

Corner the empress

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Dedicated to the memory of Elizabeth Alexandra Mary Windsor, Elizabeth II, by the Grace of God, of the United Kingdom of Great Britain and Northern Ireland and other realms and territories. Queen, Head of the Commonwealth, Defender of the Faith, etc, etc, etc.

Wythoff Nim, a.k.a. Corner the Lady, is a classic combinatorial game. A Queen is placed on an infinite chess board and two players take alternate turns, moving the Queen closer to the corner. The first player that corners the Queen wins. What happens if the Queen gets superior powers and is able to step off the diagonal or bounce against a side? We study the intriguing patterns that emerge from such games. In particular, we are interested in games in which the P -positions can be described by morphic sequences. We use the theorem-prover Walnut to prove some of our results.

1. All rise

The Queen is the most powerful piece on the chess board. She can move diagonally, horizontally, and vertically for as far as it may please her, in one go. It was not always so. Originally, she was a weak piece that could only take one diagonal step at a time. In the original Indian game Chaturanga, from which all modern versions of chess have been derived, the Queen was known as the Mantri, or Counselor. She still goes by that name in Xiangqi, the Chinese version of chess, in which she is not allowed to leave the palace. During the late Middle Ages, the Counselor changed gender and rose to prominence in European chess. The modern movement of the Queen started in Spain under the reign of Isabella I of Castile, perhaps inspired by her great political power [23]. In this paper, we will consider Queens with even greater mobility.

An ordinary chess board has only 64 squares, which are described by letters for columns and numbers for rows in the European notation (of course there also exists an English notation, but no one really understands why). Corner the Lady, or Wythoff Nim, is played on an infinite chess board on which the sun never sets (at least in the positive quadrant). Its rows and columns are

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both numbered $0, 1, 2, \dots$ for lack of letters. Anne places the Queen on any square (m, n) and Beatrix moves first. The players make alternate moves but only so that the Manhattan distance $m + n$ to the corner $(0, 0)$ decreases. The player that gets the Queen into the corner wins. The reach of the Queen is most beneficial for this game, which would take much longer with a Counselor.

To win the game, Anne could certainly place the Queen on $(0, 0)$, but that is not very entertaining. What if she moves further out in her realms? The set of all winning squares, formally known as P -positions (the complement of which are the N -positions), form a fascinating pattern that continues to be a source of inspiration for the study of combinatorial games [6]. Throughout the paper we will reserve (a_n, b_n) for the P -positions and by symmetry we may restrict ourselves to $a_n \leq b_n$. The positions are lined up by their first coordinate so that a_n is increasing. It turns out that b_n is increasing as well for all games in this paper (this does not apply to all versions of Wythoff's game; see for instance [13]) and that the sets $\{a_n : n \in \mathbb{N}\}$ and $\{b_n : n \in \mathbb{N}\}$ are complementary if we ignore the first P -position $(0, 0)$. Wythoff [22] showed for his original game that $a_n = \lfloor n\phi \rfloor$ and $b_n = a_n + n$ for the golden mean ϕ , which satisfies $\phi^2 = \phi + 1$. Holladay [11] considered what happens if the Queen can stroll slightly off-diagonal. More specifically, Holladay's Queen can move from $(x, x + c)$ to $(x', x' + d)$ if $x' < x$ and $|c - d| \leq k - 1$ for $k \in \mathbb{N}$. Let's call such a Queen a k -Queen.¹ For $k = 1$ she is the original Queen. Holladay proved that the P -positions for Corner the k -Queen satisfy $a_n = \lfloor n\psi_k \rfloor$ and $b_n = a_n + kn$ for quadratic numbers ψ_k that have continued fraction expansion $[1; k, k, k, \dots]$.

In this paper, we consider Queens that have a greater mobility than the standard Queen. A good name for games in which they occur seems to be *Corner the Empress*, since Empresses rank above Queens. Eric Duchêne and Michel Rigo introduced the idea of a morphic game [5] (see Section 4). Together with Julien Cassaigne they showed that almost all Beatty sequences (homogeneous or not) arise from the set of P -positions in certain take-away games [3]. One motivation for our present paper is a quest for more morphic games. All Empresses that we consider in this paper lead to morphic games. Some are old and some are new. All are getting cornered, which seems to be the way to go for royals in our modern age.

Our paper is a sequel to earlier work [7] in which we introduced the game of Splithoff. This game is closely related to the Queens on a Spiral problem that was solved in [4]. We left two conjectures in our earlier paper, one of which we solve in Section 3. The other conjecture, which we consider in Section 5, was recently solved by Jeffrey Shallit [20] using the automatic theorem-prover Walnut. He also settled another conjecture of ours, which arose from morphic Queens [19],

¹In fact, Holladay also considered other kinds of Queen that lead to the same P -positions.

again using Walnut. This is an elegant and powerful tool that can be used with great effect in the analysis of morphic sequences. We encourage the reader to check out the new and highly entertaining textbook [18] for many more examples of the power of Walnut.

2. The Queen Bee

Suppose the Queen has the ability to bounce, or perhaps more aptly: if Her Majesty starts out diagonally, it may please her to continue her way perpendicularly once she reaches the boundary of her domain, as long as the Manhattan distance decreases. Because of this bounciness, we shall call her the Queen Bee. The first few P -positions of Corner the Queen Bee are given in Table 1. Its defining property is that all natural numbers occur once, and only once, as an a_n or a b_n and that $b_n = 2a_n$. It turns out that Table 1 also arises from a subtraction game considered by Fraenkel [9], who calls the a_n vile and the b_n dopey.

Theorem 1. *The P -positions of Corner the Queen Bee satisfy $b_n = 2a_n$ and each natural number occurs once, and only once, as a coordinate of a P -position.*

Proof. It is not hard to see that every column and every row contains at most one P -position. To see that every column (and hence every row) contains a P -position, observe that all positions within reach of (a, y) have x -coordinate bounded by $2a$. This is immediately clear for nonbouncing moves. A bouncing move leads to a position $(a', y - a)$ such that $a' < 2a$, because the Manhattan distance decreases. Therefore, the number of P positions within reach of (a, y) is at most $2a$ and one of these positions must be reachable by infinitely many (a, y) . This is only possible by vertical moves. We conclude that the column contains a P -position. It follows that every natural number occurs once, and only once, as an a_n or a b_n .

The minimal excludant of $\{a_i, b_i : i \leq n\}$ occurs as an a or a b . It has to be a_{n+1} since we line up P -positions by increasing x -coordinate. Now consider

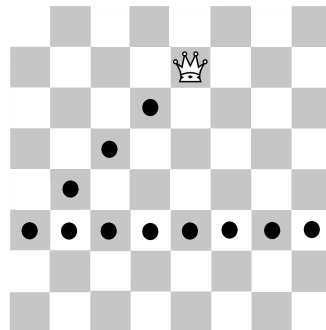


Figure 1. Possible moves of a Queen Bee that can bounce perpendicularly (black dots), if Her Majesty starts out on a diagonal.

1	3	4	5	7	9	11	12	13	15	16	17	19	20	21
2	6	8	10	14	18	22	24	26	30	32	34	38	40	42

Table 1. The first entries in the table of P -positions in Corner the Queen Bee. All coordinates satisfy $b_n = 2a_n$. The a_n are the numbers whose binary representations end with an even number of zeros. The b_n are those that end with an odd number of zeros. Notice that all positions are on diagonals $(x, x + a)$ for which a is in the top row. These are the sequences A003159 and A036554 in the OEIS.

an arbitrary $(a_{n+1}, a_{n+1} + d)$ for $0 < d < a_{n+1}$. A diagonal move bounces against the side at $(0, d)$ with $d \in \{a_i, b_i : i \leq n\}$. If d is a vile number, then by induction $(d, 2d)$ is a P -position. It is within reach of $(a_{n+1}, a_{n+1} + d)$ since $d < a_{n+1}$. If d is dopey, then the P -position $(d/2, d)$ is within reach. We conclude that $(a_{n+1}, a_{n+1} + d)$ is an N -position. It is easy to check that none of the (a_i, b_i) are within reach of $(a_{n+1}, 2a_{n+1})$. Hence, this is a P -position. \square

3. The Queen Dee

In our earlier work [7] we considered a Queen that can reflect once against a side and continue diagonally, changing her course from diagonal (constant $y - x$) to antidiagonal (constant $y + x$) if she bounces against a side of the board. We call her a Queen Dee. In this paper, we consider a 2-Queen with the same reflective property and we call her a 2-Queen Dee. A 2-Queen can take one step to the left or one step down before moving diagonally. A 2-Queen Dee can reflect off of a side; see Figure 2.

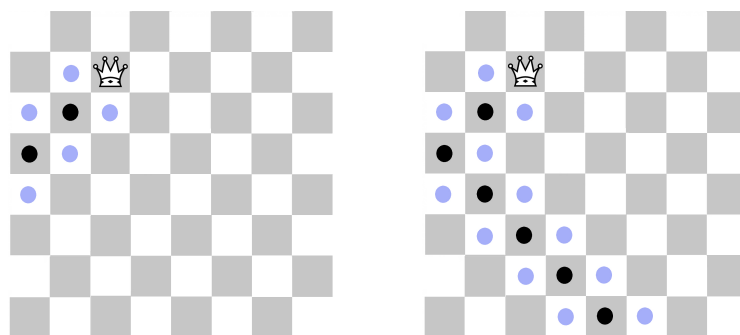


Figure 2. On the left, possible diagonal moves of a Queen (black dots) and a 2-Queen (lighter-colored dots). These moves are extended by reflection for the Queen Dee and the 2-Queen Dee, as illustrated on the right.

We showed in [7] and [14] that the P -positions in Corner the Queen Dee can be retrieved from the morphism $\tau(a) = ab$, $\tau(b) = ac$, $\tau(c) = a$, as follows. If we iterate this morphism, then the limit is the tribonacci word

$$t = \lim_{n \rightarrow \infty} \tau^n(a) = abacabaabacababacabaabacab \cdots,$$

which is an intriguing object that is studied for its own sake [16; 21]. If we delete all c 's from t then we get the word $t_c = abaabaabaababaabaabaab \cdots$, which turns out to code the P -positions (a_n, b_n) . More specifically, the location of the n -th a in t_c is equal to a_n and the location of the n -th b is equal to b_n . In this section, we complement this result by proving that the n -th P -position in Corner the 2-Queen Dee can be retrieved in the same manner from $t_b = aacaaaacaaaacaaaaca \cdots$, the tribonacci word from which all b 's have been deleted.

Recently, Jeffrey Shallit [20] gave a mechanical proof using Walnut that the P -positions of the Queen Dee are coded by the tribonacci word. It is straightforward to adapt this proof for P -positions of the 2-Queen Dee and we shall do that in Section 6. In this section we give an old-fashioned proof.

The P -positions of the 2-Queen satisfy $b_n = a_n + 2n$, i.e., they are on consecutive even diagonals. The P -positions of the 2-Queen Dee occasionally skip an even diagonal, as shown in Table 2.

In particular, the positions skip the diagonals $(x, x + 2d)$ if $2d$ is equal to $a_1 + b_1$, $a_2 + b_2$, $a_3 + b_3$, etc. It turns out that this is a defining property.

The standard algorithm to find P -positions in impartial games starts by identifying the set P_0 of all (or at least one) end-positions, which have no move. In our case, this is $(0, 0)$, the P -position (a_0, b_0) that we never list in our tables of (a_n, b_n) . Now remove all positions that have a move to P_0 , and consider the resulting game. Identify its set of end-positions and add them to get P_1 . Remove all positions that have a move to P_1 , etc. Continue until all positions are either in P_n or have been removed. The union of the P_n is the set of P -positions.

To apply this algorithm to the 2-Queen Dee we introduce some notation:

$$\begin{aligned} A_n &= \{a_i : i \leq n\}, & B_n &= \{b_i : i \leq n\}, & P_n &= \{(a_i, b_i) : i \leq n\}, \\ D_n &= \{b_i - a_i : i \leq n\}, & S_n &= \{a_i + b_i : i \leq n\}. \end{aligned}$$

These are all initial segments of sequences, of coordinates, positions, differences, and sums. To find the next P -position, we need to remove all positions with a move to P_n from the game.

Table 2 indicates that $A_n \cup B_n$ contains all integers up to $\max(A_n)$ and that $D_n \cup S_n$ contains all even integers up to $\max(D_n)$, while both D_n and S_n contain

1	2	4	5	6	7	9	10	11	13	14	15	16	18	19
3	8	12	17	20	25	29	34	39	43	48	51	56	60	65

Table 2. The first P -positions of the 2-Queen Dee are all on even diagonals $(x, x + 2d)$. Notice that $2d$ is unequal to $a_n + b_n$ for all n . These are sequences A140102 and A140103 in the OEIS.

even integers only. This is our inductive hypothesis, together with the assumption that P_n contains the first $n+1$ P -positions (if we include the origin) with increasing Manhattan distance to the origin.

Lemma 2. *Under the inductive hypothesis there is a move from (x, y) with $x \leq y$ to P_n if and only if one of the following holds: $\{x, y\} \cap (A_n \cup B_n) \neq \emptyset$ or $\{y - x - 1, y - x, y - x + 1\} \cap (D_n \cup S_n) \neq \emptyset$.*

Proof. Suppose that there exists a move from (x, y) to $(a_i, b_i) \in P_n$ (or its reflection (b_i, a_i)). If the move is horizontal or vertical, then the P -position has to share a coordinate with (x, y) . In other words, $\{x, y\} \cap (A_n \cup B_n)$ is nonempty. If the move is diagonal without a bounce, then $y - x = b_i - a_i$. If it is diagonal with a bounce, then $y - x = a_i + b_i$. Similarly, if the move is off-diagonal starting from $(x, y - 1)$ or $(x - 1, y)$ then $y - x \pm 1 = b_i - a_i$ or $y - x \pm 1 = b_i + a_i$. We conclude that a move from (x, y) to P_n is only possible if one of the two intersections $\{x, y\} \cap (A_n \cup B_n)$ or $\{y - x - 1, y - x, y - x + 1\} \cap (D_n \cup S_n)$ is nonempty.

We need to prove the converse. First observe that if (x, y) is reachable from P_n , then it has a move to P_k for $k < n$, because any position that is reachable from a P -position is an N -position. So all we have to show is that a move between (x, y) and P_n is possible if one of the two intersections is nonempty. Suppose that $\{x, y\} \cap (A_n \cup B_n) \neq \emptyset$, i.e., there exists a P -position (a_i, b_i) (or its reflection (b_i, a_i)) that is in the same row or column as (x, y) . Hence there exists a move between (x, y) and (a_i, b_i) or its reflection.

Suppose $\{x, y\} \cap (A_n \cup B_n) = \emptyset$ and $\{y - x - 1, y - x, y - x + 1\} \cap (D_n \cup S_n) \neq \emptyset$. If $y - x$ is even, then $(y - x) \in (D_n \cup S_n) \neq \emptyset$. We have that $a_i \pm b_i = y - x$ for a proper choice of the sign. If the sign is negative, then the positions are on the same diagonal and there is a move between them. If the sign is positive, then there is a diagonal move with a bounce that takes (y, x) to (a_i, b_i) . If $y - x$ is odd then either $(y - x - 1)$ or $(y - x + 1)$ is in $(D_n \cup S_n)$. By the same argument, there either exists a diagonal move (possibly with a bounce) from $(x, y - 1)$ or from $(x - 1, y)$ to some (a_i, b_i) . \square

Recall that the minimal excludant of a proper subset S of the natural numbers is denoted by $\text{mex}(S)$. For a proper subset S of the even natural numbers we define

1	3	4	6	7	9	10	12	14	15	17	18	20	21	23
2	5	8	11	13	16	19	22	25	28	31	33	36	39	42

Table 3. The first P -positions in Corner the Queen Dee. These are sequences A140100 and A140101 in the OEIS. Observe that $A140101 - A140100 = A140102$, which is the first sequence in Table 2 above, and that $A140101 + A140100 = A140103$, which is the second sequence. It immediately follows that $A140103 - A140102 = 2A140100$.

$\text{mex}_2(S)$ as the minimal excludant in $2\mathbb{N}$. For example, if $S = \{2, 4, 6, 10, 14\}$ then $\text{mex}_2(S) = 8$.

Lemma 3. *The coordinates of the $(n+1)$ -th P -position satisfy*

$$a_{n+1} = \text{mex}(A_n \cup B_n) \quad \text{and} \quad b_{n+1} - a_{n+1} = \text{mex}_2(D_n \cup S_n).$$

Proof. To find the next P -position, remove all positions with a move to P_n . The remaining positions (x, y) have empty intersections $\{x, y\} \cap (A_n \cup B_n) = \emptyset$ and $\{y - x - 1, y - x, y - x + 1\} \cap (D_n \cup S_n) = \emptyset$. We need to locate the remaining position with minimal Manhattan distance to the origin. The minimal remaining x -coordinate is $\text{mex}(A_n \cup B_n)$. The minimal remaining difference $y - x$ is $\text{mex}_2(D_n \cup S_n)$. Note that the resulting $A_{n+1}, B_{n+1}, D_{n+1}, S_{n+1}$ satisfy the inductive hypothesis. \square

There is a remarkable connection between the P -positions for Queen Dees and for 2-Queen Dees. The differences $b_n - a_n$ for the Queen Dee produce the sequence 1, 2, 4, 5, 6, 7, \dots of x -coordinates of P -positions of the 2-Queen Dee. Similarly, the sums $b_n + a_n$ produce 3, 8, 12, 17, 20, 25, \dots of y -coordinates for the 2-Queen Dee. We can retrieve the P -positions of the 2-Queen Dee from those of the Queen Dee. Conversely, rewriting these equations, we can retrieve the P -positions of the Queen Dee from those of the 2-Queen Dee. The differences $b_n - a_n$ of the 2-Queen Dee produce the sequence 2, 6, 8, 12, 14, 18, \dots of twice the x -coordinates of P -positions of the Queen Dee. Similarly, the sums produce twice the y -coordinates.

We write $d_n = b_n - a_n$ for difference and $s_n = b_n + a_n$ for sum. We reserve the Greek alphabet for the Queen Dee and Latin for the 2-Queen Dee.

Lemma 4. *The following equalities hold: $a_n = \delta_n$, $b_n = \sigma_n$, $d_n = 2\alpha_n$, $s_n = 2\beta_n$.*

Proof. We use the Greek notation $\Delta_n = \{\delta_i : i \leq n\}$ and $\Sigma_n = \{\sigma_i : i \leq n\}$. We proved in [7] that $\delta_{n+1} = \text{mex}(\Delta_n \cup \Sigma_n)$, which by induction and by Lemma 3 is equal to a_{n+1} , producing the first equality.

Unfortunately, the Greek capitals α and β are indistinguishable from the Latin capitals. Fortunately, by induction, we can identify the first n α 's with $D_n/2$ and

the first n β 's with $S_n/2$. We proved in [7] that $\alpha_{n+1} = \text{mex}(D_n/2 \cup S_n/2)$ and by Lemma 3 this implies that $\alpha_{n+1} = d_{n+1}/2$, producing the third equality.

Moving on, $b_{n+1} = a_{n+1} + d_{n+1} = \delta_{n+1} + 2\alpha_{n+1} = \alpha_{n+1} + \beta_{n+1}$ produces the second equality. Finally, $s_{n+1} = a_{n+1} + b_{n+1} = \delta_{n+1} + \sigma_{n+1} = 2\beta_{n+1}$ produces the fourth equality. \square

We conjectured in [7] that the P -positions of the 2-Queen Dee can be derived from \mathbf{t} . Our next result settles this conjecture.

Theorem 5. *The P -positions (a_n, b_n) in Corner the 2-Queen Dee are coded by \mathbf{t}_b , in the sense that a_n corresponds to the location of the n -th a , and b_n corresponds to the location of the n -th c .*

Proof. Again this follows from results in [7]. Corollary 1 in that paper states that $\delta_{n+1} - \delta_n = 1$ unless the n -th letter in \mathbf{t} is equal to b , in which case $\delta_{n+1} - \delta_n = 2$. Therefore, we can retrieve δ_n as the position of the n -th a if we code \mathbf{t} by $a \mapsto a$, $b \mapsto ac$, $c \mapsto a$. We denote this coding by κ . In particular, δ_n corresponds to the n -th a in $\kappa(\mathbf{t})$. Note that κ is equal to $\lambda \circ \tau$, where λ is the coding $a \mapsto a$, $b \mapsto \epsilon$, $c \mapsto c$. Now the result follows from

$$\mathbf{t}_b = \lambda(\mathbf{t}) = \lambda(\lim_{n \rightarrow \infty} \tau^{n+1}(a)) = \kappa(\lim_{n \rightarrow \infty} \tau^n(a)) = \kappa(\mathbf{t})$$

and the fact that by Lemma 4 the sequences δ_n and a_n are identical. \square

One may wonder what happens if we delete all a 's from \mathbf{t} . It turns out that \mathbf{t}_a codes the same table as \mathbf{t}_c .

4. Morphic games

The P -positions in Corner the Lady can be derived from the a 's and the b 's in the Fibonacci word, which is the fixed point of the morphism $a \mapsto ab$, $b \mapsto a$. The fixed point of the morphism $a \mapsto a^k b$, $b \mapsto a$ produces the P -positions for Holladay's k -Queen (here a^k denotes the block of k a 's). The period-doubling morphism $a \mapsto ab$, $b \mapsto aa$ produces the P -positions for the Queen Bee. This connection between morphisms and combinatorial games was first studied by Duchêne and Rigo [5]; see also [17]. A game is *morphic* if its P -positions can be retrieved from a recoding of a fixed point of a morphism. The k -Queen, the Queen Bee, and the 1- and 2-Queen Dee all are morphic Queens. Are there more?

Not all Bouncing Queens are morphic. In an earlier version of [7] two authors of this paper conjectured that 3- and 4-Queen Dees are morphic. However, numerical results of the other author [14] indicate otherwise.

Not all modifications of the substitutions above produce morphic Queens. The morphisms $a \mapsto ab$, $b \mapsto a$ and $a \mapsto ab$, $b \mapsto aa$, which come with the Queen and the Queen Bee, belong to the family $a \mapsto ab$, $b \mapsto a^k$ that is

1	2	4	5	7	8	9	10	12	13	15	16	17	18	20
3	6	11	14	19	22	25	28	33	36	41	44	47	50	55

Table 4. The P -positions of the $(2, 1)$ -Queen are produced by the morphism $a \mapsto aab$, $b \mapsto aa$. The positions (a_n, b_n) satisfy $b_n = 2a_n + n$. These are sequences A026367 and A026368 in the OEIS.

studied in [2] (in an equivalent form). Next in line is $a \mapsto ab$, $b \mapsto aaa$. It produces the positions $(1, 2)$, $(3, 7)$, $(4, 9)$, $(5, 11)$, $(6, 13)$. Now consider the N -position $(3, 6)$. It must be possible to move to a P -position from there, which has to be either $(0, 0)$ or $(1, 2)$. In the first case, we need a move over $(3, 6)$, which is the difference between the P -positions $(3, 7)$ and $(6, 13)$. In the second case, we need a move over $(2, 4)$, which is the difference between the P -positions $(3, 7)$ and $(5, 11)$. Hence, there is no sensible way to define a Queen in this case for which legal moves are position-independent, and so it appears that $a \mapsto ab$, $b \mapsto aaa$ does not produce a morphic Queen.

Some modified substitutions do produce morphic Queens. The morphism $a \mapsto aab$, $b \mapsto aa$ arises from the 2-Queen's morphism, just like the Queen Bee's morphism arises from the Queen's. It produces the P -positions in Table 4. These are the P -positions of $(2, 1)$ -Wythoff, a modification of Corner the Lady that was introduced by Fraenkel [8]. In this game, the Queen widens the scope of her diagonal. An ordinary Queen makes diagonal moves (x, y) with x equal to y . In $(2, 1)$ -Wythoff, we have a $(2, 1)$ -Queen who makes widened diagonal moves $x \leq y \leq 2x$, or symmetrically, $y \leq x \leq 2y$. Consecutive positions in Table 4 either differ by $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and contain each natural number once and only once. It is not hard to verify that for $k > 1$ such tables with differences $\begin{bmatrix} 1 \\ k+1 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ j+k+1 \end{bmatrix}$ correspond to the morphisms $a \mapsto a^k b$, $b \mapsto a^j$. These tables contain the P -positions of Fraenkel's $(j, 1+k-j)$ -Wythoff games (if $k \geq j$), which can be interpreted as versions of Corner the Lady for Queens with widened scopes. The first parameter j represents the scope, the second parameter represents how far the Queen can stroll off the diagonal. In particular, a $(1, k)$ -Queen is a k -Queen.

An ordinary Queen produces complementary sequences $b_n = a_n + n$. Holaday's k -Queen produces $b_n = a_n + kn$. What can be said for complementary sequences that satisfy $b_n = a_n + f(n)$ for some function f ? This problem has been studied by Kimberling [12] and some of these sequences have been entered in the OEIS (see A184117). The morphism $a \mapsto a^{k-j} b a^j$, $b \mapsto a$ produces $b_n = a_n + kn - j$ for $k > j \geq 0$. These are P -positions. Consider a k -Queen that is allowed to stroll, but with a restriction. If her initial position (x, y) satisfies $|x - y| > j$ then she cannot stroll off-diagonal to a position (u, v) with $|u - v| \leq j$. The positions $|u - v| \leq j$ form a special part of the board that can

only be entered by a horizontal or vertical move. The P -positions in this limited game are produced by $a \mapsto a^{k-j}ba^j$, $b \mapsto a$.

The interesting quality of the Queen Dee is that her P -positions are coded by the fixed point of the three letter morphism $a \mapsto ab$, $b \mapsto ac$, $c \mapsto a$. This morphism belongs to a well-studied family of morphisms, in which the 4-bonacci morphism $a \mapsto ab$, $b \mapsto ac$, $c \mapsto ad$, $d \mapsto a$ is next. In their original paper on morphic games, Duchêne and Rigo asked if it is possible to define a combinatorial game with P -positions corresponding to the 4-bonacci morphism. We can specify this. Does there exist a morphic Queen with P -positions that are coded by the fixed point of this morphism?

5. The coordinate sums of the P -positions

One of the remarkable properties of Corner the Lady is that consecutive Fibonacci numbers (F_i , F_{i+1}) occur as P -positions in this game (with i odd and starting the count from $F_1=1$ and $F_2=2$). Martin Gardner singled out this property in one of his Scientific American columns [10]. It is a consequence of the equation $a_{b_n} = a_n + b_n$ for the P -positions of Corner the Lady. This equation does not hold for Corner the Empress, but it holds approximately, as can be seen in Table 3. Its first column adds up to $1+2=3$, which is indeed equal to the first entry in the second column. However, the second column sum $3+5=8$ is not equal to the first entry of the fifth column. It is one off, 7. Further inspection indicates that $a_n + b_n$ appears to be either equal to the a entry in the b_n -th column, or is one off. In other words,

$$a_{b_n} = a_n + b_n \quad \text{or} \quad a_{b_n} = a_n + b_n - 1 \quad \text{for all } n. \quad (1)$$

In [7] we conjectured that this equation holds for Corner the Queen Dee, and Jeffrey Shallit used Walnut to prove it in [20] by the following one-liner:

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eval fokkink "?msd_trib An,a,b,ab (n>=1 & $xaut(n,a) & $yaut(n,b)
& $xaut(b,ab)) => (ab=a+b | ab=a+b-1)":
```

Walnut code is very transparent. Even if you are not familiar with the language, you can understand this statement. The final expression $(ab=a+b \mid ab=a+b-1)$ contains the conjecture. The sequences a and b are defined by means of $\$xaut(n,a)$ and $\$yaut(n,b)$. A sequence is a function from the integers to itself and the two functions for a_n and b_n are $xaut$ and $yaut$. Actually, Walnut does not write $a=\$xaut(n)$, or something like that, but treats n and a as simultaneous inputs. This is why n and a are called synchronized sequences. Since Walnut can only handle bitstrings, it needs a numeration system to deal with natural numbers. The default is the binary system, but it turns out that n and a are not synchronized in binary. The proper numeration system is the tribonacci

system, which is imported by the command `msd_trib`. The instruction `eval` commands Walnut to evaluate this statement, named after one of the authors, for all n, a, b, ab with $n \geq 1$. The statement is evaluated as TRUE.

Table 2 appears to also satisfy (1). To apply Walnut we need to implement the functions for the sequences a_n and b_n in this table. Fortunately, they can easily be derived from the synchronized sequences in [18, p. 286], which have been implemented in Walnut to reprove a theorem from [5]. We have that $a_n = A(n) - E(A(n))$ and $b_n = C(n) - E(C(n))$. The sequences A, C, E are implemented as `triba, tribc, tribe`, which we can use to define `uaut` and `vaut`:

```
def uaut "?msd_trib (s=0 & t=0) | Ex,xy $triba(s,x) & $tribe(x,xy)
& t=x-xy":
def vaut "?msd_trib (s=0 & t=0) | Ex,xy $tribc(s,x) & $tribe(x,xy)
& t=x-xy":
```

Now we can copy–paste the proof in [20] to show that the columns in Table 2 do indeed satisfy (1).

Theorem 6. *The P -positions (a_n, b_n) of the 2-Queen Dee satisfy (1).*

Proof. Walnut returns TRUE for

```
eval table2 "?msd_trib An,a,b,ab (n>=1 & $uaut(n,a) & $vaut(n,b)
& $uaut(b,ab)) => (ab=a+b | ab=a+b-1)":
```

It befits a modern Monarch to have her footsteps traced by an electronic slave. \square

Computer experiments suggested that (1) also holds for the (2, 1)-Queen, which is associated to the substitution $a \rightarrow aab, b \rightarrow aa$, as we saw above. A result of Boris Adamczewski [1] implies that $a_{b_n} - a_n - b_n$ is bounded, since the substitution is Pisot. We conjectured that it is in fact bounded by 1, and this was recently proved by Shallit [19] using Walnut.

Computer experiments indicate that $a_{b_n} - a_n - b_n$ is unbounded for the Queen Bee. We cannot use Walnut to prove or disprove this statement. Narad Rampersad and Manon Stipulanti [15] proved that the sequences a_n and b_n are not regular in this case, which implies that they cannot be synchronized.

Proposition 7. *Let $[x]$ denote the nearest integer to x , rounded up if the fractional part is $\frac{1}{2}$, and let*

$$f(x) = \sum_{j=0}^{\infty} \left\lceil \frac{x}{2^{2j+1}} \right\rceil.$$

Then $n = f(a_n)$ for the n -th P -position in Corner the Queen Bee.

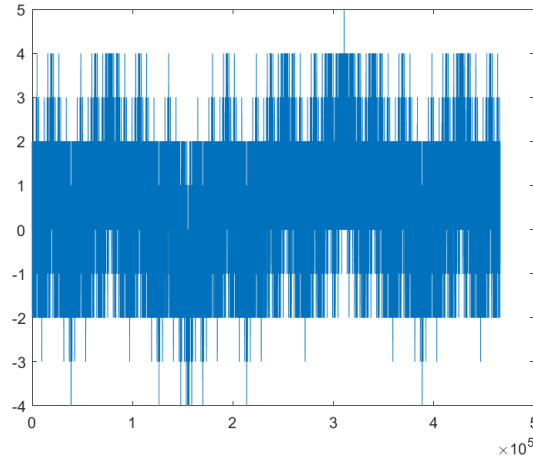


Figure 3. The difference $a_{b_n} - a_n - b_n$ plotted against n for the P -position in Corner the Queen Bee. The first difference that is equal to 5 occurs at $n = 310691$.

Proof. The sequence a_n consists of all numbers that are $4^j \bmod 2 \cdot 4^j$ for $j = 0, 1, \dots$, since these are the numbers that have a suffix of one 1 followed by $2j$ zeros. The residue class $2^{2j} \bmod 2^{2j+1}$ contains $\lfloor a_n + 2^{2j} / 2^{2j+1} \rfloor$ elements up to a_n . Therefore, a_n is the n -th element with

$$n = \sum_{j=0}^{\infty} \left\lfloor \frac{a_n + 2^{2j}}{2^{2j+1}} \right\rfloor.$$

Now notice that $\lfloor x \rfloor = \lfloor x + \frac{1}{2} \rfloor$. □

We are interested in $a_{b_n} - a_n - b_n$, which for the Queen Bee is equal to $a_{2a_n} - 3a_n$. Now a_{2a_n} has index $2a_n$ while $3a_n$ has index $f(3a_n)$ by [Proposition 7](#) (both sequences a_n and b_n are invariant under multiplication by 3). To prove $a_{2a_n} - 3a_n$ is unbounded, we need to show that $2a_n - f(3a_n)$ is unbounded. It is equal to

$$2a_n - \sum_{j=0}^{\infty} \left\lfloor \frac{3a_n}{2^{2j+1}} \right\rfloor.$$

Note that this is equal to zero if we do not round the fractions $3a_n / 2^{2j+1}$. In other words, the unboundedness of $a_{b_n} - a_n - b_n$ is equivalent to the unboundedness of

$$\sum_{j=0}^{\infty} \left\{ \frac{3a_n}{2^{2j+1}} \right\}, \quad (2)$$

where $\{x\}$ denotes the distance to the nearest integer $x - \lfloor x \rfloor$ (beware, it is not the fractional part $x - \lfloor x \rfloor$!).

Theorem 8. *The difference $a_{b_n} - a_n - b_n$ is unbounded for the P -positions of the Queen Bee.*

Proof. Define $h(x) = \sum_{j=0}^{\infty} \{x/4^j\}$. Observe that $h(4^n) = \frac{1}{3}$ and that for a given ϵ and x we have $|h(4^N + x) - h(4^N) - h(x)| < \epsilon$ if N is sufficiently large. Then

$$h(4^{n_1} + 4^{n_1+n_2} + \dots + 4^{n_1+n_2+\dots+n_k}) \approx \frac{k}{3}$$

for $n_1 < n_2 < \dots < n_k$ with sufficiently large gaps. If k is a multiple of three and if all n_i are odd, then $x = 4^{n_1} + 4^{n_1+n_2} + \dots + 4^{n_1+n_2+\dots+n_k}$ is divisible by three and $a = 2x/3$ is an entry in the sequence a_n . Since $h(x)$ is equal to the sum in (2), we see that it can be arbitrarily large. \square

6. Walnut-supported proof of Theorem 5

Jeffrey Shallit [20] showed how to use Walnut to prove that the coded tribonacci word produces the P -positions of the Queen Dee. It is straightforward to adapt his proof for the 2-Queen Dee, as follows. We say that a triple (n, x, y) with $x < y$ is good if (x, y) has no move to P_{n-1} . By Lemma 2 such a triple satisfies the following conditions:

$$\begin{aligned} \forall(k < n) \quad x &\neq a_k \wedge x \neq b_k, \\ \forall(k < n) \quad y &\neq a_k \wedge y \neq b_k, \\ \forall(k < n) \quad |(y-x) - (b_k - a_k)| &> 1, \\ \forall(k < n) \quad |(y-x) - (b_k + a_k)| &> 1. \end{aligned}$$

These constraints are first-order logic statements that can be implemented in Walnut:

```
def good "?msd_trib y>x &
  (~Ek k<n & $uaut(k,x)) & (~Ek k<n & $vaut(k,x)) &
  (~Ek k<n & $uaut(k,y)) & (~Ek k<n & $vaut(k,y)) &
  (~Ek,a,b k<n & $uaut(k,a) & $vaut(k,b) & y-x+a=b) &
  (~Ek,a,b k<n & $uaut(k,a) & $vaut(k,b) & y-x+a=b+1) &
  (~Ek,a,b k<n & $uaut(k,a) & $vaut(k,b) & y-x+a+1=b) &
  (~Ek,a,b k<n & $uaut(k,a) & $vaut(k,b) & y-x=a+b+1) &
  (~Ek,a,b k<n & $uaut(k,a) & $vaut(k,b) & y-x+1=a+b) &
  (~Ek,a,b k<n & $uaut(k,a) & $vaut(k,b) & y-x=a+b)":
```

The P -position (a_n, b_n) is the unique good position that has minimal Manhattan distance to the origin, as specified by Lemma 3. To finish the proof of Theorem 5, we need to verify that (n, a_n, b_n) satisfies:

- (1) The triple (n, a_n, b_n) is good.
- (2) If (n, x, y) is good then $a_n \leq x$.
- (3) If (n, a_n, y) is good then $b_n \leq y$.

These are three checks for Walnut:

```
eval check1 "?msd_trib An,a,b (n>=1 & $uaut(n,a) & $vaut(n,b))
                                     => $good(n,a,b)":
eval check2 "?msd_trib An,x,y (n>=1 & $good(n,x,y))
                                     => (Ea $uaut(n,a)&x>=a)":
eval check3 "?msd_trib An,a,y (n>=1 & $uaut(n,a) & $good(n,a,y))
                                     => (Eb $uaut(n,b) & y>=b)":
```

Walnut returns TRUE for all three checks.

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