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Viscosity solutions of the Hamilton–Jacobi equation on a noncompact manifold

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We study the continuous viscosity solutions of the evolutionary Hamilton–Jacobi equation

$$\partial_t U(t, x) + H(x, \partial_x U(t, x)) = 0,$$

on $[0, +\infty[\times M$, where H is a Tonelli Hamiltonian on the noncompact manifold M. We establish that all such solutions are given by the Lax–Oleinik formula. Moreover, we show that a finite everywhere Lax–Oleinik evolution is necessarily continuous and a viscosity solution on $]0, +\infty[\times M$.

The goal is also to provide a convenient reference for the evolutionary Hamilton–Jacobi equation for Tonelli Hamiltonians on noncompact manifolds.

1. Introduction

This work was started in February 2017 in Rome, following a conversation with Piermarco Cannarsa, Andrea Davini, Antonio Siconolfi and Afonso Sorrentino. We discussed the problem of the Lax–Oleinik evolution $\hat{u}$ (see Definition 8.2) of a continuous function u on a noncompact manifold. Although on a compact manifold, it was known that the Lax–Oleinik evolution of a continuous function is always locally concave and a solution of the Hamilton–Jacobi equation in evolution form, at that moment, the situation on a noncompact manifold was not clear, even assuming the continuity of the Lax–Oleinik evolution. The main problem was that it was not clear that the inf in Definition 8.2 of $\hat{u}$ was attained. After about a month, to my astonishment, I realized that no condition beyond finiteness was necessary; see Theorem 1.1.

This brought back the problem of uniqueness of a solution of the Hamilton–Jacobi equation in evolution form given an initial condition. In May 2016 in

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Shanghai, Kaizhi Wang, Lin Wang and Jun Yan, while discussing [11], brought to my attention that, contrary to my belief, the uniqueness of a continuous solution of the Hamilton–Jacobi equation in evolution form given an initial condition on a noncompact manifold was not known at that time (and therefore the Lax–Oleinik formula could not be established) unless the solution was uniformly continuous. The best results on this problem were those contained, for example, in Hitoshi Ishii's lecture notes [10], on whose methods this present work heavily relies. The difficulty here is that the maximum principle could not be applied directly, since it requires some compactness. In 2018 and 2019, I was able to show directly the Lax–Oleinik formula for arbitrary continuous solutions (see Theorem 1.2) and therefore I obtained the uniqueness as a consequence.

Beyond the new results mentioned above, the goal of this work is to provide a convenient reference for the *evolutionary* Hamilton–Jacobi equation

$$\partial_t U + H(x, \partial_x U) = 0$$

for a Tonelli Hamiltonian H on a *possibly noncompact* manifold, thus extending the results of the survey [7].

We will assume that the reader is familiar with [7], which is well adapted to our manifold setting. Other classic treatments of viscosity solutions of the Hamilton–Jacobi equation are [2; 1].

We consider a ***connected*** manifold M endowed with a ***complete*** Riemannian metric. We will denote by $\|\cdot\|_x$ the induced norm on either $T_x M$ or $T_x^* M$, the fibers above x of the tangent TM or cotangent T^*M bundle of M. We will denote by d the Riemannian distance on M obtained from the Riemannian metric. It might be useful to recall that, due to the completeness of the Riemannian metric, bounded sets for d are relatively compact. Therefore the distance d is also complete.

We endow $\mathbb{R} \times M$, $\mathbb{R} \times M \times M$, and $M \times M$ with the product Riemannian metrics, and Riemannian distances, where the Riemannian metric on $\mathbb{R}$ is the usual one.

Throughout the paper $H : T^*M \to \mathbb{R}$ will denote a continuous function which we will call the Hamiltonian.

We will study (continuous) viscosity subsolutions, supersolutions and solutions of the evolutionary Hamilton–Jacobi equation

$$\partial_t U(t, x) + H(x, \partial_x U(t, x)) = 0, \tag{1-1}$$

on a subset of $\mathbb{R} \times M$.

In fact, the main results of this work will be proved for Tonelli Hamiltonians (Definition 3.1). The statements use the (negative) Lax–Oleinik semigroup T_t^-, $t \geq 0$ (see Definition 8.1).

The main results are given in the next two theorems.

Theorem 1.1. *Assume $u : M \to [-\infty, \infty]$ is a function such that its Lax–Oleinik evolution $\hat{u} : [0, +\infty[\times M \to [-\infty, +\infty]$, $(t, x) \mapsto T_t^- u(x)$ is finite at some point (T, X), with $T > 0$ and $X \in M$. Then the function $\hat{u}$ is continuous and even locally semiconcave on $]0, T[\times M$. Moreover, the function $\hat{u}$ is a viscosity solution of the evolutionary Hamilton–Jacobi equation (1-1) on $]0, T[\times M$.*

Note that we do not assume any continuity property on u. As we already said the result above is surprising, even when u is continuous.

Theorem 1.2. *Suppose $H : T^*M \to \mathbb{R}$ is a Tonelli Hamiltonian. Assume that, for some $T > 0$ the function $U :]0, T[\times M \to \mathbb{R}$ is a continuous viscosity solution of the evolutionary Hamilton–Jacobi equation (1-1). Define $u : M \to [-\infty, \infty]$ by*

$$u(x) = \liminf_{t \to 0} U(t, x).$$

Then $U = \hat{u}$ on $]0, T[\times M \to \mathbb{R}$, where $\hat{u} : [0, +\infty[\times M \to [-\infty, +\infty]$, $(t, x) \mapsto T_t^- u(x)$ is the Lax–Oleinik evolution of u.

Obviously, Theorem 1.2 implies that continuous viscosity solutions of the evolutionary Hamilton–Jacobi equation (1-1) satisfy the Lax–Oleinik formula and also the uniqueness given a continuous boundary condition on $\{0\} \times M$.

Remark 1.3. (1) Discussions in June 2019 in Rome, with A. Davini, Hitoshi Ishii and Antonio Siconolfi pointed to the fact that the results above hold true even if H is not C^2, but still satisfies the other Tonelli conditions see 3.1.

(2) The method of this work does not allow to extend the results to the case when H is time-dependent. For example, the proof of Proposition 2.2 is not adaptable to the time-dependent case.

2. Approximation by Lipschitz subsolutions

We will assume in this section that $H : T^*M \to \mathbb{R}$ is a continuous function, which we will call the Hamiltonian. Our goal is to show that we can approximate locally continuous viscosity subsolutions of the evolutionary Hamilton–Jacobi equation (1-1) with U defined on an open subset of $\mathbb{R} \times M$ by Lipschitz viscosity subsolutions, under a coercivity condition on H.

These results are well-known when M is the Euclidean space (see Hitoshi Ishii's lectures [10] for example), but the arguments in [10] can easily be adapted to the manifold setting as we now proceed to do.

2.1. *Sup-convolution in one variable.* The usefulness of sup-convolution to improve regularity of viscosity subsolutions is already well established. As said above, our treatment in this section follows closely [10] which dealt with the Euclidean space case.

Let $u : V \to \mathbb{R}$, be a continuous function, where V is an open subset of $\mathbb{R} \times M$. Assume $K \subset V$ is compact subset. By continuity of u and compactness of K, we can find an open subset $O_1 \subset V$, with $K \subset O_1$, such that

$$m = \sup_{O_1} |u| < +\infty. \tag{2-1}$$

Again by compactness of K, we can find $\delta > 0$ and an open neighborhood $O_2 \subset O_1$ of K, with compact closure $\bar{O}_2 \subset O_1$, and such that $[t-\delta, t+\delta] \times \{x\} \subset O_1$, for every $(t,x) \in \bar{O}_2$.

For $\epsilon > 0$, we define $u_\epsilon : \bar{O}_2 \to \mathbb{R}$ by

$$u_\epsilon(t,x) = \max_{s \in [-\delta, +\delta]} u(t+s, x) - \frac{s^2}{\epsilon}. \tag{2-2}$$

Note that u_ϵ is continuous by continuity of u and compactness of $[-\delta, +\delta]$.

We summarize the properties of u_ϵ in the following proposition.

Proposition 2.1. (1) *For every $\epsilon > 0$, we have $u_\epsilon \geq u$.*

(2) *For every $0 < \epsilon < \epsilon'$, we have $u_\epsilon < u_{\epsilon'}$.*

(3) *If $(t,x) \in \bar{O}_2$, and $s_\epsilon \in [-\delta, +\delta]$ is such that $u_\epsilon(t,x) = u(t+s_\epsilon, x) - (s_\epsilon)^2/\epsilon$, then $|s_\epsilon| \leq \sqrt{2\epsilon m}$, where m is given by (2-1).*

(4) *For every $(t,x) \in \bar{O}_2$, we have $u_\epsilon(t,x) \to u(t,x)$, when $\epsilon \to 0$. The convergence is uniform on $\bar{O}_2$.*

(5) *If $\sqrt{2\epsilon m} < \delta$, for each $(t,x), (t',x) \in \bar{O}_2$, with $|t-t'| < \delta - \sqrt{2\epsilon m}$, we have*

$$|u_\epsilon(t',x) - u_\epsilon(t,x)| \leq \frac{2\sqrt{2\epsilon m} + |t-t'|}{\epsilon}|t-t'| \leq \frac{\sqrt{2\epsilon m} + \delta}{\epsilon}|t-t'|.$$

Moreover, if $\sqrt{2\epsilon m} < \delta$, for every $x \in M$, the map $t \mapsto u_\epsilon(t,x)$ is Lipschitz on every connected component of $O_2 \cap \{x\} \times \mathbb{R}$ with Lipschitz constant $\leq 2\sqrt{2m/\epsilon}$.

Proof. Parts (1) and (2) are obvious. For part (3), we notice that

$$u_\epsilon(t,x) = u(t+s_\epsilon, x) - (s_\epsilon)^2/\epsilon \geq u(t,x).$$

Therefore

$$(s_\epsilon)^2/\epsilon \leq u(t+s_\epsilon, x) - u(t,x) \leq 2\sup_{O_1} |u| = 2m.$$

For part (4), note that by part (3), we have

$$\sup_{(x,t)\in\bar{O}_2} |u_\epsilon(t,x)-u(t,x)| \le \sup\Big\{|u(t+s,x)-u(t,x)| \;\Big|\; (t,x)\in\bar{O}_2,\ |s|\le\sqrt{2\epsilon m}\Big\}.$$

By compactness of $\bar{O}_2$ and continuity of u, the right hand side of the inequality above tends uniformly on $\bar{O}_2$ to 0 as $\epsilon \to 0$.

For (5), we choose s_ϵ such that $u_\epsilon(t,x) = u(t+s_\epsilon, x) - (s_\epsilon)^2/\epsilon$. By (3), we have $|s_\epsilon| \le \sqrt{2\epsilon m}$. Therefore, we get

$$|s_\epsilon + t - t'| \le |s_\epsilon| + |t-t'| \le \sqrt{2\epsilon m} + \delta - \sqrt{2\epsilon m} = \delta.$$

Therefore, by the definition of u_ϵ, we obtain

$$u_\epsilon(t',x) \ge u(t'+(s_\epsilon+t-t'),x) - \frac{(s_\epsilon+t-t')^2}{\epsilon} = u(t+s_\epsilon,x) - \frac{(s_\epsilon+t-t')^2}{\epsilon}.$$

Subtracting this inequality from the equality $u_\epsilon(t,x) = u(t+s_\epsilon,x) - (s_\epsilon)^2/\epsilon$ yields

$$\begin{aligned} u_\epsilon(t,x) - u_\epsilon(t',x) &\le \frac{(s_\epsilon+t-t')^2}{\epsilon} - \frac{(s_\epsilon)^2}{\epsilon} \\ &= \frac{(2s_\epsilon+t-t')(t-t')}{\epsilon} \\ &\le \frac{2|s_\epsilon|+|t-t'|}{\epsilon}|t-t'| \\ &\le \frac{2\sqrt{2\epsilon m}+|t-t'|}{\epsilon}|t-t'|, \end{aligned}$$

where we used $|s_\epsilon| \le \sqrt{2\epsilon m}$, for the last inequality. By symmetry, we obtain

$$|u_\epsilon(t',x) - u_\epsilon(t,x)| \le \frac{\sqrt{2\epsilon m}+|t-t'|}{\epsilon}|t-t'|. \tag{2-3}$$

Assume t, t', x are such that $[t,t']\times\{x\} \subset O_2$. For every $\eta \in\,]0, \delta-\sqrt{2\epsilon m}[$, we can pick a monotone sequence $t = t_0, t_1, \dots, t_n = t'$, with $|t_{i+1}-t_i| \le \eta$, by applying (2-3) for t_i, t_{i+1} instead of t, t', and adding the inequalities, we obtain

$$|u_\epsilon(t',x) - u_\epsilon(t,x)| \le \frac{2\sqrt{2\epsilon m}+\eta}{\epsilon}|t-t'|.$$

We can then let $\eta \to 0$, to conclude that

$$|u_\epsilon(t',x) - u_\epsilon(t,x)| \le \frac{2\sqrt{2\epsilon m}}{\epsilon}|t-t'|. \qquad \square$$

Proposition 2.2. *Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian. Suppose $u : V \to \mathbb{R}$ is a continuous function, defined on the open subset $V \subset \mathbb{R}\times M$,* *which is a viscosity subsolution on V of the evolutionary Hamilton–Jacobi equation (1-1).*

Then, for every compact subset $K \subset V$, we can find a sequence of continuous functions $\hat{u}_n : K \to \mathbb{R}$ such that $\hat{u}_n \to u$ uniformly on K and, for all n except a finite number, the function $\hat{u}_n$ is a viscosity subsolution on the interior $\mathring{K}$ of K, not only of the evolutionary Hamilton–Jacobi equation (1-1), but also of

$$|\partial_t u(t,x)| + H(x, \partial_x u(t,x)) = C\sqrt{n}, \tag{2-4}$$

for some $C < +\infty$ independent of n. In particular, if H is coercive above each compact subset of M, then each $\hat{u}_n$ is locally Lipschitz on $\mathring{K}$.

Proof. We choose O_1, m, δ, and $O_2 \supset K$ like it is done above in the beginning of Proposition 2.1. We set $\hat{u}_n = u_{1/n} : O_2 \to \mathbb{R}$, where $u_{1/n}$ is defined by (2-2) with $\epsilon = 1/n$. Hence

$$\hat{u}_n(t,x) = \min_{s \in [-\delta, +\delta]} u(x, +s) - ns^2.$$

By part (4) of Proposition 2.1, we get the uniform convergence of $\hat{u}_n$ to u.

We pick an integer n_0 such that $\sqrt{2m/n_0} < \delta$. We now check the fact that $\hat{u}_n$ is a viscosity subsolution of both Hamilton–Jacobi equations on O_2, for all $n \geq n_0$. Assume $(t_0, x_0) \in O_2$, and that $\varphi : V \to \mathbb{R}$ is C^1 is such that $\hat{u}_n \leq \varphi$ with equality at (t_0, x_0). Since $\sqrt{2m/n} \leq \sqrt{2m/n_0} < \delta$, by Proposition 2.1(5), we know that $t \mapsto \hat{u}_n(x_, t)$ is locally Lipschitz with local Lipschitz constant $\leq 2\sqrt{2mn}$. This implies

$$|\partial_t \varphi(t_0, x_0)| \leq 2\sqrt{2mn}. \tag{2-5}$$

We now choose $s_n \in [-\delta, +\delta]$ such that

$$u(t_0 + s_n, x_0) - ns_n^2 = \hat{u}_n(t_0, x_0) = \varphi(t_0, x_0).$$

For s small enough and y close to x_0, we have $(t_0 + s, y) \in O_2$. Therefore, since $s_n \in [-\delta, +\delta]$, by the definition of $\hat{u}_n$, for s small enough and y close to x_0, we get

$$u(t_0 + s + s_n, y) - ns_n^2 \leq \hat{u}_n(t_0 + s, y) \leq \varphi(t_0 + s, y).$$

Subtracting from this inequality the equality $u(t_0 + s_n, x_0) - ns_n^2 = \varphi(t_0, x_0)$, we get

$$u(y, t_0 + s + s_n) - u(t_0 + s_n, x_0) \leq \varphi(t_0 + s, y) - \varphi(t_0, x_0).$$

Since u is a viscosity subsolution on $O_1 \ni (t_0 + s_n, x_0)$, of (1-1), we must have

$$\partial_t \varphi(t_0, x_0) + H(x_0, \partial_x \varphi(t_0, x_0)) \leq 0. \tag{2-6}$$

Therefore $\hat{u}_n$ is a viscosity subsolution of (1-1). Using the inequalities (2-5) and (2-6), we also obtain

$$|\partial_t \varphi(t_0, x_0)| + H(x_0, \partial_x \varphi(t_0, x_0)) \leq 4\sqrt{2mn}.$$

Therefore $\hat{u}_n$ is a viscosity solution of (2-4) with $C = 4\sqrt{2m}$. □

Corollary 2.3. *Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian that is coercive above each compact subset of M and convex in the momentum p; i.e., for each $x \in M$, the map $T_x^*M \to \mathbb{R}$, $p \mapsto H(x, p)$ is convex. Let $u : V \to \mathbb{R}$ be a continuous functions defined on the open subset $V \subset \mathbb{R} \times M$ which is a viscosity subsolution of the evolutionary Hamilton–Jacobi equation (1-1).*

For every open set $V' \subset V$ whose closure $\bar{V}'$ is compact and contained in V, we can approximate uniformly u on V' by a C^∞ subsolution of the evolutionary Hamilton–Jacobi equation (1-1).

Proof. By Proposition 2.2 above, we can make a first approximation by a subsolution $u_1 : V' \to \mathbb{R}$ of (1-1) that is locally Lipschitz on V'. The function $u_2 : V' \to \mathbb{R}$, $(t, x) \to u_1(t, x) - \epsilon t$ is therefore a locally Lipschitz viscosity subsolution of

$$\partial_t v + H(x, \partial_x v) = -\epsilon.$$

Note also that the variable t is bounded on the compact subset $\bar{V}'$ of $\mathbb{R} \times M$. Therefore, by choosing appropriately ϵ, we can assume u_2 uniformly as close to u_1 as we wish. We can now consider the Hamiltonian $\bar{H} : T^*(\mathbb{R} \times M)$ defined by

$$\bar{H}(t, s, x, p) = s + H(x, p),$$

where we use the identification $T^*(\mathbb{R} \times M) = T^*\mathbb{R} \times T^*M = \mathbb{R} \times \mathbb{R} \times T^*M$. The function u_2 is a locally Lipschitz viscosity subsolution of

$$\bar{H}(t, x, d_{(t,x)} v(t, x)) = -\epsilon.$$

The Hamiltonian $\bar{H}$ is convex in the momentum (s, p). We can now invoke [7, Theorem 10.6, page 1219] to approximate uniformly u_2 on V' by a C^∞ viscosity subsolution $u_3 : V' \to \mathbb{R}$ of

$$\bar{H}(t, x, d_{(t,x)} v u(t, x)) = 0.$$

This means that u_3 is both a uniform approximation of u and a viscosity subsolution of the evolutionary Hamilton–Jacobi equation (1-1). □

Corollary 2.4. *Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian that is coercive above each compact subset of M and convex in the momentum p; i.e., for each $x \in M$, the map $T_x^*M \to \mathbb{R}$, $p \mapsto H(x, p)$ is convex. If $u_1 : V \to \mathbb{R}$ and*

$u_2 : V \to \mathbb{R}$ *are continuous functions defined on the open subset* $V \subset \mathbb{R} \times M$ *which are viscosity subsolutions of*

$$\partial_t v + H(x, \partial_x v) = 0, \tag{2-7}$$

then $u = \min(u_1, u_2)$ *is also a viscosity subsolution on* V *of* (2-7).

Proof. Since H is convex, the corollary is well known when u_1 and u_2 are locally Lipschitz. In fact, since u_1 and u_2 are locally Lipschitz, they are differentiable almost everywhere and satisfy for almost every $(t, x) \in V$, the inequalities

$$\partial_t u_1(t, x) + H(x, \partial_x u_1(t, x)) \le 0 \text{ and } \partial_t u_2(t, x) + H(x, \partial_x u_2(t, x)) \le 0.$$

But the subset $D \subset V$ where the three locally Lipschitz functions u, u_1, u_2 are differentiable is of full measure and, for every $(t, x) \in D$, we have either $d_{(t,x)}u = d_{(t,x)}u_1$ or $d_{(t,x)}u = d_{(t,x)}u_2$. Therefore, $\partial_t u(t, x) + H(x, \partial_x u_(t, x)) = 0$ almost everywhere on V. Since the Hamiltonian $\bar{H}(t, x, s, p) = s + H(x, p)$ is convex in (t, p) by [7, Theorem 10.2, page 1217], we conclude that $u = \min(u_1, u_2)$ is also a viscosity subsolution on V of (2-7), when both u_1 and u_2 are locally Lipschitz.

The result for general continuous functions follows from this locally Lipschitz case and the stability of viscosity solutions (see [7, Theorem 6.1, page 1209], for example) using the approximation result obtained in Proposition 2.2. □

Corollary 2.5. *Let* $H : T^*M \to \mathbb{R}$ *be a continuous Hamiltonian that is coercive above each compact subset of* M *and convex in the momentum* p; *i.e., for each* $x \in M$, *the map* $T_x^*M \to \mathbb{R}$, $p \mapsto H(x, p)$ *is convex. Suppose the family of functions* $u_i : V \to M$, $i \in I$, *where* $V \subset \mathbb{R} \times M$ *is an open subset, is such that its infimum* $u = \inf_{i \in I} u_i$ *is continuous and everywhere finite on* V. *If each* u_i, $i \in I$ *is a viscosity subsolution* (*resp. solution*) *of the evolutionary Hamilton–Jacobi*

$$\partial_t v + H(x, \partial_x v) = 0. \tag{2-8}$$

on V, *then* u *is also a viscosity subsolution* (*resp. solution*) *of the evolutionary Hamilton–Jacobi* (2-8) *on* V.

Proof. Since the space of continuous functions $\mathcal{C}(V, \mathbb{R})$ endowed with the compact-open topology is metric and separable, we can find a sequence $(i_n)_{n \in \mathbb{N}}$, with $i_n \in I$, such that the sequence $(u_{i_n})_{n \in \mathbb{N}}$ is dense in the subset $\{u_i \mid i \in I\} \subset \mathcal{C}(V, \mathbb{R})$ for the compact open topology. Therefore

$$u = \inf_{i \in I} u_i = \inf_{n \in \mathbb{N}} u_{i_n}.$$

For $m \in \mathbb{N}$, let us define $U_m : V \to \mathbb{R}$ by

$$U_m = \min_{0 \le n \le m} u_{i_n}.$$

If each u_i, $i \in I$ is a viscosity subsolution of (2-8) on V, Corollary 2.4 implies that each U_m is also a subsolution of (2-8) on V. Note that U_m is nonincreasing in m and $U_m \searrow u$. Since we are assuming that u is finite and continuous on V, by Dini's theorem, the nonincreasing convergence $U_m \searrow u$ is uniform on every compact subset of V. Therefore by the stability theorem for viscosity solutions, the function u is a viscosity solution of (1-1) on V.

If each u_i, $i \in I$ is a viscosity solution of (2-8) on V, then $u = \inf_{i\in I} u_i$ is a supersolution of (2-8) on V; see for example [7, Proposition 8.1, page 1213]. □

2.2. *Maximum principle.*

Theorem 2.6 (maximum principle). *Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian that is continuous, coercive above each compact subset of M and convex in the momentum p. For $a < b \in \mathbb{R}$ and $K \subset M$ a compact subset, if the continuous functions $u, v : [a, b] \times K \to \mathbb{R}$ are respectively a subsolution and a supersolution of the evolutionary Hamilton–Jacobi equation (1-1) on $]a, b[\times \mathring{K}$ then the maximum of $u - v$ on $[a, b] \times K$ is achieved on $[a, b] \times \partial K \cup \{a\} \times K$. Therefore*

$$\max_{[a,b]\times K} u - v = \max_{[a,b]\times\partial K\cup\{a\}\times K} u - v.$$

Proof. It is not difficult to see that by the approximation result of Proposition 2.2, we can assume u locally Lipschitz in $\mathring{K} \times]a, b[$. As usual, for $\epsilon, \delta > 0$, we introduce the function $u_{\epsilon,\delta} : [a, b[\times K \to \mathbb{R}$ by

$$u_{\epsilon,\delta}(t, x) = u(t, x) - \epsilon(t - a) - \frac{\delta}{b - t}.$$

Note that $u_{\epsilon,\delta} \le u$ and that $u_{\epsilon,\delta}(t, x) \to -\infty$ as $t \to b$, uniformly in $x \in K$. Since $t \mapsto -\epsilon(t - a) - \delta/(b - t)$ is C^1, with derivative $t \mapsto -\epsilon - \delta/(b - t)^2 \le -\epsilon$, the function $u_{\epsilon,\delta}$ is a viscosity subsolution of

$$\partial_t u_{\epsilon,\delta} + H(x, \partial_x u_{\epsilon,\delta}) = -\epsilon, \tag{2-9}$$

on $]a, b[\times \mathring{K}$. Therefore by the doubling of variables argument (see [7, Theorem 7.1, page 1210], for example), using that $u_{\epsilon,\delta}$ is locally Lipschitz on $]a, b[\times \mathring{K}$, we conclude that $u_{\epsilon,\delta} - v$ cannot have a local maximum in $]a, b[\times \mathring{K}$. Since $u_{\epsilon,\delta}(t, x) \to -\infty$ as $t \to b$, the function $u_{\epsilon,\delta} - v$ attains its maximum at a point in $[a, b[\times \partial K \cup \{a\} \times K$. Using that $u_{\epsilon,\delta} \le u$, we obtain

$$u_{\epsilon,\delta} - v \le \max_{[a,b]\times\partial K\cup\{a\}\times K} u - v$$

on $K \times [a, b[$. Letting $\delta, \epsilon \to 0$, we obtain $u - v \le \max_{[a,b]\times\partial K\cup\{a\}\times K} u - v$ on $K \times [a, b[$. Continuity of both u and v yields

$$\max_{K\times[a,b]} u - v \le \max_{[a,b]\times\partial K\cup\{a\}\times K} u - v.$$

□

For viscosity solutions, we obtain:

Corollary 2.7. *Let $H : T^*M \to \mathbb{R}$, $(x, p) \mapsto H(x, p)$ Hamiltonian that is continuous, coercive above each compact subset of M and convex in the momentum p. For $a < b \in \mathbb{R}$ and $K \subset M$ a compact subset, assume that the two continuous functions $u, v : [a, b] \times K \to \mathbb{R}$ are viscosity solutions of the evolutionary Hamilton–Jacobi equation (1-1) on $]a, b[\times \mathring{K}$. If $u = v$ on $[a, b] \times \partial K \cup \{a\} \times K$, then $u = v$ on $[a, b] \times K$.*

3. Tonelli Hamiltonians and their Lagrangians

Definition 3.1. A Tonelli Hamiltonian H on the complete Riemannian manifold (M, g) is a function $H : T^*M \to \mathbb{R}$ satisfying the following conditions:

(1*) The function H is C^2.

(2*) (uniform superlinearity) For every $K \geq 0$, we have

$$C^*(K) = \sup_{(x,p)\in T^*M} K\|p\|_x - H(x, p) < \infty.$$

(3*) (uniform boundedness in the fibers) For every $R \geq 0$, we have

$$A^*(R) = \sup\{H(x, p) \mid \|p\| \leq R\} < +\infty.$$

(4*) (C^2 strict convexity in the fibers) For every $(x, p) \in T^*M$, the second derivative along the fibers, $\partial^2 H/\partial p^2(x, p)$, is (strictly) positive definite.

Note that both A^* and C^* are nondecreasing functions, and that (2*) implies

$$\forall (x, p) \in T^*M,\ H(x, p) \geq K\|p\| - C^*(K).$$

If M is compact, the third condition is automatically satisfied, and the second condition is equivalent to

$$\frac{H(x, p)}{\|p\|_x} \to +\infty \quad \text{as } \|p\|_x \to +\infty.$$

We thus recover the usual definition of a Tonelli Hamiltonian in the case of M compact.

We note that the uniform superlinearity implies that a Tonelli Hamiltonian is coercive.

We should emphasize that, in the noncompact case, the Tonelli condition depends on the choice of the complete Riemannian metric on M.

Example 3.2. (1) The easiest example of a Tonelli Hamiltonian is $H_0 : T^*M \to \mathbb{R}$ defined by

$$H_0(x, p) = \tfrac{1}{2}\|p\|_x^2.$$

In fact, in this case,

$$A_0^*(R) = \sup\{H_0(x, p) \mid \|p\|_x \le R\} = \tfrac{1}{2}R^2,$$
$$C_0^*(K) = \sup_{(x,p)\in T^*M} K\|p\|_x - H_0(x, p) = \sup_{(x,p)\in T^*M} K\|p\|_x - \tfrac{1}{2}\|p\|_x^2 = \tfrac{1}{2}K^2.$$

(2) Let $V : M \to \mathbb{R}$ be a C^2 function and let $X : M \to TM$ be a C^2 vector field on M. We define the Hamiltonian $H_{X,V} : T^*M \to \mathbb{R}$ by

$$H_{X,V}(x, p) = \tfrac{1}{2}\|p\|_x^2 + p(X(x)) + V(x).$$

For every $x \in M$, we have

$$\sup_{\substack{p\in T_x^*M\\ \|p\|_x=R}} H_{X,V}(x, p) = \tfrac{1}{2}R^2 + R\|X(x)\|_x + V(x).$$

Therefore

$$A_{X,V}^*(R) = \tfrac{1}{2}R^2 + \sup_{x\in M}(R\|X(x)\|_x + V(x)).$$

In particular, we get

$$A_{X,V}^*(0) = \sup_{x\in M} V(x) \text{ and } \sup_{x\in M}\|X(x)\|_x + \inf_{x\in M} V(x) \le A_{X,V}^*(1).$$

For every $x \in M$, we have

$$\begin{aligned}\sup_{\substack{p\in T_x^*M\\ \|p\|_x=R}} K\|p\|_x - H_{X,V}(x, p) &= \sup_{\substack{p\in T_x^*M\\ \|p\|_x=R}} K\|p\|_x - \tfrac{1}{2}\|p\|_x^2 - p(X(x)) - V(x)\\ &= KR - \tfrac{1}{2}R^2 + R\|X(x)\|_x - V(x).\end{aligned}$$

Therefore, for every $x \in M$, we have

$$\sup_{p\in T_x^*M} K\|p\|_x - H_{X,V}(x, p) = \tfrac{1}{2}(K + \|X(x)\|_x)^2 - V(x),$$

and

$$C_{X,V}^*(K) = \sup_{x\in M} K\|p\|_x - H_{X,V}(x, p) = \sup_{x\in M} \tfrac{1}{2}(K + \|X(x)\|_x^2) - V(x).$$

In particular, we get $-\inf_{x\in M} V(x) \le C_{X,V}^*(0)$. Therefore, the Hamiltonian $H_{X,V}$ is Tonelli if and only if $\|V\|_\infty = \sup_{x\in M} |V(x)| < +\infty$ and $\|X\|_\infty = \sup_{x\in M}\|X(x)\|_x < +\infty$.

In the sequel, we will assume that $H : T^*M \to \mathbb{R}$ is a Tonelli Hamiltonian on the complete Riemannian manifold M. We now need to introduce the (Tonelli)

Lagrangian $L : TM \to \mathbb{R}$ associated to the Hamiltonian H. It is defined by the Fenchel formula

$$L(x, v) = \sup_{p \in T_x^*M} p(v) - H(x, p) \tag{3-1}$$

Since H is Tonelli, note that the sup in the definition of L is achieved at the unique point $p \in T_x^*M$, where $v = \partial_p H(x, p)$.

Moreover, from the Fenchel formula (3-1) above, we obtain the Fenchel inequality

$$p(v) \leq L(x, v) + H(x, p) \quad \text{for all } x \in M, v \in T_xM, p \in T_x^*M, \tag{3-2}$$

with equality if and only if $v = \partial_p H(x, p)$.

This Lagrangian L is everywhere finite, and enjoys the same properties as H (see [9], for example):

(1) The Lagrangian L is at least C^2. In fact, it is as smooth as H.

(2) (uniform superlinearity) For every $K \geq 0$, we have

$$C(K) = \sup_{(x,v) \in TM} K\|v\|_x - L(x, v) < \infty. \tag{3-3}$$

(3) (uniform boundedness in the fibers) For every $R \geq 0$, we have

$$A(R) = \sup\{L(x, v) \mid \|v\| \leq R\} < +\infty. \tag{3-4}$$

(4) (C^2 strict convexity in the fibers) For every $(x, v) \in TM$, the second derivative along the fibers, $\partial^2 L/\partial v^2(x, v)$, is (strictly) positive definite.

Again (2) implies

$$\forall (x, v) \in TM, L(x, v) \geq K\|v\| - C(K). \tag{3-5}$$

A Tonelli Lagrangian on the complete Riemannian manifold (M, g) is a function $L : TM \to \mathbb{R}$ which satisfies condition (1) to (4) above. As is well-known, we can define a Hamiltonian $H : T^*M \to \mathbb{R}$ by the same Fenchel formula

$$H(x, p) = \sup_{v \in T_xM} p(v) - L(x, v).$$

Again the supremum above is attained precisely when $p = \partial_v L(x, v)$. This H is a Tonelli Hamiltonian whose associated Lagrangian is precisely L.

Example 3.3. We give the Lagrangians of the Hamiltonians in Example 3.2.

(1) The Lagrangian $L_0 : TM \to \mathbb{R}$ associated to the Tonelli Hamiltonian $H_0 : T^*M \to \mathbb{R}$ is

$$L_0(x, v) = \frac{1}{2}\|v\|_x^2,$$

and $A_0(R) = R^2/2$, $C_0(K) = K^2/2$.

(2) The Lagrangian $L_{X,V} : TM \to \mathbb{R}$ associated to the Hamiltonian $H_{X,V} : T^*M \to \mathbb{R}$ is

$$L_{X,V}(x, v) = \tfrac{1}{2}\|v - X(x)\|_x^2 - V(x) = \tfrac{1}{2}\|v\|_x^2 - \langle v, X(x)\rangle + \tfrac{1}{2}\|X(x)\|_x^2 - V(x).$$

For every $x \in M$, we have

$$\begin{aligned}\sup_{\substack{v\in T_xM\\ \|v\|_x=R}} L_{X,V}(x, v) &= \tfrac{1}{2}R^2 + R\|X(x)\|_x + \tfrac{1}{2}\|X(x)\|_x^2 - V(x)\\ &= \tfrac{1}{2}(R + \|X(x)\|_x)^2 - V(x).\end{aligned}$$

Therefore

$$A_{X,V}(R) \le \tfrac{1}{2}(R + \|X\|_\infty)^2 - \inf_{x\in M} V(x).$$

A similar computation gives

$$\begin{aligned}C_{X,V}(K) &= \tfrac{1}{2}K^2 + \sup_{x\in M}(K\|X(x)\|_x + V(x))\\ &\le \tfrac{1}{2}K^2 + K\|X\|_\infty + \sup_{x\in M} V(x).\end{aligned}$$

4. Action, minimizers, Euler–Lagrange flow

Again in the sequel, we fix a Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ on the complete Riemannian manifold (M, g) and we will denote by $L : TM \to \mathbb{R}$ its associated Tonelli Lagrangian.

We need to use the calculus of variations for Lagrangians: minimizers, extremals, Euler–Lagrange equation and flow. An introduction to these concepts can be found in [3; 5; 6], for example. We recall certain notions for the convenience of the reader and to fix notation.

Definition 4.1 (length, action). Let $\gamma : [a, b] \to M$ be an absolutely continuous curve.

- Its Riemannian length $\ell_g(\gamma)$ is

$$\ell_g(\gamma) = \int_a^b \|\dot\gamma(s)\|_{\gamma(s)}\, ds.$$

- Its action $\mathbb{L}(\gamma)$ (for L) is

$$\mathbb{L}(\gamma) = \int_a^b L(\gamma(s), \dot\gamma(s))\, ds.$$

Note that since L is bounded below by $-C(0)$ the integral above makes always sense (it can be $+\infty$). In fact, since $L + C(0) \ge 0$, we set

$$\int_a^b L(\gamma(s), \dot\gamma(s))\, ds = -C(0)(b-a) + \int_a^b L(\gamma(s), \dot\gamma(s)) + C(0) ds.$$

From the definition of the distance d on the Riemannian manifold (M, g), we also have

$$d(x, y) = \inf_{\gamma} \ell_g(\gamma),$$

where the inf is taken over all absolutely continuous curves $\gamma : [a, b] \to M$ with $\gamma(a) = x$, $\gamma(b) = y$.

Here are some basic estimates relating action of curves to their length.

Lemma 4.2. *Let $\gamma : [a, b] \to M$ be an absolutely continuous curve. For every $K \in [0, \infty[$, we have*

$$\mathbb{L}(\gamma) \geq K\ell_g(\gamma) - C(K)(b-a) \geq Kd(\gamma(a)\gamma(b)) - C(K)(b-a) \tag{4-1}$$

and

$$d(\gamma(a), \gamma(b)) \leq \ell_g(\gamma) \leq \frac{\mathbb{L}(\gamma) + C(K)(b-a)}{K}. \tag{4-2}$$

In particular, for every $\epsilon > 0$, we have

$$d(\gamma(a), \gamma(b)) \leq \ell_g(\gamma) \leq \epsilon\mathbb{L}(\gamma) + \epsilon C(1/\epsilon)(b-a). \tag{4-3}$$

Proof. We use the inequality (3-5), to obtain

$$L(\gamma(s), \dot\gamma(s)) \geq K\|\dot\gamma(s)\|_{\gamma(s)} - C(K),$$

from which it follows by integration that

$$\mathbb{L}(\gamma) \geq K\ell_g(\gamma) - C(K)(b-a).$$

Both inequalities (4-1) and (4-2) follow easily. Moreover, inequality (4-3) follows from (4-2) with $K = 1/\epsilon$. □

The estimates above yield a modulus of continuity for curves with bounded Lagrangian. Recall that a modulus of continuity is a nondecreasing function $\eta : [0, +\infty[\to [0, +\infty[$ that is continuous at 0 and satisfies $\eta(0) = 0$.

Lemma 4.3. *For every finite $K, T \geq 0$, we can find a modulus of continuity $\eta_{K,T} : [0, +\infty[\to [0, +\infty[$ such that, for every absolutely continuous curve $\gamma : [a, b] \to M$, with $b - a \leq T$ and $\mathbb{L}(\gamma) \leq K$, we have*

$$d(\gamma(t'), \gamma(t)) \leq \ell_g(\gamma\,|\,[t, t']) \leq \eta_{K,T}(|t'-t|) \quad \textit{for all } t, t' \in [a, b].$$

Proof. Since $L \geq -C(0)$, for any curve $\gamma : [a, b] \to M$, and all $a \leq t \leq t' \leq b$, we obtain

$$\begin{aligned} -C(0)(t-a) + \mathbb{L}(\gamma|[t, t']) - C(0)(b-t') &\leq \mathbb{L}(\gamma|[0, t]) + \mathbb{L}(\gamma|[t, t']) + \mathbb{L}(\gamma|[t', b]) \\ &= \mathbb{L}(\gamma). \end{aligned}$$

Therefore

$$\mathbb{L}(\gamma\,|\,[t,t']) \leq \mathbb{L}(\gamma) - C(0)(b-t') - C(0)(t-a) \leq \mathbb{L}(\gamma) + |C(0)|(b-a).$$

Hence, by (4-3) of Lemma 4.2, if $\mathbb{L}(\gamma) \leq K$ and $b-a \leq T$, for every $\epsilon > 0$, we get

$$d(\gamma(t'),\gamma(t)) \leq \ell_g(\gamma\,|\,[t,t']) \leq \epsilon(K + |C(0)|T) + \epsilon C(1/\epsilon)(t'-t).$$

It is not difficult to see that we can take modulus of continuity the function $\eta_{K,T}$ defined by

$$\eta_{K,T}(s) = \inf_{\epsilon>0} \epsilon(K + |C(0)|T) + \epsilon|C(1/\epsilon)|s. \qquad \square$$

Once action is defined, the notion of minimizer can be introduced.

Definition 4.4 (minimizer). A minimizer (for L) is a curve $\gamma : [a,b] \to M$ such that

$$\mathbb{L}(\delta) = \int_a^b L(\delta(s),\dot{\delta}(s))\,ds \geq \mathbb{L}(\gamma) = \int_a^b L(\gamma(s),\dot{\gamma}(s))\,ds,$$

for every absolutely continuous curve $\delta : [a,b] \to M$ such that $\delta(a) = \gamma(a)$ and $\delta(b) = \gamma(b)$.

It is not difficult to show that the restriction to any subinterval $[c,d] \subset [a,b]$ of a minimizer $\gamma : [a,b] \to M$ is itself a minimizer.

Examples 4.5. (1) If $L_0 : TM \to \mathbb{R}$ is given by $L_0(x,v) = \frac{1}{2}\|v\|_x^2$, then $\gamma : [a,b] \to M$ is a minimizer if and only if γ is a geodesic of M with $\ell_g(\gamma) = d(\gamma(a),\gamma(b))$. Such a minimizer satisfies

$$\mathbb{L}(\gamma) = \frac{d(\gamma(a),\gamma(b))^2}{2(b-a)}.$$

(2) (Mañé Lagrangian) Let X be a C^2 vector field on the complete Riemannian manifold M. Define the Lagrangian $L_X : TM \to \mathbb{R}$ by

$$L(x,v) = \tfrac{1}{2}\|v - X(x)\|_x^2.$$

This Lagrangian is Tonelli. Since $L \geq 0$, the solution curves of the vector field X are minimizers. In fact, they are the only minimizers for L_X with zero action.

(3) For a real number $p \geq 4$, if $L_p : TM \to \mathbb{R}$ is given by $L_p(x,v) = \frac{1}{2}\|v\|_x^2 + \frac{1}{p}\|v\|_x^p$, then L is a Tonelli Lagrangian. We note that Lagrangian $\tilde{L}_p : TM \to \mathbb{R}$ defined by $\tilde{L}_p(x,v) = \frac{1}{p}\|v\|_x^p$ is not Tonelli since $\partial^2_{v^2} L(x,0)$ is identically 0 for every $x \in M$. If $\gamma : [a,b] \to M$ is a curve, we have

$$\mathbb{L}(\gamma) = \int_a^b \tfrac{1}{2}\|\dot{\gamma}(s)\|_\gamma^2 + \tfrac{1}{p}\|\dot{\gamma}(s)\|_\gamma^p\,ds.$$

Since the functions $t \mapsto t^2$ and $t \mapsto t^p$ are strictly convex, Jensen's inequality implies

$$\begin{aligned}\frac{\mathbb{L}(\gamma)}{b-a} &\geq \frac{1}{2}\left(\frac{1}{b-a}\int_a^b \|\dot\gamma(s)\|_{\gamma(s)}\,ds\right)^2 + \frac{1}{p}\left(\frac{1}{b-a}\int_a^b \|\dot\gamma(s)\|_{\gamma(s)}\,ds\right)^p \\ &\geq \frac{1}{2}\left(\frac{d(\gamma(a),\gamma(b))}{b-a}\right)^2 + \frac{1}{p}\left(\frac{d(\gamma(a),\gamma(b))}{b-a}\right)^p,\end{aligned}$$

with equality if and only if $\|\dot\gamma(s)\|_{\gamma(s)}$ identically equals $d(\gamma(a),\gamma(b))/(b-a)$. Hence, the curve γ is a minimizer if and only if it is a length minimizing geodesic of M. Therefore the action of a minimizer $\gamma : [a,b] \to M$ is given by

$$\mathbb{L}(\gamma) = \frac{d(\gamma(a),\gamma(b))^2}{2(b-a)} + \frac{d(\gamma(a),\gamma(b))^p}{p(b-a)^{p-1}}. \qquad \square$$

Minimizers play a crucial role. Like all minima of a function, minimizers must be critical points for the action functional $\mathbb{L}$. These critical points are called extremals.

More precisely, an extremal (for L) is a curve $\gamma : [a,b] \to M$ such that the derivative $D_\gamma \mathbb{L} | \mathcal{E}_\gamma$ at γ vanishes, with

$$\mathcal{E}_\gamma = \{\delta : [a,b] \to M \mid \delta(a) = \gamma(a), \delta(b) = \gamma(b)\}.$$

By the classical calculus of variations, the curve γ is an extremal if and only if it satisfies the Euler–Lagrange equation, given in local coordinates by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}(\gamma(t),\dot\gamma(t))\right) = \frac{\partial L}{\partial x}(\gamma(t),\dot\gamma(t)). \tag{4-4}$$

This last ODE (4-4) defines a second order ODE on M. Therefore there exists a flow φ_t on TM, called the Euler–Lagrange flow, such that $\gamma : [a,b] \to M$ is an extremal if and only if its speed curve $s \mapsto (\gamma(t),\dot\gamma(t))$ is an orbit of φ_t. Moreover, for any $(x,v) \in TM$, the projected curve $\gamma_{x,v}(t) = \pi\varphi_t(x,v)$, where $\pi : TM \to M$ is the canonical projection, is an extremal with $(\gamma_{x,v}(t),\dot\gamma_{x,v}(t)) = \varphi_t(x,v)$. Hence, if two extremals have the same position and speed at a time t, then they coincide on their common interval of definition.

We now state Tonelli's theorem; see [3; 5; 6] for a proof.

Theorem 4.6 (Tonelli). *Suppose $L : TM \to \mathbb{R}$ is a Tonelli Lagrangian on the complete Riemannian manifold M. For every $t > 0$ and every $x, y \in M$, there exists an absolutely continuous curve $\gamma : [0,t] \to M$, with $\gamma(0) = x$, $\gamma(t) = y$ which is a minimizer.*

Any minimizer is as smooth as L and is a solution of the Euler–Lagrange equation.

There is a fundamental relation between the Euler–Lagrange flow for the Lagrangian $L : TM \to \mathbb{R}$ and the Hamiltonian flow of the associated Hamiltonian $H : T^*M \to \mathbb{R}$ of L. Recall that L is obtained from H by (3-1). As we already observed, it is also true, in the Tonelli case, that H can be obtained in the same way from L

$$H(x, p) = \sup_{v \in T_x M} p(v) - L(x, v). \tag{4-5}$$

Again, since L is Tonelli, the supremum in the definition of $H(x, p)$ is attained at the unique $v \in T_x M$ such that $p = \partial_v L(x, v)$. In particular, we have

$$H\,(x, \partial_v L(x, v)) = \partial_v L(x, v)(v) - L(x, v).$$

Recall that the Hamiltonian flow of H is the flow φ_t^* on T^*M obtained from the ODE on T^*M given in local coordinates by

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}.$$

The Hamiltonian H is invariant under the flow φ_t^*.

In fact the flow φ_t on TM and φ_t^* on T^*M are conjugated by the Legendre transformation $\mathcal{L} : TM \to T^*M$ given by

$$\mathcal{L}(x, x) = (x, \partial_v L(x, v)).$$

In particular, the function $H \circ \mathcal{L}$ in invariant by the Euler–Lagrange flow. Expressed in the variables (x, v), it is called the energy of the Lagrangian.

Definition 4.7. The energy $E : TM \to \mathbb{R}$ of the Lagrangian $L : TM \to \mathbb{R}$ is defined by

$$\begin{aligned} E(x, v) &= H \circ \mathcal{L}(x, v) \\ &= H(x, \partial_v L(x, v)) \\ &= \sup\nolimits_{u \in T_x M} \langle \partial_v L(x, v), u \rangle - L(x, u) \\ &= \partial_v L(x, v)(v) - L(x, v). \end{aligned} \tag{4-6}$$

As said above, E is constant along any orbit of the Euler–Lagrange flow.

Definition 4.8. Let $\gamma : [a, b] \to M$ be an extremal of L. Its energy $E(\gamma(s), \dot{\gamma}(s))$, $s \in [a, b]$, is constant along its speed curve. Therefore, we can define the energy $E(\gamma)$ for the extremal $\gamma : [a, b] \to M$ by $E(\gamma) = E(\gamma(s), \dot{\gamma}(s))$, for any $s \in [a, b]$.

We will give later estimates for speeds of extremals. We first define the minimal action to join x to y in time t.

Definition 4.9 (minimal action h_t). For $x, y \in M$ and $t > 0$, we define the minimal action $h_t(x, y)$ to join x to y in time t by

$$h_t(x, y) = \inf_{\gamma} \int_0^t L(\gamma(s)\dot{\gamma}(s))\, ds,$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0, t] \to M$, with $\gamma(0) = x$ and $\gamma(t) = y$.

We will also set $h_0(x, x) = 0$ and $h_0(x, y) = +\infty$, for $x \neq y$. These last two definitions are the natural ones in view of Lemma 4.11.

It is useful to introduce the function $\mathcal{H} : [0, +\infty[\times M \times M \to \mathbb{R}$ defined by

$$\mathcal{H}(t, x, y) = h_t(x, y).$$

Since L is bounded from below by $-C(0)$, we obtain that $\mathcal{H}(t, x, y) = h_t(x, y)$ is always finite, for $t > 0$.

By Tonelli's theorem (Theorem 4.6), for $t > 0$, the infimum in the definition of h_t is always attained. We can also use the definition of h_t to give a characterization of minimizers:

Proposition 4.10. *For any $x, y \in M$ and every $t > 0$, we can find an absolutely continuous curve $\gamma : [0, t] \to M$, with $\gamma(0) = x$, $\gamma(t) = y$ and*

$$h_t(x, y) = \mathbb{L}(\gamma) = \inf_{\gamma} \int_0^t L(\gamma(s)\dot{\gamma}(s))\, ds.$$

Any such curve is a minimizer. Moreover, an absolutely continuous curve $\delta : [a, b] \to M$ is a minimizer if and only if

$$h_{b-a}(\delta(a), \delta(b)) = \int_a^b L(\delta(s), \dot{\delta}(s))\, ds.$$

A first estimate of $h_t(x, y)$ is given by the next lemma.

Lemma 4.11. *For every $t > 0$, every $x, y \in M$ and every $K \geq 0$, we have*

$$-C(K)t + Kd(x, y) \leq h_t(x, y) \leq tA(d(x, y)/t). \tag{4-7}$$

In particular, we have $-C(0)t \leq h_t(x, y)$, $h_t(x, x) \leq A(0)t$, and $h_{d(x,y)}(x, y) \leq A(1)d(x, y)$.

Proof. A minimizing geodesic $\gamma_{x,y} : [0, t] \to M$ joining x to y has length $\ell_g(\gamma_{x,y}) = d(x, y)$ and a speed of constant norm. But integrating the speed yields the length; hence

$$\|\dot{\delta}(s)\|_{\delta(s)} = d(x, y)/t \quad \text{for } s \in [a, b].$$

By the uniform boundedness of L in the fibers (inequality (3-4)), we thus get

$$L(\gamma_{x,y}(s), \dot\gamma_{x,y}(s)) \le A\,(d(x, y)/t) \quad \text{for every } s \in [a, b],$$

and again by integration

$$\mathbb{L}(\gamma) \le tA\left(\frac{d(x, y)}{t}\right).$$

Therefore, we also obtain second inequality in (4-7).

For the first inequality of (4-7), we now observe that, by inequality (4-1) of Lemma 4.2, for any absolutely continuous curve $\gamma : [0, t] \to M$, with $\gamma(0) = x$ and $\gamma(t) = y$, we have

$$Kd(x, y) - C(K)t \le \mathbb{L}(\gamma).$$

Taking the infimum of the above inequality over all such curves γ yields the desired inequality. □

Examples 4.12. We estimate the function h_t for some examples.

(1) If $L_0 : TM \to \mathbb{R}$ is given by $L_0(x, v) = \frac{1}{2}\|v\|_x^2$, from Example 4.5(1), we obtain

$$h_t(x, y) = \frac{d(x, y)^2}{2t}.$$

(2) For a real number $p \ge 4$, if $L_p : TM \to \mathbb{R}$ is given by $L_p(x, v) = \frac{1}{2}\|v\|_x^2 + \frac{1}{p}\|v\|_x^p$, from of Example 4.5(1), we obtain

$$h_t(x, y) = \frac{d(x, y)^2}{2t} + \frac{d(x, y)^p}{pt^{p-1}}.$$

(3) If $L_{X,V} : TM \to \mathbb{R}$ is given by

$$L_{X,V}(x, p) = \tfrac{1}{2}\|v - X(x)\|_x^2 - V(x) = \tfrac{1}{2}\|v\|_x^2 - \langle v, X(x)\rangle + \tfrac{1}{2}\|X(x)\|_x^2 - V(x),$$

where $V : M \to \mathbb{R}$ is a C^2 function and X is a C^2 vector field on M. From Example 3.3(2), we know that

$$A_{X,V}(R) \le \tfrac{1}{2}(R + \|X\|_\infty)^2 - \inf_{x\in M} V(x).$$

Therefore by Lemma 4.11, we get

$$h_t(x, y) \le \frac{(d(x, y) + t\|X\|_\infty)^2}{2t} - t \inf_{x\in M} V(x).$$

Again by Example 3.3(2), we know that

$$C_{X,V}(K) \le \tfrac{1}{2}K^2 + K\|X\|_\infty + \sup_{x\in M} V(x).$$

Therefore by inequality (4-1) of Lemma 4.2, we have

$$h_t(x,y) \geq Kd(x,y) - \tfrac{1}{2}tK^2 - tK\|X\|_\infty - t\sup_{x\in M} V(x).$$

Since this is true for every $K \geq 0$, taking the supremum over all $K \geq 0$ yields

$$h_t(x,y) \geq \begin{cases} -t\sup_{x\in M} V(x) & \text{if } d(x,y) \leq t\|X\|_\infty, \\ \dfrac{(d(x,y)-t\|X\|_\infty)^2}{2t} - t\sup_{x\in M} V(x) & \text{otherwise.} \end{cases}$$

We now give some more properties of $h_t(x,y)$.

Proposition 4.13. (1) *For every $t, t' > 0$ and every $x, y \in M$, we have*

$$h_{t+t'}(x,y) = \inf_{z\in M} h_t(x,z) + h_{t'}(y,z),$$

and this infimum is attained.

(2) *If $\gamma : [a,b] \to M$ is a minimizer, for every $a', b' \in [0,t]$, with $a' < b'$, we have*

$$h_{b'-a'}(\gamma(a'),\gamma(b')) = \int_{a'}^{b'} L(\gamma(s),\dot\gamma(s))\,ds.$$

(3) *If $\gamma : [a,b] \to M$ is a minimizer, we have*

$$\begin{aligned} h_{b-a}(\gamma(a),\gamma(b)) &\geq K\ell_g(\gamma) - C(K)(b-a) \\ &\geq Kd(\gamma(a),\gamma(b)) - C(K)(b-a) \end{aligned} \tag{4-8}$$

and

$$d(\gamma(a),\gamma(b)) \leq \ell_g(\gamma) \leq \frac{h_{b-a}(\gamma(a),\gamma(b)) + C(K)(b-a)}{K}. \tag{4-9}$$

In particular, for every $\epsilon > 0$,

$$d(\gamma(a),\gamma(b)) \leq \ell_g(\gamma) \leq \epsilon h_{b-a}(\gamma(a),\gamma(b)) + \epsilon C(1/\epsilon)(b-a). \tag{4-10}$$

Proof. Part (1) follows from the following facts:

- If $\gamma : [0, t+t'] \to M$,

$$\mathbb{L}(\gamma) = \mathbb{L}(\gamma\,|\,[0,t]) + \mathbb{L}(\gamma\,|\,[t,t+t']).$$

- If $\gamma_1 : [0,t] \to M$ and $\gamma_2 : [0,t'] \to M$ are curves with $\gamma_1(t) = \gamma_2(0)$, the concatenation $\gamma_2 * \gamma_1 : [0, t+t'] \to M$, defined by

$$\gamma_2 * \gamma_1(s) = \begin{cases} \gamma_1(s) \text{ for } 0 \leq s \leq t, \\ \gamma_2(s-t) \text{ for } t \leq s \leq t+t', \end{cases}$$

is a curve joining $\gamma_1(0)$ to $\gamma_2(t')$ whose action $\mathbb{L}(\gamma_2 * \gamma_1)$ equals $\mathbb{L}(\gamma_1) + \mathbb{L}(\gamma_2)$.

Part (2) follows from Proposition 4.10, since we already observed (after Definition 4.4) that $\gamma\,|\,[a', b']$ is also a minimizer.

Parts (3) and (4) follow from inequalities (4-1), (4-2) and (4-3) in Lemma 4.2 and Proposition 4.10. □

To estimate the speed of extremals, we start with two lemmas, providing first an estimate of the partial derivative of L with respect to v, and then of the energy (Lemma 4.15).

Lemma 4.14. *For every $K \geq 0$ and every $(x, v) \in TM$, we have*

$$\|\partial_v L(x, v)\|_x \leq A(\|v\|_x + 1) + C(0),$$
$$\|\partial_v L(x, v)\|_x \|v\|_x \geq K\|v\|_x - C(K) - A(0).$$

Therefore $D(R) \to +\infty$ as $R \to +\infty$, where $D : [0, +\infty[\to [0, +\infty[$ is the function defined by

$$D(R) = \inf\left\{\|\partial_v L(x, v)\|_x \;\middle|\; v \in T_x M,\ \|v\|_x \geq R\right\}.$$

The function D is nondecreasing and $D(0) = 0$. Moreover, we have

$$\|\partial_v L(x, v)\|_x \geq D(\|v\|_x),$$

for every $(x, v) \in TM$.

Proof. By convexity of $L(x, v)$ in v, we have

$$L(x, v+u) - L(x, v) \geq \partial_v L(x, v)(u). \tag{4-11}$$

taking the sup over u with $\|u\|_x \leq 1$, we obtain

$$\|\partial_v L(x, v)\|_x \leq \max_{\|u\|_x \leq 1} L(x, v+u) - L(x, v).$$

But we know that $L \geq -C(0)$ and $\max_{\|u\|_x \leq 1} L(x, v+u) \leq A(\|v\|_x + 1)$, by inequality (3-4). Therefore we get

$$\|\partial_v L(x, v)\|_x \leq A(\|v\|_x + 1) + C(0).$$

Setting $u = -v$ in (4-11), we obtain

$$L(x, 0) - L(x, v) \geq -\partial_v L(x, v)(v),$$

from which we get

$$\begin{aligned}\|\partial_v L(x, v)\|_x \|v\|_x &\geq \partial_v L(x, v)(v)\\ &\geq L(x, v) - L(x, 0)\\ &\geq K\|v\|_x - C(K) - A(0),\end{aligned}$$

where we again used (3-4) and (3-5).

The function D is obviously nondecreasing. We then note that

$$D(0) = \inf_{(x,v)\in TM} \|\partial_v L(x,v)\| \geq 0.$$

Since L is superlinear in v, for every $x \in M$, the function $L(x,\cdot)$ achieves a minimum on T_xM, at which $\partial_v L(x,\cdot)$ vanishes. Therefore $D(0) = 0$.

We now show that $D(R) \to +\infty$ as $R \to +\infty$. Since D is nondecreasing $\lim_{R\to+\infty} D(R)$ exists in $\mathbb{R}\cup\{+\infty\}$.

Given $K \geq 0$, for any $v \in T_xM$, with $\|v\|_x \geq R$, we have

$$\|\partial_v L(x,v)\|_x \geq K - \frac{C(K)+A(0)}{\|v\|_x} \geq K - \frac{|C(K)+A(0)|}{R}.$$

Therefore $D(R) \geq K - |C(K)+A(0)|/R$, and $\lim_{R\to+\infty} D(R) \geq K$. Since $K \geq 0$ is arbitrary, we indeed get $\lim_{R\to+\infty} D(R) = +\infty$. □

Lemma 4.15. *We have*

$$A(2\|v\|_x) + 2C(0) \geq E(x,v) \geq \|\partial_v L(x,v)\|_x - A(1).$$

Therefore $E(x,v) \geq D(\|v\|_x) - A(1)$, where D is the nondecreasing function defined in Lemma 4.14.

Proof. We use again the convexity of L expressed by (4-11), with $u = v$ to obtain

$$L(x,2v) - L(x,v) \geq \partial_v L(x,v)(v).$$

Subtracting $L(x,v)$ from both sides, we get

$$L(x,2v) - 2L(x,v) \geq \partial_v L(x,v)(v) - L(x,v) = E(x,v).$$

Since $L(x,v) \geq -C(0)$ and $L(x,2v) \leq A(2\|v\|_x)$, we obtain

$$E(x,v) \leq A(2\|v\|_x) + 2C(0).$$

Since $E(x,v) = \sup_{u\in T_xM} \partial_v L(x,v)(u) - L(x,u)$, we have

$$E(x,v) \geq \sup_{\|u\|_x\leq 1} \partial_v L(x,v)(u) - L(x,u).$$

This last inequality, together with $L(x,u) \leq A(1)$, valid for $\|u\|_x \leq 1$, yields $E(x,v) \geq \|\partial_v L(x,v)\|_x - A(1)$. □

We now give the estimate on the speed of an extremal. It uses the preservation of energy along a solution of the Euler–Lagrange equation.

Proposition 4.16. *Suppose $L : TM \to \mathbb{R}$ is a given Tonelli Lagrangian. There exists a nondecreasing function $\eta : [0, +\infty[\to [0, +\infty[$ such that for every curve $\gamma : [a, b] \to M$ which satisfies the Euler–Lagrange equation, we have*

$$\sup_{t\in[a,b]} \|\dot\gamma(t)\|_{\gamma(t)} \leq \eta\Big(\inf_{t\in[a,b]} \|\dot\gamma(t)\|_{\gamma(t)}\Big).$$

Therefore

$$\sup_{t\in[a,b]} \|\dot\gamma(t)\|_{\gamma(t)} \leq \eta[\ell_g(\gamma)/(b-a)].$$

Proof. Consider the nondecreasing function D introduced in Lemma 4.14. Since $D(0) = 0$, we can introduce a nondecreasing function ζ defined on $[0, +\infty[$ by

$$\zeta(\rho) = \sup\{R \geq 0 \mid D(R) \leq \rho\}.$$

Since $D(R) \to +\infty$ as $R \to +\infty$, the function ζ is finite everywhere. We also have $\zeta(D(R)) \geq R$, since $\zeta(D(R)) = \sup\{R' \mid D(R') \leq D(R)\}$.

Consider now a solution $\gamma : [a, b] \to M$ of the Euler–Lagrange equation. Define $s_{\min}, s_{\max} \in [a, b]$ by

$$\|\dot\gamma(s_{\min})\|_{\gamma(s_{\min})} = \inf_{t\in[a,b]} \|\dot\gamma(t)\|_{\gamma(t)},$$
$$\|\dot\gamma(s_{\max})\|_{\gamma(s_{\max})} = \sup_{t\in[a,b]} \|\dot\gamma(t)\|_{\gamma(t)}.$$

By Lemma 4.15, we get

$$A(2\|\dot\gamma(s_{\min})\|_{\gamma(s_{\min})}) + 2C(0) \geq E[\gamma(s_{\min}), \dot\gamma(s_{\min})]$$

and

$$E[\gamma(s_{\max}), \dot\gamma(s_{\max})] \geq D\left(\|\dot\gamma(s_{\max})\|_{\gamma(s_{\max})}\right) - A(1).$$

We have $E[\gamma(s_{\min}), \dot\gamma(s_{\min})] = E\,(\gamma(s_{\max}), \dot\gamma(s_{\max}))$, by the conservation of energy. Therefore

$$A(2\|\dot\gamma(s_{\min})\|_{\gamma(s_{\min})}) + 2C(0) + A(1) \geq D\left(\|\dot\gamma(s_{\max})\|_{\gamma(s_{\max})}\right).$$

Since ζ is nondecreasing and $\zeta(D(R)) \geq R$, we obtain

$$\zeta\left(A(2\|\dot\gamma(s_{\min})\|_{\gamma(s_{\min})}) + 2C(0) + A(1)\right) \geq \|\dot\gamma(s_{\max})\|_{\gamma(s_{\max})}.$$

To finish the proof of the first inequality of the proposition, it suffices to define the nondecreasing everywhere finite function $\eta : [0, +\infty[\to [0, +\infty[$ by $\eta(R) = \zeta(A(2R) + 2C(0) + A(1))$.

The second inequality follows from the nondecreasing character of η and

$$(b-a) \min_{s\in[a,b]} \|\dot\gamma(s)\|_{\gamma(s)} \leq \int_a^b \|\dot\gamma(s)\|_{\gamma(s)}\, ds = \ell_g(\gamma). \qquad \square$$

Corollary 4.17. *If $L : TM \to \mathbb{R}$ is a given Tonelli Lagrangian, we can find nondecreasing functions $\bar{\eta}, \tilde{\eta} : [0, +\infty[\to [0, +\infty[$ such that any minimizer $\gamma : [a, b] \to M$ satisfies*

$$\sup_{t\in[a,b]} \|\dot{\gamma}(t)\|_{\gamma(t)} \leq \bar{\eta}\left(\frac{h_{b-a}(\gamma(a), \gamma(b))}{b-a}\right)$$

and

$$\sup_{t\in[a,b]} \|\dot{\gamma}(t)\|_{\gamma(t)} \leq \tilde{\eta}\left(\frac{d(\gamma(a), \gamma(b))}{b-a}\right).$$

Proof. By (4-9), we have

$$\ell_g(\gamma) \leq h_{b-a}(\gamma(a), \gamma(b)) + C(1)(b-a).$$

Therefore, using the function η from Proposition 4.16, since a minimizer satisfies the Euler–Lagrange equation, we obtain

$$\sup_{t\in[a,b]} \|\dot{\gamma}(t)\|_{\gamma(t)} \leq \eta\left(C(1) + \frac{h_{b-a}(\gamma(a), \gamma(b))}{b-a}\right).$$

This finishes the proof of the first inequality, with $\bar{\eta}(s) = \eta(s + C(1))$.

To prove the second one, we recall, from (4-7) in Lemma 4.11, that

$$\frac{h_{b-a}(\gamma(a), \gamma(b))}{b-a} \leq A\left(\frac{d(\gamma(a), \gamma(b))}{b-a}\right).$$

Therefore

$$\sup_{t\in[a,b]} \|\dot{\gamma}(t)\|_{\gamma(t)} \leq \bar{\eta}\left(A\left(\frac{d(\gamma(a), \gamma(b))}{b-a}\right)\right).$$

The function $t \mapsto \tilde{\eta}(t) = \bar{\eta} \circ A(t)$ is finite everywhere and nondecreasing. □

For a subset $S \subset M$, recall that its diameter $\operatorname{diam} S$, for the Riemannian distance d on M, is defined by

$$\operatorname{diam} S = \sup\{d(x, y) \mid x, y \in S\}.$$

The next result, a straightforward consequence of Corollary 4.17, provides us with the criterion for compactness of a set of minimizers.

Proposition 4.18. *Suppose $S \subset M$, with* diam *S finite, and $t_0 > 0$. Any minimizer $\gamma : [a, b] \to M$ such that $\gamma(a), \gamma(b) \in S$ and $b - a \geq t_0$ satisfies*

$$\sup_{t\in[a,b]} \|\dot{\gamma}(t)\|_{\gamma(s)} \leq \tilde{\eta}(\operatorname{diam} S/t_0),$$

where $\tilde{\eta}$ is the nondecreasing everywhere finite function from Corollary 4.17. Therefore, the set of minimizers $\gamma : [a, b] \to M$ such that $\gamma(a), \gamma(b) \in S$ and $b - a \geq t_0$ is equi-Lipschitz.

An important property of $h_t(x, y)$, namely its local semiconcavity in (x, y), is proved in [8, Theorem B.19, page 50]. It is not difficult, using the proof of Theorem B.19 in [8], to show that $\mathcal{H}(t, x, y)$ is locally semiconcave in (t, x, y) on $]0, +\infty[\times M \times M$.

Proposition 4.19. *The function $\mathcal{H}$ is locally semiconcave on $]0, +\infty[\times M \times M$. Moreover, for every compact subset $C \subset M \times M$, and every $t_0 > 0$, the family of functions $h_t : C \to \mathbb{R}$, $t \geq t_0$ is equi-semiconcave.*

Another useful reference on semiconcavity and the Hamilton–Jacobi equation is [4].

Example 4.20. If $L_0 : TM \to \mathbb{R}$ is given by $L_0(x, v) = \frac{1}{2}\|v\|_x^2$, from part (1) of Example 4.12, we obtain

$$\mathcal{H}(t, x, y) = \frac{d(x, y)^2}{2t}.$$

Therefore, from the previous proposition we obtain that d^2 is locally semiconcave on $M \times M$. Moreover, since $s \mapsto \sqrt{s}$ is C^∞ on $]0, +\infty[$, we obtain that d is locally semiconcave on $M \times M \setminus \Delta_M$, where $\Delta_M = \{(x, x) \mid x \in M\}$ is the diagonal in $M \times M$.

Since $\mathcal{H}$ is locally semiconcave, it is locally Lipschitz. Therefore, it has a derivative almost everywhere in $]0, +\infty[\times M \times M$. We proceed to express this derivative.

We need to use the notion of upper and lower differentials (called also upper and lower derivatives)–see [2; 1; 4; 6; 7] for more details on this notion and its relationship with viscosity solutions.

Notation 4.21. If $w : N \to \mathbb{R}$ is a function on the manifold N and $n \in N$, the set of upper-differentials (resp. lower-differentials) of w at N is denoted by $D^+w(n) \subset T_n^*N$ (resp. $D^-w(n) \subset T_n^*N$).

Proposition 4.22. *Since $\mathcal{H}$ is locally semiconcave on $]0, +\infty[\times M \times M$, for every $(t, x, y) \in]0, +\infty[\times M \times M$ the set of superderivatives $D^+\mathcal{H}(t, x, y) \subset T^*_{(t,x,y)}(]0, +\infty[\times M \times M = \mathbb{R} \times T_x^*M \times T_y^*M$ is not empty. If $\gamma : [0, t] \to M$ is a minimizer, with $\gamma(0) = x$ and $\gamma(t) = y$, we have*

$$(-E(\gamma), -\partial_v L(\gamma(0), \dot{\gamma}(0)), \partial_v L(\gamma(t), \dot{\gamma}(t))) \in D^+\mathcal{H}(t, x, y),$$

where $E(\gamma) = E(\gamma(s), \dot{\gamma}(s))$, $s \in [0, t]$ is the energy of the minimizer γ.

In particular, we have

$$\begin{aligned} -E(\gamma) &\in D_t^+\mathcal{H}(t, x, y), \\ -\partial_v L(\gamma(0), \dot{\gamma}(0)) &\in D_x^+\mathcal{H}(t, x, y), \\ \partial_v L(\gamma(t), \dot{\gamma}(t)) &\in D_y^+\mathcal{H}(t, x, y). \end{aligned} \tag{4-12}$$

The proof that $(-\partial_v L(\gamma(0), \dot\gamma(0)), \partial_v L(\gamma(t), \dot\gamma(t))) \in D^+h_t(x, y)$ is given in [8, Theorem B.20, page 53]. We leave it to the reader to check the superderivative in t.

Corollary 4.23. *For $(t, x, y) \in]0, +\infty[\times M \times M$, the function $\mathcal{H}$ is differentiable at $(t, x, y) \in]0, +\infty[\times M \times M$ if and only if there exists a unique minimizer $\gamma : [0, t] \to M$, with $\gamma(0) = x$ and $\gamma(t) = y$.*

Moreover, for each $(t, x, y) \in]0, +\infty[\times M \times M$, the set of superderivatives $D^+\mathcal{H}(t, x, y)$ is the convex hull of the set of covectors

$$\big(-E(\gamma), -\partial_v L(\gamma(0), \dot\gamma(0)), \partial_v L(\gamma(t), \dot\gamma(t))\big),$$

where $\gamma : [0, t] \to M$ is an arbitrary minimizer with $\gamma(0) = x$ and $\gamma(t) = y$.

Proof. If $\mathcal{H}$ is differentiable at $(t, x, y) \in]0, +\infty[\times M \times M$ and $\gamma : [0, t] \to M$ is a minimizer, with $\gamma(0) = x$ and $\gamma(t) = y$, then, by Proposition 4.22 above $\partial_y \mathcal{H}(t, x, y) = \partial_v L(\gamma(t), \dot\gamma(t))$, since L is strictly convex the speed $\dot\gamma(t)$ is completely determined by $\partial_y \mathcal{H}(t, x, y)$. Therefore, since a minimizer satisfies the Euler–Lagrange equation, the curve γ is completely determined by $\partial_y \mathcal{H}(t, x, y)$.

This proves half of the first statement of the corollary. To prove the second part, we recall that $D^+\mathcal{H}(t, x, y)$ is the convex hull of $\partial\mathcal{H}(t, x, y)$ where any point in $\partial\mathcal{H}(t, x, y)$ is a limit of a sequence of derivatives $D\mathcal{H}(t_i, x_i, y_i)$, where $(t_i, x_i, y_i) \to (t, x, y)$ as $i \to \infty$, and $\mathcal{H}$ is differentiable at each (t_i, x_i, y_i). By Proposition 4.22, the derivative $D\mathcal{H}(t_i, x_i, y_i)$ is given by a minimizer $\gamma_i : [0, t_i] \to M$ with $\gamma(0) = x_i$ and $\gamma(t_i) = y$. If $\tilde\eta$ is the nondecreasing finite everywhere function obtained in Corollary 4.17, we have

$$\|\dot\gamma_i(s)\|_{\gamma_i(s)} \le \tilde\eta\left(\frac{d(\gamma(0), \gamma(t_i))}{t_i}\right) \quad \text{for all } s \in [0, t_i].$$

Since $(t_i, x_i, y_i) \to (t, x, y)$, with $t > 0$, we have $\sup_i d(\gamma(0), \gamma(t_i))/t_i < +\infty$.

Let C be the value of this supremum. We see that the norm of the speed $\|\dot\gamma_i(s)\|_{\gamma_i(s)}$ is bounded by $\tilde\eta(C)$, independently of i and $s \in [0, t_i]$. Extracting a subsequence if necessary, we can assume that $(\gamma_i(0), \dot\gamma_i(0))$ converges to some (x, v) with $v \in T_x M$. If we call γ the solution of the Euler–Lagrange equation with $(\gamma(0), \dot\gamma(0)) = (x, v)$, we obtain that $\gamma : [0, t] \to M$ is a minimizer, with $\gamma(0) = x$ and $\gamma(t) = y$. But we have

$$D\mathcal{H}(t_i, x_i, y_i) = \big(-E(\gamma_i), -\partial_v L(\gamma_i(0), \dot\gamma_i(0)), \partial_v L(\gamma_i(t), \dot\gamma_{(}t_i))\big),$$

which tends to $\big(-E(\gamma), -\partial_v L(\gamma(0), \dot\gamma(0)), \partial_v L(\gamma(t), \dot\gamma(t))\big)$. This proves the last part of the corollary.

To finish the proof of the corollary, it suffices to show that if there is a unique minimizer $\gamma : [0, t] \to M$, with $\gamma(0) = x$ and $\gamma(t) = y$, then $\mathcal{H}$ is differentiable at (t, x, y). By what we just proved, this uniqueness condition implies that $D^+\mathcal{H}(t, x, y) = \partial\mathcal{H}(t, x, y)$ is reduced to one point. Since $\mathcal{H}$ is semiconcave, this implies that $\mathcal{H}$ is differentiable at (t, x, y). □

Corollary 4.24. *For $(t, x, y) \in]0, +\infty[\times M \times M$, the following statements are equivalent*:

(i) *The function $\mathcal{H}$ is differentiable at (t, x, y).*

(ii) *The partial derivative $\partial_x\mathcal{H}(t, x, y)$ exists.*

(iii) *The partial derivative $\partial_y\mathcal{H}(t, x, y)$ exists.*

(iv) *There exists a unique minimizer $\gamma : [0, t] \to M$, with $\gamma(0) = x$ and $\gamma(t) = y$.*

If any one of these statements is true, *we have*

$$\begin{aligned} \partial_t\mathcal{H}(t, x, y) &= -E(\gamma), \\ \partial_x\mathcal{H}(t, x, y) &= -\partial_v L(\gamma(0), \dot\gamma(0)), \\ \partial_y\mathcal{H}(t, x, y) &= \partial_v L(\gamma(t), \dot\gamma(t))), \end{aligned} \tag{4-13}$$

where $\gamma : [0, t] \to M$ is the unique minimizer with $\gamma(0) = x$ and $\gamma(t) = y$.

Proof. Of course (i) implies (ii) and (iii). From Corollary 4.23, statements (i) and (iv) are equivalent. To finish proving that (i), (ii), (iii) and (iv) are all equivalent, it remains to show that (ii) or (iii) imply (iv). We will show that (ii) implies (iv). In fact, if $\partial_x\mathcal{H}(t, x, y)$ exists and $\gamma : [0, t] \to M$ is a minimizer with $\gamma(0) = x$ and $\gamma(t) = y$, by equality (4-12) of Proposition 4.22, we have $\partial_x\mathcal{H}(t, x, y) = -\partial_v L(\gamma(0), \dot\gamma(0))$. Therefore not only the position at time 0 of γ is unique, but also its speed $\dot\gamma(0)$ is unique. Since such a minimizer γ satisfies Euler–Lagrange, we conclude that γ is unique.

The last part of the corollary follows from (4-12). □

Corollary 4.25. *We can find a nondecreasing everywhere finite function θ : $[0, +\infty[\to [0, +\infty[$ such that at every point $(t, x, y) \in]0, +\infty[\times M \times M$, where the derivative $D\mathcal{H}(t, x, y)$ exists*, *it is bounded in norm by $\theta(\mathcal{H}(t, x, y)/t)$.*

Proof. We first estimate $\partial_x\mathcal{H}(t, x, y)$. By Proposition 4.22, if $\gamma : [0, t] \to M$ is a minimizer, with $\gamma(0) = x$ and $\gamma(t) = y$, we have $\partial_x\mathcal{H}(t, x, y) = \partial_v L(\gamma(0), \dot\gamma(0))$. Therefore by Lemma 4.15, we get

$$\|\partial_x\mathcal{H}(t, x, y)\|_x \le A(\|\dot\gamma(0)\|_{\gamma(0)} + 1) + C(0).$$

Combining with Corollary 4.17, since $\mathbb{L}(\gamma) = \mathcal{H}(t, x, y)$, we obtain

$$\|\partial_x\mathcal{H}(t, x, y)\|_x \le A(\bar\eta[\mathcal{H}(t, x, y)/t] + 1) + C(0).$$

Therefore if we define the nondecreasing function $\theta_1 : [0, +\infty[\to [0, +\infty[$ by

$$\theta_1(R) = \max\left(0, A(\bar\eta[R] + 1) + C(0)\right),$$

we obtain

$$\|\partial_x \mathcal{H}(t,x,y)\|_x \le \theta_1(\mathcal{H}(t,x,y)/t).$$

In the same way, we obtain

$$\|\partial_y \mathcal{H}(t,x,y)\|_x \le \theta_1(\mathcal{H}(t,x,y)/t).$$

To estimate $\partial_t \mathcal{H}(t,x,y) = -E(\gamma(s),\dot\gamma(s))$, we use Lemma 4.15 and Corollary 4.17:

$$\begin{aligned}|\partial_t \mathcal{H}(t,x,y)| &= |E(\gamma(0),\dot\gamma(0))|\\ &\le \max\big(A(1), A(2\|\dot\gamma(0)\|_{\gamma(0)}) + 2C(0)\big)\\ &\le \max\big(A(1), A(2\bar\eta[\mathcal{H}(t,x,y)/t]) + 2C(0)\big).\end{aligned}$$

Hence, if we define the nondecreasing function $\theta_2 : [0,+\infty[\to [0,+\infty[$ by

$$\theta_2(R) = \max\big(0, A(1), A(2\bar\eta[R]) + 2C(0))\big),$$

we obtain

$$|\partial_t \mathcal{H}(t,x,y)| \le \theta_2(\mathcal{H}(t,x,y)/t).$$

Therefore

$$\|D\mathcal{H}(t,x,y)\|^2_{(t,x,y)} \le 2\theta_1(\mathcal{H}(t,x,y)/t)^2 + \theta_2(\mathcal{H}(t,x,y)/t)^2.$$

Since the functions θ_1 and θ_2 are both finite everywhere, nonnegative and nondecreasing, so is the function θ defined by

$$\theta(R) = \sqrt{2\theta_1(R)^2 + \theta_2(R)^2}.$$

This function satisfies the inequality $\|D\mathcal{H}(t,x,y)\|_{(t,x,y)} \le \theta(\mathcal{H}(t,x,y)/t)$. □

Proposition 4.26. *If we fix $y \in M$, the function $\mathcal{H}_y :]0,+\infty[\times M$, defined by*

$$\mathcal{H}_y(t,x) = \mathcal{H}(t,y,x) = h_t(y,x),$$

is a viscosity solution of

$$\partial_t \mathcal{H}_y + H(y, \partial_y \mathcal{H}_y) = 0.$$

Proof. From Proposition 4.19, we know that $\mathcal{H}_y$ is locally semiconcave. Therefore, since the Hamiltonian H is convex in p, it suffices to check the evolutionary Hamilton–Jacobi equation at every point (t,x) where $\mathcal{H}_y$ is differentiable. If (t,x) is such a point and $\gamma : [0,t] \to M$ is a minimizer with $\gamma(0) = y$, $\gamma(t) = x$, by Corollary 4.24, we have

$$\begin{aligned}\partial_x \mathcal{H}_y(t,x) &= \partial_x \mathcal{H}(t,y,x) = \partial_v L(\gamma(t),\dot\gamma(t))\\ \partial_t \mathcal{H}_y(t,x) &= \partial_t \mathcal{H}(t,y,x) = -E(\gamma(t),\dot\gamma(t)) = -H(\gamma(t), \partial_v L(\gamma(t),\dot\gamma(t))).\end{aligned}$$

Therefore

$$\begin{aligned}\partial_t \mathcal{H}_y(t,x) + H(y, \partial_y \mathcal{H}_y(t,x)) \\ &= -H(\gamma(t), \partial_v L(\gamma(t), \dot\gamma(t))) + H(\gamma(t), \partial_v L(\gamma(t), \dot\gamma(t))) \\ &= 0. \qquad \square\end{aligned}$$

5. Action and viscosity (sub)solutions

Again in the sequel, we fix a Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ on the complete Riemannian manifold (M, g) and we will denote by $L : TM \to \mathbb{R}$ its associated Tonelli Lagrangian.

A first relation between action and viscosity subsolution is given in the next Proposition 5.5. To state it, it is convenient to recall the notion of evolution domination by a Lagrangian introduced in [7, Definition 14.2, page 1232].

To do it in an appropriate way, we first recall that for a curve $\gamma : I \to M$, where I is an interval in $\mathbb{R}$, the graph $\mathrm{Graph}(\gamma) \subset \mathbb{R} \times M$ of γ is

$$\mathrm{Graph}(\gamma) = \{(t, \gamma(t)) \mid t \in I\}.$$

Definition 5.1 (evolution domination by a Lagrangian). We will say that the function $U : S \to [-\infty, +\infty]$, where $S \subset \mathbb{R} \times M$ is evolution-dominated by L on S, if, for every absolutely continuous curve $\gamma : [a, b] \to M$ with $a < b \in \mathbb{R}$ and $\mathrm{Graph}(\gamma) \subset S$ whose action $\mathbb{L}(\gamma) = \int_a^b L(\gamma(s), \dot\gamma(s))\, ds$ is finite, we have

$$U(b, \gamma(b)) \le U(a, \gamma(a)) + \int_a^b L(\gamma(s), \dot\gamma(s))\, ds. \tag{5-1}$$

We will say that such a $U : S \to [-\infty, +\infty]$ is *strongly* evolution-dominated by L on S, if for every $(t, x), (t', x') \in S$, with $t < t'$, it satisfies the stronger condition

$$U(t', x') \le U(t, x) + h_{t'-t}(x, x'). \tag{5-2}$$

Remark 5.2. (1) If $U(a, \gamma(a))$ is finite, the inequality (5-1) is equivalent to

$$U(b, \gamma(b)) - U(a, \gamma(a)) \le \mathbb{L}(\gamma) = \int_a^b L(\gamma(s), \dot\gamma(s))\, dt.$$

(2) If $S \subset \mathbb{R} \times M$ is of the form $S = I \times M$, where I is an interval in $\mathbb{R}$, then $U : I \times M \to [-\infty, +\infty]$ is evolution-dominated by L if and only if it is *strongly* evolution-dominated by L.

Proposition 5.3. *Let $U : S \to [-\infty, +\infty]$ be evolution-dominated by L on the subset $S \subset \mathbb{R} \times M$.*

(1) *Assume* $\gamma : [a, b] \to M$ *is an absolutely continuous function curve, with* $\mathrm{Graph}(\gamma) \subset S$, *whose action is finite. If* $U(t_0, \gamma(t_0)) < +\infty$ (*resp.* $U(t_0, \gamma(t_0)) > -\infty$), *then* $U(t, \gamma(t)) < +\infty$ *for* $t \in [t_0, b]$ (*resp.* $U(t, \gamma(t)) > -\infty$ *for* $t \in [a, t_0]$).

(2) *Let* $S \subset \times\mathbb{R} \to M$ *be such that* $S = I \times W$, *where* I *is an interval in* $\mathbb{R}$ *and* $W \subset M$ *is open and connected. If for some* $(x_0, t_0) \in I \times M$ *we have* $U(x_0, t_0) < +\infty$ (*resp.* $U(x_0, t_0) > -\infty$) *then* $U < +\infty$ *everywhere on* $(I \cap]t_0, +\infty[) \times W$ (*resp.* $U > -\infty$ *everywhere on* $(I \cap]-\infty, t_0[) \times W$).

(3) *If* $U : S \to [-\infty, +\infty]$ *is* ***strongly*** *evolution-dominated by* L *on* S *and, for some* $(x_0, t_0) \in S$, *we have* $U(x_0, t_0) < +\infty$ (*resp.* $U(x_0, t_0) > -\infty$) *then* $U < +\infty$ *everywhere on* $S \cap (]t_0, +\infty[\times M)$ (*resp.* $U > -\infty$ *everywhere on* $S \cap (]-\infty, t_0[\times M)$).

Proof. For part (1) we note that the evolution domination of U by L on S, for $t \in]t_0, b]$, we get

$$U(t, \gamma(t)) \leq U(t_0, \gamma(t_0)) + \int_t^{t_0} L(\gamma(s), \dot\gamma(s))\, ds.$$

Since $\int_t^{t_0} L(\gamma(s), \dot\gamma(s))\, ds$ is finite, the inequality $U(t_0, \gamma(t_0)) < +\infty$ implies $U(t, \gamma(t)) < +\infty$ for $t \in]t_0, b]$.

For part (2), since W is open and connected in the manifold M, given $t > t_0$ and $x \in$, we can find a smooth curve $\gamma : [t_0, t] \to W$ with $\gamma(t_0) = x_0$ and $\gamma(t) = x$. Since L is continuous and γ is C^1, the action $\mathbb{L}(\gamma)$ of γ is finite. Moreover $\mathrm{Graph}(\gamma) \subset I \times W$, the evolution domination condition implies $U(t, x) \leq U(t_0, x_0) + \mathbb{L}(\gamma) < +\infty$.

For part (3), it suffices to observe that, for $(t, x) \in S \cap (]t_0, +\infty[\times M)$, we have $|h_{t-t_0}(x_0, x)| < +\infty$ and the strong L domination implies $U(t, x) \leq U(t_0, x_0) + h_{t-t_0}(x_0, x)$. □

Proposition 5.4. *Suppose* $U : O \to \mathbb{R}$ *is finite-valued and evolution-dominated by* L *on the open subset* $O \subset \mathbb{R} \times M$. *Then* U *is locally bounded on* O. *The function* U *is locally strongly evolution-dominated by* L; *that is, for every* $(t_0, x_0) \in O$ *there exists a neighborhood* $V \subset O$ *of* (t_0, x_0) *such that the restriction* $U|V$ *is strongly evolution-dominated by* L *on* V.

Proof. Fix a compact neighborhood of the form $[t_0-2\delta, t_0+2\delta] \times \bar B(x_0, 3r) \subset O$ of $(t_0, x_0) \in O$. For any $x \in \bar B(x_0, 2r)$ and $t \in [t_0 - \delta, t_0 + \delta]$, the minimizing geodesic $\gamma_{x_0,x} : [t_0 - 2\delta, t] \to M$ joining x_0 to x is contained in $\bar B(x_0, 2r)$ and, by Lemma 4.11, its action $\mathbb{L}(\gamma_{x_0,x})$ is less than

$$\big(t - (t_0 - 2\delta)\big)\, A\left(\frac{d(x_0, x)}{t - (t_0 - 2\delta)}\right) \leq 3\delta A\left(\frac{2r}{\delta}\right).$$

Since the function is evolution-dominated by L on $O \supset [t_0-2\delta, t_0+2\delta] \times \bar{B}(x_0, 3r)$, we obtain

$$U(t,x) \leq U(t_0-2\delta, x_0) + \mathbb{L}(\gamma_{x_0,x}) \leq U(t_0-2\delta, x_0) + 3\delta A(2r/\delta).$$

This shows that U is bounded above on the compact neighborhood of (t_0, x_0) given by $[t_0-\delta, t_0+\delta] \times \bar{B}(x_0, 2r)$. In the same way the minimizing geodesic $\gamma_{x,x_0} : [t, t_0+2\delta] \to M$ joining x to x_0 is contained in $\bar{B}(x_0, 2r)$ and has action $\mathbb{L}(\gamma_{x,x_0})$ less than $(t_0+2\delta-t)A(d(x,x_0)/(t_0+2\delta-t)) \leq 3\delta A(2r/\delta)$. Therefore

$$U(t_0+2\delta, x_0) \leq U(t,x) + 3\delta A(2r/\delta),$$

which implies that U is bounded below on $[t_0-\delta, t_0+\delta] \times \bar{B}(x_0, 2r)$. We then set

$$K = 2\sup\bigl\{|U(t,x)| \;\big|\; (t,x) \in [t_0-\delta, t_0+\delta] \times \bar{B}(x_0, 2r)\bigr\} < +\infty.$$

Fix $(t,x), (t',x') \in [t_0-\delta, t_0+\delta] \times \bar{B}(x_0, 2r)$, with $t' < t$.

We obviously get

$$U(t',x') - U(t,x) \leq K \leq h_{t'-t}(x,x') \quad \text{for } h_{t'-t}(x,x') \geq K. \tag{5-3}$$

If $h_{t'-t}(x,x') \leq K$, pick a minimizer $\gamma : [t,t'] \to M$, with $\gamma(t) = x$, $\gamma(t') = x'$ and $h_{t'-t}(x,x') = \mathbb{L}(\gamma) \leq K$, since $|t'-t| \leq 2\delta$, from Lemma 4.3, we obtain

$$\ell_g(\gamma\,|\,[t,t']) \leq \eta_{K,2\delta}(|t'-t|),$$

where $\eta_{K,2\delta} : [0,+\infty[\to [0,+\infty[$ is a modulus of continuity; i.e., the function $\eta_{K,2\delta}$ is continuous at 0 and $\eta_{K,2\delta}(0) = 0$. Therefore, we can find $\epsilon > 0$, with $\epsilon < \delta$, such that $\eta_{K,2\delta}(s) \leq r$ for all $s \leq 2\epsilon$. Hence, if we further assume that

$$(t,x), (t',x') \in [t_0-\epsilon, t_0+\epsilon] \times \bar{B}(x_0, r),$$

we obtain $\ell_g(\gamma\,|\,[t,t']) \leq r$ and $\gamma([t,t']) \subset \bar{B}(x_0, 2r)$. Since the graph of γ is contained in $[t_0-\delta, t_0+\delta] \times \bar{B}(x_0, 2r) \subset O$ and U is evolution-dominated by L on O, we get

$$U(t',x') - U(t,x) \leq h_{t'-t}(x,x').$$

Together with (5-3), this shows that U is *strongly* evolution-dominated by L on $[t_0-\epsilon, t_0+\epsilon] \times \bar{B}(x_0, r)$. □

The reader will notice that the proof of the next proposition, giving the connection between evolution domination and viscosity subsolution, is very similar to the (standard) proof of Proposition 14.3 in [7], once we have Corollary 2.3. We provide a complete proof for the reader's convenience.

Proposition 5.5. *Let H be a Tonelli Hamiltonian on the complete Riemannian manifold M. Suppose $U : O \to \mathbb{R}$ is a continuous function defined on the open subset O. Then U is a viscosity subsolution of*

$$\partial_t U + H(x, \partial_x U) = 0, \tag{5-4}$$

on O if and only if it is evolution-dominated by L on O.

Proof. Assume that U is a viscosity subsolution of (5-4). We prove that

$$U(b, \gamma(b)) - U(a, \gamma(a)) \leq \int_a^b L(\gamma(s), \dot{\gamma}(s))\, dt, \tag{5-5}$$

holds for an absolutely continuous curve $\gamma : [a, b] \to M$, with $\mathrm{Graph}(\gamma) \subset O$.

If U is smooth, the Fenchel inequality (3-2) between L and H yields

$$\partial_x U(t, x)(v) \leq L(x, v) + H(x, \partial_x U(t, x)) \quad \text{for all } v \in T_x M.$$

Since the viscosity subsolution U of (5-4) is smooth on O, we have

$$\partial_t U(t, x) + H\,(x, \partial_x U(t, x)) \leq 0 \text{ everywhere on } O.$$

We combine the two inequalities to obtain

$$\partial_t U(t, x) + \partial_x U(t, x)(v) \leq L(x, v) \quad \text{for all } (t, x, v) \text{ with } (t, x) \in O,\ v \in T_x M.$$

Therefore, since $\mathrm{Graph}(\gamma) \subset O$ and γ is absolutely continuous, we obtain

$$\partial_t U(t, \gamma(t)) + \partial_x U(t, \gamma(t))(\dot{\gamma}(t)) \leq L(\gamma(t), \dot{\gamma}(t)) \quad \text{for almost all } s \in [a, b].$$

By integration, this proves the desired inequality.

For U just continuous, since $\gamma([a, b])$ is a compact subset, we can use Corollary 2.3 to reduce, by an approximation argument, this continuous case to the smooth case.

Let us now assume that U satisfies (5-5) for every absolutely continuous curve $\gamma : [a, b] \to M$, with $\mathrm{Graph}(\gamma) \subset O$. To prove that U is a viscosity subsolution of (5-4), consider a C^1 function $\Phi : O \to \mathbb{R}$, with $\Phi \geq U$ and $\Phi(t, x) = U(t, x)$, for some $(t, x) \in O$. If $v \in T_x M$, let $\gamma : [t-1, t] \to M$ be a smooth curve with $\gamma(t) = x$ and $\dot{\gamma}(t) = v$. Since γ is continuous and O is open for $\epsilon > 0$ small enough, we have $\mathrm{Graph}(\gamma\,|\,[t-\epsilon, t]) \subset O$. Using $\Phi \geq U$ and inequality (5-5), we get

$$\begin{aligned}\Phi(t, \gamma(t)) - \Phi(t-\epsilon, \gamma(t-\epsilon)) &\leq U(t, \gamma(t)) - U(t-\epsilon, \gamma(t-\epsilon)) \\ &\leq \int_{t-\epsilon}^{t} L(\gamma(s), \dot{\gamma}(s))\, dt.\end{aligned}$$

Dividing by ϵ and letting $\epsilon \to 0$ yields

$$\partial_t \Phi(t,x) + \partial_x \Phi(t,x)[v] \le L(x,v),$$

or equivalently

$$\partial_t \Phi(t,x) + \partial_x \Phi(t,x)[v] - L(x,v) \le 0.$$

Taking the supremum over all $v \in T_x M$, we obtain

$$\partial_t \Phi(t,x) + H(x, \partial_x \Phi(t,x)) \le 0. \qquad \square$$

6. A construction of viscosity solutions

Again in the sequel, we fix a Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ on the complete Riemannian manifold (M, g) and we will denote by $L : TM \to \mathbb{R}$ its associated Tonelli Lagrangian.

We will give a rather general way to obtain viscosity solutions on open subsets of $\mathbb{R} \times M$ of the Hamilton–Jacobi equation (1-1).

We start with a nonempty subset $K \subset \mathbb{R} \times M$. Besides being nonempty, we do not impose any other property on K. We set

$$t_{K,\inf} = \inf\{t \mid (t,x) \in K\},$$

We consider a function $U : K \to [-\infty, +\infty[$. We do not assume U continuous or even measurable; the only restriction (for convenience) is that U does not take the value $+\infty$. See Remark 6.1(1), however. We can define the function $\hat{U}$ on $]t_{K,\inf}, +\infty[\times M \to [-\infty, +\infty[$ by

$$\hat{U}(t,x) = \inf\{U(\tilde{t}, \tilde{x}) + h_{t-\tilde{t}}(\tilde{x}, x) \mid (\tilde{t}, \tilde{x}) \in K \text{ and } \tilde{t} \le t\}. \tag{6-1}$$

Note that this definition makes sense for $t > t_{K,\inf}$, since for such a t the set $\{\tilde{t} \mid (\tilde{t}, \tilde{x}) \in K \text{ and } \tilde{t} \le t\}$ is not empty.

Remark 6.1. (1) Suppose that we have a function $U : K \to [-\infty, +\infty]$, which may assume the value $+\infty$. If U is not identically $+\infty$, define K_f as

$$K_f = \{(t,x) \mid U(t,x) \ne +\infty\}.$$

Then K_f is not empty and $U_f = U \mid K_f$ never takes the value $+\infty$. We can then define $\hat{U}_f :]t_{\inf}(K_f), +\infty[\times M \to [-\infty, +\infty[$ as above by

$$\hat{U}_f(t,x) = \inf\{U(\tilde{t}, \tilde{x}) + h_{t-\tilde{t}}(\tilde{x}, x) \mid (\tilde{t}, \tilde{x}) \in K_f \text{ and } \tilde{t} \le t\}.$$

If $t_{K,\inf} = t_{K_f,\inf}$ or equivalently

$$t_{K,\inf} = \inf\{t \mid (t,x) \in K \text{ and } U(t,x) \ne +\infty\}, \tag{6-2}$$

then we have

$$\hat{U}_f(t,x)=\hat{U}(t,x)=\inf\{U(\tilde{t},\tilde{x})+h_{t-\tilde{t}}(\tilde{x},x)\mid(\tilde{t},\tilde{x})\in K \text{ and } \tilde{t}\le t\}.$$

(2) A special case of the construction above is the Lax–Oleinik evolution; see Definition 8.2 and Remark 8.4(3) below.

Theorem 6.2. *Let $U : K \to [-\infty,+\infty[$ be a function defined on the subset $K\subset\mathbb{R}\times M$.*

Define the function $\hat{U}$ on $]t_{\inf,K},+\infty[\times M\to[-\infty,+\infty[$ by

$$\hat{U}(t,x)=\inf\{U(\tilde{t},\tilde{x})+h_{t-\tilde{t}}(\tilde{x},x)\mid(\tilde{t},\tilde{x})\in K \text{ and } \tilde{t}\le t\},\tag{6-3}$$

where $t_{K,\inf}=\inf\{t\mid(t,x)\in K\}$. This function $\hat{U}$, is strongly evolution-dominated by L on $]t_{\inf,K},+\infty[\times M$. Moreover, if $\hat{U}(T,X)$ is finite for some $X\in M$ and some $T\in\,]t_{K,\inf},+\infty[$, then the function $\hat{U}$ is

(i) *finite everywhere on $]t_{K,\inf},T[\times M$;*

(ii) *bounded on every compact subset of $]t_{\inf,K},T[\times M$;*

(iii) *continuous, locally semiconcave on $]t_{K,\inf},T[\times M\setminus\bar{K}$;*

(iv) *a viscosity solution of the evolutionary Hamilton–Jacobi (1-1) on*

$$]t_{K,\inf},T[\times M\setminus\bar{K}.$$

Proof. To prove the strong evolution domination, note that for $(t,x),(t',x')\in\,]t_{\inf,K},+\infty[\times M$, with $t'<t$, if $\tilde{t}\le t'$, then $\tilde{t}\le t$. Therefore, for $(\tilde{t},\tilde{x})\in K$, with $\tilde{t}\le t'$, from (6-1), we get

$$\hat{U}(t,x)\le U(\tilde{t},\tilde{x})+h_{t-\tilde{t}}(\tilde{x},x)\le U(\tilde{t},\tilde{x})+h_{t'-\tilde{t}}(\tilde{x},x')+h_{t-t'}(x',x).$$

Again from (6-1), taking the inf over all $(\tilde{t},\tilde{x})\in K$, with $\tilde{t}\le t'$, we obtain

$$\hat{U}(t,x)\le\hat{U}(t',x')+h_{t-t'}(x',x),$$

which means that $\hat{U}$ is strongly evolution-dominated by L on $]t_{\inf,K},+\infty[\times M$.

For the rest of the proof, we assume that $\hat{U}(T,X)$ is finite for some $X\in M$ and $T\in\,]t_{K,\inf},+\infty[$.

Property (i) is a consequence of (ii). We now prove (ii). Let C be a nonempty compact subset of $]t_{\inf,K},T[\times M$. By the strong L evolution domination on $]t_{K,\inf},T[\times M$, we have

$$\hat{U}(T,X)\le\hat{U}(t,x)+h_{T-t}(x,X)\quad\text{for } t\in\,]t_{\inf,K},T[.$$

Since $(t,x)\mapsto h_{T-t}(x,X)$ is finite and continuous on $]\tilde{t},T[\times M$, which implies that it is bounded from above on the compact subset $C\subset\,]\tilde{t},T[\times M$. We conclude that $\hat{U}$ is bounded from below on C.

It remains to show that $\hat{U}$ is bounded from above on C. By compactness of C, we have $t_{\inf,C} = \inf\{t \mid (t,x) \in C\} > t_{\inf,K}$. In particular, we can find $(\tilde{t}, \tilde{x}) \in K$ with $t_{\inf,C} > \tilde{t} \geq t_{\inf,K}$. From (6-1)

$$\hat{U}(t,x) \leq U(\tilde{t},\tilde{x}) + h_{t-\tilde{t}}(\tilde{x},x) \quad \text{for } t > \tilde{t}.$$

Note that we are assuming that U does not take the value $+\infty$, hence $U(\tilde{t},\tilde{x}) < +\infty$. Since C is a compact set contained in $]\tilde{t}, T[\times M$ and $(t,x) \mapsto h_{t-\tilde{t}}(\tilde{x},x)$ is finite and continuous on $]\tilde{t}, +\infty[\times M$, this function $(t,x) \mapsto h_{t-\tilde{t}}(\tilde{x},x)$ is bounded on the compact set C. Hence $\hat{U}(t,x)$ is bounded from above on C.

To prove (iii) and (iv), we first prove a lemma.

Lemma 6.3. *Under the hypothesis of Theorem 6.2, suppose that $\hat{U}(T,X)$ is finite for some $X \in M$ and $T \in]t_{K,\inf}, +\infty[$. Assume that $\delta > 0$ and $(t_0,x_0) \in]t_{\inf,K}, T[\times M$ are such that*

$$[t_0-\delta, t_0+\delta] \times \bar{B}(x_0,\delta) \subset]t_{\inf,K}, T[\times M \setminus \bar{K}.$$

We can find $\epsilon > 0$, with $2\epsilon < \delta$, such that, for all $(t,x) \in [t_0-\epsilon, t_0+\epsilon] \times \bar{B}(x_0,\epsilon)$, we have

$$\hat{U}(t,x) = \inf\{\hat{U}(t',x') + h_{t-t'}(x',x) \mid (t',x') \in [t_0-\delta, t_0-2\epsilon] \times \bar{B}(x_0,\delta)\}.$$

Proof. For all $\epsilon > 0$, with $2\epsilon < \delta$, the inequality

$$\hat{U}(t,x) \leq \inf\{\hat{U}(t',x') + h_{t-t'}(x',x) \mid (t',x') \in [t_0-\delta, t_0-2\epsilon] \times \bar{B}(x_0,\delta)\}$$

follows from the just established strong L domination of $\hat{U}$. Therefore, it suffices to show that, we can find $\epsilon > 0$, with $2\epsilon < \delta$, such that, for all $(t,x) \in [t_0-\epsilon, t_0+\epsilon] \times \bar{B}(x_0,\epsilon)$ and all $\eta \in]0,1]$, we have

$$\inf\{\hat{U}(t',x') + h_{t-t'}(x',x) \mid (t',x') \in [t_0-\delta, t_0-2\epsilon] \times \bar{B}(x_0,\delta)\} \leq \hat{U}(t,x) + \eta.$$

From the already established part (ii), the function $\hat{U}$ is bounded on the compact subset $[t_0-\delta, t_0+\delta] \times \bar{B}(x_0,\delta)$ of $]t_{\inf,K}, T[\times M \setminus \bar{K}$. Therefore

$$A = 1 + 2\sup\{|\hat{U}(t,x)| \mid [t_0-\delta, t_0+\delta] \times \bar{B}(x_0,\delta)\} < +\infty. \tag{6-4}$$

Denote by $\eta_{A,2\delta}$ the continuity modulus provided by Lemma 4.3. Hence, for every absolutely continuous curve $\gamma : [a,b] \to M$, with $b-a \leq 2\delta$ and $\mathbb{L}(\gamma) \leq A$, we have

$$d(\gamma(t'),\gamma(t)) \leq \ell_g(\gamma\,|\,[t,t']) \leq \eta_{A,2\delta}(|t'-t|) \quad \text{for all } t,t' \in [a,b]. \tag{6-5}$$

Since $\eta_{A,2\delta}$ is a modulus of continuity, we can pick $\epsilon > 0$, with $3\epsilon < \delta$ such that

$$\eta_{A,2\delta}(\alpha) < \delta/3 \quad \text{for } 0 \leq \alpha \leq 3\epsilon. \tag{6-6}$$

Fix now $(t,x) \in\in [t_0-\epsilon, t_0+\epsilon] \times \bar{B}(x_0,\epsilon)$ and $\eta \in]0,1]$. By the definition of $\hat{U}$, (6-1), we can find $(\tilde{t},\tilde{x}) \in K$ such that

$$U(\tilde{t},\tilde{x}) + h_{t-\tilde{t}}(\tilde{x},x) \le \hat{U}(t,x)+\eta. \tag{6-7}$$

Pick a minimizer $\gamma : [\tilde{t},t] \to M$, with $\gamma(\tilde{t}) = \tilde{x}$ and $\gamma(t) = x$. Since $(\tilde{t},\gamma(\tilde{t})) \in K$, which is disjoint from $[t_0-\delta, t_0+\delta] \times \bar{B}(x_0,\delta)$, and $(t,\gamma(t)) \in [t_0-\epsilon, t_0+\epsilon] \times \bar{B}(x_0,\epsilon) \subset]t_0-\delta, t_0+\delta[\times \mathring{B}(x_0,\delta)$, we can find $s \in]\tilde{t},t[$ with $(s,\gamma(s)) \in \partial\big([t_0-\delta,t_0+\delta]\times \bar{B}(x_0,\delta)\big)$. We have

$$t_0-\delta \le s \le t \le t_0+\epsilon < t_0+\delta \text{ and } \gamma(s) \in \bar{B}(x_0,\delta). \tag{6-8}$$

Since γ is a minimizer and $\gamma(\tilde{t}) = \tilde{x}, \gamma(t) = x$, we have

$$\begin{aligned} h_{t-\tilde{t}}(\tilde{x},x) &= h_{t-\tilde{t}}(\gamma(\tilde{t}),\gamma(t)) \\ &= h_{s-\tilde{t}}(\gamma(\tilde{t}),\gamma(s)) + h_{t-s}(\gamma(s),\gamma(t)) \\ &= h_{s-\tilde{t}}(\tilde{x},\gamma(s)) + h_{t-s}(\gamma(s),x), \end{aligned}$$

which, by (6-7), implies

$$U(\tilde{t},\tilde{x}) + h_{s-\tilde{t}}(\tilde{x},\gamma(s)) + h_{t-s}(\gamma(s),\gamma(t)) \le \hat{U}(t,x)+\eta.$$

But, again by the definition (6-1) of $\hat{U}$, we have

$$\hat{U}(s,\gamma(s)) \le U(\tilde{t},\tilde{x}) + h_{s-\tilde{t}}(\tilde{x},\gamma(s)).$$

Combining the last two inequalities, we obtain

$$\hat{U}(s,\gamma(s)) + h_{t-s}(\gamma(s),\gamma(t)) \le \hat{U}(t,x)+\eta, \tag{6-9}$$

Claim We have $s \le t_0 - 2\epsilon$.

From this claim, we can finish the proof of the Lemma.

In fact, combining the claim and (6-8), we have $(s,\gamma(s)) \in [t_0-\delta, t_0-2\epsilon] \times \bar{B}(x_0,\delta)$. Therefore, using (6-9), we obtain

$$\begin{aligned} \inf\big\{\hat{U}(t',x') + h_{t-t'}(x',\gamma(t)) \mid (t',x') \in [t_0-\delta, t_0-2\epsilon] \times \bar{B}(x_0,\delta)\big\} \\ \le \hat{U}(s,\gamma(s)) + h_{t-s}(\gamma(s),\gamma(t)) \\ \le \hat{U}(t,x)+\eta. \end{aligned}$$

It remains to prove the claim. Since $(s,\gamma(s)) \in \partial\big([t_0-\delta,t_0+\delta]\times \bar{B}(x_0,\delta)\big)$ and $s < t_0+\delta$, either $s = t_0-\delta$ or $\gamma(s) \in \partial\bar{B}(x_0,\delta)$. In the first case, we get $s = t_0-\delta < t-2\epsilon$ and the claim holds. In the second case, we have $d(x_0,\gamma(s)) = \delta$. But $\gamma(t) = y \in \bar{B}(x_0,\epsilon)$, hence, using $3\epsilon < \delta$, we get

$$d(\gamma(t),\gamma(s)) \ge \delta-\epsilon > \delta/3.$$

We now observe that (6-9) implies

$$\mathbb{L}(\gamma\,|\,[s,t]) = h_{t-s}(\gamma(s),\gamma(t)) \le \hat{U}(t,x) - \hat{U}(s,\gamma(s)) + \eta \le A < +\infty, \quad (6\text{-}10)$$

since $\eta \le 1$, both (t,x), $(s,\gamma(s))$ are in $[t_0-\delta, t_0+\delta] \times \bar{B}(x_0,\delta)$ and A is given by (6-4). Furthermore, we have $s < t$ and $s, t \in [t_0-\delta, t_0+\delta]$, which yields $0 < t - s \le 2\delta$. Therefore, by the property (6-5) defining $\eta_{A,2\delta}$, we get

$$d(\gamma(t),\gamma(s)) \le \eta_{A,2\delta}(t-s).$$

Since $d(\gamma(t),\gamma(s)) > \delta/3$, it follows from by the definition of ϵ, (6-6), that $t - s > 3\epsilon$, which implies

$$s < t - 3\epsilon \le (t_0+\epsilon) - 3\epsilon = t_0 - 2\epsilon. \qquad \square$$

End of the proof of Theorem 6.2. To prove (iii) and (iv), we fix (t_0, x_0) in the open subset $]t_{K,\inf}, T[\, \times M \setminus \bar{K}$, we then pick $\delta > 0$ such that

$$[t_0-\delta, t_0+\delta] \times \bar{B}(x_0,\delta) \subset\,]t_{\inf,K}, T[\, \times M \setminus \bar{K}.$$

By Lemma 6.3, we can find $\epsilon > 0$ such that

$$\hat{U}(t,x) = \inf\{\hat{U}(t',x') + h_{t-t'}(x',x) \mid (t',x') \in [t_0-\delta, t_0-2\epsilon] \times \bar{B}(x_0,\delta)\}, \quad (6\text{-}11)$$

for all $(t,x) \in [t_0-\epsilon, t_0+\epsilon] \times \bar{B}(x_0,\epsilon)$. The map

$$[(t,x),(t',x')] \mapsto (t-t', x', x)$$

is smooth and takes values in $]0,+\infty[\, \times M \times M$ on the compact set

$$\big([t_0-\epsilon, t_0+\epsilon] \times \bar{B}(x_0,\epsilon)\big) \times \big([t_0-\delta, t_0-2\epsilon] \times \bar{B}(x_0,\delta)\big).$$

Since, by Proposition 4.19, the map $(s,x',x) \mapsto h_s(x',x)$ is locally semiconcave on $]0,+\infty[\, \times M \times M$, we conclude that $[(t,x),(t',x')] \mapsto h_{t-t'}(x',x)$ is locally semiconcave on a neighborhood of $\big([t_0-\epsilon, t_0+\epsilon] \times \bar{B}(x_0,\epsilon)\big) \times \big([t_0-\delta, t_0-2\epsilon] \times \bar{B}(x_0,\delta)\big)$. Hence, since $[t_0-\delta, t_0-2\epsilon] \times \bar{B}(x_0,\delta)$ is compact, we conclude that the family of maps

$$(t,x) \mapsto h_{t-t'}(x',x),\ (t',x') \in [t_0-\delta, t_0-2\epsilon] \times \bar{B}(x_0,\delta)$$

is uniformly locally semiconcave on a neighborhood of the compact set $[t_0-\epsilon, t_0+\epsilon] \times \bar{B}(x_0,\epsilon)$; see [8, Appendix A]. Therefore, so is the family $(t,x) \mapsto \hat{U}(t',x') + h_{t-t'}(x',x)$, $(t',x') \in [t_0-\delta, t_0-2\epsilon] \times \bar{B}(x_0,\delta)$, which by equality (6-11) implies that the finite function $\hat{U}$ is locally semiconcave (and therefore continuous) on a neighborhood $[t_0-\delta, t_0+\delta] \times \bar{B}(x_0,r)$. See [8, Proposition A.16, p. 34–35].

We then observe that by Proposition 4.26, since H does not depend on the time t, for each $(t', x') \in [t_0 - \delta, t_0 - 2\epsilon] \times \bar{B}(x_0, \delta)$, the function $(t, x) \mapsto \hat{U}(t', x') + h_{t-t'}(x', x)$ is a viscosity solution of the evolutionary Hamilton–Jacobi (1-1) on $]t_0 - 2\epsilon, +\infty[\times M$. Since we already now that $\hat{U}$ is finite and continuous on a neighborhood of $[t_0 - \epsilon, t_0 + \epsilon] \times \bar{B}(x_0, \epsilon)$, by Corollary 2.5 and equality (6-11), we conclude that $\hat{U}$ is a viscosity solution of the evolutionary Hamilton–Jacobi (1-1) on a neighborhood of $[t_0 - \epsilon, t_0 + \epsilon] \times \bar{B}(x_0, \epsilon)$. □

Proposition 6.4. *Assume $C \subset M$ is a closed subset and $a < b \in \mathbb{R}$. Suppose $U : [a, b] \times C \to \mathbb{R}$ is continuous and strongly evolution-dominated by L on $[a, b] \times C$. Set $K = \{a\} \times C \cup [a, b] \times \partial C \subset \mathbb{R} \times M$. Call $\hat{U}$ the function defined on $]a, +\infty[\times M$ by (6-1) using the restriction $U \,|\, K$:*

$$\hat{U}(t, x) = \inf\{U(t', x') + h_{t-t'}(x', x) \mid (t', x') \in K, t' < t\} \quad \text{for } t > a \text{ and } x \in M. \tag{6-12}$$

The function $\hat{\hat{U}} : [a, b] \times C \to \mathbb{R}$ defined by

$$\hat{\hat{U}}(t, x) = \begin{cases} U(a, x) & \text{for } t = a \text{ and } x \in C, \\ \hat{U}(t, x) & \text{for } t > a \text{ and } x \in C, \end{cases}$$

is continuous, strongly evolution-dominated by L and $\geq U$ on $[a, b] \times C$, with $\hat{\hat{U}} \,|\, K = \hat{U} \,|\, K$.

Moreover, this function $\hat{\hat{U}}$ is a locally semiconcave viscosity solution of the evolutionary on Hamilton–Jacobi (1-1) on $]a, b[\times \mathring{C}$.

Proof. We first note that the inequality $\hat{U} \geq U$ on $]a, +\infty[\times M$ follows from the definition of $\hat{U}$ and the strongly L evolution domination of U on $[a, b] \times C$. This obviously implies that $\hat{\hat{U}} \geq U$ on $[a, b] \times C$.

Since, by Theorem 6.2, the function $\hat{U}$ is strongly L evolution-dominated on $]a, +\infty[\times M$, we obtain that $\hat{\hat{U}}$ is strongly L evolution-dominated on $]a, b] \times C$. From the definition of $\hat{U}$, we conclude that $\hat{\hat{U}}$ is strongly L evolution-dominated on $[a, b] \times C$.

Since by Theorem 6.2, the function $\hat{U}$ is continuous on $]a, +\infty[\times M \setminus \bar{K} \supset]a, b] \times \mathring{C}$. We have to show continuity at every point of $K = \{a\} \times C \cup [a, b] \times \partial C$. Let us start with continuity at (a, x) with $x \in C$. Using that $\hat{\hat{U}} \geq U$ is strongly L evolution-dominated on $[a, b] \times C$, we get

$$U(t, y) \leq \hat{\hat{U}}(t, y) \leq \hat{\hat{U}}(a, y) + h_{t-a}(y, y) \leq U(a, y) + (t - a)A(0).$$

By continuity of U, we obtain the continuity of $\hat{\hat{U}}$ at every point of $\{a\} \times C$. It remains to show that $\hat{\hat{U}}$ is continuous at (t_0, x_0), with $a < t_0 \leq b$ and $x_0 \in \partial C$. We will show at the same time that $\hat{\hat{U}}(t_0, x_0) = \hat{U}(t_0, x_0)$. Fix $t' \in]a, t_0[$. Since

$\hat{\hat{U}} = \hat{U} \geq U$ on $]a, b] \times C$, for all $(t, x) \in]t', b] \times C$, we have

$$U(t, x) \leq \hat{\hat{U}}(t, x) = \hat{U}(t, x) \leq U(t', x_0) + h_{t-t'}(x_0, x), \tag{6-13}$$

where the last inequality follows from the definition of $\hat{U}$, since $t' < t$ and $(t', x_0) \in]a, t_0[\times \partial C \subset K$. If we apply this inequality with $(t, x) = (t_0, x_0)$, we obtain

$$U(t_0, x_0) \leq \hat{\hat{U}}(t_0, x_0) \leq U(t', x_0) + h_{t_0-t'}(x_0, x_0) \leq U(t', x_0) + A(0)(t_0 - t').$$

If we let $t' \to t_0$, by continuity of U, this last inequality yields $\hat{\hat{U}}(t_0, x_0) = U(t_0, x_0)$.

If, in equality (6-13), we keep $t' \in]a, t_0[$ fixed and we let $(t, x) \to (t_0, x_0)$, by continuity of U and h, we obtain

$$\begin{aligned} U(t_0, x_0) &\leq \liminf_{(t,x)\to(t_0,x_0)} \hat{\hat{U}}(t, x) \leq \limsup_{(t,x)\to(t_0,x_0)} \hat{\hat{U}}(t, x) \\ &\leq U(t', x_0) + h_{t-t'}(x_0, x_0) \leq U(t', x_0) + A(0)(t - t'). \end{aligned}$$

Letting again $t' \to t_0$, we conclude that $\lim_{(t,x)\to(t_0,x_0)} \hat{\hat{U}}(t, x) = U(t_0, x_0) = \hat{\hat{U}}(t_0, x_0)$. Therefore we finished both the proof of the continuity of $\hat{\hat{U}} = \hat{U}$, and the equality $\hat{\hat{U}} \mid K = U \mid K$.

The fact that $\hat{\hat{U}}$ is a locally semiconcave viscosity solution of the evolutionary Hamilton–Jacobi equation (1-1) on $]a, b[\times \mathring{C}$ follows also from Theorem 6.2, since $\hat{\hat{U}} = \hat{U}$ on $]a, b[\times \mathring{C}$. □

Theorem 6.5. *Suppose $O \subset \mathbb{R} \times M$ is an open subset. If $U : O \to \mathbb{R}$ is a continuous viscosity solution of the evolutionary Hamilton–Jacobi equation (1-1) on O, then it is locally semiconcave.*

Moreover, for every $(t, x) \in O$, we can find $(t', x') \in O$, with $t' < t$, such that

$$U(t, x) = U(t', x') + h_{t-t'}(x', x).$$

Proof. Fix $(t_0, x_0) \in O$. Since U is a viscosity solution on O, from Proposition 5.5 we obtain that U is dominated by L on O. By Proposition 5.4, we can find a neighborhood V of (t_0, x_0) in O on which U is strongly dominated by L. Without loss of generality, we can assume that $V = [t_0 - \eta, t_0 + \eta] \times \bar{B}(x_0, \eta) \subset O$, for some $\eta > 0$. We set $\mathfrak{K} = \{t_0 - \eta\} \times \bar{B}(x_0, \eta) \cup [t_0 - \eta, t_0 + \eta] \times \partial \bar{B}(x_0, \eta)$.

By Proposition 6.4, the function $\hat{U} : [t_0 - \eta, t_0 + \eta] \times \bar{B}(x_0, r) \to \mathbb{R}$ defined by

$$\hat{U}(t, x) = \begin{cases} U(t, x) & \text{if } t = a \text{ and } x \in \bar{B}(x_0, \eta), \\ \inf\{U(t', x') + h_{t-t'}(x', x) \mid (t', x') \in \mathfrak{K}, t' < t\} & \\ & \text{if } t > a \text{ and } x \in \bar{B}(x_0, \eta), \end{cases}$$

is continuous, is a locally semiconcave viscosity solution of the evolutionary on Hamilton–Jacobi (1-1) on $]t_0-\eta, t_0+\eta[\times \mathring{\bar{B}}(x_0,r)$ and satisfies $\hat{\hat{U}} = U$ on $\mathfrak{K} = \{t_0-\eta\}\times \bar{B}(x_0,\eta)\cup[t_0-\eta, t_0+\eta]\times\partial\bar{B}(x_0,\eta)$. Since $\hat{\hat{U}} = U$ on $\mathfrak{K}=\{t_0-\eta\}\times\bar{B}(x_0,\eta)\cup[t_0-\eta, t_0+\eta]\times\partial\bar{B}(x_0,\eta)$, Corollary 2.7 of the maximum principle implies $\hat{\hat{U}} = U$ on $[t_0-\eta, t_0+\eta]\times\bar{B}(x_0,r)$. But, by Proposition 6.4, the function $\hat{\hat{U}}$ is locally semiconcave on $]t_0-\eta, t_0+\eta[\times\mathring{\bar{B}}(x_0,\eta)$. This proves the first part of the theorem.

To prove the remaining part of the theorem, we use the equality $\hat{\hat{U}} = U$ on $[t_0-\eta, t_0+\eta]\times\bar{B}(x_0,r)$ and the definition of $\hat{\hat{U}}$ to find a sequence $(t'_n, x'_n)\in\mathfrak{K} = \{t_0-\eta\}\times\bar{B}(x_0,\eta)\cup[t_0-\eta, t_0+\eta]\times\partial\bar{B}(x_0,\eta)$, with $t'_n < t_0$, such that

$$U(t_0,x_0)\le U(t'_n,x'_n)+h_{t_0-t'_n}(x'_n,x_0)\to U(t_0,x_0)\quad\text{ as } n\to+\infty. \tag{6-14}$$

Since $\mathfrak{K}$ is compact, extracting if necessary, we can assume that $(t'_n, x'_n)\to(t',x')\in\mathfrak{K}$ and

$$U(t'_n,x'_n)+h_{t_0-t'_n}(x'_n,x_0)\le U(t_0,x_0)+1.$$

By continuity of U and convergence of (t'_n, x'_n), we have

$$m=\sup_n U(t_0,x_0)-U(t'_n,x'_n)+1<+\infty.$$

Therefore

$$h_{t-t'_n}(x'_n,x)\le m\quad\text{ for all } n.$$

Using the left side of the inequality (4-7) in Lemma 4.11, we obtain

$$-C(K)(t_0-t'_n)+Kd(x_0,x'_n)\le h_{t_0-t'_n}(x'_n,x_0)\le m\quad\text{ for all } n \text{ and all } K\ge 0.$$

Taking the limit as $n\to+\infty$ and reshuffling, we get

$$Kd(x_0,x')\le C(K)(t_0-t')+m\quad\text{ for all } K\ge 0.$$

We now claim that $t'<t_0$. We already know that $t'\le t_0$, since $t'_n<t_0$, for all n. Suppose then by contradiction that $t'=t_0$. The inequality above then implies

$$Kd(x_0,x')\le m\quad\text{ for all } K\ge 0.$$

From $m<+\infty$, we conclude $x_0=x'$. Hence $(t',x')=(t_0,x_0)$. This is a contradiction, since $(t',x')\in\mathfrak{K}=\{t_0-\eta\}\times\bar{B}(x_0,\eta)\cup[t_0-\eta, t_0+\eta]\times\partial\bar{B}(x_0,\eta)$. Now that we know that $t'<t_0$, using the continuity of $(s,x,y)\mapsto h_s(x,y)$ for $(s,x,y)\in]0,+\infty[\times M\times M$ we can pass to the limit in (6-14) to obtain

$$U(t_0,x_0)=U(t',x')+h_{t_0-t'}(x',x_0). \qquad\square$$

7. Calibrated curves, backward characteristics and differentiability

We again fix a Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ on the complete Riemannian manifold (M, g) and denote its associated Tonelli Lagrangian by $L : TM \to \mathbb{R}$.

Definition 7.1 (calibrated curve). Let $U : S \to [-\infty, +\infty]$ be a function defined on the subset $S \subset \mathbb{R} \times M$. A curve $\gamma : [a, b] \to M$ is said to be U-calibrated for the Lagrangian L if it is an absolutely continuous curve, with $\mathrm{Graph}(\gamma) \subset S$, whose action $\mathbb{L}(\gamma) = \int_a^b L(\gamma(s), \dot\gamma(s))\, ds$ is finite and

$$U(b, \gamma(b)) = U(a, \gamma(a)) + \mathbb{L}(\gamma) = U(a, \gamma(a)) + \int_a^b L(\gamma(s), \dot\gamma(s))\, ds.$$

Remark 7.2. (1) For such a U-calibrated curve $\gamma : [a, b] \to M$, since its action is finite, if either $U(a, \gamma(a))$ or $U(b, \gamma(b))$ is infinite they are both equal and infinite.

(2) It is not difficult to check that the property of being calibrated is stable by concatenations of curves; i.e., if $\gamma_1 : [a, b] \to M$ and $\gamma_2 : [b, c] \to M$ are U-calibrated, with $\gamma_1(b) = \gamma_2(b)$, then so is the concatenation $\gamma = \gamma_1 * \gamma_2 : [a, c] \to M$, defined by

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{for } t \in [a, b], \\ \gamma_2(t) & \text{for } t \in [b, c]. \end{cases}$$

(3) More generally, a curve $\gamma : [a, b] \to M$ is said to be piecewise calibrated if we can find a finite sequence $a = t_0 < t_1 < \cdots < t_\ell = b$ such that each restriction $\gamma \,|\, [t_i, t_{i+1}]$, $i = 0, \ldots, \ell - 1$ is U-calibrated. Of course, by part (2), any piecewise U-calibrated is U-calibrated.

(4) Suppose $u : O \to \mathbb{R}$ is a function defined on the subset $O \subset M$ and $c \in \mathbb{R}$. If we define $U : \mathbb{R} \times O \to \mathbb{R}$ by

$$U(t, x) = u(x) - ct,$$

it not difficult to see that the absolutely continuous curve $\gamma : [a, b] \to M$ is U-calibrated if and only if $\gamma([a, b]) \subset O$ and

$$u(\gamma(b)) - u(\gamma(b) = \int_a^b L(\gamma(s), \dot\gamma(s)) + c\, ds;$$

i.e., the curve γ is (u, L, c)-calibrated as defined, for example, in [6].

Definition 7.3 (local backward characteristic). Let $U : S \to [-\infty, +\infty]$ be a function defined on the subset $S \subset \mathbb{R} \times M$. A local backward U-characteristic ending at $(t, x) \in S$ is a U-calibrated curve $\gamma : [t - \epsilon, t] \to M$, with $\epsilon > 0$ and $\gamma(t) = x$.

More generally, a curve $\gamma : [a, t] \to M$ is called a local backward U-characteristic if it is a local backward U-characteristic ending at $(t, \gamma(t)) \in S$.

Theorem 6.5 obviously implies the following one:

Theorem 7.4. *Suppose $O \subset]0, +\infty[\times M$ is an open subset. If $U : O \to \mathbb{R}$ is a continuous viscosity solution of the evolutionary Hamilton–Jacobi equation (1-1) on O, then for every $(t, x) \in O$, we can find a local backward U-characteristic ending at (t, x).*

In fact, the notion of U-calibrated curve, or of local backward U-characteristic is useful when U is evolution-dominated as can be seen from Proposition 7.5 below, whose proof is quite similar to the case of stationary solutions of the Hamilton–Jacobi equation. Again the proof is given for the reader convenience. Notice that no continuity assumption has to be made on the function U which is evolution-dominated by L. Note also that by Proposition 5.5, we can apply this proposition when $U : O \to \mathbb{R}$ is continuous and a viscosity subsolution of the evolutionary Hamilton–Jacobi equation (1-1) on the open subset $O \subset \mathbb{R} \times M$.

Proposition 7.5. *Suppose that the function $U : S \to [-\infty, +\infty]$ is evolution-dominated by L on $S \subset \mathbb{R} \times M$ and $\gamma : [a, b] \to M$ is a U-calibrated curve.*

(1) *One of the following statements holds.*
  - *$U(t, \gamma(t)) = +\infty$ for every $t \in [a, b]$.*
  - *$U(t, \gamma(t)) = -\infty$ for every $t \in [a, b]$.*
  - *$|U(t, \gamma(t))| < +\infty$ for every $t \in [a, b]$.*

(2) *For any subinterval $[a', b'] \subset [a, b]$, the restriction $\gamma | [a', b']$ is also U-calibrated.*

(3) *If S is an open subset of $\mathbb{R} \times M$ and $|U(t, \gamma(t))|$ is not identically $+\infty$, the curve $\gamma : [a, b] \to M$ is a local minimizer of the action and, therefore, an extremal of L.*

Proof. Note that the action of γ is finite (as required in Definition 7.1). To prove (1), assume for example $U(t_0, \gamma(t_0)) = +\infty$ for some $t_0 \in [a, b]$, then by Proposition 5.3, we must have $U(t, \gamma(t)) = +\infty$, for $t \in [a, t_0[$. Therefore $U(a, \gamma(a)) = +\infty$. Since γ is U-calibrated, we also obtain $U(b, \gamma(b)) = +\infty$. Hence $U(t, \gamma(t)) = +\infty$ everywhere on $[a, b[$, again by Proposition 5.3. The case $U(t_0, \gamma(t_0)) = -\infty$ for some $t_0 \in [a, b]$ is similar and leads to $U(t, \gamma(t)) = -\infty$ everywhere on $[a, b]$.

To prove (2) we first observe that, since L is bounded from below, the action $\mathbb{L}(\gamma | [a', b'])$ is also finite for any subinterval $[a', b'] \subset [a, b]$. In the case where U is identically either $+\infty$ or $-\infty$, this implies the U-calibration of $\gamma | [a', b']$, for $[a', b'] \subset [a, b]$. By 1), it remains to consider the case $|U(t, \gamma(t))| < +\infty$, for every $t \in [a, b]$. In that case, from the Definition 5.1 of evolution domination, we obtain

$$U(a', \gamma(a')) - U(a, \gamma(a)) \le \int_a^{a'} L(\gamma(s), \dot\gamma(s))\, dt,$$

$$U(b', \gamma(b')) - U(a', \gamma(a')) \le \int_{a'}^{b'} L(\gamma(s), \dot\gamma(s))\, dt,$$

$$U(b, \gamma(b)) - U(b', \gamma(b')) \le \int_{b'}^{b} L(\gamma(s), \dot\gamma(s))\, dt,$$

But if we add the three inequalities above we obtain

$$U(b, \gamma(b) - U(a, \gamma(a)) \le \int_a^{b} L(\gamma(s), \dot\gamma(s))\, dt,$$

which is an equality. Therefore all three inequalities are equalities.

To prove (3), we observe that, when S is an open subset of $\mathbb{R} \times M$, any curve $\delta : [a, b] \to M$ close enough to γ (in the C^0 topology) has a graph Graph(δ) which is also included in S. If $\delta(a) = \gamma(a)$ and $\delta(b) = \gamma(b)$, the U-calibration of γ and the Definition 5.1 of evolution domination yield

$$U(b, \gamma(b)) - U(a, \gamma(a)) \le \int_a^{b} L(\delta(s), \dot\delta(s))\, dt,$$

for any absolutely continuous curve $\delta : [a, b] \to M$, with $\delta(a) = \gamma(a)$ and $\delta(b) = \gamma(b)$. But, since γ is U-calibrated, by the definition of calibration, the left side of the inequality is $\int_a^b L(\gamma(s), \dot\gamma(s))\, dt$. This proves the local minimization property. By Tonelli's theorem such a local minimizer is as smooth as L (or H) and is an extremal of L. □

Theorem 7.6. *Suppose $O \subset \mathbb{R} \times M$ is an open subset. If $U : O \to \mathbb{R}$ is a continuous viscosity solution of the evolutionary on Hamilton–Jacobi (1-1) on O, then for every $(t, x) \in O$, we can find a U-characteristic extremal $\gamma :]a, t] \to M$ ending at (t, x) and such that either $a = -\infty$ or γ extends to a continuous extremal $\gamma : [a, t] \to M$, with $(a, \gamma(a)) \in \partial O$.*

By Theorem 6.5, we can find a U-calibrated curve $\gamma : [t - \epsilon, t] \to O$, with $\gamma(t) = x$. But this curve γ is an extremal for the Lagrangian L. Therefore, we can extend γ to an extremal $\gamma :]-\infty, +\infty[\to M$. Hence Theorem 7.6 follows from the next lemma.

Lemma 7.7. *Suppose $O \subset]0, +\infty[\times M$ is an open subset. If $U : O \to \mathbb{R}$ is a continuous viscosity solution of the evolutionary Hamilton–Jacobi equation (1-1) on O. Assume that the curve $\gamma :]-\infty, +\infty[\to M$ is an extremal for L that is*

U-calibrated on an interval $[t-\epsilon, t]$ for some $\epsilon > 0$, then γ is U-calibrated on the maximal interval $]a, t]$ such that $\text{Graph}(\gamma\,|\,]a,t]) \subset O$.

Proof. Consider the maximal interval $]a, t]$ such that $\text{Graph}(\gamma\,|\,]a,t]) \subset O$. Define b as the infimum of the $s \in\,]a, t]$ such that $\gamma : [s, t]$ is U-calibrated. We have $b \le t-\epsilon$. Suppose that $b > a$, then $(b, \gamma(b)) \in O$, and by continuity of U, the restriction $\gamma\,|\,[b, t]$ is U-calibrated. By Theorem 7.4, there exists a U-calibrated curve $\tilde{\gamma} : [b-\eta, b] \to M$, with $\eta > 0$ and $\tilde{\gamma}(b) = \gamma(b)$. By Remark 7.2(2), the concatenation $\tilde{\gamma} \star \gamma : [b-\eta, t] \to M$ is also U-calibrated. By Proposition 7.5(3), this U-calibrated curve $\tilde{\gamma} \star \gamma$ is also an extremal for L. Since $\tilde{\gamma} \star \gamma = \gamma$ on $[t-\epsilon, t]$, with $t-\epsilon < t$, we must have $\tilde{\gamma} \star \gamma = \gamma$ on $[b-\eta, t]$. This implies that $\gamma\,|\,[b-\eta, t]$ is U-calibrated, which contradicts the definition of b. □

Theorem 7.8 (Lax–Oleinik). *A continuous function $U : [0, T[\, \times M \to \mathbb{R}$ that is a viscosity solution of the evolutionary Hamilton–Jacobi equation (1-1) on $]0, T[\, \times M$ satisfies the Lax–Oleinik formula*

$$U(t, x) = \inf_{y \in M} U(0, y) + h_t(y, x),$$

for all $t > 0$, $x \in M$. The infimum is achieved for all $t > 0$, $x \in M$.

Proof. Since U is a viscosity solution of the evolutionary Hamilton–Jacobi equation (1-1) on $]0, T[\, \times M$, by Proposition 5.5, it is evolution-dominated by L on $]0, T[\, \times M$. From Remark 5.2(2), it follows that U is strongly evolution-dominated by L on $]0, T[\, \times M$. By continuity of U on $[0, T[\, \times M$, we easily obtain that U is strongly evolution-dominated by L on $[0, T[\, \times M$. Therefore, we have

$$U(t, x) \le \inf_{y \in M} U(0, y) + h_t(y, x),$$

for all $t > 0$, $x \in M$. To finish the proof of the first part of the theorem, it suffices to show that, for a given $(t, x) \in\,]0, T[\, \times M$, there exists a U-calibrated curve $\gamma : [0, t] \to M$, with $\gamma(t) = x$. Since, for a curve $\gamma :]a, b] \to M$ such that $\text{Graph}(\gamma) \subset\,]0, T[\, \times M$, we must have $a \ge 0$ and $b < T$, by Theorem 7.6 applied to the open set $]0, T[\, \times M$, we can find an extremal $\gamma : [a, t] \to M$ that is U-calibrated on $]a, t]$ with $\gamma(t) = x$ and $(a, \gamma(a)) \in \partial]0, T[\, \times M = \{0\} \times M$. Hence $a = 0$, and the extremal $\gamma : [0, t] \to M$ is U calibrated by continuity of U. □

Corollary 7.9. *Suppose $U, V : [0, T[\, \times M \to \mathbb{R}$ are two continuous functions that are viscosity solutions of the evolutionary Hamilton–Jacobi equation (1-1) on $]0, T[\, \times M$. If $U \le V$ on $\{0\} \times M$, then $U \le V$ everywhere on $[0, T[\, \times M$.*

In particular, if $U = V$ on $\{0\} \times M$, then $U = V$ everywhere on $[0, T[\, \times M$.

8. The Lax–Oleinik semigroup and the Lax–Oleinik evolution

Definition 8.1. If $u : M \to [-\infty, +\infty]$ is a function and $t > 0$, the function $T_t^- u : M \to [-\infty, +\infty]$ is defined by

$$T_t^- u(x) = \inf_{y \in M} u(y) + h_t(y, x).$$

We also set $T_0^- u = u$. The (negative) Lax–Oleinik semigroup is T_t^-, $t \geq 0$.

Definition 8.2 (Lax–Oleinik evolution). If $u : M \to [-\infty, +\infty]$, we will denote by $\hat{u} : [0, +\infty[\times M \to [-\infty, +\infty]$ the function defined, for $t > 0$, by

$$\hat{u}(t, x) = T_t^- u(x) = \inf_{y \in M} u(y) + h_t(y, x)$$

and by $\hat{u}(0, x) = u(x)$.

The function $\hat{u}$ is called the (negative) Lax–Oleinik evolution of u. We note that $\hat{u} < +\infty$ on $]0, +\infty[\times M$, if u is not identically $+\infty$.

Proposition 8.3. *For any function $u : M \to [-\infty, +\infty]$, its Lax–Oleinik evolution $\hat{u} : [0, +\infty[\times M \to [-\infty, +\infty]$ is strongly evolution-dominated by L on $[0, +\infty[\times M$.*

Proof. This follows easily from the definition of $\hat{u}$ and Proposition 4.13(1). □

Remark 8.4. (1) If u is not identically $+\infty$, then $\hat{u}(t, x) < +\infty$ for all t in $]0, +\infty[$ and x in M.

(2) If $u(x_0) = -\infty$ for some $x_0 \in M$, then $\hat{u}(t, x) = -\infty$ for all t in $]0, +\infty[$ and x in M.

(3) If u is not identically $+\infty$, then the set $F_u = \{x \in M \mid u(x) \neq +\infty\}$ is not empty. If we set $K = \{0\} \times F_u$ and define $U : K \to [-\infty, +\infty[$ by $U(0, x) = u(x)$, for all $(0, x) \in K$, then $t_{K,\inf} = 0$ and $\hat{U} = \hat{u}$ on $]0, +\infty[\times M$, where $\hat{U}$ is given (see (6-1)) by

$$\begin{aligned} \hat{U}(t, x) &= \inf\{U(\tilde{t}, \tilde{x}) + h_{t-\tilde{t}}(\tilde{x}, x) \mid (\tilde{t}, \tilde{x}) \in K \text{ and } \tilde{t} \leq t\} \\ &= \inf\{u(\tilde{x}) + h_t(\tilde{x}, x) \mid \tilde{x} \in F_u\}. \end{aligned}$$

In particular, all the results given in Section 6 for functions of the type $\hat{U}$ hold for Lax–Oleinik evolutions.

Theorem 8.5. *Assume $u : M \to [-\infty, +\infty]$ is such that $\hat{u}(T, X)$ is finite for some $(T, X) \in]0, +\infty[\times M$. Then $\hat{u}$ is finite, locally semiconcave and a viscosity solution of the evolutionary Hamilton–Jacobi equation*

$$\partial_t \hat{u} + H(x, \partial_x \hat{u}) = 0,$$

on $]0, T[\times M$.

Proof. As explained above, this now follows from Theorem 6.2. □

Examples 8.6. We give some examples of Lax–Oleinik evolution.

(1) For any Tonelli Lagrangian $L : TM \to \mathbb{R}$, we know that $h_t(y, x) \geq -C(0)t$. Therefore $\hat{u}(t, x) \geq \inf_M u - C(0)t$. This implies that $\hat{u}$ is finite everywhere, on $]0, +\infty[\times M$, for any function $u : M \to]-\infty, +\infty[$ which is bounded from below and not identically equal to $+\infty$.

(2) For any Tonelli Lagrangian $L : TM \to \mathbb{R}$, if M is compact, we know that $-C(0)t \leq h_t(y, x) \leq A(\operatorname{diam} M/t)$. Since, for any function $u : M \to [-\infty, +\infty]$, we have

$$\hat{u}(t, x) = \inf_{y \in M} u(y) + h_t(y, x),$$

we obtain

$$-C(0)t + \inf_M u \leq \hat{u}(t, x) \leq A(\operatorname{diam} M/t) + \inf_M u.$$

Hence, for a compact manifold $\hat{u}$ is finite everywhere on $]0, +\infty[\times M$ if and only if u is bounded from below and not identically $+\infty$. Therefore, for compact M, the class of functions u for which $\hat{u}$ is finite on $]0, +\infty[\times M$ does not depend on M.

(3) If $A \subset M$, we define $\Xi_A : M \to \{0, +\infty\}$

$$\Xi_A(x) = \begin{cases} 0 & \text{if } x \in A, \\ \infty & \text{otherwise.} \end{cases}$$

Note that Ξ_M is identically 0, and $\Xi_\varnothing$ is identically $+\infty$. Moreover, the function Ξ_A is not identically $+\infty$ if $A \neq \varnothing$.

For a given Tonelli Lagrangian $L : TM \to \mathbb{R}$, and $A \neq \varnothing$, we obtain, from (1), that $\hat{\Xi}_A$ is finite everywhere on $]0, +\infty[\times M$, with

$$\hat{\Xi}_A(t, x) = \inf_{y \in A} h_t(y, x).$$

(4) If $M = \mathbb{R}^n$ with the Euclidean metric, we consider the Lagrangian $L_0(x, v) = \frac{1}{2}\|v\|^2$, where $\|\cdot\|$ is the usual Euclidean metric. We know from Example 4.12(1) that $h_t(x, y) = \|y - x\|^2/2t$. For $\alpha, \beta > 0$, consider the function $u_{\alpha,\beta} : M \to \mathbb{R}$ defined by

$$u_{\alpha,\beta}(x) = -\alpha\|x\|^\beta.$$

Its Lax–Oleinik evolution is given by

$$\begin{aligned} \hat{u}_{\alpha,\beta}(t, x) &= \inf_{y \in \mathbb{R}^n} -\alpha\|y\|^\beta + \frac{\|y - x\|^2}{2t} \\ &= \frac{\|x\|^2}{2t} + \inf_{y \in \mathbb{R}^n} \left(\frac{\|y\|^2}{2t} - \alpha\|y\|^\beta + \frac{\langle y, x \rangle}{t} \right). \end{aligned}$$

Therefore

(i) If $\beta > 2$ then $\hat{u}_{\alpha,\beta}$ is identically $-\infty$.

(ii) If $\beta < 2$ then $\hat{u}_{\alpha,\beta}$ is finite everywhere.

(iii) If $\beta = 2$, for $(t, x) \in M$, we have

$$\hat{u}_{\alpha,2}(t, x) \text{ is } \begin{cases} \text{finite} & \text{if } t < \alpha/2, \\ 0 & \text{if } (t, x) = (\alpha/2, 0), \\ -\infty & \text{otherwise.} \end{cases}$$

(5) If $M = \mathbb{R}^n$ with the Euclidean metric, for a real number $p \geq 4$, we consider the Lagrangian $L_p(x, v) = \frac{1}{2}\|v\|^2 + \frac{1}{p}\|v\|^p$, where $\|\cdot\|$ is the usual Euclidean metric. We know from Example 4.12(2) that $h_t(x, y) = \|y - x\|^2/2t + \|y - x\|^p/pt^{p-1}$. Therefore if, for $\beta > 0$, we consider the function

$$u_\beta(x) = -\|x\|^\beta.$$

In this case, we have

$$\hat{u}_\beta(t, x) = \inf_{y \in M} -\|y\|^\beta + \|y - x\|^2/2t + \|y - x\|^p/pt^{p-1};$$

hence $\hat{u}_\beta$ is finite everywhere if $\beta < p$ and is equal $-\infty$ everywhere for $\beta > p$.

It follows that, for a noncompact manifold M, the class of functions u for which $\hat{u}$ is finite depends on the Lagrangian.

Some of the well-known properties of the Lax–Oleinik semigroup $(T_t^-)_{t\geq 0}$ (see [6]) are given in the following proposition.

Proposition 8.7. (1) (semigroup property) *For every* $t, t \geq 0$, *we have* $T^-_{t+t'} = T_t^- \circ T^-_{t'}$. *In particular, for every* $t, t' \geq 0$ *and* $x, y \in M$,

$$\begin{aligned} T_t^- u(x) &\leq u(x) + h_t(x, x) \leq u(x) + A(0)t, \\ T^-_{t+t'} u(x) &\leq T^-_{t'} u(y) + h_t(y, x), \\ T^-_{t+t'} u(x) &\leq T^-_{t'} u(x) + h_t(x, x) \leq T^-_{t'} u(x) + A(0)t. \end{aligned}$$

(2) *For every* $u : M \to [-\infty, +\infty]$, *and every* $c \in \mathbb{R}$, *we have* $T_t^-(u + c) = T_t^-(u) + c$, *for every* $t \geq 0$.

(3) *For every* $u, v : M \to [-\infty, +\infty]$ *with* $u \leq v$ *everywhere, we have* $T_t^- u \leq T_t^- v$, *for every* $t \geq 0$.

(4) *For every* $u, v : M \to \mathbb{R}$, *we have*

$$-\|u - v\|_\infty + T_t^- v \leq T_t^- u \leq T_t^- v + \|u - v\|_\infty,$$

for every $t \geq 0$.

Here is a further observation on the Lax–Oleinik evolution.

Definition 8.8 (lower semicontinuous regularization). If $u : M \to [-\infty, +\infty]$, we define its lower semicontinuous regularization $u_- : M \to [-\infty, +\infty]$ by

$$u_-(x) = \liminf_{y \to x} u(y) = \sup_V \inf_{y \in V} u(y),$$

where the supremum is taken over all neighborhoods V of x. The function u_- is the largest lower semicontinuous function which is $\leq u$.

Proposition 8.9. *For every function $u : M \to [-\infty, +\infty]$, we have $\hat{u} = \hat{u}_-$ on $]0, +\infty[\times M$.*

Proof. Since $u_- \leq u$, we have $\hat{u}_- \leq \hat{u}$. To prove the converse inequality, it suffices to show that for $(t, x, y) \in]0, +\infty[\times M \times M$, we have

$$u_-(y) + h_t(y, x) \geq \inf_{z \in M} u(z) + h_t(z, x) = \hat{u}(x).$$

By definition of $u_-(y)$, we can find a sequence $y_n \to y$ such that $u(y_n) \to u_-(y)$. Since $h_t(\cdot, x)$ is continuous we obtain

$$u_-(y) + h_t(y, x) = \lim_{n \to +\infty} u(y_n) + h_t(y_n, x) \geq \inf_{z \in M} u(z) + h_t(z, x). \qquad \square$$

We will now consider the Lax–Oleinik evolution of Lipschitz functions.

We start with a lemma connecting the Lipschitz property with the action and the Lax–Oleinik semigroup.

Lemma 8.10. (1) *For a function $u : M \to \mathbb{R}$, and a constant $c \in \mathbb{R}$, the following two conditions are equivalent*:

- $u(x) - u(y) \leq h_s(y, x) + cs$, *for all* $s > 0$ *and* $x, y \in M$.
- $u \leq T_s^- u + cs$, *for all* $s \geq 0$.

(2) *If a function $u : M \to \mathbb{R}$, for some $c \in M$ satisfies $u \leq T_s^- u + cs$, for all $s \geq 0$, then so does $T_t^- u$ for all $t \geq 0$.*

(3) *If the function $u : M \to \mathbb{R}$ is globally Lipschitz function, with Lipschitz constant $\leq \lambda$, then $u \leq T_t^- u + C(\lambda)t$, for all $t \geq 0$, where $C(\cdot)$ is the function defined in* (3-3).

(4) *If, for some $c \in \mathbb{R}$, the function $v : M \to \mathbb{R}$ satisfies $v \leq T_t^- v + ct$, for all $t > 0$ and $x, y \in M$, then v is Lipschitz, with Lipschitz constant $\leq A(1) + c$, where $A(\cdot)$ is the function defined in* (3-4)

Proof. To prove (1), we note that the condition $u(x) - u(y) \leq h_s(y, x) + cs$ is equivalent to $u(x) \leq u(y) + h_s(y, x) + cs$. Therefore the two conditions in part (1) are equivalent since $T_s^- u(x) = \inf_{y \in M} u(y) + h_s(y, x)$.

Part (2) follows easily from parts (1), (2) and (3) of Proposition 8.7.

To prove (3), using (4-9), we note that

$$u(x) - u(y) \le \lambda d(x, y) \le h_s(y, x) + C(\lambda)s.$$

To prove (4), we note that $h_{d(x,y)}(x, y) \le A(1)d(x, y)$ by Lemma 4.11. Hence

$$v(y) - v(x) \le h_{d(x,y)}(x, y) + cd(x, y) \le [A(1) + c]d(x, y).$$

By symmetry, we conclude that the Lipschitz constant of v is $\le A(1) + c$. □

We recall that a function $u : M \to \mathbb{R}$ is said to be evolution-dominated by $L + c$ if it satisfies the equivalent properties of Lemma 8.10(1).

Proposition 8.11. *The Lax–Oleinik evolution $\hat{u}$ of any (globally) Lipschitz function $u : M \to \mathbb{R}$ is finite everywhere on $[0, +\infty[\times M$.*

Moreover, for every constant $\lambda \in [0, +\infty[$, we can find a constant Λ such that $\hat{u}$ has Lipschitz constant $\le \Lambda$ as soon as u has Lipschitz constant $\le \lambda$.

Proof. Assume that $u : M \to \mathbb{R}$ has Lipschitz constant $\le \lambda$. By Lemma 8.10(3) we have

$$u(x) \le T_s u(x) + C(\lambda)s,$$

for all $s \ge 0$. This implies that $\hat{u}$ is finite everywhere.

Lemma 8.10(2) yields

$$T_t^- u(x) \le T_s^- T_t^- u(x) + C(\lambda)s, \tag{8-1}$$

for all $t, s \in [0, +\infty[$, and $x \in M$. Therefore, by Lemma 8.10(4), the Lax–Oleinik evolution has a Lipschitz constant in x which is $\le A(1) + C(\lambda)$.

To compute the Lipschitz constant in t, we note that, by the semigroup property, we have

$$T_{t+s}^- u(x) \le T_t^- u(x) + h_s(x, x) \le T_t^- u(x) + A(0)s.$$

Combining this last equality with (8-1), we get

$$-C(\lambda)s \le T_{t+s}^- u(x) - T_t^- u(x) \le A(0)s.$$

It follows that the Lipschitz constant in t of $\hat{u}$ is $\le \max\big(|A(0)|, |C(\lambda)|\big)$. This finishes the proof of the existence of the constant Λ. □

We next extend the results obtained above to uniformly continuous function.

Corollary 8.12. *The Lax–Oleinik evolution $\hat{u} : [0, +\infty[\times M \to \mathbb{R}$ of a uniformly continuous function $u : M \to \mathbb{R}$ is finite everywhere and uniformly continuous.*

Proof. By Lemma A.1 in the Appendix, there is a sequence of Lipschitz functions $u_n : M \to \mathbb{R}$ such that $\|u - u_n\|_\infty \to 0$ as $n \to +\infty$. By Proposition 8.7(4), for every $t \ge 0$, and every $n \ge 0$, we have

$$-\|u - u_n\|_\infty + T_t^- u_n \le T_t^- u \le T_t^- u_n + \|u - u_n\|_\infty.$$

Therefore $\hat{u}$ is finite everywhere and $\|\hat{u}-\hat{u}_n\|_\infty \le \|u-u_n\|_\infty \to 0$ as $n \to +\infty$. By Proposition 8.11, each function $\hat{u}_n$ is Lipschitz. Therefore, again by Lemma A.1, the uniform limit $\hat{u}$ of the Lipschitz functions $\hat{u}_n$ is uniformly continuous. □

We then consider the case when u is Lipschitz in the large; see Definition A.2.

Corollary 8.13. *For any finite constant $K \ge 0$, we can find a finite constant κ such that any function $u : M \to \mathbb{R}$ Lipschitz in the large with constant K has a Lax–Oleinik evolution $\hat{u} : [0, +\infty[\times M \to \mathbb{R}$, which is finite everywhere and Lipschitz in the large with constant κ on $[0, +\infty[\times M \to \mathbb{R}$.*

Proof. By Proposition A.4, we can find a Lipschitz function $\varphi : X \to \mathbb{R}$, with Lipschitz constant K, such that $\|u - \varphi\|_\infty = \sup_{x\in M} |u(x) - \varphi(x)| \le K/2$.

From Proposition 8.11, the Lax–Oleinik evolution $\hat{\varphi}$ is Lipschitz with a Lipschitz constant $\le \Lambda(K)$, where $\Lambda(K)$ depends only on K. As in the proof of Corollary 8.12, by Proposition 8.7(4), we have

$$\|\hat{u} - \hat{\varphi}\|_\infty \le \|u - \varphi\|_\infty \le K/2.$$

We can now apply again Proposition A.4 of the Appendix, to conclude that $\hat{u} : [0, +\infty[\times M \to \mathbb{R}$ is finite everywhere and Lipschitz in the large with constant $\kappa = \max(\Lambda(K), K)$ on $[0, +\infty[\times M \to \mathbb{R}$. □

Of course, in Corollary 8.13 the Lax–Oleinik evolution $\hat{u}$ of the Lipschitz in the large function $u : M \to \mathbb{R}$ is, as for all Lax–Oleinik evolutions, locally Lipschitz on $]0, +\infty[\times M$, since it is everywhere finite on $]0, +\infty[\times M$. We will show in Theorem 9.5 that $\hat{u}$ is globally Lipschitz on $[t_0, +\infty[\times M$, for every $t_0 > 0$.

Our goal now is the case to give properties of $\hat{u}$ near $\{0\} \times M$ when u is just continuous or merely lower semicontinuous.

We start with a remark.

Remark 8.14. Suppose that $U : [0, +\infty[\times M \to [-\infty, +\infty]$. Denote by U^* the restriction of U to $]0, +\infty[\times M$. If $x \in M$, we can define

$$\liminf_{(t,y)\to(0,x)} U(t, y), \qquad \liminf_{(t,y)\to(0,x)} U^*(t, y) \quad \text{and} \quad \liminf_{y\to x} U(0, y),$$

where in the first case we take $(t, y) \to (0, x)$ with $t \ge 0$ and $y \in M$; in the case of U^* we take $(t, y) \to (0, x)$ with $t > 0$ and $y \in M$; and in the last case $y \to x$ with $y \in M$.

Of course we have

$$\liminf_{(t,y)\to(0,x)} U(t, y) \le \liminf_{(t,y)\to(0,x)} U^*(t, y),$$

$$\liminf_{(t,y)\to(0,x)} U(t, y) \le \liminf_{y\to x} U(0, y).$$

Since for any sequence $(t_i, y_i) \to (0, x)$, with $t_i \geq 0$ and $y_i \in M$, either $t_i = 0$ for infinitely many i, or $t_i > 0$ for infinitely many i, we conclude

$$\liminf_{(t,y)\to(0,x)} U(t, y) = \min\Big(\liminf_{(t,y)\to(0,x)} U^*(t, y), \liminf_{y\to x} U(0, y)\Big). \tag{8-2}$$

Theorem 8.15. *Let $L : TM \to \mathbb{R}$ be a Tonelli Lagrangian. If $u : M \to [-\infty, +\infty]$ is a lower semicontinuous function such that its Lax–Oleinik evolution $\hat{u}$: $[0, +\infty[\times M$, $(t, x) \mapsto T_t^- u(x)$ is finite at some $(T, X) \in]0, +\infty[\times M$, then it satisfies the following properties*:

(i) *For every $x \in M$, we have*

$$\liminf_{(t,y)\to(0,x)} \hat{u}(t, y) = \liminf_{(t,y)\to(0,x)} \hat{u}^*(t, y) = u(x).$$

Therefore the function $\hat{u}$ is lower semicontinuous on $[0, T[\times M$.

(ii) *For every $x \in M$, we have*

$$\limsup_{(t,y)\to(0,x)} \hat{u}(t, y) = \limsup_{y\to x} u(y).$$

Therefore, if u is continuous on M then $\hat{u}$ is continuous on $[0, T[\times M$.

(iii) *For every $x \in M$, both limits $\lim_{t\to 0} \hat{u}(t, x) = \lim_{t\to 0} \hat{u}^*(t, x)$ exist and*

$$\lim_{t\to 0} \hat{u}(t, x) = \lim_{t\to 0} \hat{u}^*(t, x) = u(x).$$

For every $x \in M$, the function $t \mapsto \hat{u}(t, x) + A(0)t$ is nondecreasing in t.

Proof. We first note that from Proposition 8.7(1), we have

$$\hat{u}(t, y) \leq u(y) + A(0)t. \tag{8-3}$$

This obviously implies the equality in (ii). By the lower semicontinuity of u, this also implies

$$\liminf_{(t,y)\to(0,x)} \hat{u}^*(t, y) \leq u(x).$$

Therefore, from (8-2), we conclude that

$$\liminf_{(t,y)\to(0,x)} \hat{u}(t, y) = \liminf_{(t,y)\to(0,x)} \hat{u}^*(t, y).$$

To finish the proof of (i), it remains to show that

$$\ell = \liminf_{(t,y)\to(0,x)} \hat{u}^*(t, y) \geq u(x).$$

If $\ell = +\infty$, there is nothing to prove. Therefore we assume that $\ell < +\infty$. We then choose a sequence $(t_i, y_i) \to (0, x)$, with $t_i > 0$ such that

$$\lim_{i\to+\infty} \hat{u}(t_i, y_i) = \ell.$$

We now note, again by Proposition 8.7(1), that for all $(t, y), (t', y') \in [0, +\infty[\times M$, with $t' < t$, we have

$$\hat{u}(t, y) \le \hat{u}(t', y') + h_{t-t'}(y', y). \tag{8-4}$$

In particular, we get

$$\hat{u}(t, y) \ge \hat{u}(T, X) + h_{T-t}(y, X), \tag{8-5}$$

for all $(t, y) \in [0, T[\times M$. We then use

$$\hat{u}(t_i, y_i) = \inf_{z \in M} u(z) + h_{t_i}(z, y_i)$$

to find a sequence $z_i \in M$ such that

$$\hat{u}(t_i, y_i) \le u(z_i) + h_{t_i}(z_i, y_i) \to \ell.$$

From (4-7), we know that $h_{t_i}(z, y_i) \ge -C(0)t_i \to 0$. Therefore, if z_i admits x as an accumulation point of the sequence z_i, from the lower semicontinuity of u, we would obtain

$$\ell = \lim_{i \to +\infty} u(z_i) + h_{t_i}(z, y_i) \ge \liminf_{i \to +\infty} u(z_i) \ge u(x).$$

It remains to consider the case when x is not an accumulation point of the sequence z_i. Therefore we can find $\epsilon > 0$ such that

$$d(x, z_i) > \epsilon \quad \text{for all } i. \tag{8-6}$$

Since $y_i \to x$, neglecting the first terms of the sequence, we can assume

$$d(x, y_i) < \epsilon \quad \text{for all } i. \tag{8-7}$$

For every i, we can now find a minimizer $\gamma_i : [0, t_i] \to M$, with $\gamma_i(0) = z_i$ and $\gamma_i(t_i) = y_i$. From (8-6) and (8-7), we can find $t_i' \in]0, t_i[$ such that $d(x, \gamma_i(t_i')) = \epsilon$, for all i. Since $\gamma_i : [0, t_i] \to M$ is a minimizer, with $\gamma_i(0) = z_i$ and $\gamma_i(t_i) = y_i$, we have

$$h_{t_i}(z_i, y_i) = h_{t_i'}(z_i, \gamma_i(t_i')) + h_{t_i - t_i'}(\gamma_i(t_i'), y_i).$$

Therefore

$$\begin{aligned} u(z_i) + h_{t_i}(z_i, y_i) &= u(z_i) + h_{t_i'}(z_i, \gamma_i(t_i')) + h_{t_i - t_i'}(\gamma_i(t_i'), y_i) \\ &\ge \hat{u}(t_i', \gamma_i(t_i')) + h_{t_i - t_i'}(\gamma_i(t_i'), y_i). \end{aligned}$$

Since the sequence $\gamma_i(t_i')$ is contained in the compact ball $\bar{B}(x, \epsilon)$ and $t_i \to 0 < T$, from (8-5), we get

$$\inf_i \hat{u}(t_i', \gamma_i(t_i')) = \kappa > -\infty.$$

Hence

$$u(z_i) + h_{t_i}(z_i, y_i) \geq \kappa + h_{t_i - t_i'}(\gamma_i(t_i'), y_i),$$

which implies

$$\ell = \lim_{i\to+\infty} u(z_i) + h_{t_i}(z, y_i) \geq \kappa + \lim_{i\to+\infty} h_{t_i - t_i'}(\gamma_i(t_i'), y_i). \tag{8-8}$$

For $K > 0$, we now use (4-8) and $d(x, \gamma_i(t_i')) = \epsilon$ to obtain

$$\begin{aligned} h_{t_i - t_i'}(\gamma_i(t_i'), y_i) &\geq K d(\gamma_i(t_i'), y_i) - C(K)(t_i - t_i') \\ &\geq K(\epsilon - d(x, y_i)) - C(K)(t_i - t_i'). \end{aligned}$$

Since $y_i \to x$ and $0 < t_i' < t_i \to 0$, we obtain

$$\lim_{i\to+\infty} h_{t_i - t_i'}(\gamma_i(t_i'), y_i) \geq K\epsilon.$$

Since $\epsilon > 0$ and $K > 0$ is arbitrary, we get

$$\lim_{i\to+\infty} h_{t_i - t_i'}(\gamma_i(t_i'), y_i) = +\infty.$$

This contradicts (8-8), since $\ell < +\infty$ and $\kappa > -\infty$. This finishes the proof of the equality in (i). The last part of (i) follows from this equality and the already observed continuity of $\hat{u}$ on the open subset $]0, +T[\, \times M$; see Theorem 8.5. Note that this same continuity of $\hat{u}$ on $]0, +T[\, \times M$, together with (i) and the inequality in (ii), yields also the last part of (ii).

To prove the equality in (iii), we first note, using (i), that

$$u(x) = \liminf_{(t,y)\to(0,x)} \hat{u}(t, y) \leq \liminf_{t\to 0} \hat{u}(t, x) \leq \liminf_{t\to 0} \hat{u}^*(t, x).$$

Moreover, from (8-3), we have

$$\hat{u}(t, x) \leq u(x) + A(0)t,$$

which yields

$$\limsup_{t\to 0} \hat{u}^*(t, x) \leq \limsup_{t\to 0} \hat{u}(t, x) \leq u(x).$$

The above inequalities on the lim inf's and lim sup's imply the equality in (iii). The last statement in (iii) follows from the third inequality in Proposition 8.7(1), which yields

$$\hat{u}(t + t', x) \leq \hat{u}(t, x) + A(0)t' \quad \text{for all } t, t' \geq 0. \qquad \square$$

Corollary 8.16. *Let $u : M \to [-\infty, +\infty]$ be a lower semicontinuous function, such that $\hat{u}(T, X)$ is finite for some $(T, X) \in\,]0, +\infty[\, \times M$. Then for every $(t, x) \in\,]0, T[\, \times M$, we can find a backward $\hat{u}$-characteristic $\gamma : [0, t] \to M$*

ending at (t, x). *In particular, for every* $(t, x) \in]0, T[\times M$, *we can find* $y \in M$ *such that*

$$\hat{u}(t, x) = u(y) + h_t(y, x).$$

Proof. Fix $(t, x) \in]0, T[\times M$. From Theorem 7.6, we can find an extremal $\gamma : [0, t] \to M$, with $\gamma(t) = x$, which is $\hat{u}$-calibrated on $]0, t]$; i.e., for all $s \in]0, t[$, we have

$$\hat{u}(t, x) = \hat{u}(s, \gamma(s)) + \int_s^t L(\gamma(\sigma), \dot{\gamma}(\sigma))\, d\sigma. \tag{8-9}$$

Since $(s, \gamma(s)) \to (0, \gamma(0))$ as $s \to 0$, from part (i) of Theorem 8.15, we obtain $\liminf_{s \to 0} \hat{u}(s, \gamma(s)) \geq u(\gamma(0)) = \hat{u}(0, \gamma(0))$. Hence, if we let $s \to 0$ in (8-9), we obtain

$$\begin{aligned} \hat{u}(t, x) &= \liminf_{s \to 0} \hat{u}(s, \gamma(s)) + \int_0^t L(\gamma(\sigma), \dot{\gamma}(\sigma))\, d\sigma \\ &\geq \hat{u}(0, \gamma(0)) + h_t(\gamma(0), x) \geq \hat{u}(t, x). \end{aligned}$$

Therefore all inequalities are equalities. Hence γ is $\hat{u}$-calibrated on the closed interval $[0, t]$. □

We conclude this section with a proof that all continuous viscosity solutions of the evolutionary Hamilton–Jacobi equation on an open set $]0, T[\times M$ are given by a Lax–Oleinik evolution of a unique lower semicontinuous function.

Theorem 8.17. *Assume* $U :]0, T[\times M \to \mathbb{R}$, *with* $T \in]0, +\infty]$ *is a continuous viscosity solution of the evolutionary Hamilton–Jacobi equation*

$$\partial_t U + H(x, \partial_x U) = 0,$$

on $]0, T[\times M$. *Then there exists a unique lower semicontinuous function* $u : M \to [-\infty, +\infty]$ *such that* $U = \hat{u}$ *on* $]0, T[\times M$. *In fact, we have*

$$u(x) = \liminf_{(t,y) \to (0,x)} U(t, y) = \lim_{t \to 0} U(t, x).$$

Proof. If u exists, it follows from Theorem 8.15 that we must have

$$u(x) = \liminf_{(t,y) \to (0,x)} U(t, y) = \lim_{t \to 0} U(t, x).$$

This implies the uniqueness if u exists. To prove the existence, we define $u : M \to [-\infty, +\infty]$ by

$$u(x) = \liminf_{(t,y) \to (0,x)} U(t, y).$$

This function u is lower semicontinuous. We first show that $\hat{u} \leq U$ on $]0, T[\times M$. For this, we fix $(t, x) \in]0, T[\times M$. For any $y \in M$, by definition of u, we can find a sequence $(t_i, y_i) \in]0, T[\times M$ with

$$(t_i, y_i) \to (0, y) \text{ and } U(t_i, y_i) \to u(y) \quad \text{as } i \to +\infty.$$

Since U is a viscosity solution on $]0, T[\times M$, we know from Proposition 5.5 and Remark 5.2(2), that U is strongly evolution-dominated by L. Using that $t_i \to 0 < t$, for i large, we must have

$$U(t, x) \leq U(t_i, y_i) + h_{t-t_i}(y_i, x).$$

If we let $i \to +\infty$, we obtain

$$U(t, x) \leq u(y) + h_t(y, x).$$

Since $y \in M$ is arbitrary, we conclude that $U \leq \hat{u}$ on $]0, T[\times M$.

It remains to show that $\hat{u} \leq U$ on $]0, T[\times M$. The argument is almost identical to the proof of last corollary. Fix $(t, x) \in]0, T[\times M$. From Theorem 7.6, we can find an extremal $\gamma : [0, t] \to M$, with $\gamma(t) = x$, which is U-calibrated on $]0, t]$; i.e., for all $s \in]0, t[$, we have

$$U(t, x) = U(s, \gamma(s)) + \int_s^t L(\gamma(\sigma), \dot{\gamma}(\sigma))\, d\sigma. \tag{8-10}$$

Since $(s, \gamma(s)) \to (0, \gamma(0))$ as $s \to 0$, from the definition of u, we obtain $\liminf_{s\to 0} U(s, \gamma(s)) \geq u(\gamma(0))$. Hence, if we let $s \to 0$ in (8-10), we obtain

$$\begin{aligned} U(t, x) &= \liminf_{s\to 0} U(s, \gamma(s)) + \int_0^t L(\gamma(\sigma), \dot{\gamma}(\sigma))\, d\sigma \\ &\geq u(\gamma(0)) + h_t(\gamma(0), x) \geq \hat{u}(t, x). \end{aligned}$$

□

9. Differentiability properties of the Lax–Oleinik evolution

Theorem 9.1 (differentiability theorem). *Assume that the function $U : O \to \mathbb{R}$, defined on the open subset O of $\mathbb{R} \times M$, is evolution-dominated by L. If the curve $\gamma : [a, b] \to M$, is U-calibrated for L, we have*:

(i) *If $t \in]a, b]$ then U is upper semicontinuous at $(t, \gamma(t))$ and*

$$(-E(\gamma), \partial_v L(\gamma(t), \dot{\gamma}(t)) \in D^+U(t, \gamma(t)).$$

(ii) *If $t \in [a, b[$ then U is lower semicontinuous at $(t, \gamma(t))$ and*

$$(-E(\gamma), \partial_v L(\gamma(t), \dot{\gamma}(t)) \in D^-U(t, \gamma(t)).$$

(iii) *For every $t \in [a, b]$, if the function U is differentiable at $(t, \gamma(t))$, then*

$$DU(t, \gamma(t)) = (-E(\gamma), \partial_v L(\gamma(t), \dot{\gamma}(t)) \in T^*_{(t,\gamma(t))}\mathbb{R} \times M = \mathbb{R} \times T^*_{\gamma(t)}M.$$

(iv) *If $t \in]a, b[$, then U is indeed differentiable* (*hence continuous*) *at $(t, \gamma(t))$.*

Proof. To prove (i), fix $t \in]a, b]$. By Proposition 5.4, there exists an open subset $O' \subset O$, with $(t, \gamma(t)) \in O'$, such that the restriction $U \mid O'$ is strongly evolution domination by L. By continuity of γ, we can then find $[a', b'] \subset [a, b]$, with $a' < t \leq b'$, and $(s, \gamma(s)) \in O'$, for all $s \in [a', b']$. The strong L evolution domination of $U \mid O'$ implies

$$U(s, x) - U(a', \gamma(a')) \leq h_{s-a'}(\gamma(a'), x) = \mathcal{H}(s - a', \gamma(a'), x), \tag{9-1}$$

for every $(s, x) \in O'$, with $s > a'$. Applying this inequality with $(s, x) = (t, \gamma(t))$, and using that $\gamma \mid [a', t]$ is U-calibrated, we obtain

$$\int_{a'}^{t} L(\gamma(s), \dot\gamma(s)) = U(t, \gamma(t)) - U(a', \gamma(a')) \leq \mathcal{H}(t - a', \gamma(a'), \gamma(t)).$$

But $\int_{a'}^{t} L(\gamma(s), \dot\gamma(s)) \geq \mathcal{H}(t - a', \gamma(a'), \gamma(t))$. Therefore the inequality above is an equality. Subtracting this equality from the inequality (9-1), we get

$$U(s, x) - U(t, \gamma(t) \leq \mathcal{H}(s - a', \gamma(a'), x) - \mathcal{H}(t - a', \gamma(a'), \gamma(t)). \tag{9-2}$$

Since $\mathcal{H}$ is continuous, we first obtain from this inequality (9-1) the upper semicontinuity. Moreover, inequality (9-1) together with the equality case at $(t, \gamma(t))$ implies

$$D^+_{(t,x)}\mathcal{H}(t - c, \gamma(t), \gamma(c)) \subset D^+U(t, \gamma(t)).$$

But by Proposition 4.22, we have

$$(-E(\gamma), \partial_v L(\gamma(t), \dot\gamma(t)) \in D^+_{(t,x)}\mathcal{H}(t - c, \gamma(t), \gamma(c)).$$

The proof of (ii) is similar.

To prove (iii), we recall that if $DU(t, \gamma(t))$ exists then $D^+U(t, \gamma(t)) = D^-U(t, \gamma(t)) = \{DU(t, \gamma(t))\}$.

To prove (iv), observe that, for $t \in]a, b[$, both $D^+U(t, \gamma(t)$ and $D^-U(t, \gamma(t))$ are nonempty by (i) and (ii). This implies that U is differentiable at (t, x); see for example [7, Proposition 3.3]. □

Corollary 9.2. *Assume $L : TM \to \mathbb{R}$ is a Tonelli Lagrangian. Let $U : [0, T[\times M \to \mathbb{R}$ be evolution-dominated by L.*

(i) *If U is differentiable at (t, x), then there is at most one U-calibrated $\gamma : [c, d] \to M$, with $t \in [c, d]$ and $x = \gamma(t)$.*

(ii) *If $\gamma_1 : [c_1, d_1] \to M$ and $\gamma_2 : [c_2, d_2] \to M$ are U-calibrated curves such that $\gamma_1(t) = \gamma_2(t)$, with $t \in [c_1, d_1] \cap [c_2, d_2]$ and either $t \in]c_1, d_1[$ or $t \in]c_2, d_2[$, then $\gamma_1 = \gamma_2$ on $[c_1, d_1] \cap [c_2, d_2]$.*

(iii) *If* $\gamma_1 : [0, c] \to M$ *and* $\gamma_2 : [0, d] \to M$ *are two* U*-calibrated curves, with* $c \leq d$ *such that* $\gamma_1(t) = \gamma_2(t)$, *for some* $t \in [0, c]$, *if* γ_1 *and* γ_2 *are not identical on* $[0, c]$, *then either* $t = 0$ *or* $t = c = d$.

Proof. Part (i) follows from part (iii) of Theorem 9.1, since for any such minimizer we have

$$\partial_x U(t, x) = \partial_v L(\gamma(t), \dot\gamma(t)) = \partial_v L(x, \dot\gamma(t)),$$

which shows that not only the position of the extremal γ at time t is fixed $(= x)$ but so is its speed at time t.

Part (ii) follows from part (i). In fact, if either $t \in]c_1, d_1[$ or $t \in]c_2, d_2[$, then, by part (iv) of Theorem 9.1, the function U is differentiable at (t, x).

To prove part (iii), we observe that part (ii) implies $t \in \{0, c\}$ and $t \in \{0, d\}$, which implies either $t = 0$ or $t = c = d$. □

The next corollary follows from the previous one applied to backward U-characteristics.

Corollary 9.3. *Assume* $L : TM \to \mathbb{R}$ *is a Tonelli Lagrangian. Let*

$$U :]0, T[\times M \to \mathbb{R}$$

be evolution-dominated by L.

(i) *If* U *is differentiable at* $(t, x) \in]0, T[\times M$, *then there is at most one backward* U*-characteristic* $\gamma :]0, t] \to M$ *ending at* (t, x).

(ii) *If* $\gamma :]0, a] \to M$ *is a backward* U*-characteristic, then* U *is differentiable at every* $(t, \gamma(t))$, *with* $t \in]0, a[$.

Proposition 9.4. *Let* $u : M \to [-\infty, +\infty]$ *be a lower semicontinuous function such that* $\hat u(T, X)$ *is finite for some* $(T, X) \in]0, +\infty[\times M$. *Then* $\hat u$ *is differentiable at* $(t, x) \in]0, T[\times M$ *if and only if there is a unique backward* U*-characteristic ending at* (t, x). *Moreover, the set of upper differentials* $D^+\hat u(t, x)$ *is equal to the convex hull of all covectors* $(-E(\gamma), \partial_v L(\gamma(t), \dot\gamma(t)))$, *with* $\gamma : [0, t] \to M$ *a backward* $\hat u$*-characteristic ending at* (t, x).

Proof. By Theorem 6.5, we already know that $\hat u$ is locally semiconcave. We first show that for a backward $\hat u$-characteristic $\gamma : [0, t] \to M$ ending at (t, x), we have $(-E(\gamma), \partial_v L(\gamma(t), \dot\gamma(t)) \in D^+\hat u(t, x)$.

Since γ is calibrating for $\hat u$, it is a minimizer; therefore we have

$$\mathcal{H}(t, \gamma(0), x) = \int_0^t L(\gamma(s), \dot\gamma(s))\, ds,$$

and

$$\hat u(t, x) = u(\gamma(0)) + \mathcal{H}(t, \gamma(0), x).$$

By definition of $\hat{u}$, we also have

$$\hat{u}(t', x) \leq u(\gamma(0)) + \mathcal{H}(t', \gamma(0), x'),$$

for all $(t', x') \in]0, +\infty[\times M$. This implies that $D^+_{(t,x)}\mathcal{H}(t, \gamma(0), x) \subset D^+\hat{u}(t, x)$. But by Proposition 4.22, we have $(-E(\gamma), \partial_v L(\gamma(t), \dot{\gamma}(t)) \in D^+_{(t,x)}\mathcal{H}(t, \gamma(0), x)$.

The rest of the proof is similar to the proof of Corollary 4.23. □

We now apply the results above to Lipschitz in the large functions; see A.2 for the definition.

Theorem 9.5. *Assume that $L : TM \to \mathbb{R}$ is a Tonelli Lagrangian. For every finite constant $K, t_0 > 0$, we can find a constant $\lambda < +\infty$ such that for any $u : M \to \mathbb{R}$ Lipschitz in the large with constant K, its Lax–Oleinik evolution $\hat{u}$ is finite everywhere and (globally) Lipschitz on $[t_0, +\infty[\times M$, with Lipschitz constant $\leq \lambda$.*

Proof. Since $\hat{u} = \hat{u}_-$ on $]0, +\infty[\times M$, where u_- is the lower semicontinuous regularization of u and u_- is Lipschitz in the large with the same constant by Lemma A.3, without loss of generality, we can assume that u is lower semicontinuous.

Since, from Corollary 8.13, the Lax–Oleinik evolution $\hat{u}$ is finite everywhere, from Theorem 6.5, we obtain that it is locally semiconcave. Hence, the Lax–Oleinik evolution $\hat{u}$ locally Lipschitz on $]0, +\infty[\times M$. Therefore, we need to show that the norm of derivative of $\hat{u}$ is bounded almost everywhere, on $[t_0, +\infty[\times M$, by a constant that depends only on K and t_0, but not on u.

From Corollary 8.16, for every $(t, x) \in]0, +\infty[\times M$, we can find $y \in M$ such that

$$\hat{u}(t, x) = u(y) + h_t(y, x) = u(y) + \mathcal{H}(t, y, x). \tag{9-3}$$

Since

$$\mathcal{H}(t, y, x) = h_t(y, x) \leq u(x) - u(y) + A(0)t \leq K + Kd(x, y) + A(0)t.$$

We now use the fact that $2Kd(x, y) - C(2K)t \leq h_t(y, x) = \mathcal{H}(t, y, x)$, to get $Kd(x, y) \leq \frac{1}{2}[\mathcal{H}(t, y, x) + C(2K)t]$. Combining with the inequality above, we obtain

$$\mathcal{H}(t, y, x) \leq K + \tfrac{1}{2}[\mathcal{H}(t, y, x) + C(2K)t] + A(0)t.$$

Since $\mathcal{H}(t, y, x) = h_t(y, x)$, the inequality above is equivalent to

$$\mathcal{H}(t, y, x) \leq 2K + C(2K)t + 2A(0)t.$$

Therefore, we get

$$\frac{\mathcal{H}(t, y, x)}{t} = \frac{h_t(y, x)}{t} \leq \frac{2K}{t} + C(2K) + 2A(0) \leq \frac{2K}{t_0} + C(2K) + 2A(0). \tag{9-4}$$

By its definition, the Lax–Oleinik evolution $\hat{u}$ is strongly evolution-dominated by L, as before, we have

$$\hat{u}(t', x') \leq u(y) + \mathcal{H}(t', y, x') \quad \text{for all } (t', x') \in]0, +\infty[\times M.$$

Subtracting from this last inequality the equality (9-3), we obtain

$$\hat{u}(t', x') - \hat{u}(t, x) \leq \mathcal{H}(t', y, x') - \mathcal{H}(t, y, x) \quad \text{for all } (t', x') \in]0, +\infty[\times M.$$

If $\hat{u}$ is differentiable at (t, x), since $\mathcal{H}$ is locally semiconcave, the inequality above implies that $(t', x') \mapsto \mathcal{H}(t', y, x')$ is differentiable at (t, x), with

$$\partial_t \hat{u}(t, x) = \partial_t \mathcal{H}(t, y, x) \text{ and } \partial_x \hat{u}(t, x) = \partial_x \mathcal{H}(t, y, x).$$

By Corollary 4.24, this implies that $\mathcal{H}$ is differentiable at (t, y, x). But by Corollary 4.25 and (9-4), the derivative $D\mathcal{H}(t, y, x)$ is bounded in norm by a constant depending only on $2Kt_0^{-1} + C(2K) + 2A(0)$. Therefore, the same is true for the derivative of $\hat{u}$ at (t, x). □

Appendix: Uniformly continuous and Lipschitz in the large functions

The following lemma is well-known.

Lemma A.1. *Let N be a Riemannian manifold (not necessarily complete or without boundary). Denote by d the distance on N associated to the Riemannian metric. For any function $u : M \to \mathbb{R}$, the following conditions are equivalent:*

(1) *The function u is uniformly continuous (with respect to d).*

(2) *For every $\epsilon > 0$, we can find $\lambda_\epsilon < +\infty$ such that*

$$|u(x) - u(y)| \leq \epsilon + \lambda_\epsilon d(y, x).$$

(3) *There exists a sequence of Lipschitz (for d) functions $u_n : M \to \mathbb{R}$, $n \in \mathbb{N}$ such that $u_n \to u$ uniformly on M; that is, the norm $\|u - u_n\|_\infty$ approaches 0 as $n \to +\infty$.*

Proof. The implication (3) $\Longrightarrow$ (1) is well-known.

We now prove (1) $\Longrightarrow$ (2). Since u is uniformly continuous, we can find $\alpha > 0$ such that

$$d(x, y) \leq \alpha \Longrightarrow |u(y) - u(x)| \leq \epsilon.$$

For $x, y \in N$ fixed, we can find $n \in \mathbb{N}$ such that $n\alpha \leq d(x, y) < (n+1)\alpha$. By definition of the Riemannian distance, we can find a curve $\gamma : [0, \ell] \to M$, parametrized by arc-length and such that $\gamma(0) = x$, $\gamma(\ell) = y$, and $d(x, y) \leq \ell < (n+1)\alpha$. We set $x_i = \gamma(i\alpha)$, for $i = 0, \dots, n$, and $x_{n+1} = y$. Since $d(x_i, x_{i+1}) \leq \ell_g(\gamma \,|\, [i\alpha, (i+1)\alpha]) = \alpha$, for $i = 0, \dots, n-1$, and $d(x_n, x_{n+1}) \leq$

$\ell_g(\gamma\,|[n\alpha, \ell]) = \ell - n\alpha < \alpha$, we get $|u(x_i) - u(x_{i+1})| \le \epsilon$, for $i = 0, \dots, n$. Using $n \le d(x, y)/\alpha$, this yields

$$|u(x) - u(y)| = \left|\sum_{i=0}^{n} u(x_i) - u(x_{i+1})\right| \le \sum_{i=0}^{n} |u(x_i) - u(x_{i+1})|$$
$$\le (n+1)\epsilon \le \epsilon + \frac{\epsilon}{\alpha} d(x, y).$$

This proves (2) with $\lambda_\epsilon = \epsilon/\alpha$.

It remains to prove (2) $\Longrightarrow$ (3). From (2), we get

$$u(x) - \epsilon \le u(y) + \lambda_\epsilon d(y, x).$$

Taking the infimum over y, we get

$$u(x) - \epsilon \le \inf_{y \in M} u(y) + \lambda_\epsilon d(y, x) \le u(x).$$

The function $u_\epsilon : M \to \mathbb{R}$ defined by $u_\epsilon(x) = \inf_{y \in M} u(y) + \lambda_\epsilon d(y, x)$ has Lipschitz constant $\le \lambda_\epsilon$, and satisfies $\|u_\epsilon - u\|_\infty \le \epsilon$. □

We now recall the definition of Lipschitz in the large for a function, see [12, Definition A.5].

Definition A.2. Let X be a metric space with distance d. A function $u : X \to \mathbb{R}$ is said to be Lipschitz in the large if there exists a constant $K < +\infty$ such that

$$|u(y) - u(x)| \le K + K d(x, y) \quad \text{for every } x, y \in X. \tag{A-1}$$

When the inequality above is satisfied, we say that u is Lipschitz in the large with constant K.

Lemma A.3. *Let X be a metric space with distance d. If $u : X \to \mathbb{R}$ is Lipschitz in the large with constant K, its lower semicontinuous regularization u_- is finite-valued, Lipschitz in the large with the same constant K, and $|u(x) - u_-(x)| \le K$, for every $x \in X$.*

Proof. We can find a sequence $x_i \to x$ such that $u(x_i) \to u_-(x)$. Taking the limit in inequality (A-1), with $y = x_i$, yields $|u(x) - u_-(x)| \le K$. We can also find a sequence $y_i \to y$ such that $u(y_i) \to u_-(y)$. Taking the limit in inequality (A-1), with $y = y_i$ and $x = x_i$, yields $|u_-(y) - u_-(x)| \le K + K d(x, y)$. □

Proposition A.4. *Let X be a metric space with distance d. For any function $u : X \to \mathbb{R}$ and any finite constant $K \ge 0$, the following two statements are equivalent*:

- *The function u is Lipschitz in the large with constant K.*
- *There exists a Lipschitz function $\varphi : X \to \mathbb{R}$, with Lipschitz constant K, such that $\|u - \varphi\|_\infty = \sup_{x \in M} |u(x) - \varphi(x)| \le K/2$.*

Proof. If u satisfies $|u(y) - u(x)| \leq K + Kd(x, y)$, for every $x, y \in X$. We get $-K + u(y) \leq u(x) + Kd(x, y)$. If we define the function $\varphi : X \to \mathbb{R}$ by

$$\varphi(y) = \inf_{x \in M} u(x) + \frac{K}{2} + Kd(x, y),$$

we get $-K/2 + u(y) \leq \varphi(y) \leq u(y) + K/2$. Therefore φ is finite everywhere. It is also Lipschitz with Lipschitz constant $K < +\infty$, and $\|u - \varphi\|_\infty \leq K/2$.

To prove the converse, assume $\varphi : X \to \mathbb{R}$ has Lipschitz constant $\leq K$, and $\|u - \varphi\|_\infty \leq K/2$, we have

$$\begin{aligned} |u(y) - u(x)| &\leq |u(y) - \varphi(y)| + |\varphi(y) - \varphi(x)| + |\varphi(x) - u(x)| \\ &\leq \frac{K}{2} + Kd(x, y) + \frac{K}{2} = K + Kd(x, y). \end{aligned}$$ □

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