double resonance points $r^0 = (r_1, r_2, 0)$ on Γ are those for which there exists $\hat{k} \in \mathbb{Z}^2 \setminus \{0\}$ such that $\hat{k} \cdot (r_1, r_2) = 0$. The order of r^0 is then the minimal norm of such a vector $\hat{k}$.

The proof of existence of cylinders whose projection in action lies along Γ relies on a suitable averaging of the perturbation, which necessitate to determine the zones where averaging with respect to two fast angles yield a satisfactory normal form. In the complement of these zones, where a single fast angle only is available for averaging, another process is to be used to construct the cylinders. However, the "main part" of the cylinders will come from the former process.

To make this effective, one writes the Fourier expansion of f in the form

$$f(\theta, r) = \sum_{\hat{k} \in \mathbb{Z}^2} \Big(\sum_{k_3 \in \mathbb{Z}} [f]_{(\hat{k}, k_3)}(r) e^{2i\pi k_3 \theta_3} \Big) e^{2i\pi \hat{k}\hat{\theta}},$$

where $[f]_k$ stands for the Fourier coefficient relative to $k \in \mathbb{Z}^3$. For $K \in \mathbb{N}$, we set

$$f_{>K}(\theta, r) = \sum_{\|\hat{k}\| > K} \Big(\sum_{k_3 \in \mathbb{Z}} [f]_{(\hat{k}, k_3)}(r) e^{2i\pi k_3 \theta_3} \Big) e^{2i\pi \hat{k}\hat{\theta}}.$$

When $f \in C^\kappa$ with $\kappa \geq 6$ and $p \in \{2, \ldots, \kappa - 4\}$, given a control parameter $\delta > 0$ (which will be one main parameter in the whole construction), one proves the existence of a cutoff K_δ such that

$$\| f_{>K_\delta} \|_{C^p(\mathbb{A}^3)} \leq \delta.$$

Up to a symplectic conjugacy, one can cancel the harmonics of order $< K$ when the homological equation

$$\widehat{\omega}(r) \cdot \partial_{\hat{\theta}} S(\theta, r) = f(\theta, r) - V(\theta_3, r) - f_{>K}(\theta, r).$$

can be solved, where $r \in \Gamma$ and

$$V(\theta_3, r) = \int_{\mathbb{T}^2} f(\hat{\theta}, \theta_3, r) d\hat{\theta}.$$

This yields the definition of a *finite set* $D(\delta) \subset \Gamma$ *of strong double resonance points relative to* δ, namely, those $r \in \Gamma$ for which there exist an integer vector $\hat{k} \in \mathbb{Z}^2 \setminus \{0\}$ with $\|\hat{k}\| \leq K_\delta$ such that

$$\widehat{\omega}(r) \cdot \hat{k} = 0.$$

Far enough from any strong double resonance point, averaging with respect to the angles (θ_1, θ_2) yields a one-degree-of freedom (integrable) normal form + remainder, which makes the geometry of the situation easy to analyze. This becomes irrelevant in the neighborhood of the strong double resonance points, where the main part of the normal form is a classical (nonintegrable) system on $\mathbb{T}^2$.

2. More precisely, in the neighborhood of a (closed) segment $S \subset \Gamma$ located at a distance ρ of $D(\delta)$, averaging with respect to the angles (θ_1, θ_2) yields a close to identity conjugacy Φ_ε such that, setting $\hat{\theta} = (\theta_1, \theta_2)$ and $\hat{r} = (r_1, r_2)$

$$N_\varepsilon(\theta, r) = H_\varepsilon \circ \Phi_\varepsilon(\theta, r) = h(r) + \varepsilon V(\theta_3, \hat{r}) + R(\theta, r, \varepsilon) \tag{10}$$

where R is small (depending on δ, ρ and ε) in some arbitrary C^p topology ($p \in \{2, \dots, \kappa - 4\}$ has to be chosen large enough, and so also κ, in particular to apply the KAM theorem, see below).[3] The truncated normal form

$$\tfrac{1}{2}(r_1^2 + r_2^2) + \left[\tfrac{1}{2} r_3^2 + \varepsilon V(\theta_3, \hat{r})\right] \tag{11}$$

is the skew-product of the unperturbed Hamiltonian $\frac{1}{2}(r_1^2 + r_2^2)$ with a family of "generalized pendulums", functions of $(\theta_3, r_3) \in \mathbb{A}$ and parametrized by $\hat{r}$. This is indeed a one-parameter family since $(\hat{r}, 0)$ belongs to the curve S.

Assume moreover that for $(\hat{r}, 0) \in S$ the function $V(\,\cdot\,, \hat{r})$ admits a single and nondegenerate maximum at some point $\theta_3(\hat{r})$, and, for simplicity, that $V(\theta_3(\hat{r})) = 0$. Then $O_{\hat{r}} = (\theta_3(\hat{r}), 0)$ is a hyperbolic fixed point for the Hamiltonian $\frac{1}{2} r_3^2 + \varepsilon V(\theta_3; \hat{r})$ and one immediately gets a normally hyperbolic cylinder $\mathcal{C}$ at

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[3] More precisely

$$N(\theta, r) = H \circ \Phi_\varepsilon(\theta, r) = h(r) + \varepsilon V(\theta_3, r) + \varepsilon W_0(\theta, r) + \varepsilon W_1(\theta, r) + \varepsilon^2 W_2(\theta, r),$$

where the functions $W_0 \in C^p(\mathbb{A}^3)$, $W_1 \in C^{\kappa-1}(\mathscr{W}_{\rho/4})$, $W_2 \in C^\kappa(\mathscr{W}_{\rho/4})$ satisfy

$$\|W_0\|_{C^p(\mathscr{W}_{\rho/4})} \leq \delta, \quad \|W_1\|_{C^2(\mathscr{W}_{\rho/4})} \leq c_1 \rho^{-3} \quad \|W_2\|_{C^2(\mathscr{W}_{\rho/4})} \leq c_2 \rho^{-6},$$

for suitable constants $c_1, c_2 > 0$, where ρ is the distance from the segment S to the closest strong double resonance point.

energy $\boldsymbol{e}$ for N_ε by taking the product of the torus $\mathbb{T}^2$ of the fast angles $\hat{\theta}$ with the curve

$$\{O_{\hat{r}} \mid (\hat{r}, 0) \in S\}.$$

Note that $\mathcal{C}$ is diffeomorphic to $\mathbb{T}^2 \times [0, 1]$ and that its stable and unstable manifolds are the unions for $(\hat{r}, 0) \in S$ of the products of the stable and unstable manifolds $W^{\pm}(O_{\hat{r}})$ with the torus $\mathbb{T}^2$ of fast angles.

3. We have then to choose S far enough from $D(\delta)$ so that the remainder R is small enough for the previous cylinder *together with its boundary* to persist in the system H_ε. This necessitates two steps:

- One first proves the existence of *pseudoinvariant* cylinders (that is, open cylinders which are tangent to the Hamiltonian vector field) which have transverse hyperbolic properties,[4] *but are not necessarily invariant under the flow*.
- One then proves the existence of two dimensional invariant tori inside the previous pseudoinvariant cylinder, so that two of them bound an *invariant* and genuinely normally hyperbolic cylinder.[5]

The main difficulty is to choose S not too far from $D(\delta)$, in such a way that the cylinders one obtains with the previous construction can be compared with those to be constructed below in the neighborhood of double resonance points. The main point is to prove that *pairs of KAM tori* simultaneously belong to the previous cylinders and those close to double resonance, so that one deduces that they bound part of their intersection. This proves that the "double resonance" cylinders continue the "simple resonance" ones.

This process necessitates a smallness condition of the remainder in the C^p topology with $p \leq 2$ for the normally hyperbolic persistence results of Appendix A to apply,[6] and an additional smallness condition in the C^p topology with p large enough to apply the KAM theorem and get invariant boundaries.

As a consequence, one has to make a careful choice of the parameter δ, and to make the distance ρ depend on ε in a proper way; see [44] for these technical details. The main problem is to chose this size so that the boundaries of the cylinders $\mathcal{C}$ constructed above match those which will be proved to exist inside

[4]In this noninvariant setting, the hyperbolic properties can be defined by embedding the manifold in an invariant one, after modification of the vector field. The resulting property depends on this embedding, but we will be concerned only in invariant subsets of those manifolds, limited by KAM tori, which makes our approach legitimate.

[5]This step is indeed one main difference with the other approaches to Arnold diffusion.

[6]I consider the results applied in this study as genuine persistence results, since one starts with a normally hyperbolic manifold for the normal form, which is then perturbed by the remainder -on which nothing but its size is known- and is proved to persist after perturbation.

Figure 4. The arc Γ with the low-order double resonance and the bifurcation points.

this neighborhood. Even if one could expect this size to be of the order of $\sqrt{\varepsilon}$ (which would be the optimal one), we find it efficient to use a more flexible scale and work with ε^{ν}-neighborhoods, with some constant $\nu < \frac{1}{2}$ which will be made precise in the text. The cylinders along the simple resonance segments such as S will be called *s-cylinders*, (where s stands for "pure simple resonance").

This enables us to split Γ into "s-segments" which are bounded by the neighborhoods of consecutive low order double resonance points (denoted by $\odot$ in the following picture where the curved arc is projected onto a plane). Note that one can assume without loss of generality that the extremal points of Γ are double resonance points of low order.

4. The situation is in fact slightly more complicated, due to the possible generic occurrence of *bifurcation points* for the two-phase averaged systems (11). These are the parameters $\hat{r}$ where the potential $V(\,\cdot\,,\hat{r})$ admits *two* nondegenerate global maxima instead of a single one (depicted by a $\times$ in the following figure). In the neighborhood of these points two cylinders coexist, for which we prove the existence of heteroclinic connections. We will not give more details here, since this does not yield serious additional difficulties in the construction (the arguments here are standard in transversality theory).

2.3. ***The generic hyperbolic structure of classical systems on $\mathbb{A}^2$.*** A *classical system on $\mathbb{A}^2$* is a Hamiltonian of the form

$$C(x,y) = \tfrac{1}{2}T(y) + U(x), \quad (x,y) \in \mathbb{A}^2 \tag{12}$$

where T is a positive definite quadratic form of $\mathbb{R}^2$ and U a C^{κ} potential function on $\mathbb{T}^2$, where $\kappa \geq 2$. In the sequel we require the potential U to admit a single maximum at some x^0, which is nondegenerate in the sense that the Hessian of U at x_0 is negative definite. Consequently, the lift of x^0 to the zero section of $\mathbb{A}^2$ is a hyperbolic fixed point which we denote by O. We set $\bar{e} = \operatorname{Max} U$ and we say that $\bar{e}$ is the *critical energy* for C.

Such systems appear, *up to a nonsymplectic rescaling*, in the neighborhood of a double resonance point r^0 of the initial system (8), as the main part of normal forms. The aim of this section is to depict some relevant hyperbolic properties of C, when T *is fixed* and U belongs to a dense open subset of $C^{\kappa}(\mathbb{T}^2)$, κ large enough.

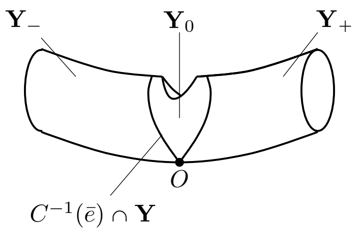


Figure 5. A singular 2-dimensional annulus.

1. Let $\pi : \mathbb{A}^2 \to \mathbb{T}^2$ be the canonical projection.

Definition 1. Let $c \in H_1(\mathbb{T}^2, \mathbb{Z})$. Let $I \subset \mathbb{R}$ be an interval. An *annulus for X^C realizing c and defined over I* is a 2-dimensional submanifold A, contained in $C^{-1}(I) \subset \mathbb{A}^2$, such that for each $e \in I$, $\mathsf{A} \cap C^{-1}(e)$ is the orbit of a periodic solution γ_e of X^C, which is hyperbolic in $C^{-1}(e)$ and such that the projection $\pi \circ \gamma_e$ on $\mathbb{T}^2$ belongs to c. We also require that the period of the orbits decreases with the energy and that for each $e \in I$, the periodic orbit γ_e admits a homoclinic orbit along which $W^{\pm}(\gamma_e)$ intersect transversely in $C^{-1}(e)$. Finally, we require the existence of a finite partition $I = I_1 \cup \cdots \cup I_n$ by consecutive intervals such that the previous homoclinic orbit varies continuously for $e \in I_i$, $1 \le i \le n$.

When I is compact, the annulus A is clearly normally hyperbolic in the usual sense (the boundary causes no trouble in this simple setting). The stable and unstable manifolds of A are well-defined, as the unions of those of the periodic solutions γ_e. Moreover, A can be continued to an annulus defined over a slightly larger interval $I' \supset I$.

2. Note that, due to the reversibility of C, the solutions of the vector field X^C occur in “opposite pairs,” whose time parametrizations are exchanged by the symmetry $t \mapsto -t$. We introduce now the second definition to be used throughout the whole paper.

Definition 2. Let $c \in H_1(\mathbb{T}^2, \mathbb{Z}) \setminus \{0\}$. A *singular annulus for X^C realizing $\pm c$* is a C^1 compact invariant submanifold $\boldsymbol{Y}$ of $\mathbb{A}^2$, diffeomorphic to the sphere S^2 minus three disjoint open discs with disjoint closures (so that $\partial \boldsymbol{Y}$ is the disjoint union of three circles), such that there exist constants $e_* < \bar{e} < e^*$ which satisfy:

- $\boldsymbol{Y} \cap C^{-1}(\bar{e})$ is the union of the hyperbolic fixed point O and a pair of opposite homoclinic orbits.
- $\boldsymbol{Y} \cap C^{-1}(]\bar{e}, e^*])$ admits two connected components $\boldsymbol{Y}_+$ and $\boldsymbol{Y}_-$, which are annuli defined over the interval $]\bar{e}, e^*]$ and realizing c and $-c$ respectively.
- $\boldsymbol{Y}_0 = \boldsymbol{Y} \cap C^{-1}([e_*, \bar{e}[)$ is an annulus realizing the null class 0.

A singular annulus, endowed with its induced dynamics, is essentially the phase space of a simple pendulum from which an open neighborhood of the elliptic fixed point has been removed.

3. We will finally need the following notion of chains of annuli for C,[7] from which we will deduce the existence and properties of the chains of cylinders near the double resonance points.

Definition 3. Let $c \in H_1(\mathbb{T}^2, \mathbb{Z})$. We say that a family $(I_i)_{1 \le i \le i_*}$ of nontrivial intervals, contained and closed in the energy interval $]\bar{e}, +\infty[$, is *ordered* when $\operatorname{Max} I_i = \operatorname{Min} I_{i+1}$ for $1 \le i \le i_* - 1$. A *chain of annuli realizing c* is a family $(\mathsf{A}_i)_{1 \le i \le i_*}$ of annuli realizing c, defined over an ordered family $(I_i)_{1 \le i \le i_*}$, with the additional property

$$W^-(\mathsf{A}_i) \cap W^+(\mathsf{A}_{i+1}) \neq \varnothing, \quad W^+(\mathsf{A}_i) \cap W^-(\mathsf{A}_{i+1}) \neq \varnothing,$$

for $1 \le i \le i_* - 1$, both intersections being transverse in their energy levels.

The last condition is equivalent to assuming that the boundary periodic orbits of A_i and A_{i+1} at energy $e = \operatorname{Max} I_i = \operatorname{Min} I_{i+1}$ admit transverse heteroclinic orbits.[8] Note that, following Definition 1, an annulus can itself be considered as a chain, whose elements are the subannuli along which the homoclinic orbits vary continuously. This slight ambiguity will cause no trouble in the construction.

4. We say that $c \in H_1(\mathbb{T}^2, \mathbb{Z}) \setminus \{0\}$ is *primitive* when the equality $c = \lambda c'$ with $c' \in H_1(\mathbb{T}^2, \mathbb{Z})$ implies $\lambda = \pm 1$. We denote by $\boldsymbol{H}_1(\mathbb{T}^2, \mathbb{Z})$ the set of primitive homology classes, by $\boldsymbol{d}$ be the Hausdorff distance for compact subsets of $\mathbb{R}^2$ and by $\Pi : \mathbb{A}^2 \to \mathbb{R}^2$ the canonical projection.

Statement II (generic hyperbolic properties of classical systems). *Let T be a quadratic form on $\mathbb{R}^2$ and for $\kappa \ge 2$, let $\mathscr{U}_0^\kappa \subset C^\kappa(\mathbb{T}^2)$ be the set of potentials with a single and nondegenerate maximum. Then there is an integer $\kappa_0 \ge 2$ such that if $\kappa \ge \kappa_0$, there exists a dense open subset*

$$\mathscr{U}(T) \subset \mathscr{U}_0^\kappa \tag{13}$$

in $C^\kappa(\mathbb{T}^2)$ such that for $U \in \mathscr{U}(T)$, the associated classical system $C = \frac{1}{2}T + U$ satisfies the following properties:

(1) *For each $c \in \boldsymbol{H}_1(\mathbb{T}^2, \mathbb{Z})$ there exists a chain $\boldsymbol{A}(c) = (\mathsf{A}_0, \dots, \mathsf{A}_m)$ of annuli realizing c, defined over ordered intervals $I_0, \dots, I_m$, with*

$$I_m = [e_P, +\infty[,$$

[7] we keep the same terminology as for the cylinders, with a slightly different sense here.

[8] But the previous formulation is more appropriate when hyperbolic continuations of the annuli are involved.

for a suitable constant e_P which we call the Poincaré energy.

(2) *Given two primitive classes $c \neq c'$, there is a $\sigma \in \{-1, +1\}$ such that the chains $\boldsymbol{A}(c) = (\mathsf{A}_i)_{0 \le i \le m}$ and $\boldsymbol{A}(\sigma c') = (\mathsf{A}'_i)_{0 \le i \le m'}$ satisfy*

$$W^-(\mathsf{A}_0) \cap W^+(\mathsf{A}'_0) \neq \varnothing \quad \textit{and} \quad W^-(\mathsf{A}'_0) \cap W^+(\mathsf{A}_0) \neq \varnothing,$$

both heteroclinic intersections being transverse in $\mathbb{A}^2$.

(3) *There exists a singular annulus $\boldsymbol{Y}$ which admits transverse heteroclinic connections with the first annulus A_0 of the chain $\boldsymbol{A}(c)$, for all $c \in \boldsymbol{H}_1(\mathbb{T}^2, \mathbb{Z})$.*

(4) *Under the canonical identification of $H_1(\mathbb{T}^2, \mathbb{Z})$ with $\mathbb{Z}^2$ and for $e > 0$, let us set, for a given primitive class $c \sim (c_1, c_2) \in \mathbb{Z}^2$:*

$$Y_c(e) = \frac{\sqrt{2e}c}{\sqrt{c_1^2 + c_2^2}} \in \mathbb{R}^2$$

Let $\boldsymbol{A}(c) = (\mathsf{A}_0, \dots, \mathsf{A}_m)$ be the associated chain and set $\boldsymbol{\gamma}_e = \mathsf{A}_m \cap C^{-1}(e)$ for e in $[e_P, +\infty[$. Then

$$\lim_{e \to +\infty} \boldsymbol{d}(\Pi(\boldsymbol{\gamma}_e), \{Y_c(e)\}) = 0.$$

We say that a chain with I_0 and I_m as in 1) is *biasymptotic to* $\bar{e} := \operatorname{Max} U$ *and to* $+\infty$. We will not only consider chains formed by nonsingular annuli, but also "generalized ones" in which we will allow a single annulus to be singular. With this terminology, one can rephrase the content of 1) and 3) of Statement II in the following concise way: for $U \in \mathscr{U}(T)$ and for each pair of classes $c, c' \in \boldsymbol{H}_1(\mathbb{T}^2, \mathbb{Z})$, there exists a generalized chain:

$$\mathsf{A}_m \leftrightarrow \cdots \leftrightarrow \mathsf{A}_1 \leftrightarrow \boldsymbol{Y} \leftrightarrow \mathsf{A}'_1 \leftrightarrow \cdots \leftrightarrow \mathsf{A}'_{m'}$$

(where $\leftrightarrow$ stands for the heteroclinic connections) which is biasymptotic to $+\infty$, and realize c and c' respectively.

In the x-plane, one therefore gets the following symbolic picture for the projection of 6 generalized chains of annuli, where the annuli are represented by fat segments, the singular annulus by a fat segment with a circle and the various heteroclinic connections are represented by $\leftrightarrow$.

The projections of the annuli on the action space are in fact more complicated than lines, they are rather 2-dimensional submanifolds with boundary, which tend to a line when the energy grows to infinity.

2.4. *Double resonance cylinders.* The dynamical structure of Hamiltonian systems at double resonance points has been widely studied, not only in the above mentioned works about Arnold diffusion, but also as an interesting problem *per se*. A complete list of these works would be unrealistic, let us only mention the

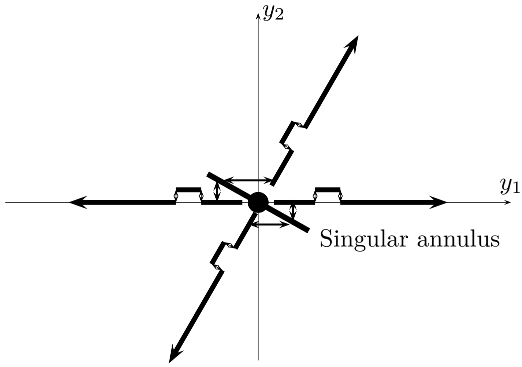


Figure 6. Projections in action of chains of annuli.

ones by G. Haller [27; 28] whose point of view is close to ours in the particular case of the intersection of a strong and a weak resonance.

1. Our point now is to construct cylinders located inside the ε^{ν}-neighborhoods of the strong double resonance points, and prove that they match the s-cylinders. Fix such a double resonance point r^0 and assume (up to a linear change of variables) that $r^0 = (\sqrt{2\boldsymbol{e}}, 0, 0)$. Hence θ_1 is the only fast angle with respect to which the averaging can be performed. This yields a normal form

$$N_\varepsilon(\theta, r) = h(r) + g_\varepsilon(\bar{\theta}, r) + R_\varepsilon(\theta, r)$$

where

$$\|g_\varepsilon - \varepsilon[f]\|_{C^p(\mathbb{T}^2 \times B(0,\varepsilon^\nu))} \le \varepsilon^{1+\sigma}, \qquad \|R_\varepsilon\|_{C^p(\mathbb{T}^3 \times B(0,\varepsilon^\nu))} \le \varepsilon^{\ell},$$

where $\sigma > 0$ and ℓ arbitrarily large. To derive this normal form in a quite flexible way, we start from Pöschel's normal form for analytic systems and apply an analytic smoothing; see [3].

The dynamical study of this normal form requires some care. To simplify we will assume here that

$$\begin{aligned} N_\varepsilon(\theta, r) &= \tfrac{1}{2} r_1^2 + \left[\tfrac{1}{2}(r_2^2 + r_3^2) + \varepsilon U(\theta_2, \theta_3)\right] + R(\theta, r, \varepsilon), \\ U(\theta_2, \theta_3) &:= \int_{\mathbb{T}} f((\theta_1, (\theta_2, \theta_3)), r^0)\, d\theta_1, \end{aligned} \tag{14}$$

where now the remainder R is extremely small (of order ε^{ℓ} with large ℓ) in some suitably chosen C^p topology over a neighborhood of r^0 of diameter ε^{ν}.[9]

[9] The complete study requires a careful analysis of mixed terms which do not appear here and whose size has to be taken into account.

2. The main role in (14) is played by the ε-dependent classical system

$$C_\varepsilon(\bar\theta,\bar r)=\tfrac12(r_2^2+r_3^2)+\varepsilon U(\theta_2,\theta_3),\qquad \bar\theta=(\theta_2,\theta_3),\bar r=(r_2,r_3).$$

To recover the unperturbed setting of Statement II, we perform the usual linear rescaling $\bar{\mathsf{r}}=\sqrt{\varepsilon}\bar r$ of the action variable only, which transforms C_ε into

$$\mathsf{C}(\bar\theta,\bar{\mathsf{r}})=\varepsilon C_\varepsilon(\bar\theta,\bar r),$$

so that the dynamics is only changed by a time dilatation, *while the geometry is preserved*. We assume that C satisfies the properties of Statement II. Let us fix a finite number of primitive homology classes c_k, $1\le k\le k_*$, and consider the associated chains $\boldsymbol{A}(c_k)$. Let $e_p(k)$ be the Poincaré energy of $\boldsymbol{A}(c_k)$ and fix $E\ge\max_k e_p(k)$, so that, setting $\boldsymbol{A}(c_k)=(\mathsf{A}_1(c_k),\dots,\mathsf{A}_{m_k}(c_k))$ the annuli

$$\mathsf{A}_1(c_k),\dots,\mathsf{A}_{m_k-1}(c_k)$$

are contained in the sublevel $\mathsf{C}\le E$, while the annulus $\mathsf{A}_{m_k}(c_k)$ intersects that level along the compact subannulus

$$\widetilde{\mathsf{A}}_{m_k}(c_k)=\mathsf{A}_{m_k}(c_k)\cap\mathsf{C}^{-1}([e_p(k),E]).$$

The previous coordinate change sends these annuli onto "homothetic" ones (parametrized by ε), contained in the sublevel $C_\varepsilon\le\varepsilon E$. Forgetting about the class c_k, let us denote them by

$$\mathsf{A}_1(\varepsilon),\dots,\mathsf{A}_{m_k-1}(\varepsilon),\widetilde{\mathsf{A}}_{m_k}(\varepsilon).\tag{15}$$

In addition, the singular annulus $\mathsf{A}^\bullet$ of C is sent onto a singular annulus $\mathsf{A}^\bullet(\varepsilon)$ of C_ε. The "length" of these annuli is of order $\sqrt{\varepsilon}$.

We can now analyze the ε-dependent truncated normal form

$$\overline{N}_\varepsilon(\bar\theta,\bar r)=\tfrac12 r_1^2+\big[\tfrac12(r_2^2+r_3^2)+\varepsilon U(\theta_2,\theta_3)\big]\tag{16}$$

on the energy level $\boldsymbol{e}$. Let $\mathsf{A}(\varepsilon)$ be an element of the family (15). Since r_1 is a first integral of $\overline{N}_\varepsilon$, taking the product of $\mathsf{A}(\varepsilon)$ with the circle $\mathbb{T}$ of the angle θ_1 gives rise to a 3-dimensional (invariant and normally hyperbolic) cylinder $\mathcal{C}(\mathsf{A}(\varepsilon))$ contained in $\overline{N}_\varepsilon^{-1}(\boldsymbol{e})$, the variable r_1 being expressed (for ε small enough) as the function of $(\bar\theta,\bar r)$ deduced from the energy relation

$$r_1=2\sqrt{\boldsymbol{e}-e},\qquad e=C_\varepsilon(\bar\theta,\bar r),\qquad(\bar\theta,\bar r)\in\mathsf{A}(\varepsilon).$$

Similarly, the singular annulus $\mathsf{A}^\bullet(\varepsilon)$ gives rise to a normally hyperbolic invariant singular cylinder $\mathcal{C}(\mathsf{A}_\bullet(\varepsilon))$ at energy $\boldsymbol{e}$ for $\overline{N}_\varepsilon$.

Since $\omega(r^0)=(\sqrt{2\boldsymbol{e}},0,0)$, the tangent space to $h^{-1}(\boldsymbol{e})$ at r^0 is the affine plane $r_1=\sqrt{2\boldsymbol{e}}$, so that one can see the variables (r_2,r_3) as natural coordinates on $h^{-1}(\boldsymbol{e})$, and the localization of the previous invariant cylinders at energy $\boldsymbol{e}$ is

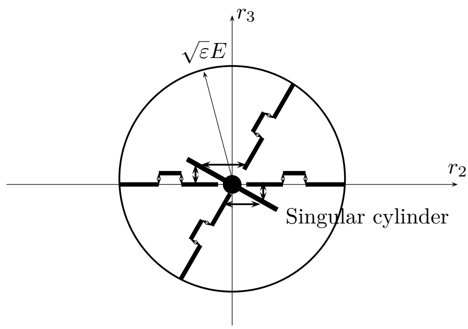


Figure 7. Projections of chains of cylinders for (16).

well-described by their projection on the (r_2, r_3)-plane. As a consequence, the arrangement of cylinders of $\overline{N}_\varepsilon$ in a zone of diameter $\sqrt{\varepsilon}E$ around r^0 is suitably deduced from the arrangement of annuli in the sublevel $C \le E$, as shown in Figure 7 (in projection on the (r_2, r_3)-plane).

Note finally that the (4-dimensional) stable and unstable manifolds of $\mathcal{C}(\mathsf{A}(\varepsilon))$ and $\mathcal{C}(\mathsf{A}^\bullet(\varepsilon))$ are the products of those of $\mathsf{A}(\varepsilon)$ and $\mathsf{A}^\bullet(\varepsilon)$ with the circle of θ_1. Consequently, the homoclinic and heteroclinic connections are the products of those of the annuli of C with the circle of θ_1. This is a degenerate situation which generically gives rise to transverse intersections when the remainder R is taken into account.

3. Once the invariant cylinders for the truncated normal form are properly determined, it remains to show their persistence in the initial system. In a similar way as for simple resonances, we take advantage of the smallness of R first to use normal hyperbolic persistence and second to show the persistence of the boundaries of the cylinders of $\overline{N}_\varepsilon$. This way we prove the existence in the initial system of a family of invariant 3-dimensional cylinders, with homoclinic and heteroclinic connections, located in an $O(\sqrt{\varepsilon})$ neighborhood of the double resonance point r^0. We call them d-cylinders.

4. So far we have described the global picture in the neighborhood of r^0. We now go back to our initial problem, which is to use (a subset of) the previous family of d-cylinders to form a chain whose extremal elements match the s-cylinders along the simple resonance Γ. To do this, due to the fact that $\Gamma \subset \{r_3 = 0\}$, we only need to consider the d-cylinders located along the r_2-axis in the previous description. Therefore we focus on the chains of annuli of C which realize the homology classes $c = (\pm 1, 0)$, whose projection lies along the r_2-axis. Taking the singular annulus into account and truncating the extremal annuli at the energy E,

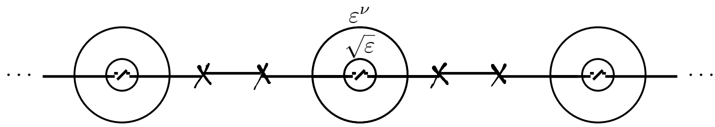


Figure 8. A chain of cylinders along Γ.

we get a generalized chain[10]

$$\widetilde{\mathsf{A}}_m^- \leftrightarrow \cdots \leftrightarrow \mathsf{A}_1^- \leftrightarrow \boldsymbol{Y} \leftrightarrow \mathsf{A}_1^+ \leftrightarrow \cdots \leftrightarrow \widetilde{\mathsf{A}}_m^+,$$

which yields the chain of cylinders

$$\mathcal{C}(\widetilde{\mathsf{A}}_m^-(\varepsilon)) \leftrightarrow \cdots \leftrightarrow \mathcal{C}(\mathsf{A}_1^-(\varepsilon)) \leftrightarrow \mathcal{C}(\boldsymbol{Y}(\varepsilon)) \leftrightarrow \mathcal{C}(\mathsf{A}_1^+(\varepsilon)) \leftrightarrow \cdots \leftrightarrow \mathcal{C}(\widetilde{\mathsf{A}}_m^+(\varepsilon)).$$

Now a crucial observation is that *both extremal cylinders* $\mathcal{C}(\widetilde{\mathsf{A}}_m^\pm(\varepsilon))$ *can be continued in a unique way over an* $O(\varepsilon^\nu)$*-neighborhood of* r^0, giving rise to "longer" cylinders $\widetilde{\mathscr{C}}_m^\pm$, still lying along the resonant line Γ. To compare these new cylinders to the s-cylinders $\mathscr{C}^\pm$ located on both sides of the double resonance point, we prove that $\widetilde{\mathscr{C}}_m^+$ and $\widetilde{\mathscr{C}}_m^-$ both contain two (essential) KAM tori, which are also contained in the s-cylinders $\mathscr{C}^+$ and $\mathscr{C}^-$ respectively. By normally hyperbolic uniqueness, this proves that the s-cylinders continue $\widetilde{\mathscr{C}}_m^\pm$ outside the ε^ν-neighborhood and completes the picture: there is a chain of cylinders and singular cylinders *passing through* the double resonance point and connecting together the two s-cylinders $\mathscr{C}^\pm$ in the ε^ν gluing zone:

$$\mathscr{C}^- \leftrightarrow \cdots \leftrightarrow \mathcal{C}(\mathsf{A}_1^-(\varepsilon)) \leftrightarrow \mathcal{C}(\boldsymbol{Y}(\varepsilon)) \leftrightarrow \mathcal{C}(\mathsf{A}_1^+(\varepsilon)) \leftrightarrow \cdots \leftrightarrow \mathscr{C}^+.$$

Applying this process for all strong double resonance points contained in Γ (and taking the bifurcations points between them into account), we construct a chain $\boldsymbol{C}$ of hyperbolic cylinders and singular cylinders whose projection $\pi(\boldsymbol{C})$ in action satisfies $d_H(\pi(\boldsymbol{C}), \Gamma) \to 0$ when $\varepsilon \to 0$ (where d_H stands for the Hausdorff distance in $\mathbb{A}^3$). This yields the following final picture for the arrangement of cylinders along the arc Γ.

5. This construction applies to each segment Γ_{k_i} of the initial broken line. To get a chain along the full broken line, we have to pass from one resonance arc to another one through the double resonance point at their intersection. For doing this, we use the full structure at this double resonance point and choose two homology classes c_1, c_2 in $\boldsymbol{H}_1(\mathbb{T}^2, \mathbb{Z})$ which correspond to the simple resonances

[10] By symmetry of C, the numbers of annuli realizing $\pm c$ are equal.

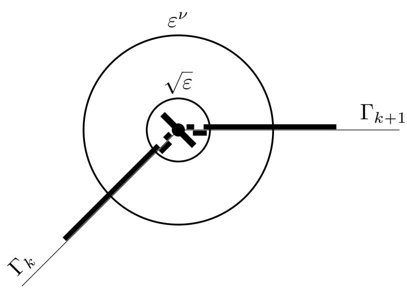


Figure 9. Transition between two arcs at a double resonance point.

arcs crossing at that point. In the same way as above, we get a chain of cylinders (with one singular cylinder) whose projection is located along both resonance arcs in an $O(\sqrt{\varepsilon})$ neighborhood of the double resonance point:

$$\mathcal{C}(\widetilde{\mathsf{A}}^-_{m_1}(c_1,\varepsilon)) \leftrightarrow \cdots \leftrightarrow \mathcal{C}(\mathsf{A}^-_1(c_1,\varepsilon)) \leftrightarrow \mathcal{C}(\boldsymbol{Y}(\varepsilon)) \leftrightarrow \mathcal{C}(\mathsf{A}^+_1(c_2,\varepsilon)) \leftrightarrow \cdots \leftrightarrow \mathcal{C}(\widetilde{\mathsf{A}}^+_{m_2}(c_2,\varepsilon)).$$

Again, we prove that the extremal cylinders $\mathcal{C}(\widetilde{\mathsf{A}}^-_{m_1}(c_1,\varepsilon))$ and $\mathcal{C}(\widetilde{\mathsf{A}}^-_{m_2}(c_2,\varepsilon))$ admit continuations to an ε^ν neighborhood of the double resonance point, and that these continuations match the s-cylinders located on both sides of the neighborhood along the simple resonance arcs.

2.5. ***Thresholds.*** The minimal regularity $\kappa_0 \geq 2$ is assumed to be large enough for our subsequent (finite number of) applications of normally hyperbolic persistence, genericity and KAM theorems to apply for $\kappa \geq \kappa_0$ in the various settings involved in the construction. We fix a Tonelli integrable Hamiltonian $h \in C^\kappa(\mathbb{R}^3)$ with $\kappa \geq \kappa_0$ together with a broken line of simple resonance arcs as in Figure 3. We outline the main steps of a proof of the existence of the lower semicontinuous threshold function $\boldsymbol{\epsilon}_0$ of Statement I. Without loss of generality, we can focus on a single resonant arc Γ and assume that:

- Γ is a graph over the plane $P=\{r_3=0\}$, so that its equation reads $r_3=r_3^*(\hat{r})$ with $\hat{r} := (r_1, r_2)$ in $\widehat{\Gamma} := \pi_P(\Gamma)$.
- The frequency vector along Γ reads $\omega(r) = (\omega_1(r), \omega_2(r), 0)$.

By compactness and convexity, the spectrum of the normal Hessian of h along Γ is bounded from below by a positive constant.

1. *Conditions for the existence of simple resonance cylinders.*

• For $\kappa \geq \kappa_0$, the C^κ one-parameter families of functions on $\mathbb{T}$ with parameter in $\widehat{\Gamma}$ which admit a single and nondegenerate maximum up to a finite number of values of the parameter, for which there are exactly two nondegenerate maxima, form an open and dense subset of $C^\kappa(\mathbb{T}\times\widehat{\Gamma})$. The averaging operator $f \mapsto \langle f \rangle_\Gamma$, where

$$\langle f(\theta_3;\hat{r})\rangle_\Gamma = \int_{\mathbb{T}^2} f(\hat{\theta},\theta^3,\hat{r},r_3^*(\hat{r}))d\hat{\theta}$$

is linear and surjective from $C^\kappa(\mathbb{A}^3,\mathbb{R})$ to $C^\kappa(\mathbb{T}\times\widehat{\Gamma})$, hence is an open mapping. Therefore there is a dense open subset $\mathcal{S}_1 \subset \mathcal{S}^\kappa$ such that for $f \in \mathcal{S}_1^\kappa$ the averaged potential $\langle f(\theta_3;\hat{r})\rangle_\Gamma$ admits a single and nondegenerate maximum $\theta_3^*(\hat{r})$ outside a finite subset of bifurcation points in $\widehat{\Gamma}$, where it admits exactly two nondegenerate maximums.

Consequently, thanks to the remark on the normal Hessian of h along Γ, over closed intervals limited by bifurcation points the hyperbolicity constant of the hyperbolic point $O(\widehat{r}) = (\theta_3^*(\hat{r}), r_3^*(\hat{r}))$ is uniformly bounded from below.[11]

• Using to a suitable $\sqrt{\varepsilon}$ rescaling, one proves (see [5] and early works by Kaloshin) that, given $f \in \mathcal{S}_1^\kappa$, the previous uniform bound yields the existence of a finite number of double resonance points (d_i) in Γ such that one gets (pseudoinvariant) simple resonance cylinders outside the union of ε^ν-neighborhoods of the fibers $\mathbb{T}^3\times\{d_i\}$ in $H_\varepsilon(\boldsymbol{e})$. The choice of κ_0 depends on the required value of $\nu < \frac{1}{2}$ (see [44]) and this statement holds for and $0 < \varepsilon < \boldsymbol{\epsilon}_1(f)$.

• For each $f \in \mathcal{S}_1$, there is an open neighborhood $\mathscr{O}(f)$ of f in $\mathcal{S}_1$ such that the set of double resonance points to be removed from the arc Γ do not depend on the choice of the perturbation $g \in \mathscr{O}$, and moreover the function $\boldsymbol{\epsilon}_1$ can be chosen so as to depend continuously on g in $\mathscr{O}$.

This process provides us with a multivalued locally continuous threshold function $\boldsymbol{\epsilon}_1 : \mathcal{S}_1^\kappa \to \mathbb{R}^{*+}$.

2. *Conditions at a double resonance point.* We fix now an open subset $\mathscr{O} \subset \mathcal{S}_1$ over which the previous two properties (double resonant points and continuity of $\boldsymbol{\epsilon}_1$) are satisfied. It is enough to consider a single double resonance point $r^0 \in \Gamma$, and one can assume its frequency vector to have the form $(\omega_1, 0, 0)$ with $\omega_1 \neq 0$. Set $\bar{\theta} := (\theta_2, \theta_3)$ and for $f \in \mathscr{O}$ let

$$U(\bar{\theta}) := \langle f \rangle_{r^0}(\bar{\theta}) = \int_{\mathbb{T}} f(\theta_1, \bar{\theta}, r^0)d\theta_1$$

[11] This point is uniquely defined by continuity at the boundaries of the interval.

be the averaged potential at r^0. The ($\sqrt{\varepsilon}$-rescaled) main part of the averaged system at r^0 reads

$$C(\bar{\theta}, \bar{r}) = \tfrac{1}{2} Q(\bar{r}) + U(\bar{\theta}), \tag{17}$$

where Q is a *fixed* quadratic form deduced from the Hessian of h at r^0. We fix a finite number of homology classes in $\boldsymbol{H}_1(\mathbb{T}^2)$ (one class in the case of a double resonant point in Int Γ and two classes in the case of a boundary point, according to the fact that one wants to construct chains following Γ in the first case and passing to another resonance arc in the second):

- Since the averaging operator $\langle\ \rangle_{r^0}$ is an open mapping $C^\kappa(\mathbb{A}^3) \to C^\kappa(\mathbb{T}^2)$, provided that κ_0 is large enough, there is a dense open subset $\mathscr{O}' \subset \mathscr{O}$ such that the classical system (17) satisfies the properties quoted in Statement II relative to the previous homology classes. In particular, there is a chain of annuli (with a single singular annulus) attached to the previous homology class (in the case of an inner double resonance point) and to the pair of classes (in the case of a boundary point).
- By (singular) normally hyperbolic persistence of the annuli, there is a multivalued locally continuous function $\boldsymbol{\epsilon}_2 : \mathscr{O}_2 \to \mathbb{R}^{*+}$ such that for $0 < \varepsilon < \boldsymbol{\epsilon}_2(g) < \boldsymbol{\epsilon}_1(g)$ there exist chains of (pseudoinvariant) cylinders (with a single singular cylinder) obtained by suspension relative to the fast angle, and then perturbation, of the chains of annuli in the averaged system
$$C_\varepsilon(\bar{\theta}, \bar{r}) = \tfrac{1}{2} Q(\bar{r}) + \varepsilon \langle g \rangle_{r^0}(\bar{\theta}).$$
- The Poincaré pseudoinvariant cylinders in these chains extend to an ε^ν-distance away from the double resonance.

This step (applied to each double resonance point in Γ) provides us with a cover $(\mathscr{O}_j)$ of $\mathcal{S}_1$ by open sets over which the function $\boldsymbol{\epsilon}_2$ is continuous and is a threshold for the existence of pseudoinvariant cylinders along Γ, and along two other resonant arcs in a small neighborhood of the boundary double resonance points.

3. *Conditions for the existence of KAM tori and invariant cylinders.* We fix now f in some $\mathscr{O}_j$. For $0 < \varepsilon < \boldsymbol{\epsilon}_2(f)$, the existence of a sufficiently large set of 2-dimensional unperturbed tori inside the pseudoinvariant cylinders (neglecting the remainders of the various normal forms) is guaranteed by usual considerations from Diophantine theory; see for instance [15]. After reducing the system (in normal form) inside the pseudoinvariant cylinders to a two dimensional discrete setting, one can apply a version of KAM theorem with vanishing torsion (which reflects the singular nature of the perturbation), deduced from Herman's work (see [29; 30]), to show the existence of 2-dimensional invariant tori close to the "boundaries" of the pseudoinvariant cylinders. These tori therefore bound genuinely invariant and normally hyperbolic tori. In the same way, one proves the existence of invariant tori inside the matching zone at an ε^{ν} distance of the double resonance points, proving that the simple resonance cylinders and the (suitably chosen) double resonance Poincaré cylinders continue one another. This provides us with a new open cover of $\mathscr{O}$ and a multivalued threshold function $0 < \boldsymbol{\epsilon}_3 < \boldsymbol{\epsilon}_2$ which is continuous on each open set of the cover and is a threshold for the existence of a family of compact invariant normally hyperbolic cylinders and singular cylinders along the arc Γ.

4. *The lower-semicontinuous threshold* $\boldsymbol{\epsilon}_0$. At this point the initial open dense set $\mathcal{S}_1$ is endowed with an open cover $(\mathscr{U}_i)_{i\in I}$ together with a threshold function $\boldsymbol{\epsilon}_3$ which is positive and continuous on each $\mathscr{U}_i$. For each i, we continue the function $(\boldsymbol{\epsilon}_3)_{|\mathscr{O}_i}$ to $\mathcal{S}^{\kappa}$ by 0 on the closed set $\mathcal{S}^{\kappa} \setminus \mathscr{U}_i$. The resulting continuation $\bar{\boldsymbol{\epsilon}}_3^{(i)}$ is therefore lower-semicontinuous on $\mathcal{S}^{\kappa}$. Applying the previous process to each arc in the initial broken line, one gets a final threshold

$$\boldsymbol{\epsilon}_0 = \mathrm{Sup}_{i\in I}\, \bar{\boldsymbol{\epsilon}}_3^{(i)},$$

which is lower-semicontinuous, positive on the dense open set $\mathcal{S}_1$ and such that *each element* in the generalized ball $\mathscr{B}^{\kappa}(\boldsymbol{\epsilon}_0)$ admits a family of compact normally hyperbolic invariant cylinders along the broken line.

5. *Connections.* We will not address in detail here the question of homoclinic and heteroclinic connections. New conditions to ensure the existence of transverse heteroclinic connections between distinct consecutive cylinders come from usual arguments from transversality theory, while the (topologically transverse) homoclinic connections require more subtle arguments (see Section 5 and [44]) from dimension theory, which finally yield the *admissible* chains along which the diffusion orbits can be proved to exist. Both type of connections require adding arbitrarily small perturbations to the elements of the generalized ball $\mathscr{B}(\boldsymbol{\epsilon}_0)$ (which is legitimized by the fact that $\mathscr{B}(\varepsilon_0)$ is open), which explains that our admissible chains exist only for a dense open set of perturbations in $\mathscr{B}(\varepsilon_0)$ in our Statement I (openness being trivial by continuity).

2.6. ***Conclusion.*** To conclude this geometric description, we want to emphasize that, while the geometric analysis is more complicated near double resonances than along simple resonance arcs, the dynamical analysis along d-cylinders is by far simpler than that along s-cylinders. Indeed, due to the existence of a global transverse intersection of the stable and unstable manifolds of a 2-dimensional annulus in the averaged classical system on $\mathbb{T}^2$, the stable and unstable manifolds of the corresponding perturbed cylinder in the initial system at fixed energy intersect transversely along a two-dimensional homoclinic annulus. After a two-dimensional reduction of the dynamics on a Poincaré section inside the cylinder (see Section 5), this yields a bisystem of *globally defined* maps (the inner and the homoclinic one) for which the existence of drifting orbits is easy to prove (see the next section for a simplified model). By contrast, the stable and unstable manifolds of an s-cylinder do coincide if the remainder of the normal form is neglected. This makes the construction of homoclinic orbits more difficult: after taking the remainder into account, they essentially come from the homoclinic intersections of invariant tori contained inside the cylinder, which yields only a locally defined homoclinic correspondence. In this case usual transversality arguments do not apply, due to the uncountable number of objects to control. Sections 4 and 5 below are devoted to this difficulty, in the general discrete case in Section 4, which is then applied in a simplified model in Section 5.

3. Diffusion orbits in the *a priori* chaotic discrete setting

The purpose of this section is to exhibit a class of symplectic diffeomorphisms of $\mathbb{A}^2 = T^*\mathbb{T}^2$, for which diffusion properties can be detected with minimal technicalities, which in addition are good models for the dynamics along double resonance cylinders.

Our framework is a discrete version of the so-called "*a priori* chaotic" setting developed in relation to Mather's work on unbounded growth of energy for nonautonomous perturbations of geodesic flows. This problem was investigated by Bolotin and Treschev [8] and Delshams, De la Llave, Seara [16]; more recently Gelfreich and Turaev systematically revisited this question in the analytic category [19]. However some significant features of our systems are rather different and make our approach both simpler and more general to some extent, since they are *far from any integrable ones*.

The main feature of our diffeomorphisms $g : \mathbb{A}^2 \circlearrowleft$ is the existence of a normally hyperbolic annulus $\mathscr{A}$ (diffeomorphic to $\mathbb{A}$) that admits a homoclinic intersection *which is itself diffeomorphic to an annulus*. This yields the existence of a natural bisystem on $\mathscr{A}$, formed by the restriction of g to $\mathscr{A}$ together with a homoclinic map defined everywhere on $\mathscr{A}$. It is then easy to show that an arbitrarily small perturbation of g puts this bisystem in a general position and allows one to apply Moeckel's theorem [48]. This yields the existence of drifting pseudoorbits, which are in turn shadowed by genuine orbits, due to normally hyperbolic shadowing properties.

3.1. ***The setting.*** We fix once and for all a closed interval $I \subset [-1, 2]$ of $\mathbb{R}$ which contains $[0, 1]$ in its interior. We work in the space $\mathscr{D}^\kappa$ of C^κ symplectic diffeomorphisms of $\mathbb{A}^2$ with support contained in $\mathbb{T}^2 \times I^2$, endowed with the natural uniform C^κ metric d_κ. The space $(\mathscr{D}^\kappa, d_\kappa)$ is complete.

We first introduce a diffusion property for diffeomorphisms in $\mathscr{D}^\kappa$ together with a specific class $\mathscr{F}^\kappa \subset \mathscr{D}^\kappa$ (uncoupled products), whose elements do not satisfy this property and which we consider as "unperturbed systems". Our main result then proves the existence of a large subset of suitable C^κ perturbations of f which admit the diffusion property.

1. Let us now give precise definitions, beginning with that of diffusion orbits.

Definition 4. Fix $\delta > 0$ and set

$$\mathscr{U}^0(\delta) = \{(\theta, r) \in \mathbb{A}^2 \mid |r_1|\} < \delta\}, \quad \mathscr{U}^1(\delta) = \{(\theta, r) \in \mathbb{A}^2 \mid |r_1 - 1| < \delta\}. \tag{18}$$

Given a diffeomorphism $g \in \mathscr{D}^\kappa$, we say that a finite orbit $x_0, \ldots, x_N$ of g is a *δ-diffusion orbit* when $x_0 \in \mathscr{U}^0$ and $x_N \in \mathscr{U}^1$.

2. The elements $f \in \mathscr{F}^\kappa$ are C^κ symplectic diffeomorphisms of $\mathbb{A}^2$ and admit the product form

$$f(\theta, r) = (f_1(\theta_1, r_1), f_2(\theta_2, r_2)), \quad (\theta, r) \in \mathbb{A}^2, \tag{19}$$

where the diffeomorphisms $f_i : \mathbb{A} \circlearrowleft$ satisfy some additional conditions:

• *Conditions on* f_1. We denote by $\mathrm{DD}(\tau)$ the set of real numbers which are Diophantine of exponent $\tau > 1$. Given $\kappa \geq 1$, we introduce the set $\mathscr{F}_1^\kappa$ of C^κ symplectic diffeomorphisms $f_1 : \mathbb{A} \circlearrowleft$ which satisfy the following conditions:

(C_1) Supp $f_1 \subset \mathbb{T} \times I$.

(C_2) *The circles* $\Gamma^0 = \mathbb{T} \times \{0\}$ *and* $\Gamma^1 = \mathbb{T} \times \{1\}$ *are invariant under* f_1, *and their rotation numbers* ρ^0, ρ^1 *are in* $\mathrm{DD}(\tau)$ *for some* $\tau > 1$.

To introduce the third condition we use the coordinate chart (θ_1, r_1) of $\mathbb{A}$ and write

$$f_1(\theta_1, r_1) = (\Theta_1(\theta_1, r_1), R_1(\theta_1, r_1)). \tag{20}$$

(C_3) *The restriction of* f_1 *to the annulus* $\mathbb{T} \times [0, 1]$ *uniformly tilts the vertical to the right*, *that is*, *there is a* $c > 0$ *such that*

$$\frac{\partial \Theta_1}{\partial r_1}(\theta_1, r_1) \geq c, \quad \forall (\theta_1, r_1) \in \mathbb{T} \times [0, 1]. \tag{21}$$

Note that, due to (C_1) and (C_2), the annulus $\mathbb{T} \times [0, 1]$ is invariant under f_1. The condition that the rotation numbers of the circles Γ^i are in $\mathrm{DD}(\tau)$ will ensure their persistence under perturbation.

• *Conditions on* f_2. We introduce the set $\mathscr{F}_2^\kappa$ of C^κ symplectic diffeomorphisms $f_2 : \mathbb{A} \circlearrowleft$ which satisfy the following conditions:

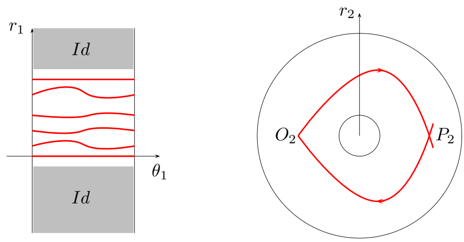


Figure 10. An unperturbed diffeomorphism.

(C_4) Supp $f_2 \subset \mathbb{T} \times I$.

(C_5) *The diffeomorphism f_2 possesses a hyperbolic fixed point O_2.*

(C_6) *The point O_2 admits a transverse homoclinic point P_2.*

We will denote by

$$\lambda(O_2) > 1 \tag{22}$$

the maximal eigenvalue of the derivative $D_{O_2} f_2$. Note that O_2 and P_2 are contained in the support of f_2.

3. We now define our set of "unperturbed" diffeomorphisms in order to guarantee additional stability properties under perturbations.

Definition 5. We define $\mathscr{F}^\kappa$ as the set of (symplectic) diffeomorphisms on $\mathbb{A}^2$ of the form (19), where $f_1 \in \mathscr{F}_1^\kappa$ and $f_2 \in \mathscr{F}_2^\kappa$ with

$$(\mathrm{Max}_{x \in \mathbb{A}} \| T_x f_1 \|)^\kappa < \lambda(O_2). \tag{23}$$

where $\|\cdot\|$ stands for the operator norm.

The domination condition (23) ensures that the invariant annulus $\mathbb{A} \times \{O_2\}$ is uniformly normally hyperbolic for f, with persistence properties in the C^κ topology and additional specific symplectic features.[12]

4. Given $\tau > 1$, when κ is large enough, for any $f \in \mathscr{F}^\kappa$ there exists an arbitrarily small $\delta > 0$ such that f does not possess any δ-diffusion orbit: classical KAM theorems in the finitely differentiable setting prove the existence of an essential invariant circle Γ for f_1 located in the zone $r_1 > 0$ (one indeed gets an infinite family of such circles), and the claim comes from the product structure of f. However, we will prove that under generic and small enough perturbations, *any* element of $\mathscr{F}^\kappa$ gives rise to a diffeomorphism which admits diffusion orbits. More precisely, our main result is the following.

[12] A weaker condition could be required, at the cost of a smoothing argument which would obscure the description.

Theorem 6. *There is a* κ_0 *such that, given* $f \in \mathscr{F}^\kappa$ *and* $\delta > 0$*, then there is an* $\varepsilon(f) > 0$ *such that the subset of all diffeomorphisms* g *in* $B^\kappa(f, \varepsilon(f))$ *which admit a* δ*-diffusion orbit is open in the* C^0 *topology and dense in* $B^{\kappa-1}(f, \varepsilon(f))$.

The existence of δ-diffusion orbits being clearly an open property in the C^0 topology, we will therefore focus on the "density". The loss of 1 derivative could be avoided using a smoothing argument that we will not describe here.

5. As already mentioned in the introduction, the proof of Theorem 6 is based on a method introduced by Moeckel in [48] to prove the existence of drifting orbits for bisystems of maps τ_0, τ_1 on the annulus. If $\Gamma_\bullet$ and $\Gamma^\bullet$ are two disjoint graphs of C^1 functions $\mathbb{T} \to \mathbb{R}$, we denote by $\boldsymbol{A}[\Gamma_\bullet, \Gamma^\bullet] \subset \mathbb{A}$ the subannulus bounded by their union.

Theorem A [48]. *Let* $\tau_0, \tau_1 : \mathbb{A} \circlearrowleft$ *be* C^1 *diffeomorphisms with compact support in* $\mathbb{T} \times I$*, where* τ_0 *is area-preserving and* τ_1 *is exact symplectic. Assume that there exist two disjoint* τ_0*-invariant* C^1 *graphs* $\Gamma_\bullet, \Gamma^\bullet$ *in* $\mathbb{T} \times I$ *and that* τ_0 *is a twist map in restriction to the annulus* $\boldsymbol{A} := \boldsymbol{A}[\Gamma_\bullet, \Gamma^\bullet]$. *Let* $\mathrm{Ess}_{\boldsymbol{A}}(\tau_i)$ *be the set of essential* τ_i*-invariant circles contained in* $\boldsymbol{A}$. *Assume that*

$$\mathrm{Ess}_{\boldsymbol{A}}(\tau_0) \cap \mathrm{Ess}(\tau_1) = \varnothing. \tag{24}$$

Then for any connected neighborhoods $U_\bullet$ *and* $U^\bullet$ *of* $\Gamma_\bullet$ *and* $\Gamma^\bullet$ *in* $\mathbb{A}$*, with* $\tau_1(\Gamma_\bullet) \subset U_\bullet$ *and* $\tau_1(\Gamma^\bullet) \subset U^\bullet$ *the bisystem* (τ_0, τ_1) *admits an orbit with first point in* $U_\bullet$ *and last point in* $U^\bullet$.

This in fact is a slight generalization of the theorem of [48], since the boundaries $\Gamma_\bullet$ and $\Gamma^\bullet$ are not assumed to be invariant under τ_1.

The next result, based on the study of the Minkowski dimension of the sets $\mathrm{Ess}_{\boldsymbol{A}}(\tau_0)$ and $\mathrm{Ess}(\tau_1)$, will provide us with the necessary tool for proving the density statement in Theorem 6.

Theorem B [48]. *Fix an integer* $p \geq 1$. *Let* $\tau_0, \tau_1 : \mathbb{A} \circlearrowleft$ *be* C^p *area-preserving diffeomorphisms with compact support in* $\mathbb{T} \times I$. *Assume that there exist two disjoint* τ_0*-invariant* C^1 *graphs* $\Gamma_\bullet, \Gamma^\bullet$ *in* $\mathbb{T} \times I$*, and that* τ_0 *is a twist map in restriction to* $\boldsymbol{A} := \boldsymbol{A}[\Gamma_\bullet, \Gamma^\bullet]$. *Assume moreover that* $(\tau_0)_{|\boldsymbol{A}}$ *has no essential invariant circle with rational rotation number. Then there exists a* C^∞ *Hamiltonian* $h : \mathbb{A} \to \mathbb{R}$ *with support in* $\mathbb{T} \times I$*, arbitrarily small in the* C^∞ *topology, such that*

$$\mathrm{Ess}_{\boldsymbol{A}}(\tau_0) \cap \mathrm{Ess}(\Phi^h \circ \tau_1 \circ \Phi^{-h}) = \varnothing, \tag{25}$$

where Φ^h *stands for the time-one map of the Hamiltonian flow generated by* h.

Note that assuming that the τ_i are area-preserving is equivalent to assuming that they are exact-symplectic a property directly related to the constructions of our bisystems in the following. The proof of Theorem B is exactly the same as in [48].

3.2. ***Proof of Theorem 6.*** Let us first informally describe the proof. The first ingredient is the choice of ε small enough so that any g in $B^\kappa(f, \varepsilon(f))$ exhibits some of the main dynamical features of f. In particular, we require that g admits a normally hyperbolic invariant annulus $\mathscr{A}_g$ close to $\mathbb{A} \times \{O_2\}$ and a homoclinic annulus $\mathscr{H}_g$ close to $\mathbb{A} \times \{P_2\}$. We then consider two diffeomorphisms of $\mathscr{A}_g$.

• The first one, φ_g, is nothing but the restriction of g to $\mathscr{A}_g$. Thanks to the domination condition (23) normally hyperbolic persistence proves that φ_g is C^κ close to f_1 (in suitable coordinates). In particular, the initial invariant circles Γ^i of f_1 will persist and give rise to essential invariant circles Γ^i_g for φ_g, which bound a compact annulus $\mathsf{A}_g \subset \mathscr{A}_g$.

• The definition of the second diffeomorphism — the *homoclinic map* ψ_g — is based on the existence of a (full) homoclinic annulus $\mathscr{H}_g$. The diffeomorphism ψ_g encodes the asymptotic properties of the associated homoclinic orbits of $\mathscr{A}_g$. More precisely, if x, y in $\mathscr{A}_g$ satisfy $y = \psi_g(x)$, then there exists an orbit $z_{-M}, \ldots, z_N$ of g, located in $\mathbb{A}^2 \setminus \mathscr{A}_g$, with z_{-M} arbitrarily close to $g^{-M}(x) = \varphi_g^{-M}(x)$ and z_N arbitrarily close to $g^N(y) = \varphi_g^N(y)$, where the integers N and M can be chosen arbitrarily large.

A key observation (introduced in [24]) is that the Poincaré recurrence theorem applies to φ_g on the compact annulus $\mathscr{A}_g$ and allows one to choose M and N so that $\varphi_g^{-M}(x)$ and $\varphi_g^N(y)$ are arbitrarily close to the initial points x and y respectively.

Using Moeckel's results, we prove that *after a small perturbation of* g the bisystem (φ_g, ψ_g) admits "drifting orbits", whose initial and final points are arbitrarily close to the boundary circles of A_g.

Finally, in view of the definition of φ_g and the asymptotic properties of ψ_g, one expects that the connecting orbits of the bisystem can be uniformly approximated by genuine orbits of g. Here, for completeness, we prove that this is the case by means of a normally hyperbolic shadowing lemma, whose idea is reminiscent of [8; 17]. Our proof closely follows the (more general) one in [23].

3.2.1. *The symplectic geometry of perturbed products.* **1.** Let us first examine the dynamical features of a diffeomorphism $f \in \mathscr{F}^\kappa(\tau)$, which are immediately deduced from the product form (19):

• The annulus $\mathscr{A} = \mathbb{A} \times \{O_2\}$ is invariant under f and diffeomorphic to $\mathbb{A}$. It is moreover κ-normally hyperbolic, due to condition (23). The stable and unstable manifolds of $\mathscr{A}$ inherit the product structure of f:

$$W^{\pm}(\mathscr{A}) = \mathbb{A} \times W^{\pm}(O_2). \tag{26}$$

These are hypersurfaces of $\mathbb{A}^2$ of class C^κ (since $W^{\pm}(O_2)$ are C^κ), and so are coisotropic in $\mathbb{A}^2$. Their characteristic leaves are the 1-dimensional submanifolds

$$\{x\} \times W^{\pm}(O_2), \quad x \in \mathbb{A}. \tag{27}$$

which coincide with the stable and unstable manifolds of the points of $\mathscr{A}$ respectively (this fact is general, see Appendix A). Let $\Pi^{\pm} : W^{\pm}(\mathscr{A}) \to \mathscr{A}$ stand for the characteristic projections, so that if $(x, w) \in W^{\pm}(x)$, then $\Pi^{\pm}(x, w) = (x, O_2)$.

• The manifolds $W^{\pm}(\mathscr{A})$ intersect transversely in $\mathbb{A}^2$ along both $\mathscr{A}$ and the homoclinic annulus

$$\mathscr{H} = \mathbb{A} \times \{P_2\}. \tag{28}$$

Moreover, for each $(x, O_2) \in \mathscr{A}$, the leaf $W^{-}((x, O_2))$ transversely intersect the manifold $W^{+}(\mathscr{A})$ at a unique point of $\mathscr{H}$, namely

$$W^{-}((x, O_2)) \cap \mathscr{H} = \{(x, P_2)\}. \tag{29}$$

One has a similar observation for the stable leaves. We denote by $\pi^{\pm}$ the restrictions of $\Pi^{\pm}$ to the annulus $\mathscr{H}$, so that

$$\pi^{\pm} : \mathscr{H} \to \mathscr{A}, \quad \pi^{\pm}(x, P_2) = (x, O_2), \quad x \in \mathbb{A}. \tag{30}$$

• Clearly $\mathscr{A}$ and $\mathscr{H}$ are symplectic submanifolds of $\mathbb{A}^2$ and $\pi^{\pm}$ are symplectic diffeomorphisms.

• There exists a pair of natural f-induced symplectic diffeomorphisms of $\mathscr{A}$. The first one is just the restriction $\varphi = f_{|\mathscr{A}}$, which here admits a natural identification with f_1. The second one is the map

$$\psi = \pi^{+} \circ (\pi^{-})^{-1}$$

which describes the homoclinic excursion of the orbits, we call it the *homoclinic map*. Clearly $\psi = \mathrm{Id}$ here.

The manifolds $W^{\pm}(\mathscr{A})$ do indeed admit a much larger intersection than $\mathscr{A} \cup \mathscr{H}$, but we neglect the other components which play no role in our construction.

The homoclinic map has been introduced in [14] and carefully studied in [17] and subsequent papers by the same authors, under the name of scattering map. We use this new terminology here to make a distinction between the homoclinic maps (or correspondences) and the heteroclinic ones, which necessarily appear when chains of cylinders are considered. While the ideas are very close, one slight difference in our (complete) work is that we perform a systematic reduction of the homo-heteroclinic map to a two-dimensional object (in order to obtain a two-dimensional bisystem), while the scattering map is usually used in a more global higher dimensional setting.

2. The symplectic features of small enough perturbations of f are immediately deduced from the symplectic normally hyperbolic persistence theorem (see Appendix A). Note in particular that even if the unperturbed annulus is noncompact, the existence and uniqueness of the perturbed one is not difficult, thanks to the compact-supported character of the perturbation.

Lemma 7. *Let $f = (f_1, f_2) \in \mathscr{F}^\kappa(\tau)$ be fixed. Then there exists $\bar\varepsilon(f) > 0$ such that for each g in $B^\kappa(f, \bar\varepsilon(f))$:*

- *There exists a (uniquely defined) symplectic normally hyperbolic g-invariant annulus $\mathscr{A}_g$ of the form*

$$\mathscr{A}_g = \{(x, a_g(x)) \mid x \in \mathbb{A}\}, \tag{31}$$

where a_g is a C^κ function $\mathbb{A} \to B^2(O_2, \alpha) \subset \mathbb{A}$ such that $\|a_g - O_2\|_{C^\kappa(\mathbb{A})} \to 0$ when $d_\kappa(g, f) \to 0$ (where $\alpha > 0$ is a suitable constant).

- *The manifolds $W^\pm(\mathscr{A}_g)$ are coisotropic with characteristic foliations*

$$(W^\pm(z))_{z \in \mathscr{A}_g},$$

and the characteristic projections $\Pi_g^\pm : W^\pm(\mathscr{A}_g) \to \mathscr{A}_g$ are $C^{\kappa-1}$.

- *There exists a (uniquely defined) symplectic homoclinic annulus*

$$\mathscr{H}_g \subset W^+(\mathscr{A}_g) \cap W^-(\mathscr{A}_g),$$

of the form

$$\mathscr{H}_g = \{(x, h_g(x)) \mid x \in \mathbb{A}\}, \tag{32}$$

where h_g is a C^κ function $\mathbb{A} \to B^2(P_2, \alpha) \subset \mathbb{A}$ such that $\|h_g - P_2\|_{C^\kappa(\mathbb{A})} \to 0$ when $d_\kappa(g, f) \to 0$.

- *The restrictions*

$$\pi_g^\pm := (\Pi_g^\pm)_{|\mathscr{H}_g} : \mathscr{H}_g \to \mathscr{A}_g \tag{33}$$

are $C^{\kappa-1}$ symplectic diffeomorphisms.

- *For each $z \in \mathscr{A}_g$, the unstable manifold $W^-(z)$ intersects $\mathscr{H}_g$ at $(\pi_g^-)^{-1}(z)$ transversely in $\mathbb{A}^2$, with an analogous property for the stable manifold.*

3.2.2. *The bisystem.* We can now introduce our bisystem on $\mathscr{A}_g$, assuming that $g \in B^\kappa(f, \bar{\varepsilon}(f))$. We first consider the restriction

$$\varphi_g : \mathscr{A}_g \circlearrowleft, \quad \varphi_g = g_{|\mathscr{A}_g}, \tag{34}$$

which is a C^κ symplectic diffeomorphism for the induced structure on $\mathscr{A}_g$. As for our second map, we set

$$\psi_g : \mathscr{A}_g \circlearrowleft, \quad \psi_g = \pi_g^+ \circ (\pi_g^-)^{-1}. \tag{35}$$

Therefore ψ_g is a $C^{\kappa-1}$ symplectic map. The next lemma (which is an application of Moser's isotopy argument) enables us to identify φ_g (and ψ_g) with a diffeomorphism of the standard annulus $\mathbb{A}$ in a proper way.

Lemma 8. *If $\bar{\varepsilon}(f)$ is small enough and $g \in B^\kappa(f, \bar{\varepsilon}(f))$, there exists a $C^{\kappa-1}$ symplectic embedding Φ_g of $\mathbb{A}$, equipped with the standard form, into $\mathbb{A}^2$ such that*:

- $\Phi_g(\mathbb{A}) = \mathscr{A}_g$.
- *The diffeomorphism $\widehat{\varphi}_g = \Phi_g^{-1} \circ \varphi_g \circ \Phi_g : \mathbb{A} \circlearrowleft$ has support in $\mathbb{T} \times I$ and tends to f_1 in the $C^{\kappa-1}$ uniform topology when $d_\kappa(g, f) \to 0$.*
- *The diffeomorphism $\widehat{\psi}_g = \Phi_g^{-1} \circ \psi_g \circ \Phi_g : \mathbb{A} \circlearrowleft$ has support in $\mathbb{T} \times I$ and tends to* Id *in the $C^{\kappa-1}$ uniform topology when $d_\kappa(g, f) \to 0$.*

The following corollary is an immediate application of the previous lemma and finitely differentiable KAM theory.

Corollary 9. *There is an $\varepsilon(f) \in]0, \bar{\varepsilon}(f)]$ such that for each diffeomorphism $g \in B^\kappa(f, \varepsilon(f))$ there exists a $C^{\kappa-1}$ symplectic embedding Φ_g of $\mathbb{A}$ into $\mathscr{A}_g$ such that the map $\widehat{\varphi}_g = \Phi^{-1} \circ \varphi_g \circ \Phi$ admits two (disjoint) essential invariant circles $\Gamma_\bullet$ and $\Gamma^\bullet$ with rotation numbers ρ^0 and ρ^1 respectively (see (C_2)), such that $\Gamma_\bullet \to \Gamma^0$ and $\Gamma^\bullet \to \Gamma^1$ in the C^0 topology when $d_\kappa(g, f) \to 0$. Moreover, the map $\widehat{\varphi}_g$ uniformly tilts the vertical over the annulus $\boldsymbol{A}_g$ bounded by $\Gamma_\bullet$ and $\Gamma^\bullet$.*

3.2.3. *The perturbative step.* We fix now a diffeomorphism $g \in B^\kappa(f, \varepsilon(f))$, where $\varepsilon(f)$ is defined in Corollary 9, and get rid of the $\hat{}$ in the previous corollary. We want to prove the existence of a perturbed diffeomorphism $\tilde{g} \in B^\kappa(f, \varepsilon(f))$, arbitrarily close to g in the $C^{\kappa-1}$ topology, for which the associated bisystem $(\varphi_{\tilde{g}}, \psi_{\tilde{g}})$ satisfies condition (24). We proceed in two steps: we first perturb g so that φ_g has no rational essential circle, and we then perturb the resulting diffeomorphism again (without perturbing φ_g) to ensure condition (24). We write ε instead of $\varepsilon(f)$ in the following.

1. *First perturbation of g: making φ_g admissible* Let J be a closed interval of $\mathbb{R}$ containing $[0, 1]$ in its interior and contained in the interior of I. Taking into account that $\mathscr{A}_g$ is of class C^κ and invariant under g, by usual perturbation techniques (see [50; 51]), there exists a C^κ diffeomorphism $g^* \in B^\kappa(f, \varepsilon)$, arbitrarily close to g in the uniform C^κ topology, which satisfies:

- The invariant annulus $\mathscr{A}_{g^*}$ coincides with $\mathscr{A}_g$.
- All periodic points of $\varphi_{g^*} = g^*_{|\mathscr{A}_{g^*}}$ in $\mathbb{T}^2 \times J^2$ are either hyperbolic or elliptic with nondegenerate Birkhoff invariant.
- The stable and unstable manifolds of the periodic orbits intersect transversely.

As a consequence, if κ is large enough to ensure the existence of invariant curves surrounding each elliptic point, one easily proves that φ_{g^*} cannot admit an essential invariant circle with rational rotation number in the compact annulus $\boldsymbol{A}_{g^*}$ defined in Corollary 9.

2. *Second perturbation of* g: *making* ψ_g *admissible* In view of the last section, replacing g with g^*, we can assume that g has no invariant circle in $\boldsymbol{A}_g$ with rational rotation number. We want now to perturb g into a new diffeomorphism $\tilde{g}$ such that

$$\mathscr{A}_{\tilde{g}} = \mathscr{A}_g, \quad \mathscr{H}_{\tilde{g}} = \mathscr{H}_g, \quad \varphi_{\tilde{g}} = \varphi_g, \quad \mathrm{Ess}_{\boldsymbol{A}_g}(\varphi_g) \cap \mathrm{Ess}(\psi_{\tilde{g}}) = \varnothing. \tag{36}$$

We first analyze the composition of g with a diffeomorphism with support localized in a small enough neighborhood of the annulus $\mathscr{H}_g$.

Lemma 10. *Let* $W_0^\pm$ *be the submanifolds (diffeomorphic to* $[0, 1] \times \mathbb{A}$*) of* $W^\pm(\mathscr{A}_g)$ *bounded by* $\mathscr{A}_g$ *and* $\mathscr{H}_g$. *Let* $\mathscr{N}$ *be a neighborhood of* $\mathscr{H}_g$ *such that*

$$\mathrm{dist}(\mathscr{A}_g, \mathscr{N}) > 0, \quad g(W_0^+) \cap \mathscr{N} = \varnothing, \quad g^{-1}(W_0^-) \cap \mathscr{N} = \varnothing. \tag{37}$$

Assume that $\boldsymbol{\chi}$ *is a diffeomorphism of* $\mathbb{A}^2$ *with support in* $\mathscr{N}$, *which leaves the annulus* $\mathscr{H}_g$ *invariant, and set* $\tilde{g} = \boldsymbol{\chi} \circ g$. *Then* $\mathscr{A}_{\tilde{g}} = \mathscr{A}_g$, $\mathscr{H}_{\tilde{g}} = \mathscr{H}_g$ *and*

$$\varphi_{\tilde{g}} = \varphi_g, \quad \psi_{\tilde{g}} = \psi_g \circ (\pi_g^- \circ \boldsymbol{\chi} \circ (\pi_g^-)^{-1}). \tag{38}$$

See [45] for a proof. We can now use Moeckel's Theorem B in order to produce our perturbation $\tilde{g}$.

Lemma 11. *There exists a diffeomorphism* $\tilde{g} \in \mathscr{D}^{\kappa-1}$, *arbitrarily close to* g *in the* $C^{\kappa-1}$ *topology such that* $\mathscr{A}_{\tilde{g}} = \mathscr{A}_g$, $\varphi_{\tilde{g}} = \varphi_g$ *and the maps* $\tau_0 = \varphi_g$ *and* $\tau_1 = \psi_{\tilde{g}}$ *satisfy condition (24) of Theorem A.*

Proof. By Theorem B there exists a C^∞ Hamiltonian $h : \mathbb{A} \to \mathbb{R}$ arbitrarily close to 0 such that φ_g and the modified diffeomorphism

$$\Phi^h \circ \psi_g \circ \Phi^{-h} \tag{39}$$

satisfy (24). In view of Lemma 10, (38), let us introduce the perturbed diffeomorphism

$$\psi_{\rm pert} = \psi_g \circ [\pi_g^- \circ \chi \circ (\pi_g^-)^{-1}] : \mathbb{A} \circlearrowleft, \tag{40}$$

where $\chi : \mathscr{H}_g \circlearrowleft$ is a diffeomorphism we want to determine (and which we then have to continue to a diffeomorphism $\boldsymbol{\chi}$ defined in a neighborhood of $\mathscr{H}_g$). We want to choose χ in order to solve the equation

$$\psi_{\rm pert} = \Phi^h \circ \psi_g \circ \Phi^{-h}. \tag{41}$$

Straightforward computation yields

$$\chi = (\pi_g^-)^{-1} \circ \psi_g^{-1} \circ \Phi^h \circ \psi_g \circ \Phi^{-h} \circ \pi_g^-. \tag{42}$$

Therefore χ is a $C^{\kappa-1}$ Hamiltonian diffeomorphism of the annulus $\mathscr{H}_g$, with compact support, which tends to Id in the $C^{\kappa-1}$ topology when $d_\kappa(g, f) \to 0$. As a consequence, there is a C^κ function $\xi : \mathbb{R} \times \mathscr{H}_g \to \mathbb{R}$, with support in $]0, 1[\times \mathscr{H}_g$ such that

$$\chi = \Phi^\xi : \mathscr{H}_g \circlearrowleft, \tag{43}$$

where Φ^ξ is the time-one map starting at 0 generated by ξ. Using the Moser isotopy argument, one proves the existence of a $C^{\kappa-1}$ symplectic diffeomorphism

$$T : \mathbb{A} \times B^2(0, \alpha) \to \mathscr{N}, \quad T(\mathbb{A} \times 0\}) = \mathscr{H}_g, \tag{44}$$

where $\mathscr{N}$ is a neighborhood of $\mathscr{H}_g$ in $\mathbb{A}^2$, α is a positive constant and the first factor is endowed with the usual symplectic structure. Fix a C^∞ bump function $\eta : B^2(0, \alpha) \to \mathbb{R}$ equal to 1 in a neighborhood of 0 and define a function $H : \mathbb{R} \times \mathbb{A} \times B^2(0, \alpha) \to \mathbb{R}$ by

$$H(t, x_1, x_2) = \eta(x_2)\xi(t, T(x_1, 0)). \tag{45}$$

Then clearly the time-one map $\boldsymbol{\chi} = T \circ \Phi^H$ leaves $\mathscr{H}_g$ invariant, with $\boldsymbol{\chi}_{|\mathscr{H}_g} = \chi$ and the support of $\boldsymbol{\chi}$ is contained in $\mathscr{N}$. Moreover, $\boldsymbol{\chi}$ tends to the identity in the $C^{\kappa-1}$ topology when h tends to 0 in the C^κ topology. Setting $\tilde{g} = \boldsymbol{\chi} \circ g$ provides us with the perturbed diffeomorphism we were looking for. □

3.2.4. *Conclusion of the proof of Theorem 6.* Fix $f \in \mathscr{F}^\kappa$ and $\delta > 0$. We assume that κ is large enough so that all the conclusions and identifications of the last sections hold. Set

$$\begin{aligned} U_\bullet &= \{(\theta_1, r_1) \in \mathbb{A} \mid r_1 \in]-\delta/4, \delta/4[\}, \\ U^\bullet &= \{(\theta_1, r_1) \in \mathbb{A} \mid r_1 \in]1-\delta/4, 1+\delta/4[\}, \\ \mathscr{U}_\bullet &= \{(\theta, r) \in \mathbb{A}^2 \mid r_1 \in]-\delta/2, \delta/2[\}, \\ \mathscr{U}^\bullet &= \{(\theta, r) \in \mathbb{A}^2 \mid r_1 \in]1-\delta/2, 1+\delta/2[\}. \end{aligned} \tag{46}$$

By Lemma 8 and Corollary 9, one can choose $\varepsilon \in]0, \varepsilon(f)[$ small enough so that for any diffeomorphism $g \in B(f, \varepsilon)$, with the notation of Lemma 8, the invariant circles $\Gamma_\bullet$ and $\Gamma^\bullet$ of φ_g satisfy

$$\Gamma_\bullet \subset U_\bullet, \quad \psi(\Gamma_\bullet) \subset U_\bullet, \quad \Gamma^\bullet \subset U^\bullet, \quad \psi(\Gamma^\bullet) \subset U^\bullet, \tag{47}$$

and are such that moreover

$$\Phi_g(U_\bullet \times \{0\}) \subset \mathscr{U}_\bullet, \quad \Phi_g(U^\bullet \times \{0\}) \subset \mathscr{U}^\bullet. \tag{48}$$

We then proved the existence of a $C^{\kappa-1}$ diffeomorphism $\tilde{g} \in B(f, \varepsilon(f))$ arbitrarily close to g in the $C^{\kappa-1}$ topology, such that the bisystem $(\varphi_{\tilde{g}}, \psi_{\tilde{g}})$ associated with $\tilde{g}$ satisfies (24). In particular, $(\varphi_{\tilde{g}}, \psi_{\tilde{g}})$ admits an orbit with first point in $\mathscr{U}_\bullet$ and last point in $\mathscr{U}^\bullet$.

The last step is to apply the normally hyperbolic shadowing lemma (see Theorem 40 in Appendix C) (with $\delta/2$ instead of δ) to the bisystem $(\varphi_{\tilde{g}}, \psi_{\tilde{g}})$. The previous orbit produces an orbit of $\tilde{g}$ with first point in $\mathscr{U}^0$ and last point in $\mathscr{U}^1$ (see Definition 4). This concludes the proof. □

4. The discrete setting for simple resonance annuli

Our objective now is to generalize the previous result to a (still discrete) case which well-adapted to the diffusion properties of the dynamics along simple resonance cylinders. The main difference with the previous model is that we have to replace the globally defined homoclinic map by a correspondence formed by a family of locally defined maps. We therefore have to introduce a local version of the Moeckel noncoincidence condition and prove that it yields the existence of drifting orbits for this type of bisystem: we require that each essential invariant circle of g admits a *splitting arc*, that is, a C^0 arc located below the invariant circle and which is sent into the invariant circle by some locally defined maps of the previous family. The question of the generic existence of such arcs for relevant examples will necessitate specific symplectic ingredients and will be examined in the next section — together with the definition of these examples. This section is extracted from the joint work [21].

4.1. ***Special twist maps and splitting arcs.*** Given $a < b$, we set $\boldsymbol{A} = \mathbb{T} \times [a, b]$ and for each $c \in [a, b]$, we write $\Gamma(c)$ for $\mathbb{T} \times \{c\}$. Given a map $f : \boldsymbol{A} \circlearrowleft$, we denote by $\mathrm{Ess}(f)$ its set of invariant essential circles.

1. We begin with the following definition for twist maps.

Definition 12. Here we say that an area-preserving twist map φ of $\boldsymbol{A}$ is *special* if φ does not admit any essential invariant circle with rational rotation number.[13]

[13] Our definition in [21] is more stringent but we will not need it in the present setting.

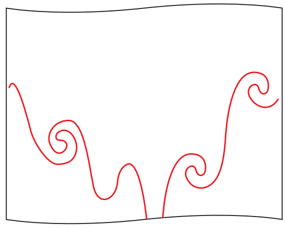


Figure 11. Positively and negatively tilted arcs.

Given an essential circle $\Gamma\subset\mathbb{T}\times]a,b[$, Γ^- (resp. Γ^+) stands for the connected component of $\boldsymbol{A}\setminus\Gamma$ located below Γ (resp. above Γ) in $\boldsymbol{A}$. In the following we will crucially use the following result.

Lemma 13. *Let φ be a special area-preserving twist map φ of $\boldsymbol{A}$. Then any two distinct elements of* $\mathrm{Ess}(\varphi)$ *are disjoint. Moreover, given an invariant essential circle $\Gamma\subset(\boldsymbol{A}\setminus\Gamma(a))$, then either Γ is the upper boundary of a Birkhoff zone of φ, or it is accumulated by a sequence of elements of* $\mathrm{Ess}(\varphi)$ *located in Γ^-.*

See Appendix B for a proof.

2. We now list the necessary definitions and results for arcs. Given $(u,v)\in\mathbb{R}^2$, let $\angle(u,v)$ be the oriented angle of (u,v) in $[0,2\pi[$. Let $f:\boldsymbol{A}\circlearrowleft$ be an area-preserving twist map. Fix a circle $\Gamma\in\mathrm{Ess}(f)$. An *arc based on* Γ is a C^0 function $\gamma:[0,1]\to\boldsymbol{A}$ such that $\gamma(0)\in\Gamma$ and $\gamma(]0,1])\in\Gamma^+$. We usually denote by $\widetilde{\gamma}$ the image $\gamma([0,1])$.

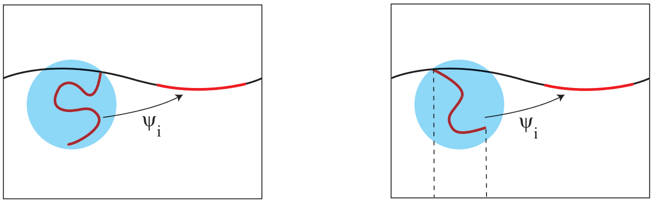
A C^1 arc based on Γ with $\gamma'(s)\neq 0$ for $s\in[0,1]$ is said to be *positively tilted* (resp. *negatively tilted*) when $\angle((0,1),\gamma'(0))\in]0,\pi[$ (resp. $\angle((0,1),\gamma'(0))\in]-\pi,0[$) and when the continuous lift to $\mathbb{R}$ of $s\mapsto\angle((0,1),\gamma'(s))$ is positive (resp. negative) over $[0,1]$.

Definition 14. Let $\varphi:\boldsymbol{A}\circlearrowleft$ be a twist map and let $\psi=(\psi_i)_{i\in I}$ be a correspondence on $\boldsymbol{A}$, where each $\psi_i:\mathrm{Dom}\,\psi_i\to\mathrm{Im}\,\psi_i$ is a local homeomorphism of $\boldsymbol{A}$. Fix $\Gamma\in\mathrm{Ess}(\varphi)$, $\Gamma\subset\boldsymbol{A}\setminus\Gamma(a)$:

- A *splitting arc* based at α for these data is an arc ζ of $\boldsymbol{A}$ whose projection on $\Gamma(a)$ has length $<\frac{1}{2}$, for which

$$\zeta(0)=\alpha,\quad \zeta(]0,1])\subset\Gamma^-;\quad \exists i\in I,\ \zeta(]0,1])\subset\mathrm{Dom}\,\psi_i,\quad \psi_i(\zeta(]0,1]))\subset\Gamma.$$

- A *right splitting arc* based at $\alpha=(\theta,r)$ is a splitting arc ζ based at α, which admits a derivative $\zeta'(0)=(u,v)$ with $u>0$, and such that $\pi(\tilde{\zeta})=[\theta,\theta+\tau]$ with $0<\tau<\frac{1}{2}$.
- A *left splitting arc* based at $\alpha=(\theta,r)$ is a splitting arc ζ based at α, which admits a derivative $\zeta'(0)=(u,v)$ with $u<0$, and such that $\pi(\tilde{\zeta})=[\theta-\tau,\theta]$ with $0<\tau<\frac{1}{2}$.



A splitting arc A right splitting arc

Figure 12. Splitting arcs.

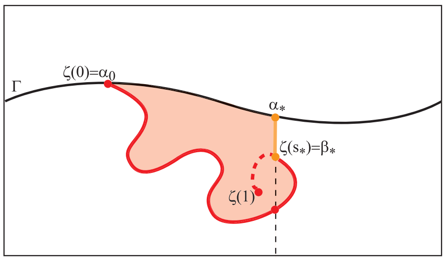


Figure 13. Domain associated to a right splitting arc.

The length $< \frac{1}{2}$ condition on an arc is there just to ensure the existence of a natural order between the projections of points located in the neighborhood of it. We will implicitly use this order in the following. One easy remark is that if ζ is a right (resp. left) splitting arc, then (up to reparametrization) the restriction $\zeta_{|[0,s]}$ with $0 < s \leq 1$ is also a right (resp. left) splitting arc, so that the previous condition is not restrictive.

Given a point $\alpha = (\theta_0, r_0)$ in $\boldsymbol{A}$, we denote by

$$V^-(\alpha) = \{(\theta_0, r) \mid r \in [a, r_0]\}$$

the vertical below α in $\boldsymbol{A}$.

Definition 15. Let $\Gamma \in \mathrm{Ess}(\varphi)$, $\Gamma \subset \boldsymbol{A} \setminus \Gamma(a)$, be the graph of the continuous function $\gamma : \mathbb{T} \to [a, b]$ and $\alpha_0 \in \Gamma$. Let ζ be a right splitting arc based on Γ at $\alpha_0 = \zeta(0)$, let α_* be a point in Γ such that

$$\pi(\alpha_0) < \pi(\alpha_*) < \mathrm{Max}_{s \in [0,1]} \pi(\zeta(s)),$$

and let $\beta_* = \zeta(s_*)$ be the point in $V^-(\alpha_*) \cap \tilde{\zeta}$ with maximal r-coordinate. Let C be the Jordan curve formed by the concatenation of the arcs $\zeta([0, s_*])$, $[\beta_*, \alpha_*]$, and $[\alpha_*, \alpha_0] \subseteq \Gamma$. We denote by $D(\zeta_{|[0,s_*]})$ the connected component of the complement of C contained in Γ^-. We say that $D(\zeta_{|[0,s_*]})$ is a *domain associated with* ζ. We define a domain associated with a left splitting arc similarly.

The first obvious property of the domains defined above is the obvious following remark.

Lemma 16. *For any $x \in D(\zeta)$, the vertical $V^-(x)$ below x intersects $\zeta(]0,1[)$.*

The crucial property is the following.

Lemma 17. *Consider an essential circle $\Gamma \in \mathrm{Ess}(\widehat{\varphi})$ contained in $\boldsymbol{A} \setminus \Gamma(a)$, and a right (resp. left) splitting arc ζ based on Γ. Consider an essential circle $\Gamma_\bullet \subset \boldsymbol{A}$ such that $\tilde{\zeta}$ is contained in the domain $\Gamma_\bullet^+$ above $\Gamma_\bullet$. Let D be a domain associated to ζ. Let η be a negatively (resp. positively) tilted arc with $\eta(0) \in \Gamma_\bullet$, $\eta(]0,1]) \subset \Gamma_\bullet^+ \cap \Gamma^-$, and $\eta(1) \in D$. Then $\eta(]0,1[) \cap \zeta(]0,1]) \neq \varnothing$.*

The proof is an immediate consequence of Lemmas 34 and 38.

4.2. ***Existence of pseudoorbits for bisystems of correspondences.*** We can now state and prove a generalization of Moeckel's theorem to bisystems of correspondences on a two-dimensional annulus, which has to be seen as a Poincaré section of a compact hyperbolic invariant cylinder. We prove the existence of pseudoorbits "drifting from the bottom to the top of the annulus". We do not present here the more complete formalism of [21] which is adapted to the case of pseudoorbits drifting along chains of heteroclinically connected annuli.

1. We first need to make the definition of an orbit of a polysystem more precise. Let A be some set and consider a set $f = \{f_i \mid i \in I\}$ of locally defined maps $f_i : \mathrm{Dom}\, f_i \to A$. We say that a finite sequence $(x_n)_{0 \le n \le n_*-1}$ of points of A is a *finite orbit of f, of length $n_* \ge 1$*, when there exists a sequence $\omega = (i_n)_{0 \le n \le n_*-1} \in I^{n_*}$ such that, for $0 \le n \le n_* - 1$,

$$x_{n+1} = f_{i_n}(x_n),$$

and we write

$$x_{n_*} = f^\omega(x_0).$$

We formally consider the point x_0 as being the 0-length orbit of x_0.

Given a subset $B \subset A$, we set

$$f^\omega(B) = \bigcup_{x \in B_\omega} f^\omega(x)$$

where B_ω is the subset of B formed by the points x such that $f^\omega(x)$ is well-defined.

The *full orbit of $B \subset A$* under f is the subset of A formed by the union of all $f^\omega(B)$ for all sequences (of any length) ω (so that in particular B is contained in its full orbit under f).

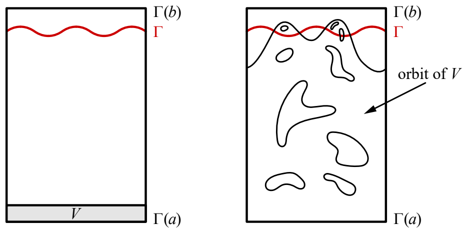


Figure 14. The setting of Theorem 19.

2. To deal with the notions of right and left splitting arcs in a similar way, we will need the following result of symmetrization of a polysystem; see [21] for a proof.

Lemma 18. *Let A be a metric space endowed with a finite Borel measure, positive on the nonempty open subsets of A. Let φ be a measure-preserving homeomorphism of A and let $(\psi_i)_{i\in I}$ be a polysystem on A, where* $\operatorname{Dom}\psi_i$ *is open and the map* $\psi_i : \operatorname{Dom}\psi_i \to \operatorname{Im}\psi_i$ *is a homeomorphism, for all $i \in I$. Fix a nonempty open subset $V \subset A$. Let U_f and U_g be the full orbit of V under the polysystems*

$$f = (\varphi, \psi = (\psi_i)_{i\in I}) \quad \textit{and} \quad g = (\varphi, \varphi^{-1}, \psi = (\psi_i)_{i\in I})$$

respectively. Then U_f is contained and dense in U_g.

3. The main result of this section is the following.

Theorem 19. *Let $\varphi : \mathbf{A} \circlearrowleft$ be a special twist map and let $\psi = (\psi_i)_{i\in I}$ be a correspondence on $\mathbf{A}$. Assume that for each element $\Gamma \in \operatorname{Ess}(\varphi)$ there is a right or left splitting arc based on Γ. Fix $\Gamma \in \operatorname{Ess}(\varphi) \setminus \{\Gamma(a), \Gamma(b)\}$ together with a neighborhood V of $\Gamma(a)$ in $\mathbf{A}$. Then given $\delta > 0$, the full orbit of V under the polysystem $f = (\varphi, \psi = (\psi_i)_{i\in I})$ intersects $\Gamma(b-\delta)^+$.*

Given $\nu > 0$, we define a *ν-ball* of $\mathbb{T} \times \mathbb{R}$ as a subset $B = B_\theta \times B_r$ where B_θ and B_r are intervals of $\mathbb{T}$ and $\mathbb{R}$ respectively, such that

$$\operatorname{length} B_r > \nu \operatorname{length} B_\theta. \tag{49}$$

The *center of B* is (a_θ, a_r), where a_θ, a_r are the mid-points of B_θ and B_r. Given a topological space E and $A \subset B \subset E$ with A connected, $\operatorname{CC}(B, A)$ stands for the connected component of B containing A.

Proof. We assume for example that φ tilts the vertical to the right, the other case being exactly similar:

• We assume without loss of generality that V is open in $\mathbf{A}$, and connected. Let U be the full orbit of the open set V under the symmetrized polysystem

$g = (\varphi, \varphi^{-1}, \psi = (\psi_i)_{i\in I})$ on $\boldsymbol{A}$. Note that $\varphi(U) = U$ and $\psi_i(U) \subset U$. Set $U_c = \mathrm{CC}(U, \Gamma(a))$. Then U_c is open and contains V, so $\varphi(U_c) = U_c$. Thanks to Lemma 18, it is enough to prove that U_c intersects $\Gamma(b-\delta)^+$.

Let us assume by contradiction that U_c is contained in $\Gamma(b-\delta)^-$.

- Set $O = \boldsymbol{A} \setminus \overline{U_c}$, so that O is open, contains $\Gamma(b)$, and $O \cap V = \varnothing$. Moreover, since $\varphi(\overline{U_c}) = \overline{U_c}$,

$$\varphi(O) = \boldsymbol{A} \setminus \varphi(\overline{U_c}) = \boldsymbol{A} \setminus \overline{U_c} = O.$$

Then $\varphi(\mathrm{CC}(O, \Gamma(b))) = \mathrm{CC}(O, \Gamma(b))$ and so $\varphi(\overline{\mathrm{CC}(O, \Gamma(b))}) = \overline{\mathrm{CC}(O, \Gamma(b))}$. Let

$$U = \boldsymbol{A} \setminus \overline{\mathrm{CC}(O, \Gamma(b))},$$

so that U is open and $\varphi(U) = U$, and set finally

$$\mathcal{U} = \mathrm{CC}(U, \Gamma(a)), \tag{50}$$

hence $\mathcal{U}$ is open, connected and $\varphi(\mathcal{U}) = \mathcal{U}$. Moreover clearly

$$\bar{\mathcal{U}} \subset \boldsymbol{A} \setminus \mathrm{CC}(O, \Gamma(b)), \tag{51}$$

and

$$U_c \subset \mathcal{U}, \tag{52}$$

since $\overline{O} = \boldsymbol{A} \setminus \mathrm{Int}(\overline{U_c}) \subset \boldsymbol{A} \setminus U_c$, so $\overline{\mathrm{CC}(O, \Gamma(b))} \subset \boldsymbol{A} \setminus U_c$ and $U_c \subset \boldsymbol{A} \setminus \overline{\mathrm{CC}(O, \Gamma(b))} = U$, which proves our claim since $\Gamma(a) \subset U_c$.

- Let us prove that $\Gamma := \mathrm{Fr}\,\mathcal{U}$ is a Lipschitz graph over $\mathbb{T}$, invariant under φ, by the Birkhoff theorem (see Appendix B). By local connectedness of $\boldsymbol{A}$, one readily proves that $\mathrm{Int}\,\overline{\mathcal{U}} = \mathcal{U}$, since $\mathcal{U}$ is a connected component of the complement of the closure of an open set. Moreover $\varphi(\mathcal{U}) = \mathcal{U}$. Let now $\boldsymbol{S}$ be the quotient of $\boldsymbol{A}$ by the identification of each boundary circle to one point, so that $\boldsymbol{S}$ is homeomorphic to S^2. Up to this quotient, $\mathcal{U}$ is a connected component of the complement in $\boldsymbol{S}$ of a compact connected subset, so is homeomorphic to a disk. Going back to the initial space $\boldsymbol{A}$ proves that $\mathcal{U}$ is homeomorphic to $\mathbb{T} \times [0, 1[$. So by the Birkhoff theorem, $\Gamma = \partial\mathcal{U}$ is a Lipschitz graph over $\mathbb{T}$, invariant under φ; see [48] for more details.

- We now prove that $\Gamma \subset \overline{U_c}$, and so $\Gamma \subseteq \mathrm{Fr}(U_c) = \mathrm{cl}(U_c) \setminus U_c$. Assume that $x \in \Gamma$ is not in $\overline{U_c}$, so that there exists a small ball $B(x, \varepsilon)$ with $B(x, \varepsilon) \cap \overline{U}_c = \varnothing$. Let z be some point on the vertical through x, located under Γ and inside $B(x, \varepsilon)$. Let us show that the semivertical σ over z in $\boldsymbol{A}$ is disjoint from $\overline{U_c}$. First $\Gamma \cap \sigma = \{x\}$, since Γ is a graph, so that $\sigma = [z, x] \cup [x, \xi]$, with $\xi \in \Gamma(b)$. Clearly $[z, x] \subset B(x, \varepsilon)$ so $[z, x] \cap \overline{U_c} = \varnothing$, and $]x, \xi] \cap \overline{\mathcal{U}} = \varnothing$ since $\Gamma = \partial\mathcal{U}$ is a graph. Since $U_c \subset \mathcal{U}$, this proves that $\sigma \cap \overline{U}_c = \varnothing$.

As a consequence $\sigma \cup \Gamma(b)$ is a connected set which satisfies $(\sigma \cup \Gamma(b)) \cap \overline{U_c} = \varnothing$. Therefore
$$(\sigma \cup \Gamma(b)) \subset \mathrm{CC}(O, \Gamma(b))$$
and thus $(\sigma \cup \Gamma(b)) \cap \overline{\mathcal{U}} = \varnothing$ by (51). This is a contradiction since $x \in \Gamma \subset \overline{\mathcal{U}}$. Therefore $\Gamma \subset \overline{U_c}$.

• Since Γ is an invariant essential circle for the special twist map φ, there are only two possibilities:

– Γ is the upper boundary of a Birkhoff zone.

– Γ is accumulated from below by essential invariant circles in the Hausdorff topology.

We will prove that both possibilities yield a contradiction with the initial assumption that $U_c \cap \Gamma(b) = \varnothing$.

• Assume first that Γ is the upper boundary of a Birkhoff zone $\mathscr{Z}$ and let Γ_* be the lower boundary of $\mathscr{Z}$. Let ν be the Lipschitz constant of Γ_*. Since Γ_* is a graph and U_c is open, connected, contains V and $\overline{U_c} \cap \Gamma \neq \varnothing$, then $U_c \cap \Gamma_* \neq \varnothing$. So there exists a ν-ball $\boldsymbol{B} \subset U_c$ centered on Γ_*.

We assumed that there exists a right or left splitting arc ζ based at some point α of Γ. Let D be its associated domain. By restricting ζ if necessary, one can moreover assume without loss of generality that $D \subset \mathscr{Z}$. We introduce the closed connected set $X = \Gamma \cup \tilde{\zeta}$, where $\tilde{\zeta} = \zeta([0, 1])$, which disconnects the annulus $\boldsymbol{A}$ since it contains Γ.

Assume first that ζ is a right splitting arc. By Proposition 39, there exist $z_0 \in \boldsymbol{B}$ and $n \in \mathbb{N}$ such that $z_n := \varphi^n(z_0) \in D$. Then, by Lemma 17 there exists a *positively* tilted arc based on Γ_* and ending at z_n which does not intersect X.

By Lemma 35 there exists a *negatively* tilted arc γ with image in $\boldsymbol{B}$ based on Γ_* and ending at z_0. Therefore, by Lemma 37, $\gamma_n := f^n \circ \gamma$ is a *negatively* tilted arc based on Γ_* and ending at z_n.

Assume that the image $\widetilde{\gamma}_n$ does not intersect X, then by Lemma 38 the vertical $V^-(z_n)$ does not intersect X, which contradicts Lemma 16. Therefore $\widetilde{\gamma}_n \cap X \neq \varnothing$, thus $\widetilde{\gamma}_n \cap \tilde{\zeta} \neq \varnothing$.

If now ζ is a left splitting arc, we use φ^{-1} instead of φ. This first yields a $z_0 \in \boldsymbol{B}$ such that $z_{-n} := \varphi^{-n}(z_0) \in D$, then a negatively tilted arc based on Γ_* and ending at z_{-n} which does not intersect X, and a positively tilted arc, still denoted by γ_n, based on Γ_* and ending at z_{-n}. As above, this proves that $\widetilde{\gamma}_n \cap \tilde{\zeta} \neq \varnothing$.

As a consequence, $U_c \cap \tilde{\zeta} \neq \varnothing$ since $\widetilde{\gamma}_n \subset U_c$, and therefore there is a small open ball $B \subset U_c$ centered on $\zeta(]0, 1])$ and, by definition, an index $i \in I$ such that $B \subset \operatorname{Dom} \psi_i$. Thus $\psi_i(B)$ is an open set which intersects Γ, and therefore also U_c since $\Gamma \subset \overline{U_c}$. This proves that $\psi_i(B) \subset U_c$ by connectedness, so that

U_c contains points strictly above the circle Γ. This is a contradiction with the construction of $\Gamma = \mathrm{Fr}\,\mathcal{U}$ and the inclusion $U_c \subset \mathcal{U}$, which ensures that all points of U_c are located below Γ.

• Assume now that Γ is accumulated from below by an increasing sequence $(\Gamma_m)_{m\geq 1}$ of essential invariant circles for φ. Let ζ be a splitting arc based on Γ. Let S_m be the closed strip limited by Γ_m and Γ_{m+1}. For m large enough, $S_m \cap \tilde{\zeta}$ contains a C^0 curve ℓ which intersects both Γ_m and Γ_{m+1}. Now $\Gamma \subset \overline{U_c}$, so that $U_c \cap S_m$ contains a C^0 curve ℓ' which also intersects both Γ_m and Γ_{m+1}. Therefore, by Lemma 36, there exists an integer n such that $\varphi^n(U_c) \cap \ell \neq \varnothing$, and so by invariance of U_c under φ, $U_c \cap \ell \neq \varnothing$. Since $\ell \subset \tilde{\zeta} \subset \mathrm{Dom}\,\psi_i$ for some $i \in I$, there exists a ball $B \subset U_c$ centered on $\ell \subset \tilde{\zeta}$ and contained in $\mathrm{Dom}\,\psi_i$. This yields the same contradiction as in the previous paragraph. □

Slightly more involved assumptions and arguments show that the full orbit of V intersects each pair of disjoint essential circles located in $\mathbb{T} \times]a, b[$, which enables us in [21] to prove the existence of orbits drifting along chains of heteroclinically connected annuli (and cylinders). We show in the next section how the present result can be implemented in a model of a single s-resonance cylinder.

5. Diffusion orbits along simple resonance cylinders

In this section we introduce an example of *a priori* unstable perturbation of an integrable Hamiltonian on the annulus $\mathbb{A}^3$, which admits a normally hyperbolic 3-dimensional cylinders with coinciding stable and unstable manifolds. To deal with this degenerate situation, symplectic geometry reveals itself to be crucial at two levels.

First, to prove the existence of homoclinic solutions for the essential invariant tori located inside the cylinders. Then, to reduce the problem to a two-dimensional setting and use the result of the previous section, we introduce a Poincaré section (diffeomorphic to $\mathbb{A}$) of the flow in the cylinder and we assume that the unperturbed system induces a twist return map — not necessarily close to any integrable one. The homoclinic intersections then enable us to construct a homoclinic correspondence (a family of locally defined diffeomorphisms) on the annulus, which breaks each essential invariant circle of the twist map (due to the existence of a splitting arc).

The second crucial resort to symplectic geometry is to prove the genericity of the existence of these splitting arcs. Our approach consist proving a general result on the existence of homoclinic intersections, which would be violated if some invariant circle would not admit a splitting arc.

The results of this section give an account of a work joint work in progress with L. Lazzarini, devoted to the complete description of a simple example illustrating the methods of [43].

5.1. ***Setting and main result.*** We write $\theta = (\theta_0, \theta_1, \theta_2)$ and $r = (r_0, r_1, r_2)$ for the angle and action variables in $\mathbb{A}^3$. We set $\hat{\theta} = (\theta_0, \theta_1)$, $\hat{r} = (r_0, r_1)$.

1. Given an integer $\kappa \geq 2$, we denote by $\mathscr{G}^\kappa$ the affine subspace of Hamiltonians on $\mathbb{A}^3$ of the form

$$G(\theta, r) = r_0 + g(\theta_0, \theta_1, \theta_2, r_1, r_2), \quad (\theta, r) \in \mathbb{A}^3, \tag{53}$$

where g is of class C^κ and satisfies

$$\|g\|_\kappa := \sum_{k=0}^{\kappa} \mathrm{Sup}_{x\in\mathbb{A}^3} \|D^k g(x)\| < +\infty. \tag{54}$$

We endow $\mathscr{G}^\kappa$ with the uniform distance induced by the previous norm and we denote by $B^\kappa(G, r)$ the associated open ball centered at $G \in \mathscr{G}^\kappa$, with radius r.

2. We introduce a subset of $\mathscr{G}^\kappa$ formed by the "unperturbed" Hamiltonians of the form

$$\begin{aligned} F(\theta, r) &= F_1(\theta_0, \theta_1, r_0, r_1) + F_2(\theta_2, r_2), \\ F_1(\theta_0, \theta_1, r_0, r_1) &= r_0 + f_1(\theta_0, \theta_1, r_1), \end{aligned} \tag{55}$$

where the Hamiltonians F_i satisfy a set of additional conditions:

• **Conditions on** F_1**.** The level $F_1^{-1}(0)$ is a cylinder which admits the global coordinates $(\theta_0, \theta_1, r_1)$. To set out our first conditions we focus on the dynamics generated by F_1 on $F_1^{-1}(0)$ only. For each $\theta_0^* \in \mathbb{T}$, the surface

$$\Sigma^{\theta_0^*} = \{\theta_0 = \theta_0^*\} \cap F_1^{-1}(0) \subset \mathbb{A}^2$$

is a global section for the restriction of X_{F_1} to $F_1^{-1}(0)$, since $\dot{\theta}_0 = 1$. Moreover, the standard Liouville form $\boldsymbol{\lambda}$ on $\mathbb{A}^3$ induces on $\Sigma^{\theta_0^*}$ the form $r_1 d\theta_1$, so that (θ_1, r_1) are global exact-symplectic coordinates on $\Sigma^{\theta_0^*}$. We denote by $\varphi^{\theta_0^*}$ the (exact-symplectic) Poincaré map induced on $\Sigma^{\theta_0^*}$ by the flow Φ_{F_1}. In the coordinates chart (θ_1, r_1) we write

$$\varphi^{\theta_0^*}(\theta_1, r_1) = (\Theta_1^{\theta_0^*}(\theta_1, r_1), R_1^{\theta_0^*}(\theta_1, r_1)). \tag{56}$$

The maps $\varphi^{\theta_0^*}$ are clearly pairwise conjugated. We now list the conditions to be satisfied by the Hamiltonians F_1:

(C_1) For each $\theta_0^* \in \mathbb{T}$, the circles $\Gamma(0) = \mathbb{T} \times \{0\}$ and $\Gamma(1) = \mathbb{T} \times \{1\}$ in $\Sigma^{\theta_0^*}$ (relative the coordinates (θ_1, r_1)) are invariant under $\varphi^{\theta_0^*}$, and their rotation numbers ν_0 and ν_1 are Diophantine.[14]

(C_2) There is a constant $\mu > 0$ such that for all $\theta_0^* \in \mathbb{T}$,

$$\frac{\partial \Theta_1^{\theta_0^*}}{\partial r_1}(\theta_1, r_1) \geqslant \mu$$

for all $(\theta_1, r_1) \in \mathbb{T} \times [0, 1]$.

By (C_1), the 2-dimensional tori $T(i) = \mathbb{T}^2 \times \{r_1 = i\} \subset F_1^{-1}(0)$ are invariant, and they bound a compact invariant cylinder $\boldsymbol{C} \subset F_1^{-1}(0)$. Moreover, by (C_2), the map $\varphi^{\theta_0^*}$ induces is a twist map of $\Sigma^{\theta_0^*} \cap \boldsymbol{C} = \mathbb{T} \times [0, 1]$, with twist constant μ independent of θ_0^*.

• **Conditions on F_2.** On the last factor $\mathbb{A}$, endowed with the coordinates (θ_2, r_2), we introduce the following conditions to be satisfied by the Hamiltonians F_2.

(C_3) The vector field X_{F_2} possesses a hyperbolic fixed point O_2, with $F_2(O_2) = 0$.

(C_4) The fixed point O_2 admits a homoclinic orbit $\boldsymbol{\zeta}$ for X_{F_2} and there exists an open interval $I \subset \mathbb{R}$ such that $\boldsymbol{\zeta}$ transversely intersects $\boldsymbol{\sigma} = \left\{\frac{1}{2}\right\} \times I$ at exactly one point, that we denote by P_2. Moreover the map $r_2 \mapsto F_2\left(\frac{1}{2}, r_2\right)$ is a diffeomorphism from I_2 onto its image.

(C_5) Let λ_{O_2} stand for the positive eigenvalue of $T_{O_2} X_{F_2}$. Then there is a $p > 0$ such that

$$\lambda_{O_2} > p[\mathrm{Max}_{\hat{x} \in \mathbb{A}^2} \| T_{\hat{x}} X_{\boldsymbol{F}_1} \|]. \tag{57}$$

In the sequel, we denote by $\mathscr{F}^\kappa(p)$, or $\mathscr{F}^\kappa$ for short, the set of Hamiltonians F of the form (55) *which satisfy conditions* (C_1)–(C_5).[15]

Note in particular that when $\varepsilon = 0$, the Arnold Hamiltonian is in $\mathscr{F}^\omega$, so that our study extends Arnold's one.

3. As in Section 3, it is not difficult to prove that given $F \in \mathscr{F}^\kappa(p)$ with p large enough for the normally hyperbolic persistence to hold and $\kappa \geq \kappa_0$ large enough for the KAM theorem to apply,[16] there is a $\delta_0 \in]0, 1[$ such that no orbit of X_F can intersect both zones $\{r_1 < \delta_0\}$ and $\{r_1 > 1 - \delta_0\}$. This motivates the following definition.

Definition 20. Fix $\delta > 0$. Given a Hamiltonian $G \in \mathscr{G}^\kappa$, we say that a solution $\gamma(t) = (\theta(t), r(t))$ of the system generated by G on $\mathbb{A}^3$ is a *δ-diffusion solution*

[14] These rotation numbers are independent of θ_0^*.

[15] The choice of μ is quite innocuous.

[16] Both values being dependent.

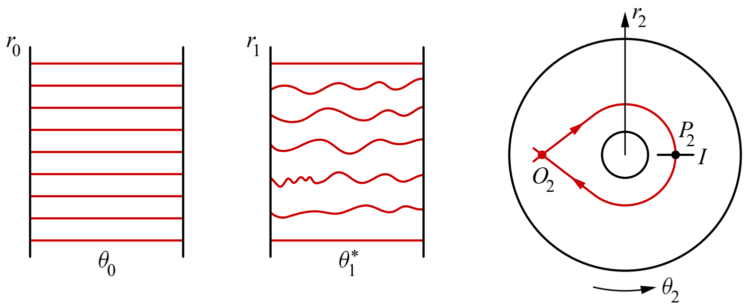


Figure 15. An unperturbed system.

when it is defined on some interval $[0, T]$ and satisfies

$$r_1(0) < \delta, \quad r_1(T) > 1 - \delta. \tag{58}$$

The main result of this section is the following.

Theorem 21. *There exist $p > 0$ and $\kappa_0 \geq p$ such that for $\kappa \geq \kappa_0$ and $F \in \mathscr{F}^\kappa(p)$, then for any $\delta \in]0, 1[$ there is an $\varepsilon > 0$ such that the set of Hamiltonians in the ball $B^\kappa(F, \varepsilon) \subset \mathscr{G}^\kappa$ which admit a δ-diffusion solution is dense for the induced C^κ topology and open for the C^2 topology.*

The existence of δ-diffusion solutions is clearly an open condition in the C^1 topology for Hamiltonians of $\mathscr{G}^\kappa$. So the main task to prove Theorem 21 is to ensure that the conditions introduced in [21], which yield the existence of diffusion solutions and are encoded in the notion of "good cylinders" below (see Theorem 31), are satisfied for a dense subset of $B^\kappa(F, \varepsilon)$ in the C^κ topology.

5.2. ***Geometry and dynamics of the perturbed systems.*** The proof of the following lemma is a simple application of the normally hyperbolic persistence theorem, the normally hyperbolic symplectic theorem of Appendix A and the KAM theorem in the version of Herman [29].

Lemma 22. *Fix $\kappa \geq \kappa_0 \geq p$ (large enough) and $F \in \mathscr{F}^\kappa(p)$. Let $\boldsymbol{R} = \boldsymbol{I}_0 \times \boldsymbol{I}_1$ be a fixed rectangle (with $I_1 \subset \mathrm{Int}\, \boldsymbol{I}_1$) in and fix $\rho > 0$. Then there is an $\varepsilon_0(F) > 0$ such that for any Hamiltonian $G \in B^\kappa(F, \varepsilon_0(F))$ the following properties are satisfied*:

(1) *There exists a normally hyperbolic symplectic invariant annulus for X_G, of the form*

$$\mathcal{A}_G = \{(\theta, r) \in \mathbb{A}^3 \mid (\theta_2, r_2) = a_G(\hat{\theta}, \hat{r})\} \tag{59}$$

where $a_G : \mathbb{A}^2 \to \mathbb{A}$ is a C^4 function whose image is contained in a small ball centered at O_2, which satisfies

$$\|a_G - O_2\|_{C^p(\mathbb{T}^2 \times \boldsymbol{R})} \to 0 \quad \textit{when} \quad \|G - F\|_{C^\kappa(\mathbb{A}^3)} \to 0. \tag{60}$$

Moreover, $\mathcal{A}_G$ *admits* $(\hat{\theta}, \hat{r}) \in \mathbb{T}^2 \times \mathbb{R}^2$ *as global coordinates* (*nonsymplectic in general*).

(2) *The level* $G^{-1}(0)$ *intersects* $\mathcal{A}_G \cap (\mathbb{T}^2 \times \boldsymbol{R})$ *transversely in* $\mathbb{A}^3$ *and*

$$\mathcal{C}_G := \mathcal{A}_G \cap (\mathbb{T}^2 \times \boldsymbol{R}) \cap G^{-1}(0) \tag{61}$$

is a 3-*dimensional submanifold of* $\mathbb{A}^3$, *diffeomorphic to* $\mathbb{T}^2 \times \mathbb{R}$, *with coordinates* $(\theta_0, \theta_1, r_1)$. *There is an open interval* $I_1 \subset \boldsymbol{I}_1$ *containing* $[0, 1]$ *such that the domain* $(\theta_0, \theta_1, r_1) \in \mathbb{T}^2 \times I_1$ *is well-defined in* $\mathcal{C}_G$.

(3) *For any* $\theta_0^* \in \mathbb{T}^2$, *the surface*

$$\Sigma_G^{\theta_0^*} = \mathcal{C}_G \cap \{\theta_0 = \theta_0^*\} \tag{62}$$

is a global symplectic section of the Hamiltonian flow Φ_G *restricted to* $\mathcal{C}_G$, *with coordinates* (θ_1, r_1) (*nonsymplectic in general*). *The domain* $(\theta_1, r_1) \in \mathbb{T} \times I_1$ *is well-defined in* $\Sigma_G^{\theta_0^*}$, *for any* $\theta_0^* \in \mathbb{T}$.

(4) *For any* θ_0^*, *in the coordinates* (θ_1, r_1), *the Poincaré return map* $\varphi_G^{\theta_0^*}$ *associated with* $\Sigma_G^{\theta_0^*}$ *converges to the map* $\varphi_F^{\theta_0^*}$ *in the compact-open* C^p *topology when* $\|G - F\|_{C^\kappa} \to 0$.

(5) *For any* θ_0^*, *the map* $\varphi_G^{\theta_0^*}$ *leaves invariant two* (*uniquely defined*) *essential circles* $\Gamma_G^{\theta_0^*}(0)$ *and* $\Gamma_G^{\theta_0^*}(1)$ *in* Σ_G, *with rotation numbers* ν_0 *and* ν_1 (*see* (C_1)), *which moreover satisfy*

$$\Gamma_G^{\theta_0^*}(0) \subset \{|r_1| < \rho\}, \qquad \Gamma_G^{\theta_0^*}(1) \subset \{|1 - r_1| < \rho\}. \tag{63}$$

(6) *For any* θ_0^*, *let* $\boldsymbol{A}_G^{\theta_0^*}$ *be the compact annulus bounded by* $\Gamma_G^{\theta_0^*}(0)$ *and* $\Gamma_G^{\theta_0^*}(1)$ *inside* $\Sigma_G^{\theta_0^*}$. *Then the restriction of* $\varphi_G^{\theta_0^*}$ *to* $\boldsymbol{A}_G^{\theta_0^*}$ *is a twist map*, *with twist constant* $\geq \mu/2$. *We denote by* $\mathrm{Ess}(\varphi_G^{\theta_0^*})$ *the set of essential invariant circles of this restriction. Each element of* $\mathrm{Ess}(\varphi_G^{\theta_0^*})$ *is a* $2/\mu$-*Lipschitz-continuous graph relative to the coordinates* (θ_1, r_1).

(7) *Set* $\varphi_G := \varphi_G^0$. *For each* $\Gamma \in \mathrm{Ess}(\varphi_G)$ *we set* $T(\Gamma) = \Phi_G(\mathbb{R} \times \Gamma)$, *so that* $T(\Gamma)$ *is a* 2-*dimensional invariant torus. We set*

$$\mathrm{Tess}(G) = \{T(\Gamma) \mid \Gamma \in \mathrm{Ess}(\varphi_G)\}.$$

(8) *For* $x \in \mathcal{A}_G$, *let* $W^{\pm}(x)$ *be the local invariant manifolds attached to* x. *Let* $W^{\pm}(\mathcal{C}_G)$ *and* $W^{\pm}(\mathcal{A}_G)$ *be those attached to* $\mathcal{C}_G$ *and* $\mathcal{A}_G$ (*defined as the unions of the previous ones*). *Then* $W^{\pm}(\mathcal{A}_G)$ *are coisotropic and their characteristic foliations coincide with their foliations* $\{W^{\pm}(x) \mid x \in\}$.

(9) *For* $x \in \mathcal{A}_G$, $W^{\pm}(x)$ *intersect transversely* $\boldsymbol{\Lambda} = \left\{\theta_2 = \frac{1}{2}\right\}$ *in* $\mathbb{A}^3$ *and for* $x \in \mathcal{C}_G$, $W^{\pm}(x)$ *intersect transversely* $\boldsymbol{\Delta}_G$ *in* $G^{-1}(0)$. *Both intersections are singletons.*

We set

$$\mathcal{A}_G^{\pm} := W^{\pm}(\mathcal{A}_G) \cap \mathbf{\Lambda}, \quad \mathcal{C}_G^{\pm} := W^{\pm}(\mathcal{C}_G) \cap \mathbf{\Delta}_G,$$

The first intersection is transverse in $\mathbb{A}^3$, *while the second one is transverse in* $G^{-1}(0)$.

(10) *The annuli* $\mathcal{A}_G$ *admits* $(\hat{\theta}, \hat{r})$ *as coordinates, while the cylinders* $\mathcal{C}_G$) *admit* $(\hat{\theta}, r_1)$ *as coordinates. Due to the particular choice of the section* $\mathbf{\Lambda}$, *the induced Liouville form on* $\mathscr{A}_G^{\pm}$ *reads*

$$r_0 d\theta_0 + r_1 d\theta_1,$$

so that $(\hat{\theta}, \hat{r})$ *are exact-symplectic coordinates.*[17]

(11) *We denote by* $\Pi_G^{\pm} : W^{\pm}(\mathcal{A}_G) \to \mathcal{A}_G$ *the characteristic projections, by* $\pi_G^{\pm} : \mathcal{A}_G^{\pm} \to \mathcal{A}_G$ *the restrictions of* $\Pi_G^{\pm}$ *to* $\mathcal{A}_G^{\pm}$ *and by* $j_G^{\pm} := (\pi_G^{\pm})^{-1} : \mathcal{A}_G \to \mathcal{A}_G^{\pm}$. *The maps* $\pi_G^{\pm}$ *and* $j_G^{\pm}$ *are exact-symplectic and converge to* $\pi_F^{\pm}$ *and* $j_F^{\pm}$ *in the compact-open* C^{p-1} *topology when* $\|G - F\|_\kappa \to 0$. *As a consequence,* $\pi_G^{\pm} \circ j_G^{\pm}$ *converge to* Id *in the compact-open* C^{p-1} *topology.*

We denote by $\boldsymbol{C}_G$ the compact cylinder bounded in $\mathcal{C}_G$ by $T(\Gamma_G(0))$ and $T(\Gamma_G(1))$, so that

$$\boldsymbol{C}_G = \Phi_G(\mathbb{R} \times \boldsymbol{A}_G). \tag{64}$$

In the following we will implicitly assume that κ_0 and p are large enough for the previous conclusions to hold true.

5.3. ***Homoclinic intersections.*** This section is devoted to the existence of homoclinic intersections for tori in Tess(G), where G is a small enough perturbation of an element of $\mathscr{F}^\kappa$.

Proposition 23. *Fix* $F \in \mathscr{F}^\kappa$, $\kappa \geq \kappa_0$. *Then there is a positive* $\varepsilon_1(F) < \varepsilon_0(F)$ (*where* $\varepsilon_0(F)$ *was defined in Lemma 22*) *such that for any* $G \in B^\kappa(F, \varepsilon_1(F))$, *and for any torus* $T \in$ Tess(G)

$$\#(W^-(T) \cap W^+(T) \cap \mathbf{\Delta}_G) \geq 3.$$

Under specific convexity assumptions on G, this could easily be deduced from Fathi's weak KAM theory, but we deal here with more general Hamiltonians and we want a purely symplectic proof based on Lagrangian intersection arguments. The main difficulty here is that the tori of Tess(G) are not smooth, so that we need to generalize the standard notion of Lagrangian manifold. There are several ways for doing this, see [1], here we adopt Herman's one since our tori are Lipschitz-continuous graphs. The Lagrangian character of such a graph amounts to saying that the induced Liouville form is closed in the sense of distributions.

[17] This property will be crucial in the following.

We will take advantage of the Lipschitzian character of the tori of at each step, and since the proof is not completely standard, we will give all details for the sake of completeness.

Here we only give a sketch of proof in the regular case. Assume that T is smooth (at least C^2) and, with the notation of Lemma 22, set

$$T^{\pm} := j_G^{\pm}(T) \subset \mathcal{A}_G^{\pm} \cap \mathbf{\Delta}_G.$$

• We identify $\mathbf{\Delta}_G$ with $\mathbb{A}^2$ by using the global symplectic chart $(\widehat{\theta}, \widehat{r}) \in \mathbb{A}^2$ of $\mathbf{\Delta}_G$. A simple application of the usual implicit function theorem proves that for G close enough to F in $\mathscr{G}^\kappa$, the tori $T^{\pm}$ are graphs over the base $\mathbb{T}^2$.

• The torus T is clearly Lagrangian in $\mathcal{A}_G$ (transport of an isotropic curve by the Hamiltonian flow) and, since the maps $j_G^{\pm}$ are exact-symplectic for the induced structures on $\mathcal{A}_G$ and $\mathcal{A}_G^{\pm}$, the tori $T^{\pm}$ are Lagrangian in $\mathcal{A}_G^{\pm}$. They are therefore isotropic and contained in $\mathbf{\Delta}_G$, so $T^{\pm}$ are Lagrangian in $\mathbf{\Delta}_G$. As a consequence,

$$T^{\pm} = \alpha^{\pm}(\mathbb{T}^2),$$

where $\alpha^{\pm} : \mathbb{T}^2 \to \mathbb{A}^2$ are closed 1-form on $\mathbb{T}^2$.

• By compactness, $T^+ \cap T^-$ is nonempty if the form $\alpha = \alpha^+ - \alpha^-$ is exact. Thus all we need is to check that the class $[\alpha] \in H^1(\mathbb{T}^2, \mathbb{Z})$ vanishes on $H_1(\mathbb{T}^2, \mathbb{Z})$. This can be done by comparing the integrals of $\alpha^{\pm}$ along two closed curves c_1 and c_2 in $\mathbb{T}^2$ generating $H_1(\mathbb{T}^2, \mathbb{Z})$. But since the induced Liouville form $\iota^*\boldsymbol{\lambda}$ satisfies $(\alpha^{\pm})^*(\iota^*\boldsymbol{\lambda}) = \alpha^{\pm}$, where ι is the inclusion $\mathbf{\Delta}_G \subset \mathbb{A}^3$, we may equivalently compare the integrals of the ambient Liouville form $\boldsymbol{\lambda}$ along $c_i^{\pm} = (\iota \circ \alpha^{\pm})(c_i)$, for $i = 1, 2$.

• The key observation is that the cycles $c_i^{\pm}$ belong to $\mathcal{A}_G^{\pm}$, and these two annuli are exchanged by the exact symplectic map $j_G^+ \circ (j_G^-)^{-1}$. This yields the equalities:

$$\int_{c_i^-} \boldsymbol{\lambda} = \int_{(j_G^+ \circ (j_G^-)^{-1})(c_i^-)} \boldsymbol{\lambda} = \int_{c_i^+} \boldsymbol{\lambda},$$

where the first one comes from the exactness of $j_G^+ \circ (j_G^-)^{-1}$ and the second one from the fact that j_G^+ and $(j_G^-)^{-1}$ are close to the identity relative to the charts $(\widehat{\theta}, \widehat{r})$ in their respective domains, so that $(j_G^+ \circ (j_G^-)^{-1})(c_i^-)$ is homotopic to c_i^+ in T^+.

• As a consequence $[\alpha]$ vanishes on $H_1(\mathbb{T}^2, \mathbb{Z})$ and α is exact. This ends the sketch of proof in the regular case.

5.4. ***Generic properties of $\boldsymbol{C}_G$ and $\boldsymbol{W}^{\pm}(\boldsymbol{C}_G)$.*** **1.** The following statement is the continuous version of the one in Section 3.2.3. It is now based of the global flow-box theorem (Appendix D), together with the methods introduced by Robinson in [50; 51]; see also [29].

Proposition 24. *For $\kappa \geq \kappa_0$, the subset of all Hamiltonians G in $B^\kappa(F, \varepsilon_1(F))$ such that no circle in $\mathrm{Ess}(\varphi_G)$ has rational rotation number is a residual subset of $B^\kappa(F, \varepsilon_1(F))$.*

2. The following result is also an application of the global flow-box theorem, it is also a continuous version of the perturbative result used in the discrete setting. Given $G \in B^\kappa(F, \varepsilon_1(F))$, we examine the perturbations of the asymptotic manifolds of $\boldsymbol{C}_G$ and their characteristic foliations that come from the perturbation of G.

Proposition 25. *Fix $G \in B^\kappa(F, \varepsilon_1(F))$. Then there exists a compact neighborhood K of $\boldsymbol{C}_G^- \cup \boldsymbol{C}_G^+$ in $\boldsymbol{\Delta}_G$, which satisfies*

$$\Phi_G([-2, 2] \times K) \cap \boldsymbol{C}_G = \varnothing, \tag{65}$$

such that for any pair of C^∞ Hamiltonian diffeomorphisms $\phi^- : \boldsymbol{\Delta}_G \circlearrowleft$ and $\phi^+ : \boldsymbol{\Delta}_G \circlearrowleft$ with support in $\mathrm{Int}\, K$, there exists a C^κ Hamiltonian $\widetilde{G} \in B^\kappa(F, \varepsilon_1(F))$ which coincides with G outside a compact subset of $\Phi_G(]-1, 0[\times K) \cup \Phi_G(]0, 1[\times K)$, so that $\boldsymbol{C}_{\widetilde{G}} = \boldsymbol{C}_G$, which satisfies

$$\begin{aligned} \phi^-(\boldsymbol{C}_G^-) &= \boldsymbol{C}_{\widetilde{G}}^-, \quad j_{\widetilde{G}}^- = \phi^- \circ j_G^-, \\ \phi^+(\boldsymbol{C}_G^+) &= \boldsymbol{C}_{\widetilde{G}}^+, \quad j_{\widetilde{G}}^+ = \phi^+ \circ j_G^+. \end{aligned} \tag{66}$$

Moreover one can choose $\widetilde{G}$ so that $\|\widetilde{G} - G\|_\kappa \to 0$ when $\phi^\pm$ are generated by Hamiltonians which tend to 0 in the C^∞ topology.

3. Our first application of the previous result ensures the generic transversality of the intersection $\mathcal{C}_G^+ \cap \mathcal{C}_G^-$, it is based only on standard transversality arguments.

Proposition 26. *The set $\mathscr{G}_0$ of hamiltonians $G \in B^\kappa(F, \varepsilon_1(F))$ such that the intersection $\mathcal{C}_G^+ \cap \mathcal{C}_G^-$ is transverse in $\boldsymbol{\Delta}_G$ in the neighborhood of $\boldsymbol{C}_G^+ \cap \boldsymbol{C}_G^-$ is open and dense in $B^\kappa(F, \varepsilon_1(F))$.*

The existence of homoclinic intersections hence immediately yields the following.

Corollary 27. *There is a neighborhood $\mathscr{O}$ of $\boldsymbol{C}_G^- \cap \boldsymbol{C}_G^+$ in $\boldsymbol{\Delta}_G$ such that $\mathscr{I}_G \cap \mathscr{O}$ is a 2-dimensional submanifold of $\boldsymbol{\Delta}_G$.*

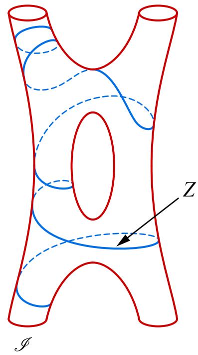


Figure 16. Intersection and singular curve.

5.5. ***Reduction to the 2-dimensional setting.*** The last corollary gives us now the possibility to recover the two-dimensional discrete setting introduced in Section 4.

1. The submanifold $\mathscr{I}_G$ is an interesting example of intersection of two transverse coisotropic submanifolds of a symplectic manifold. We were unable to find a systematic study of the generic singularities of such intersections, so let us quote here one remarkable genericity property. We first introduce the vector fields along $\mathcal{C}_{\tilde{G}}^{\pm}$ defined by

$$X_G^{\pm} = X_G - (j_G^{\pm})_\star X_G \tag{67}$$

One easily proves that they are tangent to the leaves of the characteristic foliations of $W^{\pm}(\mathcal{C}_G)$.

We introduce the following sets:

- $Z_G = \{x \in \mathscr{I}_G \mid T_x W^-(\mathcal{C}_G) \cap T_x W^+(\mathcal{C}_G)$ is Lagrangian$\}$.
- $Z_G^+ = \{x \in \mathscr{I}_G \mid X_G^+(x) \in T_x W^-(\mathcal{C}_G)\}$.
- $Z_G^- = \{x \in \mathscr{I}_G \mid X_G^-(x) \in T_x W^+(\mathcal{C}_G)\}$.

Note that Z_G is precisely the complement of the symplectic locus inside $\mathscr{I}_G$. The striking fact is the following remark.

Lemma 28. *The sets $Z_G^{\pm}$ and Z_G coincide, and the set $\mathscr{G}_1$ of Hamiltonians $G \in \mathscr{G}_0$ such that Z_G is a 1-dimensional submanifold of $\mathscr{I}_G$ in the neighborhood of $\boldsymbol{C}_G^- \cap \boldsymbol{C}_G^+$ is open and dense in $B^\kappa(F, \varepsilon_1(F))$.*

The points of the singular locus Z_G are precisely those where the characteristic projections $\Pi_G^{\pm}$ restricted to the intersection $W^+(\mathcal{C}_G) \cap W^-(C_G)$ are not local diffeomorphisms (by definitions of $Z_G^{\pm}$). These points necessitate special care in our construction, and we will forget about them in the following.

2. Recall that

$$\Sigma_G = C_G \cap \{\theta_0 = 0\}$$

is a two dimensional annulus, which contains two disjoint invariant circles $\Gamma_0(G)$ and $\Gamma_1(G)$ bounding a compact invariant annulus $\boldsymbol{A}_G$. We therefore have a first global return (twist) map φ_G at our disposal and we need to construct a homoclinic correspondence to get the bisystem of Section 4. This correspondence would ideally be obtained by transport of small pieces of the intersection $\mathscr{I}_G$ on $\mathscr{C}_G$ by the characteristic projections, and then on Σ_G by the Hamiltonian flow inside C_G, which is not possible in the neighborhood of points of Z_G. Let us introduce the subset

$$\Xi \subset \mathscr{I}$$

of all homoclinic points corresponding to the essential invariant tori contained in $\boldsymbol{C}_G$, and, for simplicity, assume that

$$\Xi \cap Z_G = \varnothing.$$

By compactness, one can find a cover $(D_\alpha)_{\alpha \in A}$ of Ξ by small discs contained in $\mathscr{I} \setminus Z_G$, such that moreover, setting $D^\pm = \pi^\pm(D) \subset \boldsymbol{C}_G$:

- There are C^1 functions $\boldsymbol{\tau}^\pm : D^\pm \to \mathbb{R}$ such that $\Phi_G^{\boldsymbol{\tau}^\pm}(D^\pm) \subset \Sigma_G$ and $\Phi_G^{\boldsymbol{\tau}^\pm}$ are C^1 diffeomorphisms onto their images.

Definition 29 (homoclinic diffeomorphisms and correspondences). Given a small disc D satisfying the previous constraint, we define the *homoclinic diffeomorphism* ψ_D attached to D by

$$\begin{aligned} \psi_D : \operatorname{Dom} \psi_D &\to \operatorname{Im} \psi_D \\ x &\mapsto \Phi_G^{\boldsymbol{\tau}^+} \circ \Pi^- \circ (\Pi^+_{|D})^{-1} \circ (\Phi_G^{\boldsymbol{\tau}^-})^{-1}(x). \end{aligned} \tag{68}$$

where

$$\operatorname{Dom} \psi_D := \Phi_G^{\boldsymbol{\tau}^-}(D^-) \subset \Sigma_G, \quad \operatorname{Im} \psi_D := \Phi_G^{\boldsymbol{\tau}^-}(D^+) \subset \Sigma_G.$$

We define a *homoclinic correspondence* for Σ_G as a family of homoclinic diffeomorphisms $\psi = (\psi_\alpha := \psi_{D_\alpha})_{\alpha \in A}$ attached to a cover $(D_\alpha)_{\alpha \in A}$ of Ξ by small discs in $\mathscr{I} \setminus Z_G$.

Note that we do not require the supports of the homoclinic diffeomorphisms in a homoclinic correspondence to be pairwise disjoint. Given a homoclinic correspondence $\psi = (\psi_\alpha)_{\alpha \in A}$, we define the associated set $\mathscr{X}^-$ of (negative) transported homoclinic points as

$$\mathscr{X}^- = \bigcup_{\alpha \in A} \psi_\alpha(\Xi \cap D_\alpha).$$

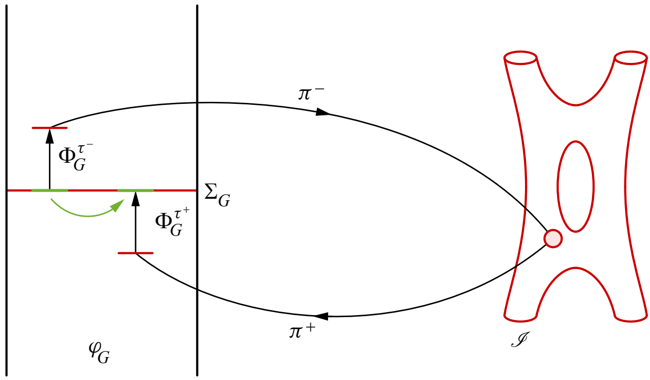


Figure 17. A homoclinic diffeomorphism.

3. *Good cylinders.* We now come to our main definition and result; see [21].

Definition 30. We say that the compact invariant cylinder $\boldsymbol{C}_G$ defined in (64) is a *good cylinder* when the return map φ_G attached to the section $\boldsymbol{A}_G$ is a simple twist map, and when there exists a homoclinic correspondence $\psi_G = (\psi_\alpha)_{\alpha\in A}$ of $\mathcal{C}_G$ such that for any $\Gamma \in \mathrm{Ess}(\varphi_G)$ there exists a (right or left) splitting arc based on Γ for the correspondence ψ_G.

Theorem 31. *Fix $F \in \mathscr{F}^\kappa$, $\kappa \geq \kappa_0$. Then there is an $\boldsymbol{\epsilon}(F) \in]0, \varepsilon_1(F)]$ such that the set of Hamiltonians G in the ball $B^\kappa(F, \boldsymbol{\epsilon}(F))$ for which $\boldsymbol{C}_G$ is a good cylinder is dense in $B^\kappa(F, \varepsilon)$.*

The results of Section 4 (Theorem 19) immediately enables us to deduce Theorem 21 from Theorem 31.

4. *Idea of the proof of Theorem 31.* The genericity results of Section 5.4 are taken for granted. It therefore remains to show the following proposition, where, given $G, \widetilde{G}$ in $\mathscr{G}^\kappa$, we say that $\widetilde{G}$ is G-admissible when $\widetilde{G}$ coincides with G outside a neighborhood of $\mathscr{A}_G$.

Proposition 32. *Fix $F \in \mathscr{F}^\kappa$ with $\kappa \geq \kappa_0$ and $G \in \mathscr{G}_1(F)$. Then for any $\alpha > 0$, there exists an admissible Hamiltonian $\widetilde{G} \in \mathscr{G}_1(F)$ with*

$$\|\widetilde{G} - G\|_\kappa < \alpha \tag{69}$$

such that $\boldsymbol{C}_{\widetilde{G}} = \boldsymbol{C}_G$ is a good cylinder for $\widetilde{G}$.

We require the admissibility condition to ensure that the inner map $\varphi_{\widetilde{G}}$ and φ_G coincide, and therefore admit the same set of essential invariant circles. To prove Proposition 32, we have to find an arbitrarily small perturbation $\widetilde{G}$ of G such that each invariant circle of $\mathrm{Ess}(\varphi_G)$ admits a (right or left) splitting arc.

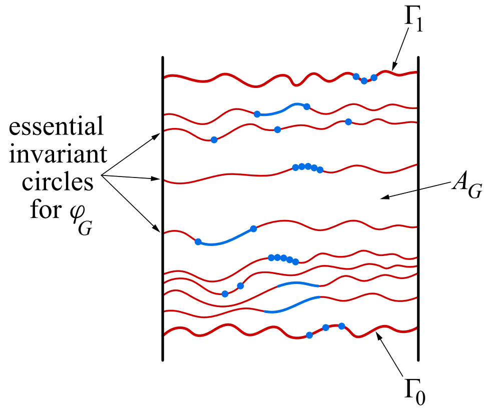


Figure 18. Essential circles and homoclinic points for φ_G.

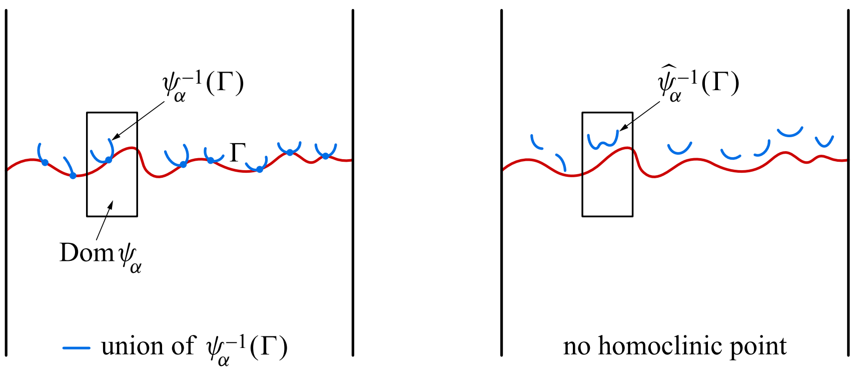


Figure 19. Disjunction of the arcs.

The idea is essentially symplectic and is based on the existence of (transported) homoclinic points on each invariant circle of $\varphi_{\widehat{G}}$ for each $\widehat{G}$ in $\mathscr{G}_1(F)$ (by construction and Proposition 23).

We argue by contradiction. The main ingredient is the possibility to construct an arbitrarily small and admissible perturbation $\widehat{G} \in \mathscr{G}_1(F)$ of G whose set of transported homoclinic points is *totally disconnected*.[18] We then prove that if $\varphi_{\widehat{G}}$ admits an invariant circle Γ with no splitting arc, then it is possible to perturb $\widehat{G}$ another time — still inside $\mathscr{G}_1(F)$ — to ensure that the new homoclinic correspondence satisfies

$$\psi_\alpha^{-1}(\Gamma) \cap \Gamma = \varnothing, \quad \forall \alpha \in A.$$

[18]This is done by local arguments of Minkowski dimension and a convergent sequence of Hamiltonian perturbations of the homoclinic correspondence, of the same type as Moeckel's ones.

Consequently the circle Γ do not posses (transported) homoclinic points, which contradicts Proposition 23. The disconnectedness of the set of homoclinic points is used to produce local perturbations of the homoclinic correspondence by composition with local Hamiltonian diffeomorphisms which increase the action r in a very small neighborhood of each homoclinic point, and decrease the action where the arcs a far from the circle Γ. The proof will full details will appear in the joint work with L. Lazzarini.

6. Broadening the scope

The previous presentation has to be seen as a first introduction to the geometric approach to Arnold diffusion, whose methods, results and scope can be improved on by using recent developments in symplectic topology, two-dimensional dynamics and control theory. We briefly discuss the first two points in this section.

6.1. ***Symplectic topology.*** We refer to [2] for a survey of the origins of the questions in symplectic topology and to Gromov's seminal paper [26], and Laudenbach and Sikorav [38] for seminal results in Lagrangian intersection problems. Here we will a similar result, in its most basic form proved by Lalonde and Sikorav [37].

One main difficulty in the application of the methods presented in Section 5 to the *a priori* stable case comes from the essentially singular perturbations involved in this setting. The absence of hyperbolicity in the unperturbed system makes the embedding of the cylinders very delicate, in the sense that they are graphs of function whose C^1 norm tends to infinity when the size of the perturbation tends to 0. This makes in turn very difficult the detection of the graph properties of the essential circles contained in these cylinders.[19] This difficulty can be overcome by a very careful analysis of the location of these objects (as in [33]) or by cutting the cylinders into smaller and smaller pieces (as in [44]). However a way to get rid of the graph constraint in the proof of existence of Lagrangian intersection would be much more satisfactory, and this is precisely the content of another famous Arnold conjecture — which could perhaps have been inspired by the present problem.

Let us recall one first result in the direction of the Arnold intersection conjecture. Let M and L be compact manifolds of the same dimension. Endow T^*M with its usual Liouville form $\boldsymbol{\lambda}$ and set $\boldsymbol{\Omega} = d\boldsymbol{\lambda}$. Recall that an embedding $j : L \to T^*M$ is said to be exact when $j^*\lambda$ is an exact form. An embedded submanifold of T^*M is said to be exact when it is the image of an exact embedding.

[19] As is the also case for the usual invariant objects from weak KAM theory.

Theorem [37]. *Let L and L' be two exact submanifolds of $T^*\mathbb{T}^n$. Then $L \cap L'$ is nonempty.*

This enables us to some extent to relax the graph assumption for the essential circles in the previous section. Indeed, given an essential smooth torus T contained in the hyperbolic cylinder $\boldsymbol{C}_G$, without any torsion assumption on return map, one can introduce a Weinstein tubular neighborhood $N \sim T^*(j^-(T))$ of the image $j_G^-(T) \subset \boldsymbol{\Delta}_G$. Then, since $j_{\widetilde{G}}^{\pm}$ are exact-symplectic for the induced Liouville form on $\mathscr{A}_{\widetilde{G}}^{\pm}$, one can prove that $j^+(T)$ is an exact submanifolds of N (for the usual exact structure), hence the previous theorem proves that the intersection $j^-(T) \cap j^+(T)$ is nonempty (where $j^-(T)$ is identified with the zero section of N).

To conclude in the case of Lipschitz tori, it suffices to prove the existence of two sequences $(T_n^{\pm})_{n\in\mathbb{N}}$ of tori of Δ with $T_n^+ \cap T_n^- \neq \varnothing$ which converge to $j_{\widetilde{G}}^{\pm}(T)$ in the C^0 topology. To see this, one can first perform a smoothing of the initial torus $T = \Phi_{\widetilde{G}}(\mathbb{R} \times \Gamma)$ by symplectic plumbing of the transport of a smoothed invariant circle Γ, and then to perform a smoothing of $j_{\widetilde{G}}^{\pm}$ by means of their generating Hamiltonians. We expect this strong result to enable us to give simpler proofs of the existence of homoclinic orbits in the *a priori* stable case, as well as to deal with a larger set of perturbations of the completely integrable Hamiltonian h.

6.2. ***Two-dimensional dynamics without convexity.*** One can also expect new results for diffusion without the convexity assumption on h, using the generic transitivity result of [35; 36]. Let us give an example which mimics the *a priori* chaotic setting of Section 3, *without any twist condition*. Let $\mathscr{D}^\kappa$ be the group of C^κ symplectic diffeomorphisms of the product $S = S_1 \times S_2$, where (S_i, ω_i) are compact symplectic smooth surfaces, equipped with the symplectic form $\omega = \omega_1 \oplus \omega_2$. Let $\mathscr{F}^\kappa \subset \mathscr{D}^\kappa$ be the subset formed by the product diffeomorphisms of the form

$$f(x_1, x_2) = (f_1(x_1), f_2(x_2)), \quad x_i \in S_i, \tag{70}$$

satisfying the following conditions:

(C1) Both f_1, f_2 are symplectic.

(C2) f_2 admits a hyperbolic fixed point O_2.

(C3) The Lyapunov exponents of f_2 at O_2 dominate the Lyapunov exponents of f_1 on S_1.

(C4) f_2 has a transverse homoclinic point P_2 for O_2 in S_2.

Then the following result (from a current joint work with M. Gidea) holds true: there is a κ_0 such that for $\kappa \geq \kappa_0$, for every $f \in \mathscr{F}^\kappa$ there exists an $\varepsilon_0 > 0$

(depending on f) such that for every diffeomorphism g in a residual subset $R^\kappa(f, \varepsilon_0)$ of the ball $B^\kappa(f, \varepsilon_0)$, there exists an orbit $(x_1^n, x_2^n)_{n\in\mathbb{N}}$ of g such that the projected sequence $(x_1^n)_{n\in\mathbb{N}}$ is dense in S_1.

An extension to this result to the discrete setting for *a priori* unstable systems as in section IV is a very challenging question which has deep consequences for diffusion for perturbations of nonconvex completely integrable Hamiltonians.

Appendix A. Normal hyperbolicity and symplectic geometry

We refer to [4; 9; 31] for general definitions an results on normal hyperbolicity. Here we limit ourselves to a very simple class of systems which admit a normally hyperbolic invariant (noncompact) submanifold, which serves us as a model from which all other definitions and properties will be deduced.

1. The following statement is a simple version of the persistence theorem for normally hyperbolic manifolds well-adapted to our setting, whose germ can be found in [6]. We limit ourselves to the case of 1-dimensional stable and unstable directions, which is the only one we have to deal with in this paper. We fix an integer $m \geq 1$ and endow $\mathbb{R}^{m+2}$ with the coordinates (x, u, s), with $x \in \mathbb{R}^m$, $(u, s) \in \mathbb{R}^2$.

Theorem (the normally hyperbolic persistence theorem). *Fix $m \geq 1$ and consider a vector field of class C^1 V_0 on $\mathbb{R}^{m+2}$ of the form*

$$V_0(x, u, s) = (X(x, u, s), \lambda_u(x)u, -\lambda_s(x)s), \quad (x, u, s) \in \mathbb{R}^{m+2}. \tag{71}$$

Assume moreover that there exists $\lambda > 0$ such that for $x \in \mathbb{R}^m$:

$$\lambda_u(x) \geq \lambda, \quad \lambda_s(x) \geq \lambda. \tag{72}$$

Fix a constant $R > 0$ and set $O_R = \{(x, u, s) \in \mathbb{R}^{m+2} \mid \|(u, s)\| < R\}$ and assume that

$$\|\partial_x X\|_{C^0(O_R)} < \lambda. \tag{73}$$

Then there exist constants $\delta_* > 0$, $c_* > 0$, $C > 0$, such that if V_r is a C^1 vector field on $\mathbb{R}^{m+2}$ such that

$$\|V_r\|_{C^1(\mathbb{R}^{m+2})} \leq \delta_*, \tag{74}$$

setting $V = V_0 + V_r$, the following assertions hold:

- *The maximal invariant set for V contained in O_R is an m-dimensional manifold $\mathcal{A}(V)$ which admits the graph representation*:

$$\mathcal{A}(V) = \{(x, u = U(x), s = S(x)) \mid x \in \mathbb{R}^m\},$$

where U and S are C^1 maps $\mathbb{R}^m \to \mathbb{R}$ such that

$$\|(U, S)\|_{C^0(\mathbb{R}^m)} \leq c_* \|V_r\|_{C^0}. \tag{75}$$

• *The maximal positively invariant set for V contained in O_R is an $(m+1)$-dimensional manifold $W^+(\mathcal{A}(V))$ which admits the graph representation*:

$$W^+(\mathcal{A}(V)) = \{(x, u = U^+(x, s), s) \mid x \in \mathbb{R}^m, s \in]-R, R[\},$$

where $U^+ : \mathbb{R}^m \times]-R, R[\to \mathbb{R}$ is a C^1 map such that

$$\|U^+\|_{C^0(\mathbb{R}^m)} \leq c_* \|V_r\|_{C^0}. \tag{76}$$

• *The maximal negatively invariant set for V contained in O_R is an $(m+1)$-dimensional manifold $W^-(\mathcal{A}(V))$ which admits the graph representation*:

$$W^-(\mathcal{A}(V)) = \{(x, u, s = S^-(x, u)) \mid x \in \mathbb{R}^m, u \in]-R, R[\},$$

where $S^- : \mathbb{R}^m \times]-R, R[\to \mathbb{R}$ is a C^1 map such that

$$\|S^-\|_{C^0(\mathbb{R}^m)} \leq c_* \|V_r\|_{C^0}. \tag{77}$$

• *The manifolds $W^\pm(\mathcal{A}(V))$ admit C^0 foliations $(W^\pm(x))_{x \in \mathcal{A}(V)}$ such that for $w \in W^\pm(x)$*

$$\mathrm{dist}(\Phi^t(w), \Phi^t(x)) \leq C \exp(\pm\lambda t), \quad t \geq 0. \tag{78}$$

• *If moreover V_0 and V_r are assumed to be of class C^p, $p \geq 1$, and if*

$$p \|\partial_x X\|_{C^0(O_R)} < \lambda \tag{79}$$

then the functions U, S, U^+, S^- are of class C^p and there is a constant C_p, depending only on V_0, such that U, S U^+, S^+ tend to 0 in the C^p compact-open topology when V_r tends to 0 in the C^p topology.

• *Assume moreover that the vector fields V_0, V_r are $\boldsymbol{L}$-periodic in x, where $\boldsymbol{L}$ is a lattice in $\mathbb{R}^m$. Then their flows and the manifolds $\mathcal{A}(V)$ and $W^\pm(\mathcal{A}(V))$ pass to the quotient $(\mathbb{R}^m/\boldsymbol{L}) \times \mathbb{R}^2$*

The last statement will be applied in the case where $m = 2\ell$ and $\boldsymbol{L} = \mathbb{Z}^\ell \times \{0\}$, so that the quotient $\mathcal{A}(V)$ is diffeomorphic to the annulus $\mathbb{A}^\ell$.

2. The following result describes the symplectic geometry of our system in the case where V is a Hamiltonian vector field. We keep the notation of the previous theorem.

Theorem (the symplectic normally hyperbolic theorem). *Endow $\mathbb{R}^{2m+2}$ with a symplectic form Ω such that there exists a constant $C > 0$ such that for all $z \in O_R$*

$$|\Omega(v, w)| \leq C \|v\| \|w\|, \quad \forall v, w \in T_z M. \tag{80}$$

Let H_0 be a C^2 Hamiltonian on $\mathbb{R}^{2m+2}$ whose Hamiltonian vector field V_0 satisfies (71) and (72), and consider a Hamiltonian $H = H_0 + P$. Then if the vector field V generated by H satisfies (73) and (74) the following properties hold:

- *The manifold $\mathcal{A}(V)$ is Ω-symplectic.*
- *The manifolds $W^{\pm}(\mathcal{A}(V))$ are coisotropic and the 1-dimensional stable and unstable foliations $(W^{\pm}(x))_{x\in\mathcal{A}(V)}$ coincide with the characteristic foliations of $W^{\pm}(\mathcal{A}(V))$.*
- *If H is C^{p+1} and condition (79) is satisfied, then the manifolds $\mathcal{A}(V)$, $W^{\pm}(\mathcal{A}(V))$ are of class C^p and the foliations $(W^{\pm}(x))_{x\in\mathcal{A}(V)}$ are of class C^{p-1}.*

Appendix B. A reminder on twist maps

We refer to the appendix of [29] and [40; 41] for more details and proofs about the Birkhoff theory of twist maps. Let $a < b$ be fixed. We set

$$\boldsymbol{A} = \mathbb{T} \times [a, b], \quad \Gamma(a) = \mathbb{T} \times \{a\}, \quad \Gamma(b) = \mathbb{T} \times \{b\}.$$

The closure of a subset $E \subset \boldsymbol{A}$ will be indifferently denoted by $\operatorname{cl} E$ or $\bar{E}$, and its interior will be denoted by $\operatorname{Int} E$. The set $\operatorname{Fr} E = \operatorname{cl} E \setminus \operatorname{Int} E$ is the frontier of E. A disk is an open connected and simply connected subset of $\boldsymbol{A}$.

Here we say that $f : \boldsymbol{A} \to \boldsymbol{A}$ is a *twist map* when it is a C^1 diffeomorphism, preserves $\Gamma(a)$ and $\Gamma(b)$ and tilts the vertical, that is, $f(\theta, r) = (\Theta, R)$ with

$$\partial_r \Theta(\theta, r) > 0 \quad \text{or} \quad \partial_r \Theta(\theta, r) < 0, \quad \forall (\theta, r) \in \boldsymbol{A}.$$

Then f tilts the vertical *to the right* in the former case and *to the left* in the latter one. A continuous map $f : \boldsymbol{A} \to \boldsymbol{A}$ is said to be *area-preserving* when it leaves invariant a Radon measure which is positive on the open subsets of $\boldsymbol{A}$. An essential circle in $\boldsymbol{A}$ is a C^0 curve which is homotopic to $\Gamma(a)$.

Theorem (Birkhoff). *Let $f : \boldsymbol{A} \to \boldsymbol{A}$ be an area-preserving twist map. Then there exists $\nu > 0$ such that any essential circle invariant under f is the graph of some ν-Lipschitz function $\ell : \mathbb{T} \to [a, b]$.*

The second result from Birkhoff's theory we need is the following.

Theorem (Birkhoff). *Let $f : \boldsymbol{A} \to \boldsymbol{A}$ be an area-preserving twist map. Assume that U is an open subset of $\boldsymbol{A}$ homeomorphic to $\mathbb{T} \times [0, 1[$, with $\Gamma(a) \subset U$, such that $f(U) \subset U$ and such that U is the interior of its closure. Then the frontier* Fr *U is an invariant essential circle.*

One easily deduces from the first Birkhoff theorem that the set $\operatorname{Ess}(f)$ of essential invariant circles of f, endowed with the Hausdorff topology, is compact.

Given $\Gamma \in \mathrm{Ess}(f)$ with $\Gamma = \mathrm{Graph}(\ell)$, we set

$$\Gamma^+ = \{(\theta, r) \in \boldsymbol{A} \mid r > \ell(\theta)\}, \quad \Gamma^- = \{(\theta, r) \in \boldsymbol{A} \mid r < \ell(\theta)\}. \tag{81}$$

By the Poincaré theory, every $\Gamma \in \mathrm{Ess}(f)$ admits a rotation number in $\mathbb{T}$ for $f_{|\Gamma}$. One can choose a common lift to $\mathbb{R}$ for the rotation number of all circles, which yields a function $\rho : \mathrm{Ess}(f) \to \mathbb{R}$. This function is continuous and increasing, in the sense that if $\Gamma_i = \mathrm{Graph}\,\ell_i$, $i = 1, 2$ are invariant with $\ell_1 \leq \ell_2$, then $\rho(\ell_1) \leq \rho(\ell_2)$. Moreover, $\rho(\ell_1) < \rho(\ell_2)$ when $\ell_1 < \ell_2$.

Definition 33. Let $f : \boldsymbol{A} \to \boldsymbol{A}$ be an area-preserving twist map of the annulus $\boldsymbol{A}$. Let $\ell_\bullet$ and $\ell^\bullet$ be two functions $\mathbb{T} \to]a, b[$ whose graphs $\Gamma(a)$ and $\Gamma^\bullet$ are in $\mathrm{Ess}(f)$. Then one says that the set

$$\mathscr{B} = \{(\theta, r) \mid \theta \in \mathbb{T}, \ell_\bullet(\theta) \leq r \leq \ell^\bullet(\theta)\}$$

is a *Birkhoff zone* when that there is no element $\Gamma = \mathrm{Graph}\,\ell \in \mathrm{Ess}(f)$ such that $\ell_\bullet \leq \ell \leq \ell^\bullet$ and $\ell \neq \ell_\bullet$, $\ell \neq \ell^\bullet$.

We now prove Lemma 13, which was used in Section 3

Proof of Lemma 13. The main property of a special twist map f, coming from the fact that no element of $\mathrm{Ess}(f)$ has rational rotation, is that two distinct elements of $\mathrm{Ess}(f)$ are disjoint; see [34], Section 13.2. As a consequence, the rotation number $\rho : \mathrm{Ess}(f) \to \mathbb{R}$ is a homeomorphism onto its image $\mathcal{R} = \rho(\mathrm{Ess}(f))$, by compactness of $\mathrm{Ess}(f)$. The boundaries of the Birkhoff zones are sent by ρ on the boundaries of the maximal intervals in the complement $\mathrm{Rot} \setminus \rho(\mathrm{Ess}(f))$, where $\mathrm{Rot} = [\rho(\Gamma(a)), \rho(\Gamma(b))]$ is the rotation interval of f. Our claim easily follows. □

We can now state a second easy lemma on special twist maps and domains associated with right or left splitting arcs.

Lemma 34. *Consider an essential circle $\Gamma \in \mathrm{Ess}(\varphi)$, $\Gamma \subset \boldsymbol{A} \setminus \Gamma(a)$, and a right (resp. left) splitting arc ζ based on Γ, with domain $D(\zeta)$. Consider an essential circle $\Gamma(a) \subset \boldsymbol{A}$ such that $\tilde{\zeta}$ is contained in the domain $\Gamma(a)^+$ above $\Gamma(a)$. Then for $x \in D(\zeta)$ there exists a positively (resp. negatively) tilted arc γ with $\gamma(0) \in \Gamma(a)$ and $\gamma(1) = x$, whose image does not intersect the union $\Gamma \cup \tilde{\zeta}$.*

The following easy result on negatively tilted arcs is used several times in our constructions.

Lemma 35. *Let Γ be an essential circle of $\boldsymbol{A}$ which is the graph of a ν-Lipshitz function $\ell : \mathbb{T} \to [0, 1]$, and let B be a ν-ball centered on Γ. Then for any $z \in \Gamma^+ \cap B$, there exists a negatively tilted arc based on Γ and ending at z, whose image is contained in B.*

The proof of the following lemma is immediate.

Lemma 36. *Let $f : \boldsymbol{A} \to \boldsymbol{A}$ be an area-reserving twist map. Let $\Gamma^{\pm}$ be two non-intersecting essential invariant circles contained in $\boldsymbol{A}$. Then for any continuous curves C and C' which intersect both circles $\Gamma^{\pm}$, the positive orbit of C under f intersects C'.*

We refer to [39] for the proofs of the following two results from Birkhoff's theory.

Lemma 37. *Let $f : \boldsymbol{A} \to \boldsymbol{A}$ be an area-preserving twist map and let Γ be an essential invariant circle for f. The inverse image $f^{-1} \circ \gamma$ of a positively tilted arc γ based on Γ is a positively tilted arc based on Γ. The direct image $f \circ \gamma$ of a negatively tilted arc γ emanating from Γ is a negatively tilted arc based on Γ.*

Given a point $x \in \boldsymbol{A}$, we define the lower vertical $V^-(x)$ as the vertical segment joining a point of the lower boundary of $\boldsymbol{A}$ to x.

Lemma 38. *Let $f : \boldsymbol{A} \to \boldsymbol{A}$ be an area-preserving twist map. Let $\Gamma \in \mathrm{Ess}(f)$. Let X be a connected closed subset of $\boldsymbol{A}$ which disconnects the annulus $\boldsymbol{A}$ and such that $X \subset \Gamma^+$. Let $x \in \boldsymbol{A}$ be such that there exists a positively tilted arc γ and a negatively tilted arc η, both based on Γ and ending at x, such that the images of γ and η do not intersect X. Then the vertical $V^-(x)$ does not intersect X.*

The following strong connecting lemma appeared with a different proof in [22].

Proposition 39. *Let $f : \boldsymbol{A} \to \boldsymbol{A}$ be a (not necessarily special) area-preserving twist map. Let $\Gamma(a)$ and $\Gamma^\bullet$ be the boundary components of some Birkhoff zone of instability for f. Fix a pair of open sets $V_\bullet$, $V^\bullet$ which intersect $\Gamma(a)$ and $\Gamma^\bullet$ respectively, with moreover $V_\bullet \subset (\Gamma^\bullet)^-$. Then there exist a point $z \in V_\bullet$ and an integer $n \geq 0$ such that $f^n(z) \in V^\bullet$. Moreover the integer n can be chosen arbitrarily large.*

Proof. Set

$$U = \bigcup_{n \geq 0} f^n(\Gamma(a)^- \cup V_\bullet) = \Gamma(a)^- \cup \Big(\bigcup_{n \geq 0} f^n(V_\bullet) \Big),$$

so that U is a connected and f-invariant neighborhood of $\partial_\bullet \boldsymbol{A}$, which satisfies

$$U \subset (\Gamma^\bullet)^-.$$

Hence the frontier $\Gamma := \mathrm{Fr}\,\mathcal{U}$ of its associated filled subset is in $\mathrm{Ess}(f)$ and satisfies $\Gamma(a) \leq \Gamma \leq \Gamma^\bullet$. Therefore $\Gamma = \Gamma(a)$ or $\Gamma = \Gamma^\bullet$. The former equality is impossible by construction, so $\Gamma = \Gamma^\bullet$.

As a consequence, $\Gamma^\bullet \subset \mathrm{Fr}\,\mathcal{U} \subset \mathrm{Fr}\,U$, so there exists an integer $n \geq 0$ such that

$$f^n(V_\bullet) \cap V^\bullet \neq \varnothing,$$

which proves our claim. Finally, observe that by choosing arbitrarily small open subsets $W_\bullet \subset V_\bullet$, $W^\bullet \subset V^\bullet$ and applying the previous result to the pair $W_\bullet$, $W^\bullet$, one can ensure that the integer n can be chosen arbitrarily large. □

Appendix C. Normally hyperbolic shadowing

For the convenience of the reader, we add a proof of the normally hyperbolic shadowing lemma, whose main ingredient is the Poincaré recurrence theorem and which closely follows [23; 24]. Let d stand for the product metric on $\mathbb{A}^2$.

Theorem 40. *Fix $f \in \mathscr{F}^\kappa$ with κ so that the statements of the last section hold. Fix g in $B^\kappa(f, \varepsilon(f))$ and fix an orbit $x_0, \dots, x_n$ of the polysystem (φ_g, ψ_g) on $\mathscr{A}_g$. Then for any $\delta > 0$ there is an orbit $z_0, \dots, z_N$ of g in $\mathbb{A}^2$ such that $d(z_0, x_0) < \delta$ and $d(z_N, x_n) < \delta$. One can moreover choose z_0 so that for each $i \in \{0, \dots, n\}$, there is an $m(i)$ with*

$$d(g^{m(i)}(z_0), x_i) < \delta. \tag{82}$$

Since φ_g has compact support and preserves the symplectic area on $\mathscr{A}_g$, by the Poincaré recurrence theorem almost every point of $\mathscr{A}_g$ is positively and negatively recurrent for φ_g. In the following we use *recurrent* as a shorthand for *positively and negatively recurrent*.

The other main tool of the proof is the following λ-lemma.

Lemma (normally hyperbolic inclination lemma). *Fix $f \in \mathscr{F}^\kappa$ with κ so that the statements of the last section hold, and fix g in $B(f, \varepsilon(f))$. Let $(j_x)_{x \in \mathscr{A}_g}$ be a continuous family of C^1 parametrizations of the local unstable manifolds attached to $\mathscr{A}_g$, that is, a C^0 map $j : \mathscr{A}_g \times [-1, 1] \to W^-(\mathscr{A}_g)$ such that, setting $j_x = j(x, \cdot)$,*

$$j_x(0) = x, \quad j_x([-1, 1]) \subset W^-(x), \tag{83}$$

and j_x is C^1. Then for any C^1 submanifold Δ of $\mathbb{A}^2$ which intersects $W^+(\mathscr{A}_g)$ transversely in $\mathbb{A}^2$ at some point $\xi \in W^+(x)$, there exist a sequence $(\Delta_n)_{n \in \mathbb{N}}$ such that

$$\xi \in \Delta_n \subset \Delta \quad \forall n \in \mathbb{N}, \tag{84}$$

and for $n \in \mathbb{N}$, a C^1 diffeomorphism $\ell_n : [-1, 1] \to g^n(\Delta_n)$ such that

$$\lim_{n \to \infty} \|\ell_n - j_{g^n(x)}\|_{C^0} = 0. \tag{85}$$

We refer to [52] for a proof with detailed estimates in the compact setting, which directly applies here thanks to our compactness assumption on the support of g.

Proof of Theorem 40. We will write φ, ψ instead of φ_g, ψ_g. Fix an orbit $x_0, \dots, x_n$ of the polysystem (φ, ψ) on $\mathscr{A}_g$ and fix $\delta > 0$. We fix a tubular neighborhood $\mathscr{N}$ of $\mathscr{A}_g$ in $\mathbb{A}^2$ such that $\mathscr{N} \cap W^-(\mathscr{A}_g)$ is invariant by g^{-1} and for each $z \in \mathscr{N} \cap W^-(\mathscr{A}_g)$ with $z \in W^-(y)$

$$d(g^{-1}(z), g^{-1}(y)) < d(z, y). \tag{86}$$

Setting $\tau_0 = \varphi$ and $\tau_1 = \psi$, by definition, there exists a sequence $\omega_0, \dots, \omega_{n-1}$ in $\{0, 1\}$ such that, for $0 \le j \le n-1$,

$$x_{j+1} = \tau_{\omega_j}(x_j). \tag{87}$$

Choose $r > 0$ small enough so that if $D_0 = \mathscr{A}_g \cap B(x_0, r)$ and if

$$D_{j+1} = \tau_{\omega_j}(D_j) \quad \text{for} \quad 0 \le j \le n-1, \tag{88}$$

then $D_j \subset \mathscr{A}_g \cap B(x_j, \delta/2)$ for $0 \le j \le n$ (which is possible by continuity of both maps τ_j).

We will prove the existence of an orbit $(y_j)_{1 \le j \le n}$ of (τ_0, τ_1) associated with the same sequence (ω_j), such that the point y_j belongs to D_j and is recurrent for $\tau_0 = \varphi$, and the existence of a sequence of balls $(B_j)_{0 \le j \le n}$ of $\mathbb{A}^2$ which satisfy the following two properties:

(C_j) For $0 \le j \le n$, B_j is centered at some point $z_j \in W^-(y_j) \cap \mathscr{N}$ and $B_j \subset B(y_j, \delta/2)$.

(T_j) For $0 \le j \le n-1$, $\exists m_j > 0$ such that $g^{m_j}(B_j) \subset B_{j+1}$.

We will construct these objects backwards, by finite induction. It is enough to prove that given some recurrent point $y_{j+1} \in D_{j+1}$ together with a ball B_{j+1} satisfying (C_{j+1}), one can find a recurrent point $y_j \in D_j$, a ball B_j satisfying (C_j) and a positive m_j which satisfies (T_j).

3. Assume first that $x_{j+1} = \varphi(x_j)$, so $D_{j+1} = \varphi(D_j)$. By assumption, the point $y_{j+1} \in D_{j+1}$ is recurrent for φ, hence the point $y_j = \varphi^{-1}(y_{j+1})$ is in D_j and is recurrent for φ too. By (C_{j+1}), the ball B_{j+1} is centered at some $z_{j+1} \in W^-(y_{j+1})$. By our assumption on $W^-(\mathscr{A}_g) \cap \mathscr{N}$ and since g coincides with φ on $\mathscr{A}_g$, setting $z_j = g^{-1}(z_{j+1})$,

$$d(z_j, y_j) = d(g^{-1}(z_{j+1}), g^{-1}(y_{j+1})) < d(z_{j+1}, y_{j+1}) < \tfrac{\delta}{2}. \tag{89}$$

Therefore, by continuity of g, there exists a ball B_j centered at z_j and contained in $B(y_j, \delta/2)$ such that $g(B_j) \subset B_{j+1}$.

4. Assume now that $x_{j+1} = \psi(x_j)$, so that $D_{j+1} = \psi(D_j)$. Let R_j and R_{j+1} be the full-measure subsets of D_j and D_{j+1} formed by the recurrent points for φ. Since ψ is measure preserving, $R_{j+1} \cap \psi(R_j)$ is a full measure subset of D_{j+1}. Therefore, there exists a recurrent point $\bar{y}_j \in R_j$ such that $\bar{y}_{j+1} = \psi(\bar{y}_j)$ is recurrent, and so close to y_{j+1} that, by continuity of the unstable foliation, the leaf $W^-(\bar{y}_{j+1})$ intersects the ball B_{j+1}. By definition of ψ and by the last item in Lemma 7, the submanifold $\Delta = W^-(\bar{y}_j)$ intersects $W^+(\mathscr{A}_g)$ transversely in $\mathbb{A}^2$ at some point $\xi \in W^+(\bar{y}_{j+1})$. Apply the inclination lemma to Δ in the neighborhood of ξ, together with the positive recurrence property of $\bar{y}_{j+1}$: there exists an arbitrarily large integer m' such that $g^{m'}(\Delta)$ intersects B_{j+1}. Fix

$$z \in g^{m'}(\Delta) \cap B_{j+1}, \tag{90}$$

then

$$g^{-m'}(z) \in \Delta \subset W^-(\bar{y}_j). \tag{91}$$

Now, by definition of $W^-(\bar{y}_j)$ and since $\bar{y}_j$ is negatively recurrent, there is an (arbitrarily large) integer m'' such that

$$d(g^{-m''}(g^{-m'}(z)), g^{-m''}(\bar{y}_j)) < \delta/2 \quad \text{and} \quad g^{-m''}(\bar{y}_j) \in D_j. \tag{92}$$

Set $y_j = g^{-m''}(\bar{y}_j)$, so that y_j is recurrent and the point $z_j = g^{-(m''+m')}(z) \in W^-(y_j)$ satisfies

$$d(y_j, z_j) < \delta/2 \quad \text{and} \quad g^{(m''+m')}(z_j) = z \in B_{j+1}. \tag{93}$$

Hence by continuity there exists a ball B_j centered at z_j such that conditions (C_j) and (T_j) are satisfied.

5. As a consequence, there exists a sequence of integers $(m_i)_{1\leq\leq n}$ such that for $1 \leq i \leq n$

$$g^{m_i} \circ \cdots \circ g^{m_1}(B_0) \subset B_i.$$

By construction, any $z_0 \in B_0$ satisfies our statement. □

Appendix D. A global Hamiltonian flow-box theorem

We refer to [46] for the necessary definitions and results in symplectic geometry. The proof of the following global form of the Hamiltonian flow-box theorem is immediate.

Lemma 41. *Let (M^{2m}, Ω) be a symplectic manifold with Poisson bracket $\{\cdot,\cdot\}$, and fix a Hamiltonian $H \in C^\infty(M)$ with complete vector field X_H.*

- *Let Λ be a codimension 1 submanifold of M, transverse to X_H, such that there exists an open interval $I \subset \mathbb{R}$ with $0 \in I$, for which the restriction of Φ_H to $I \times \Lambda$ is an embedding. Set*

$$\mathscr{D} = \Phi_H(I \times \Lambda) \tag{94}$$

and let $T : \mathscr{D} \to \mathbb{R}$ be the transition time function defined by

$$\Phi_H(-T(x), x) \in \Lambda, \quad \forall x \in \mathscr{D}. \tag{95}$$

Then T is C^∞, $\{H, T\} = 1$ and $\Lambda = T^{-1}(0)$, so X_T is tangent to Λ.

• *Assume moreover that there exist an open interval J and $\boldsymbol{e}_0 \in J$ such that, setting*

$$\Lambda_{\boldsymbol{e}_0} = H^{-1}(\boldsymbol{e}_0) \cap \Lambda,$$

the flow of X_T is defined on $J \times \Lambda_{\boldsymbol{e}_0}$ and satisfies

$$\Lambda = \Phi_T(J \times \Lambda_{\boldsymbol{e}_0}). \tag{96}$$

Then the form $\Omega_{\boldsymbol{e}_0}$ induced by Ω on $\Lambda_{\boldsymbol{e}_0}$ is symplectic, and the map

$$\begin{aligned} \boldsymbol{\chi} : (I \times J) \times \Lambda_{\boldsymbol{e}_0} &\to \mathscr{D} \\ ((t, \boldsymbol{e}), x) &\mapsto \Phi_H(t, \Phi_T(\boldsymbol{e} - \boldsymbol{e}_0, x)) \end{aligned} \tag{97}$$

is a C^∞ symplectic diffeomorphism on its image, where $(I \times J) \times \Lambda_{\boldsymbol{e}_0}$ is equipped with the form

$$(d\boldsymbol{e} \wedge dt) \oplus \Omega_{\boldsymbol{e}_0}. \tag{98}$$

Moreover

$$H \circ \boldsymbol{\chi}((t, \boldsymbol{e}), x) = \boldsymbol{e}, \quad \forall (t, \boldsymbol{e}, x) \in (I \times J \times \Lambda_{\boldsymbol{e}_0}), \tag{99}$$

and

$$\boldsymbol{\chi}^*(X_H) = \frac{\partial}{\partial t}. \tag{100}$$

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References

[1] M.-C. Arnaud and J. Xue, "A C^1 Arnold–Liouville theorem", pp. 1–31 Astérisque **416**, Société Mathématique de France, Paris, 2020.

[2] V. Arnold, "First steps in symplectic topology", *Russian Mathematical Surveys* **41**:6 (1986), 1.

[3] S. Barbieri, J.-P. Marco, and J. E. Massetti, "Flexibility and analytic smoothing in averaging theory", preprint, 2021. arXiv 2105.08365

[4] P. Berger and A. Bounemoura, "A geometrical proof of the persistence of normally hyperbolic submanifolds", *Dyn. Syst.* **28**:4 (2013), 567–581.

[5] P. Bernard, "The dynamics of pseudographs in convex Hamiltonian systems", *J. Amer. Math. Soc.* **21**:3 (2008), 615–669.

[6] P. Bernard, "Large normally hyperbolic cylinders in a priori stable Hamiltonian systems", *Ann. Henri Poincaré* **11**:5 (2010), 929–942.

[7] P. Bernard, V. Kaloshin, and K. Zhang, "Arnold diffusion in arbitrary degrees of freedom and normally hyperbolic invariant cylinders", *Acta Math.* **217**:1 (2016), 1–79.

[8] S. Bolotin and D. Treschev, "Unbounded growth of energy in nonautonomous Hamiltonian systems", *Nonlinearity* **12**:2 (1999), 365–388.

[9] M. Chaperon, "The Lipschitzian core of some invariant manifold theorems", *Ergodic Theory Dynam. Systems* **28**:5 (2008), 1419–1441.

[10] C.-Q. Cheng, "The genericity of Arnold diffusion in nearly integrable Hamiltonian systems", *Asian J. Math.* **23**:3 (2019), 401–438.

[11] C.-Q. Cheng and J. Yan, "Arnold diffusion in Hamiltonian systems: a priori unstable case", *J. Differential Geom.* **82**:2 (2009), 229–277.

[12] A. Delshams and G. Huguet, "Geography of resonances and Arnold diffusion in a priori unstable Hamiltonian systems", *Nonlinearity* **22**:8 (2009), 1997–2077.

[13] A. Delshams and R. G. Schaefer, "Arnold diffusion for a complete family of perturbations", *Regul. Chaotic Dyn.* **22**:1 (2017), 78–108.

[14] A. Delshams, R. de la Llave, and T. M. Seara, "A geometric approach to the existence of orbits with unbounded energy in generic periodic perturbations by a potential of generic geodesic flows of $\mathbf{T}^2$", *Comm. Math. Phys.* **209**:2 (2000), 353–392.

[15] A. Delshams, R. de la Llave, and T. M. Seara, "A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model", **179**:844 (2006), viii+141 in.

[16] A. Delshams, R. de la Llave, and T. M. Seara, "Orbits of unbounded energy in quasi-periodic perturbations of geodesic flows", *Adv. Math.* **202**:1 (2006), 64–188.

[17] A. Delshams, R. de la Llave, and T. M. Seara, "Geometric properties of the scattering map of a normally hyperbolic invariant manifold", *Adv. Math.* **217**:3 (2008), 1096–1153.

[18] A. Fathi, "Weak KAM theorem in Lagrangian dynamics", preliminary version, number 10, 2008, tinyurl.com/FathiKAM.

[19] V. Gelfreich and D. Turaev, "Arnold diffusion in a priori chaotic symplectic maps", *Comm. Math. Phys.* **353**:2 (2017), 507–547.

[20] M. Gidea and R. de la Llave, "Topological methods in the instability problem of Hamiltonian systems", *Discrete Contin. Dyn. Syst.* **14**:2 (2006), 295–328.

[21] M. Gidea and J.-P. Marco, "Diffusion along chains of normally hyperbolic cylinders", preprint, 2017. arXiv 1708.08314

[22] M. Gidea and C. Robinson, "Diffusion along transition chains of invariant tori and Aubry–Mather sets", *Ergodic Theory Dynam. Systems* **33**:5 (2013), 1401–1449.

[23] M. Gidea, R. de la Llave, and T. M-Seara, "A general mechanism of diffusion in Hamiltonian systems: qualitative results", *Comm. Pure Appl. Math.* **73**:1 (2020), 150–209.

[24] M. Gidea, R. de la Llave, and T. M. Seara, "A general mechanism of instability in Hamiltonian systems: skipping along a normally hyperbolic invariant manifold", *Discrete Contin. Dyn. Syst.* **40**:12 (2020), 6795–6813.

[25] A. B. Givental, B. A. Khesin, J. E. Marsden, A. N. Varchenko, V. A. Vassiliev, O. Y. Viro, and V. M. Zakalyukin (editors), *Instability of dynamical systems with several degrees of freedom*, Springer, Berlin, Heidelberg, 2009.

[26] M. Gromov, "Pseudo holomorphic curves in symplectic manifolds", *Invent. Math.* **82**:2 (1985), 307–347.

[27] G. Haller, "Diffusion at intersecting resonances in Hamiltonian systems", *Phys. Lett. A* **200**:1 (1995), 34–42.

[28] G. Haller, "Universal homoclinic bifurcations and chaos near double resonances", *J. Statist. Phys.* **86**:5-6 (1997), 1011–1051.

[29] M.-R. Herman, *Sur les courbes invariantes par les difféomorphismes de l'anneau*, *Vol.* 1, Astérisque **103**, Société Mathématique de France, Paris, 1983. With an appendix by Albert Fathi.

[30] M.-R. Herman, *Sur les courbes invariantes par les difféomorphismes de l'anneau*, *Vol.* 2, Astérisque **144**, Société Mathématique de France, Paris, 1986.

[31] M. W. Hirsch, C. C. Pugh, and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics **583**, Springer, 1977.

[32] V. Kaloshin and M. Saprykina, "An example of a nearly integrable Hamiltonian system with a trajectory dense in a set of maximal Hausdorff dimension", *Comm. Math. Phys.* **315**:3 (2012), 643–697.

[33] V. Kaloshin and K. Zhang, *Arnold diffusion for smooth systems of two and a half degrees of freedom*, Annals of Mathematics Studies **208**, Princeton University Press, 2020.

[34] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications **54**, Cambridge University Press, 1995.

[35] A. Koropecki and M. Nassiri, "Transitivity of generic semigroups of area-preserving surface diffeomorphisms", *Math. Z.* **266**:3 (2010), 707–718.

[36] A. Koropecki and M. Nassiri, "Erratum to: Transitivity of generic semigroups of area-preserving surface diffeomorphisms [MR2719428]", *Math. Z.* **268**:1-2 (2011), 601–604.

[37] F. Lalonde and J.-C. Sikorav, "Sous-variétés lagrangiennes et lagrangiennes exactes des fibrés cotangents", *Comment. Math. Helv.* **66**:1 (1991), 18–33.

[38] F. Laudenbach and J.-C. Sikorav, "Persistance d'intersection avec la section nulle au cours d'une isotopie hamiltonienne dans un fibré cotangent", *Invent. Math.* **82**:2 (1985), 349–357.

[39] P. Le Calvez, "Propriétés dynamiques des régions d'instabilité", *Ann. Sci. École Norm. Sup.* (4) **20**:3 (1987), 443–464.

[40] P. Le Calvez, "Propriétés dynamiques des difféomorphismes de l'anneau et du tore", *Astérisque* 204 (1991), 131.

[41] P. Le Calvez, "Drift orbits for families of twist maps of the annulus", *Ergodic Theory Dynam. Systems* **27**:3 (2007), 869–879.

[42] V. Mandorino, "Generic transitivity for couples of Hamiltonians", *J. Dyn. Control Syst.* **19**:4 (2013), 593–610.

[43] J.-P. Marco, "Arnold diffusion for cusp-generic nearly integrable convex systems on $\mathbb{A}^3$", preprint, 2016. arXiv 1602.02403

[44] J.-P. Marco, "Chains of compact cylinders for cusp-generic nearly integrable convex systems on $\mathbb{A}^3$", preprint, 2016. arXiv 1602.02399

[45] J.-P. Marco, "Twist maps and Arnold diffusion for diffeomorphisms", pp. 473–495 in *Variational methods*, edited by M. Bergounioux et al., Radon Ser. Comput. Appl. Math. **18**, De Gruyter, Berlin, 2017.

[46] D. McDuff and D. Salamon, *Introduction to symplectic topology*, Oxford Mathematical Monographs, Oxford University Press, New York, 1995.

[47] D. N. Mèzer, "Arnold diffusion, I: Announcement of results", *Sovrem. Mat. Fundam. Napravl.* **2** (2003), 116–130.

[48] R. Moeckel, "Generic drift on Cantor sets of annuli", pp. 163–171 in *Celestial mechanics* (Evanston, IL, 1999), edited by A. Chenciner et al., Contemp. Math. **292**, Amer. Math. Soc., Providence, RI, 2002.

[49] M. Nassiri and E. R. Pujals, "Robust transitivity in Hamiltonian dynamics", *Ann. Sci. Éc. Norm. Supér.* (4) **45**:2 (2012), 191–239.

[50] R. C. Robinson, "Generic properties of conservative systems", *Amer. J. Math.* **92** (1970), 562–603.

[51] R. C. Robinson, "Generic properties of conservative systems, II", *Amer. J. Math.* **92** (1970), 897–906.

[52] L. Sabbagh, "An inclination lemma for normally hyperbolic manifolds with an application to diffusion", *Ergodic Theory Dynam. Systems* **35**:7 (2015), 2269–2291.

[53] D. Treschev, "Evolution of slow variables in a priori unstable Hamiltonian systems", *Nonlinearity* **17**:5 (2004), 1803–1841.

[54] J.-C. Yoccoz, "Travaux de Herman sur les tores invariants", exposé 754, pp. 311–344 in *Séminaire Bourbaki*, 1991/92, Astérisque **206**, Société Mathématique de France, Paris, 1992.

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