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Denjoy subsystems and horseshoes

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We introduce a notion of weak Denjoy subsystem (WDS) that generalizes the Aubry–Mather–Cantor sets to diffeomorphisms of manifolds. We explain how a rotation number can be associated to such a WDS. Then we build in any horseshoe a continuous one parameter family of such WDS that is indexed by its rotation number. Looking at the inverse problem in the setting of Aubry–Mather theory, we also prove that for a generic conservative twist map of the annulus, the majority of the Aubry–Mather sets are contained in some horseshoe that is associated to a Aubry–Mather set with a rational rotation number.

1. Introduction and main results

All the dynamicists know the famous Poincaré sentence about periodic orbits:

> *Ce qui nous rend ces solutions périodiques si précieuses*, *c'est qu'elles sont*, *pour ainsi dire*, *la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable.*

But a periodic orbit for a dynamical system $f : X \to X$ is simply a finite invariant subset and the dynamics restricted to this set cannot be very complicated. What is more interesting is the dynamics close to such a periodic orbit, that may give rise to various rich phenomena. For example, for a symplectic diffeomorphism of a surface, two kinds of restricted dynamics to invariant Cantor sets can exist close to the periodic orbits, that are:

- Horseshoes close to hyperbolic periodic points (see [27]);[1] since the work of Katok in [16], they are known to be the evidence of positive topological entropy. Moreover, they contain a dense set of periodic points.

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[1] We will define them precisely later in the article.

- Aperiodic Aubry–Mather sets close to elliptic periodic points (see [3; 8; 22]); they are known to have zero topological entropy and contain no periodic points.

Although we will later focus on some specific horseshoes, we give here a general definition of horseshoe.

Definition. Let $f : M \to M$ be a surface diffeomorphism. A *horseshoe for* f is a f-invariant subset $H \subset M$ such that the dynamics $f_{|H}$ is C^0 conjugate to the one of a nontrivial transitive subshift with finite type. A horseshoe for f is a σ_2*-horseshoe* when the dynamics $f_{|H}$ is C^0 conjugate to the shift with two symbols.

Example. The first horseshoe was introduced by S. Smale in [27] close to a transversal homoclinic intersection of a hyperbolic periodic point. This horseshoe is hyperbolic. Burns and Weiss extended this in [7] to the case of topologically transversal homoclinic intersection. Le Calvez and Tal use purely topological horseshoes for 2-dimensional homeomorphisms in [20].

The category of aperiodic Aubry–Mather set was recently extended in [2] to the notion of so-called Denjoy subsystem by P. Le Calvez and the author. We recall the definition given in [2].

Definition. Let $f : M \to M$ be a C^k diffeomorphism of a manifold M. A C^k (*resp. Lipschitz*) *Denjoy subsystem for* f is a triplet (K, γ, h) where:

- $\gamma : \mathbb{T} \to M$ is a C^k (resp. bi-Lipschitz) embedding.
- $h : \mathbb{T} \to \mathbb{T}$ is a Denjoy example with invariant compact minimal set $K \subset \mathbb{T}$.
- $f(\gamma(K)) = \gamma(K)$.
- $\gamma \circ h_{|K} = f \circ \gamma_{|K}$.

Remarks.
- In this definition, $\gamma(\mathbb{T})$ is not necessarily invariant.
- Observe the importance of γ to fix the regularity of $\gamma(K)$.
- For $k = 0$, what we call a C^0-diffeomorphism is in fact a homeomorphism and in this case we just require that γ is a continuous embedding.
- The embedding is also useful to define a circular order on the Cantor set $\gamma(K)$.

Example. There exists different notions of Aubry–Mather sets for the exact symplectic twist maps of the annulus; see [3] and [22]. We will follow [4] and for us, an Aubry–Mather set is a well ordered compact set that contains only minimizing orbits in a variational setting; see, e.g., [3; 4]. Let us recall some results that are contained in [4] and [1] and that we will use. We fix an exact symplectic twist map f of the infinite annulus and a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$. Then:

- Every Aubry–Mather set is a partial Lipschitz graph.
- Every Aubry–Mather set $\mathcal{A}$ has a rotation number $\rho(\mathcal{A}) \in \mathbb{R}$.
- For every $r \in \mathbb{R}\backslash\mathbb{Q}$, there exists a unique maximal (for $\subset$) Aubry–Mather set $\mathcal{A}_r$ with rotation number r that contains every Aubry–Mather set with the same rotation number.
- For every $r = \frac{p}{q} \in \mathbb{Q}$, there exist two Aubry–Mather set $\mathcal{A}_r^{\pm}$ with rotation number r that are maximal (for $\subset$) among the Aubry–Mather sets with the same rotation number. They are such that: $\forall x \in \tilde{\mathcal{A}}_r{}^{+}, \pi_1 \circ F^q(x) \geq \pi_1(x)+p$ (resp. $\forall x \in \tilde{\mathcal{A}}_r{}^{-}, \pi_1 \circ F^q(x) \leq \pi_1(x)+p$) where $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is the first projection.
- If $(\mathcal{A}_n)$ is a sequence of Aubry–Mather sets such that the sequence of rotation numbers $(\rho(\mathcal{A}_n))$ converges to some $r \in \mathbb{R}$, then $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ is relatively compact and any limit point of $(\mathcal{A}_n)$ for the Hausdorff distance is an Aubry–Mather set with rotation number r.

The Aubry–Mather sets $\mathcal{A}_r$ that have an irrational rotation number and that are not a complete graph always contain a Lipschitz Denjoy subsystem $\mathcal{C}_r$.

We noticed that an important advantage of γ is to define a circular order along $\gamma(K)$. But to do that, we only need the embedding restricted to K. That is why we introduce now a new notion, the one of *weak Denjoy subsystem* that extends the one of Denjoy subsystem. This notion is similar to the one of Denjoy set that was introduced by J. Mather in [23].

Definition. Let $f : M \to M$ be a homeomorphism of a manifold M. A *weak Denjoy subsystem for f* (in short WDS) is a triplet (K, j, h) where:

- $h : \mathbb{T} \to \mathbb{T}$ is a Denjoy example with invariant minimal set $K \subset \mathbb{T}$.
- $j : K \to M$ is a homeomorphism onto its image.
- $f(j(K)) = j(K)$.
- $j \circ h_{|K} = f \circ j$.

When j is bi-Lipschitz or a C^k embedding (in the Whitney sense), we speak of Lipschitz or C^k weak Denjoy subsystem for f. Two WDS (K_1, j_1, h_1) and (K_2, j_2, h_2) are *equivalent* if $j_1(K_1) = j_2(K_2)$.

The restriction of a Denjoy subsystem to its nonwandering set is always a WDS. On a surface, we have the reverse implication.

Proposition 1.1. *Let (K, j, h) be a WDS of a surface homeomorphism. Then there exists a C^0 Denjoy subsystem (K, γ, h) such that $\gamma_{|K} = j$.*

Remarks.

- This result is specific to the case of surfaces because it uses a classical result on extension of homeomorphisms between Cantor sets of surfaces.
- We do not know about such a result with more regularity: Lipschitz, C^1 (a kind of Whitney extension theorem for diffeomorphisms). Observe that we proved in [2] that there exists no C^2 Denjoy subsystem.

Remark. Let us recall that a *circular order relation* on a set X is a relation $\prec$ that is defined on the triplets of points of X such that:

- If $x, y, z \in X$, we have $x \prec y \prec z$ or $z \prec y \prec x$; we use the notation $[x, z]_\prec = \{y \in X; x \prec y \prec z\}$.
- If $x \neq z$, the two previous lines of inequalities are simultaneously satisfied if and only if $x = y$ or $y = z$.
- If $x \prec y \prec z$, then $y \prec z \prec x$.
- If $x \prec y \prec z$ and $x \prec z \prec t$ then $x \prec y \prec t$.

If $\prec$ is a circular order on X, the inverse order $- \prec$ is defined by

$$\forall x, y, z \in X, \quad x(- \prec)y(- \prec)z \Leftrightarrow z \prec y \prec x.$$

Notations.

- If (K, j, h) is a WDS, we denote by $\prec_K$ the circular order on $j(K)$ that is deduced from the one of $K \subset \mathbb{T}$ via the map j.
- The *graph* $\mathcal{G}(\prec_K)$ of this order relation is the set of the triplets $(a, b, c) \in (j(K))^3$ such that $j^{-1}(a) \prec j^{-1}(b) \prec j^{-1}(c)$ where $\prec$ is the usual order on $\mathbb{T}$. This graph $\mathcal{G}(\prec_K)$ is then a closed subset of $(j(K))^3$ and then of $(M)^3$. Observe that for every $a, c \in j(K)$, $\mathcal{G}(\prec_K, a, c) = \{b \in j(K); (a, b, c) \in \mathcal{G}(\prec_K)\}$ is a nonempty compact subset of M, called an *interval* of $\mathcal{G}(\prec_K)$.

Remark. We have $\mathcal{G}(\prec_K, a, a) = j(K)$ and for $a \neq c$, $\mathcal{G}(\prec_K, a, c)$ contains at least a and c. Moreover, we have $\mathcal{G}(\prec_K, a, c) = \{a, c\}$ if and only if $\{a, c\}$ is one gap of the Cantor set.[2]

The first theorem we will prove allows us to extend Poincaré's notion of rotation number to WDS, or more precisely to the classes of equivalence of WDS.

Theorem 1.2. *Let (K_1, j_1, h_1) and (K_2, j_2, h_2) be two equivalent WDS for a same homeomorphism $f : M \to M$ of a manifold M. Then:*

- *There exists a homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ h_1 = h_2 \circ h$.*
- *We have $\prec_{K_1} = \prec_{K_2}$ or $\prec_{K_1} = - \prec_{K_2}$, hence the two orders have the same intervals.*

[2]Observe that in this case, a and c are α and ω-asymptotic under the dynamics.

Corollary 1.3. *The map ρ defined on the set of WDS with values in $\mathbb{T}/x \sim -x$ that associates to any WDS (K, γ, h) the rotation number of h modulo its sign is such that if (K_1, γ_1, h_1) and (K_2, γ_2, h_2) are equivalent, then $\rho(K_1, \gamma_1, h_1) = \rho(K_2, \gamma_2, h_2)$.*

Let us endow the set of WDS with a topology that focus on their order relation.

Notations. We endow M with a Riemannian metric d and M^3 is endowed with the natural sup distance associated to d that is denoted by d_∞. Then D (resp. D_∞) is the associated Hausdorff distance on the set of nonempty compact subsets of M (resp. M^3).

Definition. Let $f : M \to M$ be a homeomorphism of a manifold M. Let (K_i, j_i, h_i) be two weak Denjoy subsystems for f. We denote by $\mathcal{G}(\prec_{K_i}) \subset M^3$ (resp. $\mathcal{G}(- \prec_{K_i})$) the graph of $\prec_{K_i}$ (resp. $- \prec_{K_i}$).

We define a distance δ on the set of the weak Denjoy subsystems for f by the following equality.

We have

$$\delta((K_1,j_1,h_1),(K_2,j_2,h_2))$$
$$=\max\left\{D(j_1(K_1),j_2(K_2)),\min\{D_\infty(\mathcal{G}(\prec_{K_1}),\mathcal{G}(\prec_{K_2})),D_\infty(\mathcal{G}(-\prec_{K_1}),\mathcal{G}(\prec_{K_2}))\}\right\}.$$

Proposition 1.4. *The map that associates to every WDS its rotation number is continuous.*

Remark. The previous result extends a result that is well-known in the setting of well-ordered sets for twist maps.

Horseshoes and WDS are different but in general, it is believed that, up to some entropy restriction, horseshoes dynamics contain every dynamics (via symbolic dynamics).[3]

We will prove that every horseshoe contains many WDS, and even a continuous 1-parameter family (D_ρ) continuously depending on its rotation number ρ where ρ is in a nontrivial interval of $\mathbb{T}/x \sim -x$ of irrational numbers.

Theorem 1.5. *Let $f : M \to M$ be a C^k diffeomorphism and let $\mathcal{H}$ be a horseshoe for f. Then exists $N \geq 1$ and a continuous map $D : r \in (\mathbb{T}\backslash\mathbb{Q})/x \sim -x \mapsto (K_r, j_r, h_r)$ such that:*

- *$D(r) = (K_r, j_r, h_r)$ is a continuous WDS with rotation number r for f^N.*
- *$j_r(K_r) \subset \mathcal{H}$.*

Moreover, if $\mathcal{H}$ is a σ_2-horseshoe, we have $N = 1$.

[3] This is not completely correct because, for example, the dynamics of an odomoter cannot be embedded in a horseshoe even if it has zero entropy: it is an isometry and the horseshoe is expansive.

Different authors before us built embedding of Denjoy dynamics into horseshoes. The first one is certainly [12], that embeds some Denjoy dynamics into the abstract horseshoe by using Sturmian sequences (this article follows the seminal work of Hedlund and Morse in [25]). In [15], the authors build an uncountable family of Denjoy dynamics in a given horseshoe. If we analyze their construction, for every irrational rotation number, they build an uncountable family of weak Denjoy subsystems with two holes that are not conjugate together (see Markley, [21], for a characterization of conjugated Denjoy examples). In [6], Boyland used a distance different from the one we use (his distance uses the Hausdorff distance D in M and also a distance on the set of Borel probability measures) and proved that for every irrational rotation number and every integer $N \geq 1$, there is a N-dimensional topological disc of weak Denjoy subsystems having this rotation number in every horseshoe. He also explained a general method to obtain all the weak Denjoy subsystem of a horseshoe. In [5], looking for special invariant measures of the angle doubling on the circle, Bousch uses the one side shift on $\{0, 1\}^{\mathbb{N}}$ and the unique invariant measure with support in a Cantor set analogous to the one we build.

Remarks.
- The continuous WDS that we will embed in the horseshoe are WDS that have only a pair of orbits that are ω asymptotic (and then α-asymptotic because we have a Denjoy dynamics), i.e., that correspond to a Denjoy example with exactly one orbit of a wandering interval (we will say one gap).
- Observe that the shift dynamics is expansive. Hence we cannot embed in it a WDS with a infinite countable number of gaps: one of these gaps would have all its orbit with diameter less than the expansivity constant, which is impossible.
- But it is possible to embed a family of WDS with a finite number p of gaps in a $\sigma_{\sup\{2,p\}}$-horseshoe.
- A similar method to embed Cantor sets with an interval of rotation numbers was proposed by K. Hockett and P. Holmes for dissipative twist maps in [15]. Here we proved a more general statement (for WDS) and also prove a continuity result.

Corollary 1.6. *Let $f : M^{(2)} \to M^{(2)}$ be a C^k diffeomorphism of a surface and let $\mathcal{H}$ be a horseshoe for f. Then exists $N \geq 1$ and a map $D : r \in (\mathbb{T}\backslash\mathbb{Q})/x \sim -x \mapsto (K_r, \gamma_r, h_r)$ such that:*

- *$D(r) = (K_r, \gamma_r, h_r)$ is a continuous Denjoy subsystem with rotation number r for f^N.*
- *The map $W : r \in (\mathbb{T}\backslash\mathbb{Q})/x \sim -x \mapsto (K_r, \gamma_{r|K_r}, h_r)$ is continuous.*

- $\gamma_r(K_r) \subset \mathcal{H}$.

Moreover, if $\mathcal{H}$ is a σ_2-horseshoe, we have $N = 1$.

Remarks.
- We do not know how to choose continuously the embedding γ_r or at least its image. But we do not need that to describe the dynamics on $\gamma_r(K_r)$.
- In the setting of the Aubry–Mather theory, the map that associates to any Aubry–Mather set $\mathcal{A}_r$ the graph that linearly interpolates $\mathcal{A}_r$ is in fact continuous when we endow the set of functions with the C^0 distance, and we will see that the Aubry–Mather sets with an irrational rotation number are actually contained in some horseshoes in the generic case. But the Aubry–Mather set do not continuously depend on the rotation number $r \in \mathbb{R}\backslash\mathbb{Q}$, so even in the case of the Aubry–Mather sets we do not know if we can interpolate in a continuous way by a curve.

We now focus on Aubry–Mather theory and address the inverse problem: are the WDS that appear in a natural way in symplectic 2-dimensional dynamics contained in some horseshoe?

Theorem 1.7. *Let $f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ be an exact symplectic twist map and let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of f. Assume that $\mathcal{A}_r^+$ (resp. $\mathcal{A}_r^-$) is uniformly hyperbolic for some rational number $r \in \mathbb{Q}$. Let $\mathcal{V}_r$ be a neighborhood of $\mathcal{A}_r^+$ (resp. $\mathcal{A}_r^-$). Then there exists a horseshoe $\mathcal{H}_r^+$ (resp. $\mathcal{H}_r^-$) for some f^N and $\varepsilon > 0$ such that:*

- *$\mathcal{H}_r^+$ (resp. $\mathcal{H}_r^-$) contains $\mathcal{A}_r^+$ (resp. $\mathcal{A}_r^-$) and is contained in $\mathcal{V}_r$.*
- *Every Aubry–Mather set with rotation number in $(r, r+\varepsilon)$ (resp. $(r-\varepsilon, r)$) is contained in $\mathcal{H}_r^+$ (resp. $\mathcal{H}_r^-$).*
- *Every point in $\mathcal{H}_r^+$ (resp. $\mathcal{H}_r^-$) has no conjugate points, i.e., has its orbit that is locally minimizing.*

Remarks.
- It is well known that the topological entropy of a twist map restricted to the union of all its hyperbolic Aubry–Mather sets is zero and has zero Hausdorff dimension; see [9]. Theorem 1.7 implies that for an open and dense subset of conservative twist diffeomorphisms (in any reasonable topology), there exists an invariant set $\mathcal{K}$ of points with no conjugate points such that the dynamics restricted to $\mathcal{K}$ has positive topological entropy and positive Hausdorff dimension.
- In [18], a transitive set that contains all the Aubry–Mather sets is built by P. Le Calvez. But this set is very different from the one we build here, because it contains in general orbits with conjugate points and is far from every Aubry–Mather set $\mathcal{A}_r^{\pm}$. Moreover, it is not a horseshoe.

- Observe that no WDS (K, j, h) that is contained in a hyperbolic horseshoe is C^1. Indeed, the endpoints a, b of every gap are α and ω-asymptotic and their orbits are dense in $j(K)$. But for n large enough, $f^n(a)$ and $f^n(b)$ are in the same local stable manifold and then oriented in the stable direction and $f^{-n}(a)$ and $f^{-n}(b)$ are in the same local unstable manifold and the oriented in the unstable direction. Hence, close to any point in $j(K)$, we find points such that the geodesic that joins them is either along the stable or the unstable direction. So j cannot be C^1. In the Aubry–Mather setting, it is Lipschitz.
- As we noticed before, a weak Denjoy subsystem (K, j, h) that is contained in a horseshoe has a finite number of gaps. When the horseshoe is uniformly hyperbolic with an expansivity constant equal to ε and j is k-bi-Lipschitz, it can be proved that the number of gaps is at most $\frac{k}{\varepsilon}$.

A remarkable result of P. Le Calvez asserts that general Aubry–Mather sets of general exact symplectic twist diffeomorphisms are uniformly hyperbolic; see[19]. Joint with Theorem 1.7, this implies the following corollary.

Corollary 1.8. *There exists a dense G_δ subset $\mathcal{G}$ of the set of C^k exact symplectic twist diffeomorphisms (for $k \geq 1$) such that for every $f \in \mathcal{G}$, there exist an open and dense subset $U(f)$ of $\mathbb{R}$ and a sequence $(r_n)_{n\in\mathbb{N}}$ in $U(f) \cap \mathbb{Q}$ such that every minimizing Aubry–Mather set with rotation number in $U(f)$ is hyperbolic and contained in a horseshoe associated to a minimizing hyperbolic Aubry–Mather set whose rotation number is r_n.*

Remark. Observe that in [11], Goroff gives an example where the union of all the Aubry–Mather sets is uniformly hyperbolic.

An open problem is the possible extension of Theorem 1.7 in a relaxed setting. Hence we rise the following questions.

Question (A. Fathi). Without assuming hyperbolicity, are the Aubry–Mather sets that are Cantor contained in some (nonhyperbolic) horseshoe?

Another question concerns the dynamics that are not necessarily twist diffeomorphisms.

Question. For a (possibly generic) symplectic diffeomorphism, is any WDS contained in some horseshoe?

It is possible to build C^1 or C^2 examples that have WDS that are not contained in horseshoes (examples that have a C^1 invariant curve on which the dynamics is Denjoy, see [14]), but our question concerns higher differentiability.

1A. *Notations.* For any hyperbolic periodic point x of a C^1 diffeomorphism, we denote by $W^s(x, f)$ or $W^s(x)$ (resp. $W^u(x, f)$ or $W^u(x)$) its stable (resp. unstable) submanifold and by $W^s_{\text{loc}}(x, f)$ or $W^s_{\text{loc}}(x)$ (resp. $W^u_{\text{loc}}(x, f)$ or $W^u_{\text{loc}}(x)$) its local stable (resp. unstable) submanifold. We adopt exactly the same notations for not necessarily periodic points that belong to some hyperbolic set.

Also we mention that the annulus is $\mathbb{A} = \mathbb{T} \times \mathbb{R}$, that its tangent space is $\mathbb{A} \times \mathbb{R}^2$ and that the tangent space at every point is endowed with its usual Euclidean norm. Moreover, we use the notation $\pi_1 : \mathbb{A} \to \mathbb{T}$ for the first projection as well as its lift $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$.

2. Proof of Proposition 1.1

We assume that (K, j, h) is a weak Denjoy subsystem of a surface homeomorphism. If we embed $\mathbb{T}$ in $\mathbb{R}^2$, then K is a Cantor set that is a subset of $\mathbb{R}^2$.

The main argument of the proof is a result that is contained in Chapter 13 of [24].

Theorem. *Every homeomorphism between two Cantor subsets of $\mathbb{R}^2$ can be extended so as to give a homeomorphism of $\mathbb{R}^2$ onto itself.*

Corollary 2.1. *Let C be a Cantor subset that is contained in a topological open disc D. For every $\delta > 0$, there exists a finite number of disjoint topological discs $D_1, \dots, D_n$ with diameter less than δ such that*

$$C \subset \bigcup_{1 \le i \le n} D_k \subset \bigcup_{1 \le i \le n} \bar{D}_k \subset D.$$

Let us prove the corollary. If C is a Cantor set that is contained in an open disc D, there exists a homeomorphism $h : D \to \mathbb{R}^2$ such that $h(C)$ is the triadic Cantor set $C_0 \subset \mathbb{R} \times \{0\}$. We can decrease slightly D in such a way that $C \subset D' \subset \bar{D}' \subset D$ and h is restricted to the closed topological disc $\bar{D}'$. For every $\varepsilon > 0$, there exists a covering of C_0 by a finite number of topological discs $d_1, \dots, d_n$ that are contained in $h(D')$ and have diameter less than ε; indeed, this result is well known for the triadic subset in the real line and we just have to choose ε less than the distance between C_0 and $\mathbb{R}^2 \backslash h(D')$ and thicken the intervals into topological discs. Because h^{-1} is uniformly continuous, we deduce that for every $\varepsilon > 0$, there exists a finite covering of C by a finite number of disjoint discs that are contained in D' and have diameter less than ε. □

There exists $\eta > 0$ (that is less than the radius of injectivity of the Riemannian metric on $j(K)$) such that every set with diameter less than η that intersects $j(K) = C$ is contained in some topological disc. As $C = j(K)$ is a Cantor set, it is (uniformly) homeomorphic to the triadic Cantor set. Hence, there exists a closed partition of C into a finite number of sets $C_1, \dots, C_p$ that are open an closed in C and have diameter less than η. As the diameter of every C_j is less than η, there exists a topological disc B_j that contains the Cantor set C_j. We introduce the notation $\delta = \min\{\min\{d(C_i, C_j); i \neq j\}, \eta\}$. Then we can apply Corollary 2.1: there exists a finite number of disjoint topological discs $D_1^j, \dots, D_{n_j}^j$ with diameter less than $\frac{\delta}{2}$ such that $C_j \subset \bigcup_{1\leq k\leq n_j} D_k^j \subset \bigcup_{1\leq k\leq n} \bar{D}_k^j \subset B_j$. For every $i \neq j$, D_k^j intersects C_j and has diameter less than $d(C_i, C_j)/2$. We deduce that if $(j,k) \neq (j',k')$, then $D_k^j \cap D_{k'}^{j'} = \varnothing$. We have found a covering of C by disjoint discs. We can join them to obtain a topological disc D in M that contains C. There exists a homeomorphism $\Phi : D \to \mathbb{R}^2$.

Then the Cantor subset $\Phi \circ j(K)$ of $\mathbb{R}^2$ is homeomorphic to the Cantor subset K of $\mathbb{R}^2$.

We deduce that there exists a homeomorphism $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ that extends the homeomorphism $\Phi \circ j : K \to \Phi \circ j(K)$.[4] Then $\gamma = \Phi^{-1} \circ \psi : \mathbb{T} \to D \subset M$ is a simple continuous curve and (K, γ, h) is a Denjoy subsystem that extends (K, j, h).

3. Proof of Theorem 1.2 and Corollary 1.3

3A. ***Proof of the first point of Theorem 1.2 and of Corollary 1.3.*** Let (X, d) be a metric space. We associate to any continuous dynamical system $F : X \to X$ an equivalence relation $\mathcal{R}_F$ that is defined by

$$x\mathcal{R}_F y \Leftrightarrow \lim_{k\to+\infty} d(F^k x, F^k y) = 0.$$

Observe that if $H : X \to Y$ is a homeomorphism and if X is compact, then we have

$$x\mathcal{R}_F y \Leftrightarrow H(x)\mathcal{R}_{H\circ F\circ H^{-1}} H(y).$$

Hence $X/\mathcal{R}_F$ is compact if and only if $Y/\mathcal{R}_{H\circ F\circ H^{-1}}$ is compact. We denote by $p_F : X \to X/\mathcal{R}_F$ the projection.

Because of Poincaré classification of circle homeomorphisms (see for example [17]), for every orientation preserving homeomorphism h of the circle with an irrational rotation number, we have $\mathcal{R}_h = \mathcal{R}_{h^{-1}}$. Moreover, for such an orientation preserving homeomorphism of the circle with irrational rotation number, the relation is closed and it is also true for the restriction to any invariant compact subset. In this case, the quotient space, that corresponds to a closed equivalence relation on a compact space, is also compact. We then consider a semiconjugation k between the orientation preserving homeomorphism h of the

[4] Observe that this is specific to the 2-dimensional setting and that there exists some homeomorphisms between two Cantor subsets of $\mathbb{R}^3$ that cannot be extended to a homeomorphism of $\mathbb{R}^3$, see Theorem 5 of chapter 18 of [24].

circle with irrational rotation number and a rotation R_α, i.e., k is nondecreasing continuous map onto the circle such that

$$k \circ h = R_\alpha \circ k.$$

Then $k : \mathbb{T} \to \mathbb{T}$ is continuous and we have

$$\forall x, y \in \mathbb{T}, \quad k(x) = k(y) \Leftrightarrow x \mathcal{R}_h y.$$

We denote by K_h the unique nonempty minimal h-invariant compact subset (then K_h is $\mathbb{T}$ or a Cantor subset) and we denote by G_h the set of points of K_h that are $\mathcal{R}_h$ related to another point of $\mathbb{T}$. In other words, G_h is the union of the endpoints of the gaps of the set K_h. Then there exists a unique map $\bar{k} : K_h/\mathcal{R}_h \to \mathbb{T}$ such that $\bar{k} \circ p_h = k$. The definition of the quotient topology implies that $\bar{k}$ is continuous and it is then a homeomorphism from $K_h/\mathcal{R}_h$ to $\mathbb{T}$. Moreover, there exists a unique map $\bar{h} : K_h/\mathcal{R}_h \to K_h/\mathcal{R}_h$ that is the quotient dynamics and that satisfies

$$\bar{h} \circ p_h = p_h \circ h;$$

we have then

$$\bar{k} \circ \bar{h} = R_\alpha \circ \bar{k},$$

i.e., $\bar{k}$ is a conjugation between $\bar{h}$ and R_α.

Let us consider two WDS (K_1, j_1, h_1) and (K_2, j_2, h_2) for the same homeomorphism $f : M \to M$ of a manifold M such that $C = j_1(K_1) = j_2(K_2)$. Let k_i be a semiconjugation between h_i and a rotation R_{a_i}, i.e.,

$$k_i \circ h_i = R_{a_i} \circ k_i.$$

As $f_{|C} = j_i \circ h_i \circ j_i^{-1}$, then $C/\mathcal{R}_f$ is homeomorphic to $K_i/\mathcal{R}_{h_i}$ and so to $\mathbb{T}$. We denote by $p : C \to C/\mathcal{R}_f$ the projection and by $\bar{f} : C/\mathcal{R}_f \to C/\mathcal{R}_f$ the reduced dynamics; see Figure 1.

Then the map $k_i \circ j_i^{-1} : C \to \mathbb{T}$ is a continuous surjection such that

$$k_i \circ j_i^{-1}(x) = k_i \circ j_i^{-1}(y) \Leftrightarrow p(x) = p(y).$$

Hence, there exists a unique homeomorphism $\ell_i : C/\mathcal{R}_f \to \mathbb{T}$ such $\ell_i \circ p = k_i \circ j_i^{-1}$. We have then for all $\bar{x} = p(x) \in C/\mathcal{R}_f$

$$\begin{aligned} R_{a_i} \circ \ell_i(\bar{x}) &= R_{a_i} \circ k_i \circ j_i^{-1}(x) \\ &= k_i \circ h_i \circ j_i^{-1}(x) \\ &= k_i \circ j_i^{-1} \circ (j_i \circ h_i \circ j_i^{-1})(x) \\ &= k_i \circ j_i^{-1} \circ f(x) \\ &= \ell_i(\bar{f}(\bar{x})). \end{aligned}$$

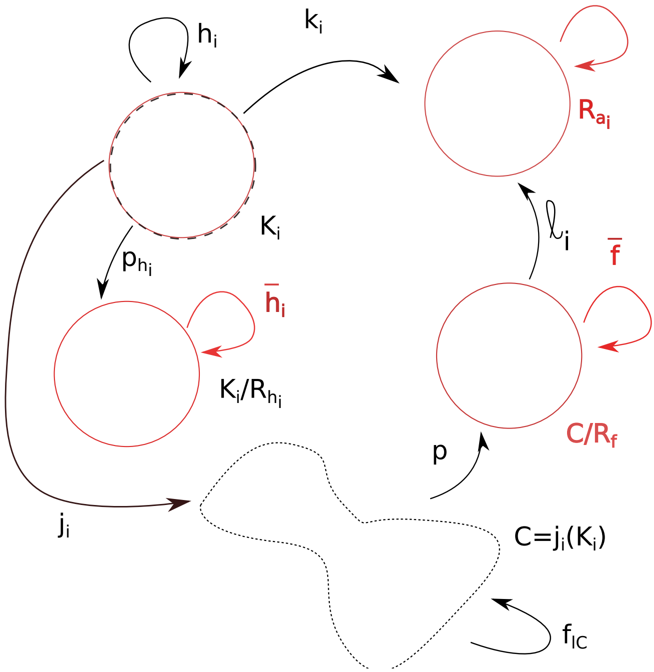


Figure 1. Two WDS, (K_1, j_1, h_1) and (K_2, j_2, h_2) for the same homeomorphism $f : M \to M$ of a manifold M such that $C = j_1(K_1) = j_2(K_2)$.

We deduce that

$$R_{a_1} = \ell_1 \circ \bar{f} \circ \ell_1^{-1} = (\ell_1 \circ \ell_2^{-1}) \circ R_{a_2} \circ (\ell_1 \circ \ell_2^{-1})^{-1}.$$

As R_{a_1} and R_{a_2} are conjugate, we have $a_1 = \pm a_2$. More precisely, $a_1 = a_2$ when the conjugation preserves the orientation (and then is $(x \mapsto x + C)$) and $a_1 = -a_2$ when the conjugation reverses the orientation (and then is $(x \mapsto C - x)$). This gives Corollary 1.3 but doesn't end the proof of the first point of Theorem 1.2.

To finish the proof of this point, let us observe that

$$j_1(G_{h_1} \cap K_1) = j_2(G_{h_2} \cap K_2) = \{x \in C; \exists y \in C; y \neq x, y\mathcal{R}_f x\}$$

is the set of the endpoints of the gaps of $f_{|C}$ (gaps are pairs of points that are ω-asymptotic). We denote this set by C_0.

Thus we have $k_i(G_{h_i}) = k_i \circ j_i^{-1}(C_0) = \ell_i \circ p(C_0)$. We deduce that $k_1(G_{h_1}) = \ell_1 \circ \ell_2^{-1}(k_2(G_{h_2}))$. As $\ell_1 \circ \ell_2^{-1}$ is either a translation $x \mapsto x + C$ or a symmetry $x \mapsto C - x$, there exists $C \in \mathbb{R}$ such that either $k_1(G_{h_1}) = C + k_2(G_{h_2})$ or

$k_1(G_{h_1}) = C - k_2(G_{h_2})$. In other words, the image by k_1 of the union of the gaps of K_{h_1} is the image by a translation or a symmetry of the image by k_2 of the union of the gaps of K_{h_2}. As explained in [13] and [21], this is equivalent to the fact that h_1 and h_2 are conjugated.

3B. ***Proof of the second point of Theorem 1.2.*** For the second point, we know that $j_i : K_i \to C$ defines the order $\prec_{K_i}$. If we identify points that are ω-asymptotic, we obtain a reduced order relation $\bar{\prec}_{K_i}$ on $K_i/\mathcal{R}_{h_i}$ and $C/\mathcal{R}_f$ and $\bar{j}_i : K_i/\mathcal{R}_{h_i} \to C/\mathcal{R}_f$ is an order preserving homeomorphism. As there are only two possible orientations on the circle, we deduce for the two reduced order relations on $C/\mathcal{R}_f$ that either they are equal or they are reverse. To deduce the result for the nonreduced relation, we have just to note that there is only one way to define the closed order relation $\prec_{K_i}$ on C whose reduced relation is $\bar{\prec}_{K_i}$.

4. Proof of Proposition 1.4

Let us begin by explaining some results on the symbolic dynamics of WDS. If (K, j, h) is a WDS for f, we can encode the dynamics in the following noninjective way.[5] Let $x_0 \in j(K)$ be a point of $j(K)$. We consider the interval I_0 of $j(K)$ of the points $y \in j(K)$ such that x_0, y and $f(x_0)$ are in this order for $\prec_K$. We decide that $x_0 \in I_0$ but $f(x_0) \notin I_0$. We denote by $I_1 = j(K)\backslash I_0$ the complement of I_0 in $j(K)$. Then we consider the map that associates to every point $x \in j(K)$ its *itinerary*

$$\mathcal{I}(x) = (n_k(x))_{k\in\mathbb{Z}}$$

where $f^k(x) \in I_{n_k(x)}$. When x_0 is the right end of a gap (a gap is the image by j of the two endpoints of a wandering interval of h) of $j(K)$, I_0 and I_1 are closed and open in $j(K)$ and then $\mathcal{I}$ is continuous.[6]

We assume that x_0 is indeed the right end of a gap of $j(K)$ and we denote by $\mathcal{K}$ the set $\mathcal{I}(K)$. As the Denjoy example is semiconjugate to the rotation with angle $\alpha = \rho(h)$, $\mathcal{I}(x_0)$ is nothing else than the Sturmian sequence that is associated to the rotation R_α, i.e., (see [10]) $n_k(x) = 0$ if and only if $k\alpha \in [0, \alpha)$.

Let us now consider a WDS (K_1, j_1, h_1) that is close to (K, j, h) for the topology that we defined before. Let (x_1, x_0) be the gap whose x_0 is the right end in $j(K)$. Then the interval $\mathcal{G}(\prec_K, x_1, x_0) = \{x_0, x_1\}$ has only two points. As $\mathcal{G}(\prec_{K_1})$ is close to $\mathcal{G}(\prec_K)$ for the Hausdorff distance, there exists two points $y_1, y_0 \in j(K_1)$ that are close to x_1, x_0 and such that $\mathcal{G}(\prec_{K_1}, y_1, y_0)$ is contained

[5] Observe that this is not necessarily the encoding that is given by the subshift of finite type on the horseshoe when this WDS is contained in some horseshoe.

[6] We will prove in Section 5 that when h is a Denjoy example with one gap, then $\mathcal{I}$ is in fact a homeomorphism on its image.

in a neighborhood of $\mathcal{G}(\prec_K, x_1, x_0)$. As we know that $y_1, y_0 \in \mathcal{G}(\prec_{K_1}, y_1, y_0)$, that y_0 is close to x_0 and that y_1 is close to x_1, this implies that $\mathcal{G}(\prec_{K_1}, y_1, y_0)$ is close to $\mathcal{G}(\prec_K, x_1, x_0)$ for the Hausdorff distance. Then we write

$$\mathcal{G}(\prec_{K_1}, y_1, y_0) = \mathcal{G}_0 \cup \mathcal{G}_1$$

where the points of $\mathcal{G}_0$ are close to x_0 and the points of $\mathcal{G}_1$ are close to x_1. Observe that $\mathcal{G}(\prec_{K_1}, y_1, y_0)$ is an interval for $\prec_{K_1}$, where $\prec_{K_1}$ define a (noncircular) total order. Hence we can define $z_1 = \sup \mathcal{G}_1$ and $z_0 = \inf \mathcal{G}_0$. Then $\{z_1, z_0\}$ is a gap of K_1 that is close to $\{x_1, x_0\}$ for the Hausdorff topology. We then associate to z_1 its itinerary exactly as we did for x_1. Let us fix $N \geq 1$. Then if (K_1, j_1, h_1) is close enough to (K, j, h), the two itineraries between $-N$ and N match. But these itineraries determine the first terms of the continued fraction of the two rotations numbers of h_1, h (see [10]). Because they coincide up to the order N, we deduce that $\rho(h_1)$ is close to $\rho(h)$ and then that the rotation number map is continuous.

5. Proof of Theorem 1.5 and Corollary 1.6

5A. ***Proof of Theorem 1.5.*** We will use the following notions.

Definition. A *n-cylinder* in Σ_2 is a set of sequences $(u_k)_{k\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ such that $u_{-n} = \delta_{-n}; \dots, u_0 = \delta_0; \dots; u_n = \delta_n$ where the δ_is are fixed in $\{0, 1\}$. Defining $d((u_k)_{k\in\mathbb{Z}}, (v_k)_{k\in\mathbb{Z}}) = \max_{k\in\mathbb{Z}}|u_k - v_k|/(|k| + 1)$, observe that a n-cylinder is exactly a closed ball with radius $1/(n + 2)$. A n-word of u is a sequence of n successive terms of u.

Let $f : M \to M$ be a C^k diffeomorphism and let $\mathcal{H}$ be a horseshoe for f. Then there exists a transitive subshift with finite type $\sigma_A : \mathcal{K} \to \mathcal{K}$ that is defined on some shift invariant compact subset $\mathcal{K}$ of Σ_p such that $f_{|\mathcal{H}}$ is C^0 conjugate to σ_A. Then there exists a σ_A-invariant compact subset $\mathcal{K}_0 \subset \mathcal{K}$ and $N \geq 1$ such that $\sigma^N_{A|\mathcal{K}_0}$ is C^0 conjugate to σ_2. Hence we just need to prove the theorem for a σ_2-horseshoe to deduce the general statement. We assume that $f_{|\mathcal{H}} = k \circ \sigma_2 \circ k^{-1}$.

Let $h_\alpha : \mathbb{T} \to \mathbb{T}$ be a Denjoy example with minimal Cantor set C_α such that:

- $\mathbb{T}\backslash C_\alpha$ is the orbit of one interval $I_\alpha = (a_\alpha, b_\alpha)$.
- The rotation number of h is α.

We consider two disjoint segments $I_0(\alpha)$ and $I_1(\alpha)$ in $\mathbb{T}$ such that:

- One endpoint of $I_j(\alpha)$ is in I_α and the other one is in $h_\alpha(I_\alpha)$.
- $I_0(\alpha)$ joins I_α to $h_\alpha(I_\alpha)$ in the direct sense.

Let $k_\alpha : \mathbb{T} \to \mathbb{T}$ be a semiconjugation between h_α and R_α, i.e., $k_\alpha \circ h_\alpha = R_\alpha \circ k_\alpha$. Then, the intervals $I_0(\alpha)$ and $I_1(\alpha)$ are mapped on intervals $K_0 = [0, \alpha]$ and

$K_1 = [\alpha, 1]$. As α is irrational, if $(n_k)_{k\in\mathbb{Z}} \in \Sigma_2$ is any sequence of 0 and 1, there exists at most one $\theta \in \mathbb{T}$ such that, for every $k \in \mathbb{Z}$, we have $\theta + k\alpha \in K_{n_k}$. Let us now consider two points $\theta_1 \neq \theta_2$ in C_α such that for every $k \in \mathbb{Z}$, $h_\alpha^k(\theta_1)$ and $h_\alpha^k(\theta_2)$ belong to a same interval $I_{n_k}(\alpha)$. Then for every $k \in \mathbb{Z}$, the points $k_\alpha \circ h_\alpha^k(\theta_1) = k_\alpha(\theta_1) + k\alpha$ and $k_\alpha \circ h_\alpha^k(\theta_2) = k_\alpha(\theta_2) + k\alpha$ belong to the same interval K_{n_k} and so $k_\alpha(\theta_1) = k_\alpha(\theta_2)$, i.e., θ_1 and θ_2 are the two endpoints of some gap of the Cantor set C_α. So there exists $k \in \mathbb{Z}$ such that $h_\alpha^k(\theta_1)$ and $h_\alpha^k(\theta_2)$ are the two endpoints of I_α for example $I_\alpha = (h_\alpha^k(\theta_1), h_\alpha^k(\theta_2))$. But this implies that $h_\alpha^k(\theta_1) \in I_1(\alpha)$ and $h_\alpha^k(\theta_2) \in I_0(\alpha)$ and this contradicts that for every $k \in \mathbb{Z}$, $h_\alpha^k(\theta_1)$ and $h_\alpha^k(\theta_2)$ belong to a same interval $I_{n_k}(\alpha)$. So we have proved that if we use the notation for $\theta \in C_\alpha$ that $h_\alpha^k(\theta) \in I_{n_k(\theta)}$, then the map $\ell_\alpha : C_\alpha \to \Sigma_2$ defined by $\ell_\alpha(\theta) = (n_k(\theta))_{k\in\mathbb{Z}}$ is injective. As the $I_k(\alpha) \cap C_\alpha$ are open (and closed) in C_α, this map is also continuous and then is a homeomorphism onto its image. This provides a homeomorphism from C_α onto $\ell_\alpha(C_\alpha) \subset \Sigma_2$ such that

$$\forall\theta \in C_\alpha, \ell_\alpha \circ h_\alpha(\theta) = \sigma_2 \circ \ell_\alpha(\theta).$$

The WDS with rotation number $\alpha \in \left[0, \frac{1}{2}\right)\backslash\mathbb{Q}$ that we consider is then $(C_\alpha, j_\alpha = k \circ \ell_\alpha, h_\alpha)$.

Observe that $\ell_\alpha(b_\alpha) = (n_k(b_\alpha))_{k\in\mathbb{Z}}$ is the Sturmian sequence that is associated to the rotation R_α. Let us recall that if $u = (u_k)_{k\in\mathbb{Z}}$ is a Sturmian sequence, then for every $n \geq 1$, there are exactly $n + 1$ n-words in u. As $h_{\alpha|C_\alpha}$ is minimal, the orbit of $\ell_\alpha(b_\alpha)$ under σ_2 is dense in $\ell_\alpha(C_\alpha)$. Now let us fix $\alpha_0 \in [0, 1/2)\backslash\mathbb{Q}$ and $n \geq 1$. There exists $N \geq 1$ such that all the m-words in $\ell_{\alpha_0}(b_{\alpha_0})$ with $m \leq 2n + 1$ are contained in $(n_k(b_{\alpha_0}))_{k\in[-N,N]}$. If α is close enough to α_0, $(n_k(b_\alpha))_{k\in[-N,N]}$ is equal to $(n_k(b_{\alpha_0}))_{k\in[-N,N]}$. As $\ell_\alpha(b_\alpha) = (n_k(b_\alpha))_{k\in\mathbb{Z}}$ is Sturmian, this implies that all the m-words in $\ell_\alpha(b_\alpha)$ with $m \leq 2n+1$ are contained in $(n_k(b_\alpha))_{k\in[-N,N]} = (n_k(b_{\alpha_0}))_{k\in[-N,N]}$, which means that the distance between the σ_2 orbits of $\ell_\alpha(b_\alpha)$ and $\ell(b_{\alpha_0})$ is less than $1/(n + 2)$. This implies that $\ell_\alpha(C_\alpha)$ is $\frac{1}{n}$-close to $\ell_{\alpha_0}(C_{\alpha_0})$. Hence $j_\alpha(C_\alpha) = k(\ell_\alpha(C_\alpha))$ is close to $j_{\alpha_0}(C_{\alpha_0}) = k(\ell_{\alpha_0}(C_{\alpha_0}))$.

Now we want to prove that $\mathcal{G}(\prec_{C_\alpha})$ is close to $\mathcal{G}(\prec_{C_{\alpha_0}})$. In a equivalent way, we can work in Σ_2 instead of $\mathcal{H}$ and assume that the graphs of $\mathcal{G}(\prec_{C_\alpha})$ and $\mathcal{G}(\prec_{C_{\alpha_0}})$ are in $(\Sigma_2)^3$. Then the intersection of the n cylinder $C(\delta_{-n}, \ldots, \delta_0, \ldots, \delta_n) = \{(u_k)_{k\in\mathbb{Z}}; \forall k \in [-n, n], u_k = \delta_k\}$ with $\ell_\alpha(C_\alpha)$ is an interval for the order $\prec_{C_\alpha}$, that is before encoding the intersection of intervals $\bigcap_{k=-n}^{k=n} h_\alpha^{-k}(I_{\delta_k})$. This interval is nonempty if and only if $(\delta_i)_{i\in[-n,n]}$ is a $(2n + 1)$-word of the Sturmian sequence $(n_k(b_{\alpha_0}))_{k\in\mathbb{Z}}$ for α_0. Now let us fix $n \geq 1$. There exists $N \geq 1$ such that all the admissible $(2n + 1)$-words of $(n_k(b_{\alpha_0}))_{k\in\mathbb{Z}}$ are contained in the sequence $(n_k(b_{\alpha_0}))_{k\in[-N,N]}$. There exists a neighborhood V of α_0 in $\mathbb{T}$ such that, for every $\alpha \in V$, we have:

- $\forall k \in [-N, N], n_k(b_\alpha) = n_k(b_{\alpha_0})$.
- The intervals

$$C(n_{k-n}(b_\alpha), \dots, n_k(b_\alpha), \dots, n_{k+n}(b_\alpha)) \cap \ell_\alpha(C_\alpha)$$

  and

$$C(n_{k-n}(b_{\alpha_0}), \dots, n_k(b_{\alpha_0}), \dots, n_{k+n}(b_{\alpha_0})) \cap \ell_\alpha(C_{\alpha_0})$$

  for $n - N \le k \le N - n$ (that are $\frac{1}{n}$-close to each other) are in the same order, for $\prec_{K_\alpha}$ for the first ones and for $\prec_{K_{\alpha_0}}$ for the second ones, because it is the order of this intervals for the two rotations.

We deduce that $\mathcal{G}(\prec_{C_\alpha})$ is $\frac{1}{n}$-close to $\mathcal{G}(\prec_{C_{\alpha_0}})$.

5B. ***Proof of Corollary 1.6.*** It is a corollary of Proposition 1.1 and Theorem 1.5.

6. Proof of Theorem 1.7 and Corollary 1.8

Definition. Let $f : \mathbb{A} \to \mathbb{A}$ be a diffeomorphism. Then f is an exact symplectic twist map if:

- The diffeomorphism f is isotopic to identity.
- If $\lambda = \pi_2 d\pi_1$ is the Liouville 1-form on $\mathbb{A}$, then $f^*\lambda - \lambda$ is exact.
- If $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of f, for every $x \in \mathbb{R}$, the map $y \in \mathbb{R} \mapsto \pi_1 \circ f(x, y)$ is a C^1 diffeomorphism onto $\mathbb{R}$.

6A. ***Proof of Theorem 1.7.*** We assume that $f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ is an exact symplectic twist map and that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is one of its lifts. We assume that $\mathcal{A}_r^+$ is uniformly hyperbolic for some rational number $r \in \mathbb{Q}$. We want to prove that there exists a horseshoe $\mathcal{H}_r^+$ for some f^n and $\varepsilon > 0$ such that:

- $\mathcal{H}_r^+$ contains $\mathcal{A}_r^+$.
- Every Aubry–Mather set with rotation number in $(r, r + \varepsilon)$ is contained in $\mathcal{H}_r^+$.
- Every point in $\mathcal{H}_r^+$ has no conjugate points, i.e., has its orbit that is locally minimizing.

We write $r = \frac{p}{q}$ as an irreducible fraction. As $\mathcal{A}_r^+$ is a compact uniformly hyperbolic set, it has a finite number of q-periodic points. We denote them by $x_1, \dots, x_n$ in the usual cyclic order along $\mathbb{T}$ (for the first projection). Then $\mathcal{A}_r^+$ is the union of these periodic points and some heteroclinic orbits between these heteroclinic points; see, e.g., [4]. Moreover, such heteroclinic orbit for f^q that is contained in $\mathcal{A}_r^+$ can only connect an x_k to x_{k+1} (with $x_{n+1} = x_1$). If two heteroclinic orbits in $\mathcal{A}_r^+$ connect the same x_k to x_{k+1}, we can choose an order on the union of these two orbits $(y_j)_{j\in\mathbb{Z}}$ and $(z_j)_{j\in\mathbb{Z}}$ for f^q such that

$$x_k < \cdots < y_{j-1} < z_{j-1} < y_j < z_j < y_{j+1} < z_{j+1} < \cdots < x_{k+1}.$$

Let $\varepsilon > 0$ be an expansivity constant for $f^q_{|\mathcal{A}_r^+}$ and let K be a Lipschitz constant of the Aubry–Mather set $\mathcal{A}_r^+$ as a graph. Then for some j we have $d(y_j, z_j) \geq \varepsilon$. Hence the distance between the projections of y_j, z_j on the first factor is more than $\varepsilon/(1+K)$ for some j. Of course we can use the same argument for any finite set of heteroclinic orbits $(y^1_j)_{j\in\mathbb{Z}}, \dots, (y^N_j)_{j\in\mathbb{Z}}$ connecting x_k to x_{k+1} in $\mathcal{A}_r^+$. We have

$$x_k \cdots < y^1_{j-1} < \cdots < y^N_{j-1} < y^1_j < \cdots < y^N_j < \cdots < x_{k+1},$$

and we find N integers $j_1, \dots, j_N$ such that (with the convention $y_j^{N+1} = y^1_{j+1}$)

$$\forall i \in \{0, N\}, d(y^i_{j_i}, y^{i+1}_{j_i}) \geq \varepsilon.$$

Then the intervals $(\pi_1(y^1_{j_1}), \pi_1(y^2_{j_1})), \dots, (\pi_1(y^N_{j_N}), \pi_1(y^{N+1}_{j_N}))$ are disjoint intervals in $\mathbb{T}$ with length larger or equal to $\varepsilon/(K+1)$. This implies that $N \leq (K+1)/\varepsilon$. Hence $\mathcal{A}_r^+$ is a hyperbolic set that is the union of periodic orbits and of a *finite* number of heteroclinic orbits. Moreover, there always exists at least a heteroclinic connection in $\mathcal{A}_r^+$ between two adjacent periodic points in $\mathcal{A}_r^+$ (see [4]). Hence $\mathcal{A}_r^+$ is a cycle of transverse heteroclinic intersections with period q (see definition in the Appendix).

We introduce the notation $p : \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}$ for the usual projection. When $E \subset \mathbb{T} \times \mathbb{R}$, we denote by $\tilde{E} = p^{-1}(E)$ its lift.

Let us fix a neighborhood $\mathcal{N}$ of $\mathcal{A}_r^+$. Then $\mathcal{A}_r^- \backslash \mathcal{N}$ is finite because $\mathcal{A}_r^-$ is the union of $\mathcal{A}_r^- \cap \mathcal{A}_r^+$ and the union of a finite number of orbits that are homoclinic to $\mathcal{A}_r^- \cap \mathcal{A}_r^+$. For every $x \in \tilde{\mathcal{A}}_r^- \backslash \widetilde{\mathcal{N}}$, we have $\pi_1 \circ F^q(x) < \pi_1(x) + p$. Then $\varepsilon = \min\{\pi_1(x) + p - \pi_1 \circ F^q(x); x \in \tilde{\mathcal{A}}_r^- \backslash \widetilde{\mathcal{N}}\}$ is a positive number. We introduce the open set

$$\mathcal{U} = p\left(\left\{x \in \mathbb{R}^2; \pi_1(x) + p - \pi_1 \circ F^q(x) > \tfrac{\varepsilon}{2}\right\}\right)$$

that contains $\mathcal{A}_r^- \backslash \mathcal{N}$. Then $\mathcal{N} \cup \mathcal{U}$ is a neighborhood of $\mathcal{A}_r^+ \cup \mathcal{A}_r^-$. As the rotation number map is continuous and as the union of minimizing orbits is closed, there exists $\eta > 0$ such that every Aubry–Mather set with rotation number in $(r-\eta, r+\eta)$ is in $\mathcal{N} \cup \mathcal{U}$. If moreover $\mathcal{A}$ is an Aubry–Mather set with rotation number in $(r, r+\eta)$, then we have

$$\forall x \in \tilde{\mathcal{A}}, \quad \pi_1 \circ F^q(x) > \pi_1(x) + p.$$

Hence $\mathcal{A} \cap \mathcal{U} = \varnothing$ and thus $\mathcal{A} \subset \mathcal{N}$. We have the proved that there exists $\eta > 0$ such that every Aubry–Mather set with rotation number in $(r, r+\eta)$ is contained in $\mathcal{N}$.

We then use Section A3 of the Appendix. There exists $N \geq 1$ and a neighborhood $\mathcal{N}$ of the cycle of transverse heteroclinic intersections with period q $\mathcal{A}_r^+$, such that the maximal f^{qN} invariant set contained in $\mathcal{N}$ is a horseshoe $\mathcal{H}_r^+$ for f^{qN} (see Definition). This horseshoe then satisfies the two first points of Theorem 1.7.

Moreover, observe that along $\mathcal{A}_r^+$, there exists a Df invariant field of half-lines (the half Green bundles g_+ of G_+, see [1]) transverse to the vertical fiber, that is a subset of the unstable bundle along $\mathcal{A}_r^+$. By continuity of the unstable bundle along any hyperbolic set, we can extend g_+ to the whole $\mathcal{H}_r^+$ into a field of half-line that are contained in the unstable bundle. If $\mathcal{N}$ is small enough, this field as well as its first qN images by Df is also transverse to the vertical. This implies the last point of Theorem 1.7.

6B. ***Proof of Corollary 1.8.*** We use the results of P. Le Calvez that are in [19]. We consider the G_δ subset $\mathcal{G}$ of the set of C^k symplectic twist diffeomorphisms f whose elements satisfy the following conditions:

- If x is a periodic point for f with smallest period q, none of the eigenvalues of $Df^q(x)$ is a root of unity.
- All the heteroclinic intersections between invariant manifolds of hyperbolic periodic points are transverse.

It is proved in [19] that all the Aubry–Mather sets that have a rational rotation number are hyperbolic. By Theorem 1.7, for every $r \in \mathbb{Q}$, there exists an open interval $(r-\varepsilon_r, r+\varepsilon_r)$ such that every Aubry–Mather set with rotation number in this interval is contained in the horseshoe $\mathcal{H}_r^+$ or the horseshoe $\mathcal{H}_r^-$. This gives the conclusion of the corollary for

$$U(f) = \bigcup_{r\in\mathbb{Q}} (r-\varepsilon_r, r+\varepsilon_r).$$

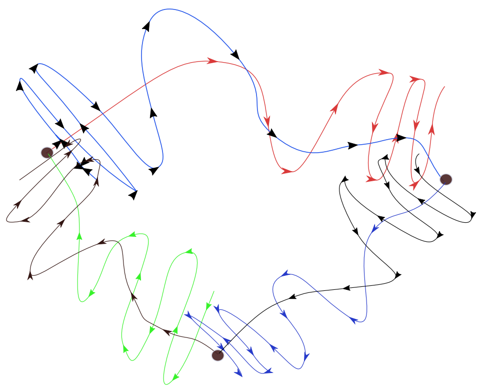
Appendix: On horseshoes

In this section, we will be interested in some horseshoes that are related to the heteroclinic intersections. Generally, authors look at what happens close to one homoclinic point associated to a periodic point (in [7], the authors also consider heteroclinic connections for two fixed points). But to apply our results to Aubry–Mather sets, we will need to study the horseshoes that can be built by using a (circular) family of periodic points and heteroclinic intersections. Let us explain this now.

A1. ***Introduction to heteroclinic horseshoes.*** We will consider heteroclinic cycles. For a diffeomorphism $f : M \to M$ of a surface, we will call a q-periodic point x a *saddle* if the two eigenvalues λ, μ of $Df^q(x)$ are positive and such that $\mu < 1 < \lambda$.

Definition. Let $f : M \to M$ be a surface diffeomorphism. A *cycle of transverse heteroclinic intersections with period* 1 is determined by:

- A finite cyclically ordered set of saddle hyperbolic fixed points $x_{n+1} = x_1, \dots, x_n$ with an orientation on each submanifold $W^s(x_i)$ and $W^u(x_i)$.
- For every $k \in [1, n]$ a nonzero finite number n_k of transverse heteroclinic points $y_1^k, \dots, y_{n_k}^k$ in $W^u(x_k, f) \cap W^s(x_{k+1}, f)$ such that $x_k, y_1^k, \dots, y_{n_k}^k$ are in this order along $W^u(x_k, f)$ and $y_1^k, \dots, y_{n_k}^k, x_{k+1}$ also along $W^s(x_{k+1}, f)$.[7] Moreover, they define different orbits:



Definition. Let $f : M \to M$ be a surface diffeomorphism and let $q \geq 1$ be an integer. A *cycle of transverse heteroclinic intersections with period q* is determined by:

- A finite cyclically ordered set of saddle hyperbolic q-periodic points $x_{nq+1} = x_1, \dots, x_{nq}$ such that this order is preserved by f with an orientation on each submanifold $W^s(x_i)$ and $W^u(x_i)$; we assume that every set $\{x_i, x_{i+n}, \dots x_{i+(q-1)n}\}$ is an orbit.
- For every $k \in [1, qn]$ a nonzero finite number n_k of transverse heteroclinic points $y_1^k, \dots, y_{n_k}^k$ in $W^u(x_k, f) \cap W^s(x_{k+1}, f)$ such that $x_k, y_1^k, \dots, y_{n_k}^k$ are in this order along $W^u(x_k, f)$ and $y_1^k, \dots, y_{n_k}^k, x_{k+1}$ also along $W^s(x_{k+1}, f)$.[8] Moreover, they define different orbits.

[7] This implies that the y_i^k are all on a same branch of $W^u(x_k, f)$ and $W^s(x_{k+1}, f)$.

[8] This implies that the y_i^k are all on a same branch of $W^u(x_k, f)$ and $W^s(x_{k+1}, f)$.

- We also assume $n_{k+n} = n_k$, that x_k and x_{n+k} are on a same orbit and that y_j^k and y_j^{n+k} are on the same orbit.

Notation. Now we consider a cycle of transverse heteroclinic intersections $\mathcal{H}$ with period q for f that is given by the x_k and the y_j^k as before. We denote by $K(\mathcal{H})$ the union of the orbits of the x_k and the y_j^k.

Remark. Observe that $K(\mathcal{H})$ is a f-invariant compact set that is uniformly hyperbolic. We denote by E the tangent bundle TM. By [28], we can translate the hyperbolicity condition by using some cones. This is an open condition and we can extend these cones to a compact neighborhood $\mathcal{V}$ of $K(\mathcal{H})$ such that:

- There exists a continuous splitting $E = E^1 \oplus E^2$ on $\mathcal{V}$ that coincides with $E = E^s \oplus E^u$ on $K(\mathcal{H})$ and two norms $|\cdot|_i$ on E^i such that
$$C_x = \{v = v_1 + v_2, v_1 \in E_x^1, v_2 \in E_x^2, |v_1|_{1,x} \leq |v_2|_{2,x}\};$$
the family $(C_x)_{x\in\mathcal{V}}$ is the associated cone field; the dual cone field is the family $(C_x^*)_{x\in\mathcal{V}}$ defined by $C_x^* = E_x \backslash \operatorname{int} C_x$.
- For some constant $c > 1$, we have for every $x \in \mathcal{V}$, $v_1 \in E_x^1$ and $v_2 \in E_x^2$
$$c^{-1}\|v_1 + v_2\| \leq \max\{|v_1|_{1,x}, |v_2|_{2,x}\} \leq c\|v_1 + v_2\|_x.$$
- There exists an integer $m \geq 1$ and a constant $\mu > 1$ so that:
  (1) For $x \in \mathcal{V}$, $Df(C_x) \subset \widetilde{C}_{\mu,f(x)}$ where
  $$\widetilde{C}_{\lambda,x} = \{v = v_1 + v_2 \in E_x;\ \mu|v_1|_{1,x} \leq |v_2|_{2,x}\}.$$
  (2) For $x \in \mathcal{V}$, for $v \in C_x$, $\|Df^m(v)\|_{f^m(x)} \geq \mu.\|v\|_x$.
  (3) For $x \in \mathcal{V}$, for $v \in C_x^*$, $\|Df^{-m}(v)\|_{f^{-m}(x)} \geq \mu.\|v\|_x$.

We define
$$\mathcal{K}(\mathcal{V}) = \bigcap_{k\in\mathbb{Z}} f^k(\mathcal{V}).$$

Then $\mathcal{K}(\mathcal{V})$ is compact and hyperbolic. Let $\varepsilon > 0$ be a constant of expansivity, i.e., such that
$$\forall x, y \in \mathcal{K}(\mathcal{V}), (\forall k \in \mathbb{Z}, d(f^k x, f^k y) < \varepsilon) \Rightarrow x = y.$$

Choosing possibly a smaller neighborhood, we can assume that the diameter of every connected component of $\mathcal{V}$ is smaller than ε, and also that $\mathcal{V}$ has a finite number N of connected components that all meet $K(\mathcal{V})$.

We denote by $\mathcal{C}_1, \dots, \mathcal{C}_N$ the connected components of $\mathcal{V}$ and define the itinerary function $H : \mathcal{K}(\mathcal{V}) \to \Sigma_N$ by $f^k(x) \in \mathcal{C}_{H(x)_k}$. Hence the k-th component of $H(x)$ corresponds to the connected component of $\mathcal{V}$ that contains $f^k x$. Then

H is continuous. Because of the expansiveness property, H is injective, so H is a homeomorphism from $\mathcal{K}(\mathcal{V})$ onto $H(\mathcal{K}(\mathcal{V})) \subset \Sigma_N$ such that

$$\forall x \in \mathcal{K}(\mathcal{V}), \quad \sigma \circ H(x) = H \circ f(x).$$

But in fact, we are looking for dynamics that are actually conjugate to a transitive subshift of finite type. In order to build such dynamics, we will be more precise for the choice of $\mathcal{V}$ in Section A3.

A2. *Rectangles partition.* Here we explain how a good family of rectangles, called a rectangles partition, is useful to build a locally maximal invariant hyperbolic sets. We introduce geometric Markov partition, that are reminiscent from the Markov partition and that are studied in [26], but as we didn't find the exact setting that we use elsewhere, we give some details.

We assume that $f : M \to M$ is a C^1 diffeomorphism and that $\mathcal{V} \subset M$ is an open set endowed with two continuous families of open symmetric cones, the unstable one $x \in \mathcal{V} \mapsto C^u(x) \subset T_xM$ and the stable one $x \in \mathcal{V} \mapsto C^s(x) \subset T_xM$ such that, if we denote the closure of a set A by $\bar{A}$, we have for a constant $\lambda \in (0, 1)$:

- $\forall x \in \mathcal{V} \cap f^{-1}(\mathcal{V})$, $Df(\overline{C^u}(x)) \subset C^u(f(x))$ and $Df(C^s(x)) \supset \overline{C^s}(f(x))$.
- $\forall x \in \mathcal{V}, \forall v \in C^u(x)$, $\|Df(x)v\| \geq \frac{1}{\lambda}\|v\|$ and $\forall x \in \mathcal{V}, \forall v \in C^s(x)$,
$$\|Df(x)v\| \leq \lambda\|v\|.$$
- $\forall x \in \mathcal{V}$, $C^u(x) \cap C^s(x) = \{\vec{0}\}$.

Definition.
- A C^1-embedding $\gamma : [a, b] \to \mathcal{V}$ define a *unstable* (*resp. stable*) *curve* if $\forall t \in [a, b]$, $\gamma'(t) \in C^u(\gamma(t))$ (resp. $\forall t \in [a, b]$, $\gamma'(t) \in C^s(\gamma(t))$).
- A *rectangle* R is given by an embedding $\Phi_R : [0, 1]^2 \to R \subset \mathcal{V}$ such that for every $t \in [0, 1]$, $\Phi_R(\{t\} \times [0, 1])$ (resp. $\Phi_R([0, 1] \times \{t\})$) defines a stable (resp. unstable) curve.
- Then the *stable* (*resp. unstable*) boundary of R is $\partial^s R = \Phi_R(\{0, 1\} \times [0, 1])$ (resp. $\partial^u R = \Phi_R([0, 1] \times \{0, 1\})$).
- A rectangle R' is a *stable* (*resp. unstable*) *subrectangle* of a rectangle R if $R' \subset R$ and $\partial^u R' \subset \partial^u R$ (resp. $\partial^s R' \subset \partial^s R$).

Remarks.
(1) Observe that a stable curve is always transversal to an unstable curve, and that when their mutual intersection with some rectangle is nonempty, then it is a point.
(2) To a given rectangle R, we can associate different embeddings Φ_R and then different stable and unstable foliations $\mathcal{F}^s(R)$ and $\mathcal{F}^s(R)$.
(3) The stable and unstable boundaries are independent from the embedding.

(4) When $\gamma \subset \mathcal{V}$ is a unstable (stable) curve, every connected component of $f(\gamma) \cap \mathcal{V}$ (resp. $f^{-1}(\gamma) \cap \mathcal{V}$) is also an unstable (resp. stable) curve.

Let us now introduce the notion of rectangles partition that we will use.

Definition. A *rectangles partition* is a finite set $\{\mathcal{R}_1, \ldots, \mathcal{R}_m\}$ of disjoint rectangles of $\mathcal{V}$ such that, if we use the notation $\mathcal{R}_{jk} = f(\mathcal{R}_j) \cap \mathcal{R}_k$, we have:

- For every $j, k \in \{1, \ldots, m\}$, either $\mathcal{R}_{jk} = \varnothing$ or $\mathcal{R}_{jk}$ is an unstable subrectangle of $\mathcal{R}_k$. When $\mathcal{R}_{jk} \neq \varnothing$, we use the notation
$$\mathcal{R}_j \xrightarrow{f} \mathcal{R}_k,$$
and we say that we have a *transition* from $\mathcal{R}_j$ to $\mathcal{R}_k$.
- When $\mathcal{R}_{jk} \neq \varnothing$, then $f(\partial^u \mathcal{R}_j) \cap \partial^u \mathcal{R}_k = \varnothing$ and $f(\partial^s \mathcal{R}_j) \cap \partial^s \mathcal{R}_k = \varnothing$.

An *admissible sequence* is then $(i_k)_{k \in \mathbb{Z}} \in \{1, \ldots, m\}^{\mathbb{Z}} = \Sigma_m$ such that
$$\forall k \in \mathbb{Z}, \mathcal{R}_{i_k} \xrightarrow{f} \mathcal{R}_{i_{k+1}}.$$

Remark. Observe that $\mathcal{R}_j \xrightarrow{f} \mathcal{R}_k$ if and only if $\mathcal{R}_k \xrightarrow{f^{-1}} \mathcal{R}_j$ (the stable boundary for f^{-1} is then the unstable one for f).

Notation. We denote by $\Lambda(\mathcal{R}_1, \ldots, \mathcal{R}_m)$ the maximal invariant set that is contained in $\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_m$, i.e.,
$$\Lambda(\mathcal{R}_1, \ldots, \mathcal{R}_m) = \bigcap_{k \in \mathbb{Z}} f^k(\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_m).$$

Observe that this set is hyperbolic. Hence there exist a stable and an unstable submanifold at every of its points. We even have the following result.

Proposition A.1. *If $x \in \Lambda(\mathcal{R}_1, \ldots, \mathcal{R}_m) \cap \mathcal{R}_{i_0}$, then the connected component of $W^s(x) \cap \mathcal{R}_{i_0}$ (resp. $W^u(x) \cap \mathcal{R}_{i_0}$) that contains x is a stable (resp. unstable) curve that joins the two connected components of $\partial^u \mathcal{R}_{i_0}$ (resp. $\partial^s \mathcal{R}_{i_0}$).*

Proof. As $\Lambda(\mathcal{R}_1, \ldots, \mathcal{R}_m)$ is hyperbolic, there exists $\varepsilon > 0$ such that for every $x \in \Lambda(\mathcal{R}_1, \ldots, \mathcal{R}_m)$, the length of every branch of $W^s(x)$ is greater than ε. We denote by $\mathcal{M} > 0$ a lower bound of the length of the stable curves contained in one $\mathcal{R}_{j_0}$ that join the two components of $\partial^u \mathcal{R}_{j_0}$. Then we choose $N \geq 1$ such that $\frac{\varepsilon}{\lambda^N} > \mathcal{M}$. Then if j_0 is such that $f^N(x) \in \mathcal{R}_{j_0}$, the curve $f^{-N}(W^s(f^N(x)) \cap \mathcal{R}_{j_0})$ is contained in $W^s(x)$ and crosses the two connected components of $\partial^u \mathcal{R}_{i_0}$. This gives the wanted result. $\square$

Different versions of the following proposition exist in different settings. We will provide a proof for the convenience of the reader.

Proposition A.2. *Let $\{\mathcal{R}_1, \dots, \mathcal{R}_m\}$ be a rectangle partition for f in $\mathcal{V}$. Let $(i_k)_{k\in\mathbb{Z}} \in \Sigma_m$ be a sequence. The two following assertions are equivalent*:

- $(i_k)_{k\in\mathbb{Z}}$ *is an admissible sequence.*
- *There exists a unique point $x \in \mathcal{R}_{i_0}$ such that*

$$\forall k \in \mathbb{Z}, \quad f(x) \in \mathcal{R}_{i_k}.$$

Proof. We just prove the direct implication, the only one that is nontrivial. Hence we assume that $(i_k)_{k\in\mathbb{Z}}$ is an admissible sequence.

We begin by proving the existence of x. For every $n \in \mathbb{N}$, we introduce the notation

$$D_n^s = \bigcap_{k=0}^{n} f^{-k}(\mathcal{R}_{i_k}) \quad \text{and} \quad D_n^u = \bigcap_{k=0}^{n} f^{k}(\mathcal{R}_{i_{-k}}).$$

Then $(D_n^u)_{n\in\mathbb{N}}$ (resp. $(D_n^s)_{n\in\mathbb{N}}$) is a decreasing sequence of unstable (resp. stable) rectangles of $\mathcal{R}_{i_0}$. Hence $(K_n)_{n\in\mathbb{N}} = (D_n^u \cap D_n^s)_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty compact subsets of $\mathcal{R}_{i_0}$. Their intersection contains at least one point x, and this point satisfies

$$\forall k \in \mathbb{Z}, \quad f^k(x) \in \mathcal{R}_{i_k}.$$

We now want to prove the unicity of x. We introduce the notation

$$D_\infty^u = \bigcap_{n\in\mathbb{N}} D_n^u \quad \text{and} \quad D_\infty^s = \bigcap_{n\in\mathbb{N}} D_n^s.$$

Lemma A.3. *The set D_∞^u (resp. D_∞^s) is an unstable curve that joins the two connected components of $\partial^s R_{i_0}$ (resp. $\partial^u R_{i_0}$). More precisely, if $\{x\} = D_\infty^u \cap D_\infty^s$, then $D_\infty^u \subset W^u(x)$ and $D_\infty^s \subset W^s(x)$.*

Let us prove Lemma A.3. We just prove the result for D_∞^s. As every D_n^s is a stable rectangle, D_∞^s is a connected compact set that joins the two connected components of $\partial^u R_{i_0}$. To prove that it is a (at least continuous) curve, we just need to prove that it intersects every leaf of the unstable foliation $\mathcal{F}^u(\mathcal{R}_{i_0})$ of $\mathcal{R}_{i_0}$ at most once. So let $\mathcal{L}^u$ be an unstable leaf of $\mathcal{R}_{i_0}$ and let x, y be two points of $D_\infty^s \cap \mathcal{L}^u$. We denote by $\mathcal{L}^u[x, y]$ the arc of $\mathcal{L}^u$ that has for endpoints x and y. Observe that $\mathcal{L}^u[x, y] \subset \mathcal{R}_{i_0}$ Then for every $n \in \mathbb{N}$, the connected component $\mathcal{L}_n$ of $f^n(\mathcal{L}^u) \cap \mathcal{R}_{i_n}$ that contains $f^n(\mathcal{L}^u[x, y])$ is an unstable curve of $\mathcal{R}_{i_n}$.[9] Let $\mathcal{B}$ a common upper bound of the lengths of the unstable leaves that are contained in some rectangle of the Markov partition (observe that these curves are uniformly Lipschitz graphs in the charts $\Phi_{\mathcal{R}_i}$). Then we have length$(\mathcal{L}_n) \le \mathcal{B}$ and we

[9] Observe that the endpoints of this curve are indeed in $\mathcal{R}_{i_n}$ and hence by the point (4) of the remark, $f^n(\mathcal{L}^u[x, y]) \subset \mathcal{R}_{i_n}$.

deduce $\forall n \in \mathbb{N}$, $\text{length}(\mathcal{L}^u[x, y]) = \text{length}(f^{-n}(\mathcal{L}_n)) \leq \lambda^n \mathcal{B}$. So $x = y$ and D^s_∞ intersects every unstable leaf at most once, and so exactly once because D^s_∞ is a connected set that joins the two connected components of $\partial^u R_{i_0}$.

Moreover, observe that D^s_∞ contains the connected component C^s of $W^s(x) \cap \mathcal{R}_{i_0}$ that contains x. This implies that $D^s_\infty = C^s$ is a smooth stable curve (see Proposition A.1). □

A3. ***Precise construction of heteroclinic horseshoes.*** We use the same notations as in Section A1.

Remark. As explained before, we want to build an invariant set that is close (for the Hausdorff distance) to $K(\mathcal{H})$. That is why we need to use all the heteroclinic intersections that are in $K(\mathcal{H})$ in our construction. Another approach could be to use the transitivity of the relation $\mathcal{R}$ defined on q-periodic points by: $x\mathcal{R}y$ if $W^s(x, f)$ and $W^u(y, f)$ have a transverse heteroclinic intersection. This implies that every periodic point in $K(\mathcal{H})$ has a homoclinic intersection and thus we could use directly Smale's method (see [27]) to build a homoclinic horseshoe. Unfortunately, a neighborhood of this homoclinic orbit is not necessarily a neighborhood of the whole $K(\mathcal{H})$ and so this horseshoe is in general not close to $K(\mathcal{H})$ for the Hausdorff distance, so doesn't give us what we want.

Theorem A.4. *There exists $N \geq 1$ and a neighborhood $\mathcal{N}$ of the cycle $K(\mathcal{H})$ of transverse heteroclinic intersections with period q, such that the maximal f^{qN} invariant set contained in $\mathcal{N}$ is a horseshoe Λ for f^{qN} (see Definition).*

As $K(\mathcal{H})$ is (uniformly) hyperbolic, we can chose a neighborhood $\mathcal{V}$ of $K(\mathcal{H})$, a constant $\lambda \in (0, 1)$ and two continuous families of open symmetric cones (see Section A1) the unstable one $x \in \mathcal{V} \mapsto C^u(x) \subset T_xM$ and the stable one $x \in \mathcal{V} \mapsto C^s(x) \subset T_xM$ such that, if we denote the closure of a set A by $\bar{A}$, we have:

- $\forall x \in \mathcal{V} \cap f^{-1}(\mathcal{V})$, $Df(\overline{C^u}(x)) \subset C^u(f(x))$ and $Df(C^s(x)) \supset \overline{C^s}(f(x))$.
- $\forall x \in \mathcal{V}$, $\forall v \in C^u(x)$, $\|Df(x)v\| \geq \frac{1}{\lambda}\|v\|$ and $\forall x \in \mathcal{V}$, $\forall v \in C^s(x)$,
$$\|Df(x)v\| \leq \lambda\|v\|.$$
- $\forall x \in \mathcal{V}$, $C^u(x) \cap C^s(x) = \{\vec{0}\}$.

Notation. For every x_k, we denote by $B^s(x_k)$ the branch of $W^s(x_k)$ that contains the y_i^{k-1}s and by $B^u(x_k)$ the branch of $W^u(x_k)$ that contains the y_i^ks. Then we choose a small (curved) rectangle R_k with two sides on $B^s(x_k)$ and $B^u(x_k)$; see Figure 2.

We denote by δ_k^u and δ_k^s the size of R_k along $B^u(x_k)$ and $B^s(x_k)$.

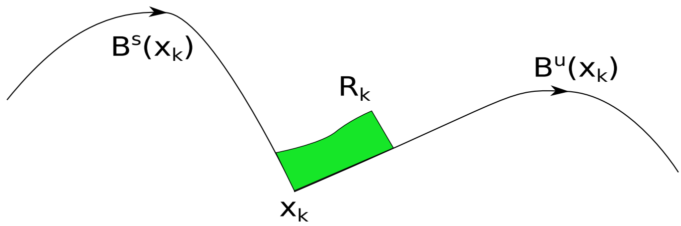


Figure 2. A small (curved) rectangle R_k with two sides on $B^s(x_k)$ and $B^u(x_k)$.tw.

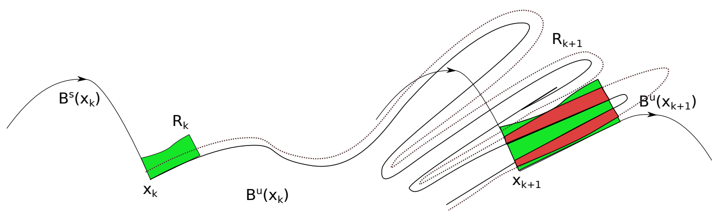


Figure 3. The subrectangles of R_{k+1} which are connected components of $f^{qN_k}(R_k) \cap R_{k+1}$ that meets $W^s_{\mathrm{loc}}(x_{k+1})$ at some point of the orbit of y_i^k.

Then we look at the Poincaré map for f^q from R_k onto R_{k+1}. Adjusting the quantities δ^u and δ^s, we can find some N_k such that $f^{qN_k}(R_k) \cap R_{k+1}$ contains the union of a finite numbers of unstable rectangles. There are two cases:

- When $n = q = 1$, there are $n_0 + 1$ rectangles: R_0^0 that contains x_0 and R_0^1, $R_0^2, \ldots, R_0^{n_0}$ such that R_0^i is a connected component of $f^{qN_0}(R_0) \cap R_0$ that meets $W^s_{\mathrm{loc}}(x_0)$ at some point of the orbit of y_i^0.
- When $nq > 1$, there are n_k unstable subrectangles of R_{k+1} that we denote by $R^1_{k+1}, R^2_{k+1}, \ldots, R^{n_k}_{k+1}$ such that R^i_{k+1} is a connected component of $f^{qN_k}(R_k) \cap R_{k+1}$ that meets $W^s_{\mathrm{loc}}(x_{k+1})$ at some point of the orbit of y_i^k;[10] see Figure 3.

[10]Observe that $f^{qN_k}(R_k) \cap R_{k+1}$ can have other connected components, for example connected components that correspond to other heteroclinic intersections. We just work with some chosen heteroclinic points.

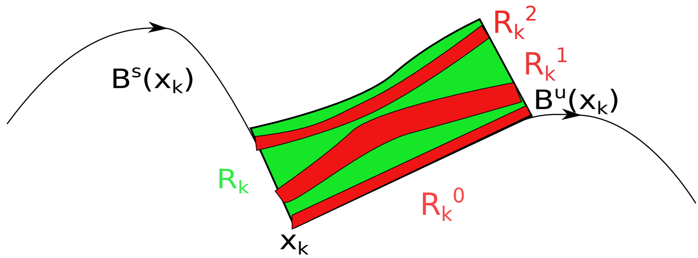


Figure 4. The connected component of $R_k \cap f^{qN}(R_k)$ that contains x_k.

When we decrease δ_k^u or δ_{k+1}^s, then N_k increases and when we decrease δ_k^s or δ_{k+1}^u, then N_k doesn't change. Hence, if we possibly decrease the δ_k^u's, we can assume that all the N_k are equal to some constant integer that we denote by N. Let us denote by R_k^0 the connected component of $R_k \cap f^{qN}(R_k)$ that contains x_k and let us prove that it is disjoint from the R_k^i for $1 \le i \le n_k$. There are two cases:

- There is only one fixed point in the heteroclinic cycle, i.e., $q = n = 1$; in this case the rectangles R_1^i are different connected components of $R_1 \cap f^N(R_1)$ and so they are disjoint.
- If not, as the different R_k are disjoint, in particular $f^{qN}(R_k)$ and $f^{qN}(R_{k-1})$ are disjoint and every unstable rectangle that is contained in $R_k \cap f^{qN}(R_k)$ is disjoint from $\bigcup_{i=1}^{n_{k-1}} R_k^i$; see Figure 4.

We introduce the notation $\mathcal{T}_k = \bigcup_{i=0}^{n_k} R_k^i$ and consider now the f^{qN}-invariant set

$$\Lambda = \bigcap_{j\in\mathbb{Z}} f^{jqN}\left(\bigcup_{k=1}^{qn} \mathcal{T}_k\right).$$

Then the R_k^js with $1 \le k \le nq$ and $0 \le j \le n_k$ define a rectangle partition for $f_{|\mathcal{V}}^{qN}$ and the following transitions occur:[11]

- $\forall i \in [0, n_k]$, $R_k^i \xrightarrow{f^{qN}} R_k^0$.
- $\forall i \in [0, n_k], \forall j \in [1, n_{k+1}]$, $R_k^i \xrightarrow{f^{qN}} R_{k+1}^j$.

We denote by A the associated matrix. Observe that for every R_k^i, R_h^j, then R_k^i can be connected to R_h^j by a succession of such transitions. We deduce from Proposition A.2 that $f_{|\Lambda}^{Nq}$ is conjugate to the subshift associated to A, that is

[11] We do not know if other transitions occur.

transitive. In particular, $f^{Nq}_{|\Lambda}$ is mixing, has an infinity of periodic points and has positive topological entropy.

Remarks.

- If we decrease the constants δ^u_k and δ^s_k, then we increase N but this is not a problem because we just add some iterations of f^q that are close to the periodic orbits where we know exactly how the dynamics looks like. An advantage is that decreasing sufficiently these constants, we can be sure that $\bigcup_{j=0}^{qN} f^j\big(\bigcup_{k=1}^{qn} \mathcal{T}_k\big)$ is contained in a small neighborhood of the heteroclinic cycle $K(\mathcal{H})$. So in this case, the Hausdorff distance between $K(\mathcal{H})$ and the invariant set $\bigcup_{j=1}^{qN} f^j(\Lambda)$ is also as small as we want.
- Being defined by a rectangle partition, the set Λ is a locally maximal invariant set by f^{qN}.

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