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Improving upper and lower bounds of the number of games born by day 4

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In combinatorial game theory, the lower and upper bounds of the number of games born by day 4 have been recognized as $3.0 \cdot 10^{12}$ and 10^{434}, respectively. We improve the lower bound to $10^{28.2}$ and the upper bound to $4.0 \cdot 10^{184}$.

1. Introduction

The main result of this study is improving the upper and lower bounds on the total number of canonical forms born by day 4 under normal play convention.

We recall the following definitions and theorems for canonical forms.

Definition 1. For any game G,

- a left option A is a *dominated option* if another left option B satisfies $A \leq B$,
- a right option H is a dominated option if another right option I satisfies $H \geq I$,
- a left option A is a *reversible option* if a right option A^R of A satisfies $A^R \leq G$, and
- a right option H is a reversible option if a left option H^L of H satisfies $H^L \geq G$.

Theorem 2 (removing dominated options). *Let $G = \{A, B, C, \dots \mid H, I, J, \dots\}$, and let G' be obtained from G by removing A. If A is a dominated option, then $G = G'$.*

Theorem 3 (bypassing reversible options). *Let $G = \{A, B, C, \dots \mid H, I, J, \dots\}$ and assume that A^R, a right option of A, satisfies $A^R \leq G$. Let G'' be obtained from G by replacing A with all left options of A^R. Then $G = G''$.*

These theorems also hold for the set of right options.

Theorem 4. *For any game G, there is a game G' such that $G = G'$, and G' has no dominated options nor reversible options.*

We call such a game a *canonical form* of G.

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Theorem 5. *For two games G and H, assume that $G = H$. Let G' and H' be canonical forms of G and H, respectively. Then G' and H' are isomorphic.*

These are well-known results. For more details of these theorems, see [2].

Games with game tree heights less than or equal to n are called games born by day n, and the set of all canonical forms born by day n is denoted by $\mathbb{G}_n$. So far, the total number of canonical forms born by day 0, 1, 2, 3 have been recognized as 1, 4, 22, 1474, respectively. Meanwhile, the total number of canonical forms born by day 4 is vast, and only wide lower and upper bounds $3 \cdot 10^{12}$ and 10^{434} are known. Improving these upper and lower bounds was discussed as an open problem in [2] and [1] and has created much attention. In this study, we used algebraic properties and programming to improve the upper and lower bounds and obtained a new upper bound, $4.0 \cdot 10^{184}$, and a new lower bound, $10^{28.2}$.

1.1. ***Early results.*** Early results of counting the number of games are summarized as a note in [2, Chapter 3, Section 1]. According to the note, the total number of canonical forms born by day 3 and the total numbers of dicotic games born by day 4, reduced games born by day 4, and hereditarily transitive games born by day 4 are known, whereas the total number of canonical forms born by day 4 is unknown. As indicated in the note, upper and lower bounds of general $|\mathbb{G}_n|$ were obtained in [3]. The upper bound is given in the following three forms, with the lower forms having more complicated inequalities but stricter upper bounds:

$$|\mathbb{G}_{n+1}| \le 2^{|\mathbb{G}_n|+1} + |\mathbb{G}_n|,$$
$$|\mathbb{G}_{n+1}| \le |\mathbb{G}_n| + 2^{|\mathbb{G}_n|} + 2,$$
$$|\mathbb{G}_{n+1}| \le |\mathbb{G}_n| + \left(|\mathbb{G}_{n-1}|^2 + \tfrac{5}{2}|\mathbb{G}_{n-1}| + 2\right)2^{|\mathbb{G}_n|-2|\mathbb{G}_{n-1}|}.$$

Substituting $|\mathbb{G}_3| = 1474$ and $|\mathbb{G}_2| = 22$ into the third equation and examining the values specifically, we obtain

$$|\mathbb{G}_4| \le 1474 + \left(22^2 + \tfrac{5\cdot 22}{2} + 2\right)2^{1474-2\cdot 22} < 10^3 \cdot 10^{0.3011\cdot 1430} < 10^{434}.$$

The lower bound is given in the following two forms, with the lower form having a more complicated inequality but a stricter lower bound:

$$|\mathbb{G}_{n+1}| \ge 2^{\frac{1}{2}|\mathbb{G}_n|/|\mathbb{G}_{n-1}|},$$
$$|\mathbb{G}_{n+1}| \ge (8|\mathbb{G}_{n-1}| - 4)(2^{(|\mathbb{G}_n|-2)/(2|\mathbb{G}_{n-1}|-1)} - 1).$$

Substituting $|\mathbb{G}_3| = 1474$ and $|\mathbb{G}_2| = 22$ into the second equation, we obtain

$$|\mathbb{G}_4| \ge (8 \cdot 22 - 4) \cdot (2^{(1474-2)/(2\cdot 22-1)} - 1) > 171 \cdot 10^{10.304} > 3.0 \cdot 10^{12}.$$

Thus, we have

$$3.0 \cdot 10^{12} < |\mathbb{G}_4| < 10^{434},$$

and the gap between the two bounds is vast.

The rest of the paper is organized as follows. Section 2 studies how the number of antichains can be used for bounding the number of canonical forms. Section 3 studies properties of $\mathbb{G}_3$ for bounding the number of antichains in $\mathbb{G}_3$. Using these properties, Section 4 presents improved upper and lower bounds of $|\mathbb{G}_4|$. Section 5 provides a conclusion. In the Appendix, all the elements of $\mathbb{G}_3$ used in this improvement are summarized in tables.

2. Using antichains to bound $|\mathbb{G}_n|$

In this study, we improve the upper and lower bounds of $|\mathbb{G}_4|$ by using the number of antichains in $\mathbb{G}_3$.

First, we prepare the following lemmas. Note that $n \geq g$ for any game $g \in \mathbb{G}_n$.

Lemma 6. *Assume that $s \in \mathbb{G}_n$. If $n-1 < s$, then $s = n$.*

Proof. Because $n-1-s < 0$, s has a left option s^L such that $s^L \geq n-1$. Since $s^L \in \mathbb{G}_{n-1}$, $s^L = n-1$ and this yields that the set of left options of s is $\{n-1\}$, because s is canonical.

Next, assume that s has a right option $s^R \in \mathbb{G}_{n-1}$. Then $n-1-s$ has a left option $n-1-s^R \geq 0$, which contradicts the fact that $n-1-s < 0$. Thus, the set of right options of s is the empty set. That is, $s = \{n-1 \mid \} = n$. □

Lemma 7. *Assume that $n > 1$. We also assume that $S = \{s_1, s_2, \ldots, s_k\}$ is an antichain of games born by day n. That is, for any i and j, $s_i, s_j \in \mathbb{G}_n$ and $s_i \ngeq s_j$ holds. Then $\{n \mid s_1, s_2, \ldots, s_k\}$, $\{s_1, s_2, \ldots, s_k \mid -n\}$, $\{n-1 \mid s_1, s_2, \ldots, s_k\}$, and $\{s_1, s_2, \ldots, s_k \mid -(n-1)\} \in \mathbb{G}_{n+1}$ are distinct canonical forms. Moreover, if S, S' are distinct antichains in $\mathbb{G}_n$ with $S, S' \notin \{\{n\}, \{n-1\}, \{-n\}, \{-n+1\}\}$, then the four canonical forms obtained for S are pairwise distinct from those obtained for S'.*

Proof. Consider $\{n-1 \mid s_1, s_2, \ldots, s_k\} \in \mathbb{G}_{n+1}$. If this is not a canonical form, then there is a dominated or reversible option in the left or right options.

First, there is only one left option, $n-1$, which is not a dominated option. The right options also have no dominated option because the set of all right options is an antichain. Further, the left option $n-1$ has no right option; therefore, this is not a reversible option.

The remaining case is that a right option, s_i, is a reversible option. If s_i is a reversible option, then s_i has a left option s_i^L satisfying $\{n-1 \mid s_1, s_2, \ldots, s_k\} \leq s_i^L$. However, considering $\{n-1 \mid s_1, s_2, \ldots, s_k\} - s_i^L$, this game has a left option $n-1-s_i^L$. Because s_i is born by day n, s_i^L is born by day $n-1$. Thus, $n-1 \geq s_i^L$.

This yields $n-1-s_i^L \geq 0$ and $\{n-1 \mid s_1, s_2, \ldots, s_k\} - s_i^L \not\leq 0$, which means $\{n-1 \mid s_1, s_2, \ldots, s_k\} \leq s_i^L$ does not happen. Other cases are proved similarly.

Next, let $S = \{s_1, s_2, \ldots, s_k\}$ and $S' = \{s'_1, s'_2, \ldots, s'_l\}$ be subsets of $\mathbb{G}_n$ and $S, S' \notin \{\{n\}, \{n-1\}, \{-n\}, \{-n+1\}\}$. Then

$$\{n \mid s_1, s_2, \ldots, s_k\} - \{s'_1, s'_2, \ldots, s'_l \mid -n\} \not\leq 0,$$

because $n - \{s'_1, s'_2, \ldots, s'_l \mid -n\} \geq 0$;

$$\{n \mid s_1, s_2, \ldots, s_k\} - \{n-1 \mid s'_1, s'_2, \ldots, s'_l\} \not\leq 0,$$

because $n - \{n-1 \mid s'_1, s'_2, \ldots, s'_l\} \geq 0$;

$$\{n \mid s_1, s_2, \ldots, s_k\} - \{s'_1, s'_2, \ldots, s'_l \mid -(n-1)\} \not\leq 0,$$

because $n - \{s'_1, s'_2, \ldots, s'_l \mid -(n-1)\} \geq 0$;

$$\{s_1, s_2, \ldots, s_k \mid -n\} - \{n-1 \mid s'_1, s'_2, \ldots, s'_l\} \not\geq 0,$$

because $-n - \{n-1 \mid s'_1, s'_2, \ldots, s'_l\} \leq 0$;

$$\{s_1, s_2, \ldots, s_k \mid -n\} - \{s'_1, s'_2, \ldots, s'_l \mid -(n-1)\} \not\geq 0,$$

because $-n - \{s'_1, s'_2, \ldots, s'_l \mid -(n-1)\} \leq 0$. Finally, consider

$$\{n-1 \mid s_1, s_2, \ldots, s_k\} - \{s'_1, s'_2, \ldots, s'_l \mid -(n-1)\} \\ = \{n-1 \mid s_1, s_2, \ldots, s_k\} + \{n-1 \mid -s'_1, -s'_2, \ldots, -s'_l\}.$$

Let $G_1 = \{n-1 \mid s_1, s_2, \ldots, s_k\}$ and $G_2 = \{s'_1, s'_2, \ldots, s'_l \mid -(n-1)\}$. If $s'_i < n-1$ for all i, then $n-1-G_2 \geq 0$ and $G_1 \neq G_2$. Therefore, assume that there exists s'_i such that $s'_i \not< n-1$. From Lemma 6, if $s'_i > n-1$ then $s'_i = n$ and $S' = \{n\}$. Thus, $s'_i \not> n-1$. The game $G_1 - G_2$ has two left options $n-1-G_2$ and $G_1 + (n-1)$. Suppose for a contradiction that $G_1 - G_2 = 0$. Then a right option of G_1 satisfies $s_a = -(n-1)$ and a left option of G_2 satisfies $s'_c = n-1$. In addition, since $S \neq \{-(n-1)\}$ and $S' \neq \{n-1\}$, another right option of G_1 satisfies $s_b \not\gtreqless -(n-1)$ and another left option of G_2 satisfies $s'_d \not\gtreqless n-1$. Then $G_1 - G_2$ has a right option $s_b - G_2$. Here, $s_b + n - 1 \not\gtreqless 0$ because $s_b \not\gtreqless -(n-1)$ and for any left option $s_b^L \in \mathbb{G}_{n-1}$ of s_b, $s_b^L - G_2 \not\geq 0$ because $s_b^L - s'_c = s_b^L - (n-1) \leq 0$. This contradicts the fact that $G_1 - G_2 = 0$ and we obtain $\{n-1 \mid s_1, s_2, \ldots, s_k\} \neq \{s'_1, s'_2, \ldots, s'_l \mid -(n-1)\}$.

Thus, the forms are distinct. □

Lemma 8. *For any integers $i \geq j$, $\{i \mid j\}$ is a canonical form.*

Proof. Obviously, there are no dominated options. Suppose for a contradiction that the left option i is reversible. In particular i has a right option, so necessarily $i < 0$ and that right option is $i+1$, and hence $\{i \mid j\} \geq i+1$. However, $\{i \mid j\} < i+1$, because $i-(i+1) < 0$, $j-(i+1) < 0$ and $\{i \mid j\} - (i+2) \not\geq 0$. We have a contradiction, so the left option i is not reversible. An analogous proof shows that the right option j is not reversible either, so $\{i \mid j\}$ is a canonical form. □

Let $a(n)$ be the number of antichains in $\mathbb{G}_n$.

Theorem 9. *For all* $n \geq 2$, $4a(n) + 2n^2 - 5n - 3 \leq |\mathbb{G}_{n+1}| \leq a(n)^2$.

Proof. The upper bound is trivial since both sets of options must be antichains. For the lower bound, from Lemma 8, we have all games $\{i \mid j\}$ with $-n \leq j \leq i \leq n$, where i and j are integers. Then we have all canonical forms given by Lemma 7 with $S \neq \{i\}$, so in total we get $\frac{1}{2}(2n+1)(2n+2) + 4(a(n) - (2n+1)) = 4a(n) + 2n^2 - 5n - 3$. □

In this paper, we establish lower and upper bounds for $a(3)$. Note that if $n \geq 3$, then $2n^2 - 5n - 3 \geq 0$ and $a(n)$ is much larger than $2n^2 - 5n - 3$. Thus, we will use $4a(n)$ as a lower bound instead of $4a(n) + 2n^2 - 5n - 3$, for simplicity.

3. Division of $\mathbb{G}_3$

3.1. *Stratification of* $\mathbb{G}_3$. In this section, to improve upper and lower bounds of $|\mathbb{G}_4|$, we divide $\mathbb{G}_3$ by two methods which have good properties. First, by using CGSuite (Version 0.7), we found all the elements of $\mathbb{G}_3$ and divided them using Algorithm 1.

```
S ← G_3
i ← 0
while |S| > 0 do
  i ← i + 1
  U_i ← ∅
  for all s ∈ S do
    if ∀t ∈ S (s ≮ t) then
      U_i = U_i ∪ {s}
    end if
  end for
  S ← S \ U_i
end while
```

Algorithm 1. Algorithm determining stratification of $\mathbb{G}_3$.

As a result, the elements of $\mathbb{G}_3$ are divided into sets $U_1, U_2, \ldots, U_{45}$. We call them a *stratification* of $\mathbb{G}_3$. Each pair of elements in the same set is incomparable and each element in U_i $(i > 1)$ has at least one larger element in U_{i-1}. For reference, the results for $\mathbb{G}_2$ using a similar process are shown in Figure 1.

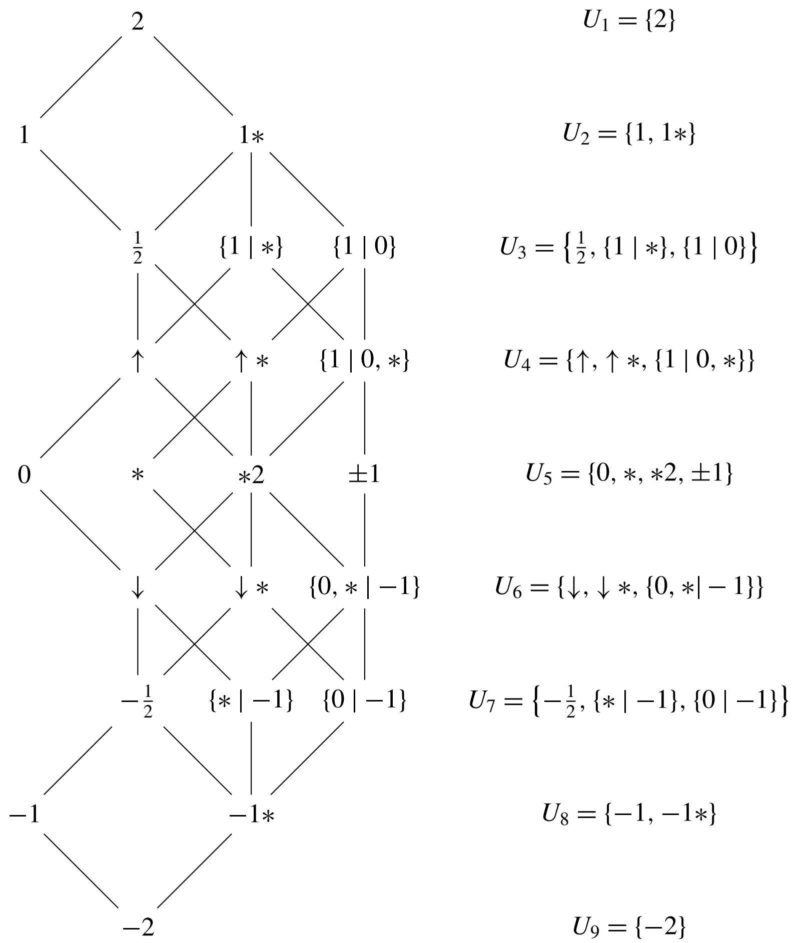

Figure 1. 22 games born by day 2 and their stratification.

For $\mathbb{G}_3$, the number of elements is enormous, so a schematic is shown in Figure 2. Elements connected by lines have an order, with the upper element being larger than the lower element. Elements not directly or indirectly connected to each other are not comparable.

For disjoint sets A and B, we write $A \oplus B = A \cup B$. Then $\mathbb{G}_3 = U_1 \oplus U_2 \oplus \cdots \oplus U_{45}$. Here, every U_i is an antichain by Algorithm 1. The set with the highest number of elements is U_{23} with 86 elements. In addition, when $1 \le i \le 22$, $|U_i| < |U_{i+1}|$, and when $23 \le i \le 45$, $|U_i| > |U_{i+1}|$. Further, by observing the calculated results, we obtain the following result.

Lemma 10. *The stratification is upper and lower symmetric. That is, for any* $u \in U_i$, $-u \in U_{46-i}$ *holds.*

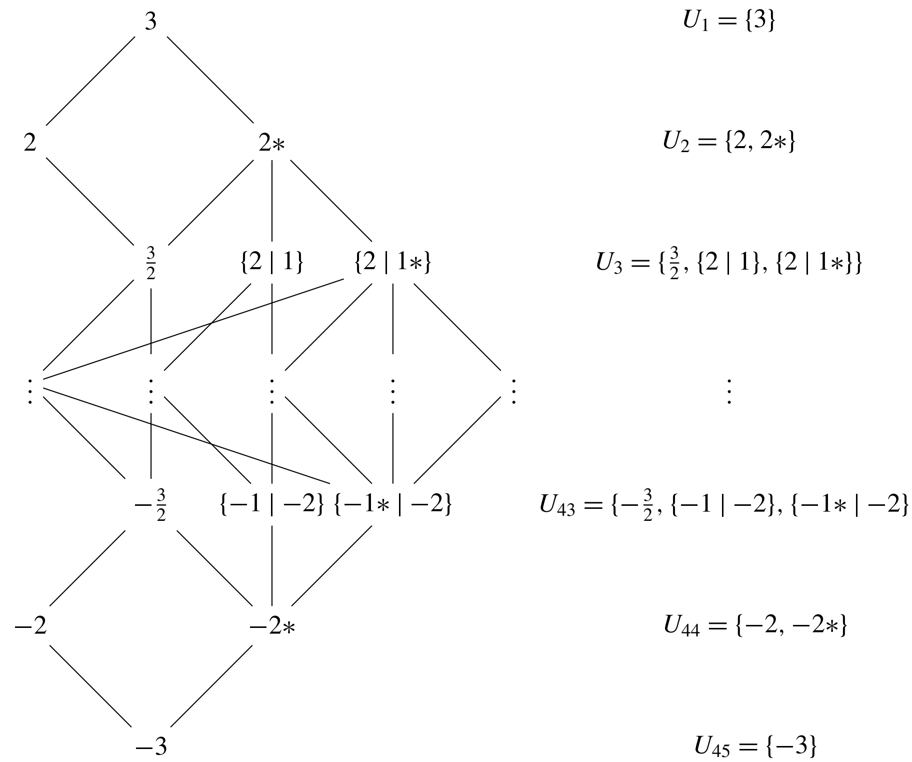

Figure 2. 1474 games born by day 3 and their stratification.

Since this result is obtained by observation, not by algebraic proof, it is not clear whether it holds for $n > 3$. If it can be proved in an algebraic way as well, the result may be extended to the general case.

Next, for each i, we constructed bipartite graph $\mathcal{G}_i = (V_1 \oplus V_2, E)$ as follows:

(1) Let $|V_1| = |U_i|$ and $|V_2| = |U_{i+1}|$, and let v_{1j} be the vertex corresponding to $s_j \in U_i$, and v_{2k} the vertex corresponding to $t_k \in U_{i+1}$.

(2) For any $s_j \in U_i$, $t_k \in U_{i+1}$, if $s_j > t_k$ then $(v_{1j}, v_{2k}) \in E$, and otherwise, $(v_{1j}, v_{2k}) \notin E$.

As an example, Figure 3 shows $\mathcal{G}_3$. Here, we examined the maximum matching of each $\mathcal{G}_i$ and obtained the following result.

Lemma 11. *Let $M(\mathcal{G})$ be the number of elements of a maximum matching of graph $\mathcal{G}$. Then $M(\mathcal{G}_i) = \min(|U_i|, |U_{i+1}|)$.*

That is, any element in the set with a smaller number of elements is always included in the maximum matching.

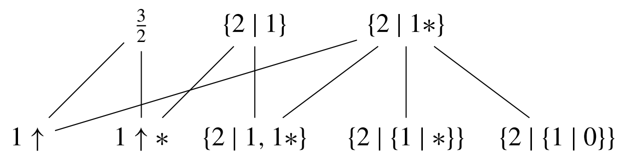

Figure 3. $\mathcal{G}_3$.

3.2. ***Chain division of*** $\mathbb{G}_3$**.** Next, we divide $\mathbb{G}_3$ by another way.

Definition 12. $S = T_1 \oplus T_2 \oplus \cdots \oplus T_m$ is a *chain division* of S by $T_1, T_2, \ldots, T_m$ if for any i, every $s, t \in T_i$ $(s \neq t)$ satisfies $s > t$ or $s < t$.

For example, Figure 4 shows a chain division of $\mathbb{G}_2$ by T_1, T_2, T_3, and T_4.

Next, we consider the chain division of $\mathbb{G}_3$. We regard every element in $\mathbb{G}_3$ as a vertex and the order of elements as edges. Here, we consider deleting all edges except for the matching obtained from Lemma 11. Then, from every element in $U_1, U_2, \ldots, U_{21}$, or U_{22}, one can reach an element in U_{23} through some edges. Further, from Lemma 10, there is an upper and lower symmetry, and therefore, from every element in $U_{24}, U_{25}, \ldots, U_{44}$, and U_{45}, one can also reach an element in U_{23} through corresponding edges. Thus, there exists as many chains as elements in U_{23}.

Theorem 13. *There is a chain division of* $\mathbb{G}_3$ *by* 86 *sets.*

We call the sets $T_1, T_2, \ldots, T_{86}$.

We calculated using programming and obtained a chain division. We also obtained the following result using the pigeonhole principle and the fact that U_{23} has 86 elements.

Lemma 14. *The antichain in* $\mathbb{G}_3$ *with the largest number of elements has* 86 *elements.*

4. Improving upper and lower bounds of $|\mathbb{G}_4|$

4.1. ***Improving the lower bound.*** From Theorem 9 and Lemma 14, we immediately obtain the following result.

Corollary 15. $$4 \cdot 2^{86} = 2^{88} \leq |\mathbb{G}_4|.$$

Since $2 > 10^{0.3}$, we have $10^{26.4} < 2^{88} \leq |\mathbb{G}_4|$ and have succeeded in significantly improving the previously known lower bound $3 \cdot 10^{12}$.

4.2. ***Further improvement of the lower bound.*** This result can be improved by scrutinizing the number of antichains. For any element u in U_{22}, we checked the number of elements that are smaller than u and included in U_{23}. Then there are 9, 25, 33, 14 elements which have 2, 3, 4, 5 smaller elements in U_{23}, respectively.

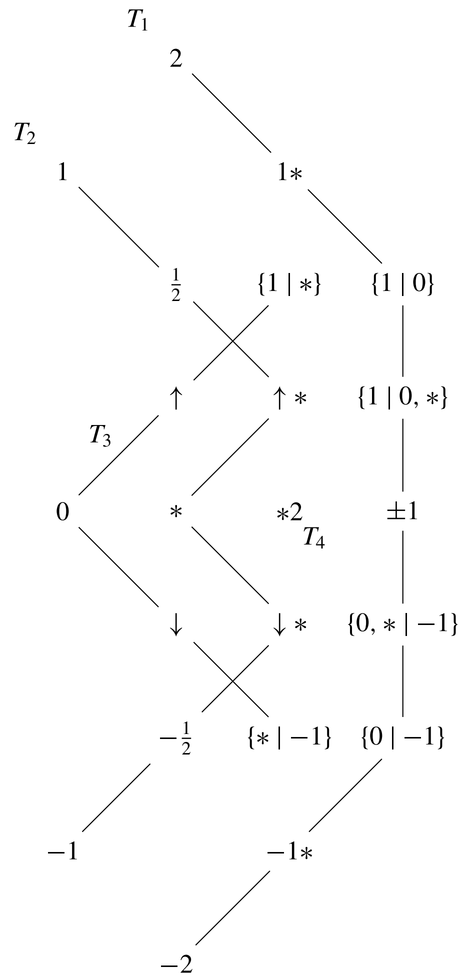

Figure 4. Chain division of games born by day 2.

Therefore, the number of antichains which have one element from U_{22} and some (possibly 0 as well) elements from U_{23} is $9 \cdot 2^{84} + 25 \cdot 2^{83} + 33 \cdot 2^{82} + 14 \cdot 2^{81} = (9 \cdot 8 + 25 \cdot 4 + 33 \cdot 2 + 14)2^{81} = 252 \cdot 2^{81}$.

From Lemma 10, U_{22} and U_{24} have a sign-reversed relationship of elements, and for every element in U_{23}, its inverse is also in U_{23}. Therefore, the number of antichains with one element from U_{24} and some (possibly 0 as well) elements from U_{23} is the same.

We also consider the number of antichains which have two elements from U_{22} and some (possibly 0 as well) elements from U_{23}. It is at least $\frac{1}{2}(9 \cdot 8) \cdot 2^{82} + 9 \cdot 25 \cdot 2^{81} + \frac{1}{2}(25 \cdot 24) \cdot 2^{80} + 9 \cdot 33 \cdot 2^{80} + 9 \cdot 14 \cdot 2^{79} + 25 \cdot 33 \cdot 2^{79} + 25 \cdot 14 \cdot 2^{78} + \frac{1}{2}(33 \cdot 32) \cdot 2^{78} + 33 \cdot 14 \cdot 2^{77} + \frac{1}{2}(14 \cdot 13) \cdot 2^{76}$. By calculating, this is

$(2304+7200+4800+4752+1008+6600+1400+2112+924+91)\cdot 2^{76} = 31191\cdot 2^{76}$.

We can also use upper and lower symmetry. Therefore, calculating the total number obtained so far yields $2^{86}+2\cdot 252\cdot 2^{81}+2\cdot 31191\cdot 2^{76} = 39767\cdot 2^{77} > 2^{15}\cdot 2^{77} = 2^{92}$. Thus, the following holds.

Corollary 16. $$4\cdot 2^{92} = 2^{94} < |\mathbb{G}_4|.$$

Thus, $10^{28.2} < |\mathbb{G}_4|$.

4.3. ***Improving the upper bound.*** Next, we consider the upper bound. In a canonical form, the sets of left and right options are both antichains. Therefore, by using chain division, the following lemma holds.

Lemma 17. *For any game $G \in \mathbb{G}_4$, let the set of games in left options of G be $S = \{s_1, s_2, \ldots, s_m\}$. Then, for any i, j $(i \neq j)$, if $s_i \in T_x$, $s_j \in T_y$ then $x \neq y$. This is also true for the set of games in right options.*

Proof. If $s_i \in T_x$, $s_j \in T_y$, and $x = y$, then $s_i < s_j$ or $s_i > s_j$ holds, which is a contradiction. □

From this lemma, the following holds.

Lemma 18. $$a(3) \le (|T_1|+1)\times(|T_2|+1)\times\cdots\times(|T_{86}|+1).$$

Proof. In any antichains in $\mathbb{G}_3$, there is at most one element in $|T_i|$ for each $1 \le i \le 86$. Thus, $a(3) \le (|T_1|+1)\times(|T_2|+1)\times\cdots\times(|T_{86}|+1)$. □

The square of this upper bound on $a(3)$ equals about 3.7979×10^{202}. From Theorem 9, we have the following result.

Corollary 19. $$|\mathbb{G}_4| < 3.8\cdot 10^{202}.$$

4.4. ***Further improvement of the upper bound.*** We also consider the further improvement of this upper bound. For each $u \in T_i$, count how many elements in T_j $(i < j)$ are incomparable to u, and let $t_{i,j}$ be the maximum of all these numbers. That is, $t_{i,j} = \max(|\{t' \in T_j \mid t \not\gtrless t'\}|)_{t\in T_i}$. In addition, let $S_i = |T_i|\times(t_{i,i+1}+1)\times(t_{i,i+2}+1)\times\cdots\times(t_{i,86}+1)$. Then S_i is an upper bound of the number of antichains that do not include any elements in T_k $(k < i)$ and include an element in T_i. Therefore, $|\mathbb{G}_4| \le \left(\sum_{i=1}^{86} S_i + 1\right)^2$.

By calculating this, we obtained the following corollary. The values of S_i are shown in Table 1. Here, $(|T_8|+1)\times(|T_9|+1)\times\cdots\times(|T_{86}|+1) < 1.0\cdot 10^{91} < S_2$. Since $\sum_{i=k}^{86} S_i \le (|T_k|+1)\times(|T_{k+1}|+1)\times\cdots\times(|T_{86}|+1)$, we have $|\mathbb{G}_4| \le \left(\sum_{i=1}^{7} S_i + 10^{91}\right)^2 < 4.0\cdot 10^{184}$.

Corollary 20. $$|\mathbb{G}_4| < 4.0\cdot 10^{184}.$$

Therefore, from Corollaries 16 and 20, we obtain the following theorem, significantly improving previously known upper and lower bounds.

S_1	S_2	S_3	S_4	S_5	S_6	S_7
$4.0 \cdot 10^{90}$	$1.8 \cdot 10^{92}$	$1.5 \cdot 10^{89}$	$1.5 \cdot 10^{87}$	$5.5 \cdot 10^{86}$	$3.0 \cdot 10^{87}$	$3.0 \cdot 10^{83}$

Table 1. Bounds on values of S_i.

Theorem 21. $$10^{28.2} < |\mathbb{G}_4| < 4.0 \cdot 10^{184}.$$

5. Conclusion

In this study, we significantly improved the upper and lower bounds of the number of games born by day 4. We obtained this result by using some algebraic properties of combinatorial game theory. As the improvement of upper and lower bounds has long remained unsolved, this result is an important development for all aspects of the game as an algebraic object and aspects analyzing the game itself. However, there remains a gap in the width of the upper and lower bounds, and we will continue trying to improve them. Specifically, we improved the upper and lower bounds of $|\mathbb{G}_4|$ by calculating the length of chains, the number of chains, and the lower bounds of the number of antichains in $\mathbb{G}_3$. As this method can be generalized, by calculating the length of chains, the number of chains, and the lower bounds of the number of antichains in $\mathbb{G}_n$, we also try improving the upper and lower bounds of $|\mathbb{G}_{n+1}|$. The methods applied on $\mathbb{G}_3$ could also be applied on $\mathbb{G}_2$, using the divisions of $\mathbb{G}_2$ from Figures 1 and 4, thus yielding bounds for $\mathbb{G}_3$. Doing this, we get $208 \leq |\mathbb{G}_3| \leq 451584$. We can see that the lower bound is much closer to the true value $|\mathbb{G}_3| = 1474$, so it is likely that our lower bound on $|\mathbb{G}_4|$ is also closer to the true value than our upper bound. This research is also related to counting and algorithms, and we will continue to contribute to combinatorial game theory and related fields by improving our methods and conducting research on applications of the results obtained.

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Appendix: $\mathbb{G}_3$ and its stratification

Tables 2, 3, and 4 show every element in $\mathbb{G}_3$ and its stratification. Because $U_{24}, \ldots, U_{45}$ can be constructed by using upper and lower symmetry, we omit them. The i-th element from the left in each row belongs T_i in the chain division we used. An Excel file integrating these tables is available here: https://sites.google.com/site/kokisuetsugu2/games-born-by-day-3.

U_1	3
U_2	$2, 2*$
U_3	$\frac{3}{2}, \{2\mid 1\}, \{2\mid 1*\}$
U_4	$1\uparrow, 1\uparrow *, \{2\mid 1, 1*\}, \{2\mid\mid 1\mid *\}, \{2\mid\mid 1\mid 0\}$
U_5	$1*2, 1*, \{2\mid \frac{1}{2}\}, 1+\text{Tiny}_{1*}, 1+\text{Tiny}_{1}, \{2\mid 1, \{1\mid *\}\}, \{2\mid 1, \{1\mid 0\}\},$ $\{2\mid \{1\mid 0\}, \{1\mid *\}\}$
U_6	$\{1, 1*\mid \frac{1}{2}\}, 1\downarrow *, \{2\mid \frac{1}{2}, \{1\mid *\}\}, \{1, 1*\mid 1, \{1\mid *\}\}, \{1, 1*\mid 1, \{1\mid 0\}\},$ $\{2\mid \frac{1}{2}, \{1\mid *\}\}, \{2\mid \frac{1}{2}, \{1\mid 0\}\}, \{2\mid 1, \{1\mid 0\}, \{1\mid *\}\}, \{2\mid\mid 1\mid 0, *\}$
U_7	$\{1, 1*\mid \frac{1}{2}, \{1\mid *\}\}, \{1\mid \frac{1}{2}\}, \{2\mid \frac{1}{2}, \{1\mid 0\}, \{1\mid *\}\}, \{1\mid 1, \{1\mid *\}\},$ $\{1\mid 1, \{1\mid 0\}\}, \{2\mid \uparrow\}, \{1, 1*\mid \frac{1}{2}, \{1\mid 0\}\}, \{1, 1*\mid 1, \{1\mid 0\}, \{1\mid *\}\},$ $\{1\mid\mid 1\mid 0, *\}, \{2\mid \uparrow *\}, \{2\mid 1, \{1\mid 0, *\}\}, \{2\mid \pm 1\}$
U_8	$\{1, 1*\mid \uparrow\}, \{1\mid \frac{1}{2}, \{1\mid *\}\}, \{2\mid \uparrow, \{1\mid 0\}\}, \{1\mid 1, \{1\mid 0\}, \{1\mid *\}\}, \{1\mid \frac{1}{2}, \{1\mid 0\}\},$ $\{2\mid 0\}, \{1, 1*\mid \uparrow *\}, \{1, 1*\mid \frac{1}{2}, \{1\mid 0\}, \{1\mid *\}\}, \{1, 1*\mid 1, \{1\mid 0, *\}\},$ $\{2\mid \uparrow *, \{1\mid *\}\}, \{2\mid \frac{1}{2}, \{1\mid 0, *\}\}, 1+\text{Tiny}_{2}, \{2\mid 1, \pm 1\}, \{2\mid *\}$
U_9	$\{1, 1*\mid 0\}, \{1\mid \uparrow\}, \{2\mid \uparrow, \{1\mid 0, *\}\}, \{1\mid \frac{1}{2}, \{1\mid 0\}, \{1\mid *\}\}, \{1\mid \uparrow *\},$ $\{2\mid 0, \{1\mid 0\}\}, \{1, 1*\mid \uparrow *, \{1\mid *\}\}, \{1, 1*\mid \uparrow, \{1\mid 0\}\}, \{1\mid 1, \{1\mid 0, *\}\},$ $\{2\mid \uparrow, \uparrow *\}, \{1, 1*\mid \frac{1}{2}, \{1\mid 0, *\}\}, 1, \{1, 1*\mid 1, \pm 1\}, \{1, 1*\mid *\},$ $\{2\mid \frac{1}{2}, \pm 1\}, \{2\mid \uparrow *, \{1\mid 0, *\}\}, \{2\mid *, \{1\mid *\}\}$
U_{10}	$\{1, 1*\mid 0, \{1\mid 0\}\}, \{1, \{1\mid 0\}\mid 0\}, \{2\mid 0, \{1\mid 0, *\}\}, \{1\mid \frac{1}{2}, \{1\mid 0, *\}\},$ $\{1\mid \uparrow *, \{1\mid *\}\}, \{2\mid 0, \uparrow *\}, \{1, 1*\mid \uparrow, \uparrow *\}, \{1\mid \uparrow, \{1\mid 0\}\}, \{1\mid 1, \pm 1\},$ $\{2\mid \uparrow, \uparrow *, \{1\mid 0, *\}\}, \{1, 1*\mid \uparrow, \{1\mid 0, *\}\}, 1\downarrow, \{1, 1*\mid \frac{1}{2}, \pm 1\},$ $\{1, \{1\mid *\}\mid *\}, \{2\mid \uparrow, \pm 1\}, \{1, 1*\mid \uparrow *, \{1\mid 0, *\}\}, \{1, 1*\mid *, \{1\mid *\}\},$ $\{2\mid *, \uparrow\}, \{2\mid \uparrow *, \pm 1\}, \{2\mid *, \{1\mid 0, *\}\}$
U_{11}	$\{1, \{1\mid 0\}\mid 0, \{1\mid 0\}\}, \{1\mid 0\}, \{2\mid 0, \uparrow *, \{1\mid 0, *\}\}, \{1\mid \frac{1}{2}, \pm 1\}, \{1\mid \uparrow, \uparrow *\},$ $\{1, 1*\mid 0, \uparrow *\}, \{1, 1*\mid \uparrow, \uparrow *, \{1\mid 0, *\}\}, \{1\mid \uparrow, \{1\mid 0, *\}\}, \frac{3}{4},$ $\{2\mid \uparrow, \uparrow *, \pm 1\}, \{1, 1*\mid 0, \{1\mid 0, *\}\}, \{1*\mid \frac{1}{2}\}, \{1, 1*\mid \uparrow, \pm 1\}, \{1\mid *\},$ $\{2\mid 0, \pm 1\}, \{1\mid \uparrow *, \{1\mid 0, *\}\}, \{1, \{1\mid *\}\mid *, \{1\mid *\}\}, \{1, 1*\mid *, \uparrow\},$ $\{1, 1*\mid \uparrow *, \pm 1\}, \{1, 1*\mid *, \{1\mid 0, *\}\}, \{2\mid 0, *\}, \{2\mid *, \uparrow, \{1\mid 0, *\}\},$ $\{2\mid *2\}, \{2\mid *, \pm 1\}$
U_{12}	$\{1, \{1\mid 0\}\mid 0, \{1\mid 0, *\}\}, \{1\mid 0, \{1\mid 0\}\}, \{2\mid 0, *2\}, \frac{1}{2}*, \{1\mid \uparrow, \uparrow *, \{1\mid 0, *\}\},$ $\{1, \{1\mid 0\}\mid 0, \uparrow *\}, \{1, 1*\mid 0, \uparrow *, \{1\mid 0, *\}\}, \{1\mid \uparrow, \pm 1\}, \frac{1}{2}, \{2\mid *, \uparrow, \pm 1\},$ $\{1, 1*\mid 0, \pm 1\}, \{1*\mid \uparrow *\}, \{1*\mid \uparrow\}, \{1\mid *, \{1\mid *\}\}, \{2\mid 0, \uparrow *, \pm 1\},$ $\{1\mid \uparrow *, \pm 1\}, \{1, \{1\mid *\}\mid *, \uparrow\}, \{1, 1*\mid 0, *\}, \{1, 1*\mid \uparrow, \uparrow *, \pm 1\},$ $\{1, \{1\mid *\}\mid *, \{1\mid 0, *\}\}, \{2\mid 0, *, \{1\mid 0, *\}\}, \{1, 1*\mid *, \uparrow, \{1\mid 0, *\}\},$ $\{1, 1*\mid *2\}, \{1, 1*\mid *, \pm 1\}, \{2\mid \pm 1, *2\}, \{2\mid *, *2\}$
U_{13}	$\{1\mid 0, \{1\mid 0, *\}\}, \{1\mid 0, \uparrow *\}, \{2\mid 0, *, *2\}, \{\frac{1}{2}\mid \uparrow\}, \{1\mid *2\}, \{1, \{1\mid 0\}, \{1\mid *\}\mid 0, *\},$ $\{1, \{1\mid 0\}\mid 0, \uparrow *, \{1\mid 0, *\}\}, \{1\mid \uparrow, \uparrow *, \pm 1\}, \frac{1}{4}, \{2\mid *, \pm 1, *2\}, \{1, \{1\mid 0\}\mid 0, \pm 1\},$ $\{1*\mid \uparrow, \uparrow *\}, \{1*\mid 0\}, \{1\mid *, \uparrow\}, \{2\mid 0, \pm 1, *2\}, \{\frac{1}{2}\mid \uparrow *\},$ $\{1, \{1\mid *\}\mid *, \uparrow, \{1\mid 0, *\}\}, \{1, 1*\mid 0, *, \{1\mid 0, *\}\}, \{1, 1*\mid 0, \uparrow *, \pm 1\},$ $\{1\mid *, \{1\mid 0, *\}\}, \{2\mid 0, *, \pm 1\}, \{1, 1*\mid *, \uparrow, \pm 1\}, \{1, 1*\mid 0, *2\}, \{1, \{1\mid *\}\mid *, \pm 1\},$ $\{1, 1*\mid \pm 1, *2\}, \{1, 1*\mid *, *2\}, \{2\mid \downarrow\}, \{1*\mid *\}, \{2\mid\mid 0, *\mid -1\}, \{2\mid \downarrow *\}$

Table 2. Elements in U_i $(1 \le i \le 13)$.

U_{14}	$\{1\mid 0,\pm 1\}, \{1\mid 0,\uparrow *,\{1\mid 0,*\}\}, \{2\mid 0,\downarrow *\}, \{\frac{1}{2}\mid \uparrow,\uparrow *\}, \{1,\{1\mid *\}\mid *,*2\}, \{1,\{1\mid *\}\mid 0,*\}, \{1,\{1\mid 0\}\mid 0,*2\}, \{1\mid \pm 1,*2\}, \Uparrow *, \{2\mid *,\{0,*\mid -1\}\},$ $\{\frac{1}{2},\{1\mid 0\}\mid 0\}, \{1*\mid *,\uparrow\}, \{1*\mid 0,\uparrow *\}, \{1,\{1\mid 0\}\mid 0,*\}, \{2\mid 0,\{0,*\mid -1\}\}, \Uparrow, \{1,\{1\mid *\}\mid *,\uparrow,\pm 1\}, \{1,\{1\mid 0\},\{1\mid *\}\mid 0,*,\{1\mid 0,*\}\},$ $\{1,\{1\mid 0\}\mid 0,\uparrow *,\pm 1\}, \{1\mid *,\uparrow,\{1\mid 0,*\}\}, \{2\mid 0,*,\pm 1,*2\}, \{1,1*\mid 0,*,\pm 1\}, \{1,1*\mid 0,\pm 1,*2\}, \{1\mid *,\pm 1\}, \{1,1*\mid *,\pm 1,*2\}, \{1,1*\mid 0,*,*2\},$ $\{1,1*\mid \downarrow\}, \{\frac{1}{2},\{1\mid *\}\mid *\}, \{1,1*\Vert 0,*\mid -1\}, \{1,1*\mid \downarrow *\}, \{2\mid \downarrow,\pm 1\}, \{2\mid *,\downarrow\}, \{1*\mid *2\}, \{2\mid \downarrow *,\pm 1\}$
U_{15}	$\{\frac{1}{2}\mid 0\}, \{1\mid 0,\uparrow *,\pm 1\}, \{1,1*\mid 0,\downarrow *\}, \{\frac{1}{2},\{1\mid *\}\mid *,\uparrow\}, \{1,\{1\mid *\}\mid *,\pm 1,*2\}, \{1,\{1\mid 0,*\}\mid 0,*\}, \{1\mid 0,*2\}, \{1\Vert 0,*\mid -1\}, \{0\mid \uparrow,\uparrow *\},$ $\{2\mid \downarrow *,\{0,*\mid -1\}\}, \{0,\{1\mid 0\}\mid 0\}, \{1*\mid *,*2\}, \{1*\mid 0,*2\}, \{1,\{1\mid 0\}\mid 0,*,\{1\mid 0,*\}\}, \{2\mid 0,*,\{0,*\mid -1\}\}, \{1\mid *\Vert *\}, \{1,\{1\mid 0\},\{1\mid *\}\mid 0,*,\pm 1\},$ $\{1,\{1\mid *\}\mid 0,*,\{1\mid 0,*\}\}, \{\frac{1}{2},\{1\mid 0\}\mid 0,\uparrow *\}, \{1\mid *,*2\}, \{1,1*\mid 0,*,\pm 1,*2\}, \{1*\mid 0,*\}, \{1,\{1\mid 0\}\mid 0,\pm 1,*2\}, \{1\mid *,\uparrow,\pm 1\}, \{1,1*\mid *,\{0,*\mid -1\}\},$ $\{1,\{1\mid 0\},\{1\mid *\}\mid 0,*,*2\}, \{1,\{1\mid 0\}\mid \downarrow\}, \{\frac{1}{2}\mid *\}, \{1,1*\mid 0,\{0,*\mid -1\}\}, \{1,\{1\mid *\}\mid \downarrow *\}, \{1,1*\mid \downarrow,\pm 1\}, \{1,1*\mid *,\downarrow\}, \{\frac{1}{2}\mid *2\},$ $\{1,1*\mid \downarrow *,\pm 1\}, \{2\mid 0,\downarrow *,\pm 1\}, \{2\mid \downarrow,\{0,*\mid -1\}\}, \{2\mid *,\downarrow,\pm 1\}, \{2\mid \downarrow,\downarrow *\}, \{1*\Vert 0,*\mid -1\}$
U_{16}	$\{\frac{1}{2}\mid 0,\uparrow *\}, \{1\mid 0,\pm 1,*2\}, \{1,\{1\mid 0\},\{1\mid *\}\mid 0,\downarrow *\}, \{\frac{1}{2}\mid *,\uparrow\}, \{\frac{1}{2},\{1\mid *\}\mid *,*2\}, \{1\mid 0,*\}, \{1\mid \downarrow\}, \{1,\{1\mid *\}\mid *,\{0,*\mid -1\}\}, \{0,\{1\mid 0\}\mid 0,\uparrow *\},$ $\{2\Vert 0\mid -1\}, \uparrow^{[2]} +*, \{1*\mid *,\{0,*\mid -1\}\}, \{1*\mid 0,\{0,*\mid -1\}\}, \{1,\{1\mid 0,*\}\mid 0,*,\{1\mid 0,*\}\}, \{1,1*\mid 0,*,\{0,*\mid -1\}\}, \{1\mid *\Vert *,\uparrow\},$ $\{1,\{1\mid 0\}\mid 0,*,\pm 1\}, \{1,\{1\mid *\}\mid 0,*,\pm 1\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid 0,*\}, \{1\mid \downarrow *\}, \{1,\{1\mid 0\},\{1\mid *\}\mid 0,*,\pm 1,*2\}, \{1*\mid 0,*,*2\}, \{\frac{1}{2},\{1\mid 0\}\mid 0,*2\},$ $\{1\mid *,\pm 1,*2\}, \{1,1*\mid \downarrow *,\{0,*\mid -1\}\}, \{1,\{1\mid *\}\mid 0,*,*2\}, \{1,\{1\mid 0\}\mid \downarrow,\pm 1\}, \uparrow^{[2]}, \{1,\{1\mid 0\}\mid 0,\{0,*\mid -1\}\}, \{1,\{1\mid *\}\mid \downarrow *,\pm 1\},$ $\{1,1*\mid \downarrow,\{0,*\mid -1\}\}, \{1,\{1\mid 0\},\{1\mid *\}\mid *,\downarrow\}, \uparrow *3, \{1,1*\mid 0,\downarrow *,\pm 1\}, \{2\mid 0,\downarrow *,\{0,*\mid -1\}\}, \{2\mid *,\downarrow,\{0,*\mid -1\}\}, \{1,1*\mid *,\downarrow,\pm 1\},$ $\{1,1*\mid \downarrow,\downarrow *\}, \{\frac{1}{2}\Vert 0,*\mid -1\}, \{1*\mid \downarrow\}, \{1*\mid \downarrow *\}, \{2\mid \downarrow,\downarrow *,\pm 1\}, \{2\Vert *\mid -1\}, \{2\mid -\frac{1}{2}\}, \{1,\{1\mid 0\}\mid 0,*,*2\}$
U_{17}	$\{0,\uparrow *\mid 0,\uparrow *\}, \{\frac{1}{2}\mid 0,*2\}, \{1,\{1\mid 0\},\{1\mid *\}\mid 0,\downarrow *,\pm 1\}, \{\uparrow\mid *,\uparrow\}, \{\frac{1}{2},\{1\mid *\}\mid \downarrow *\}, \{1\mid 0,*,\{1\mid 0,*\}\}, \{1\mid \downarrow,\pm 1\}, \{1\mid *,\{0,*\mid -1\}\},$ $\{\{1\mid 0\},\{1\mid *\}\mid 0,*\}, \{1,1*\Vert 0\mid -1\}, \uparrow *, \{1*\mid 0,*,\{0,*\mid -1\}\}, \{\frac{1}{2},\{1\mid 0\}\mid 0,\{0,*\mid -1\}\}, \{1,\{1\mid 0,*\}\mid 0,*,\pm 1\},$ $\{1,\{1\mid 0\},\{1\mid *\}\mid 0,*,\{0,*\mid -1\}\}, \{1\mid *\Vert *,*2\}, \{\frac{1}{2},\{1\mid 0\}\mid 0,*\}, \{1,\{1\mid *\}\mid 0,*,\pm 1,*2\}, \{\frac{1}{2},\{1\mid *\}\mid 0,*\}, \{1,\{1\mid 0\}\mid 0,\downarrow *\},$ $\{1,\{1\mid 0\},\{1\mid *\}\mid *,\downarrow,\pm 1\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid 0,*,*2\}, \{0,\{1\mid 0\}\mid 0,*2\}, \{\frac{1}{2}\mid *,*2\}, \{1,\{1\mid *\}\mid \downarrow *,\{0,*\mid -1\}\}, \{1,\{1\mid *\}\mid 0,\downarrow *\}, \{\frac{1}{2},\{1\mid 0\}\mid \downarrow\},$ $\uparrow, \{1\mid 0,\{0,*\mid -1\}\}, \{1\mid \downarrow *,\pm 1\}, \{1,\{1\mid 0\}\mid \downarrow,\{0,*\mid -1\}\}, \{1,\{1\mid *\}\mid *,\downarrow\}, \{0\Vert 0,*\mid -1\}, \{1,1*\mid 0,\downarrow *,\{0,*\mid -1\}\}, \{2\mid 0,\{0\mid -1\}\},$ $\{2\mid \downarrow,\downarrow *,\{0,*\mid -1\}\}, \{1,1*\mid *,\downarrow,\{0,*\mid -1\}\}, \{1,\{1\mid 0\},\{1\mid *\}\mid \downarrow,\downarrow *\}, \{\frac{1}{2},\{1\mid *\}\mid *,\{0,*\mid -1\}\}, \{1*\mid \downarrow,\{0,*\mid -1\}\}, \{1*\mid 0,\downarrow *\},$ $\{1,1*\mid \downarrow,\downarrow *,\pm 1\}, \{1,1*\Vert *\mid -1\}, \{1,1*\mid -\frac{1}{2}\}, \{1,\{1\mid 0,*\}\mid 0,*,*2\}, \{1*\mid *,\downarrow\}, \{1*\mid \downarrow *,\{0,*\mid -1\}\}, \{1,\{1\mid 0\}\mid *,\downarrow\},$ $\{1,\{1\mid 0\}\mid 0,*,\pm 1,*2\}, \{2\mid *,\{*\mid -1\}\}, \{2\mid \pm 1,-\frac{1}{2}\}, \{2\mid -1\}$
U_{18}	$\{\uparrow *,\{1\mid *\}\mid 0,*\}, \{0,\uparrow *\mid 0,*2\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid 0,\downarrow *\}, \{\uparrow,\{1\mid 0\}\mid 0,*\}, \{1\mid *\Vert \downarrow *\}, \{1\mid 0,*,*2\}, \{\frac{1}{2}\mid \downarrow\}, \{1,\{1\mid 0\}\mid 0,*,\{0,*\mid -1\}\},$ $\{\{1\mid 0\},\{1\mid *\}\mid 0,*,*2\}, \{1,\{1\mid *\}\Vert 0\mid -1\}, \{0,*\mid 0,\uparrow *\}, \{1*\mid *,\downarrow,\{0,*\mid -1\}\}, \{\frac{1}{2}\mid 0,\{0,*\mid -1\}\}, \{1\mid 0,*,\pm 1\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid 0,*,\{0,*\mid -1\}\},$ $\{\uparrow\mid *,*2\}, \{\frac{1}{2},\{1\mid 0\}\mid 0,*,*2\}, \{1,\{1\mid *\}\mid 0,*,\{0,*\mid -1\}\}, \{\frac{1}{2},\{1\mid 0,*\}\mid 0,*\}, \{1,\{1\mid 0\}\mid 0,\downarrow *,\pm 1\}, \{1,\{1\mid *\}\mid *,\downarrow,\pm 1\}, \{\frac{1}{2},\{1\mid *\}\mid 0,*,*2\},$ $\{0,\{1\mid 0\}\mid 0,\{0,*\mid -1\}\}, \{\frac{1}{2}\mid *,\{0,*\mid -1\}\}, \{1\mid \downarrow *,\{0,*\mid -1\}\}, \{1,\{1\mid *\}\mid 0,\downarrow *,\pm 1\}, \{0,\{1\mid 0\}\mid \downarrow\}, \{0\mid *,\uparrow\}, \{1\mid \downarrow,\{0,*\mid -1\}\},$ $\{\frac{1}{2}\mid \downarrow *\}, \{\frac{1}{2},\{1\mid 0\}\mid \downarrow,\{0,*\mid -1\}\}, \{1,\{1\mid *\}\mid \downarrow,\downarrow *\}, \{1\mid *\Vert *,\{0,*\mid -1\}\}, \{1,\{1\mid 0\},\{1\mid *\}\mid 0,\downarrow *,\{0,*\mid -1\}\}, \{1,1*\mid 0,\{0\mid -1\}\},$ $\{2\mid \downarrow,\{0\mid -1\}\}, \{1,\{1\mid 0\},\{1\mid *\}\mid *,\downarrow,\{0,*\mid -1\}\}, \{1,\{1\mid 0\},\{1\mid *\}\mid \downarrow,\downarrow *,\pm 1\}, \{\frac{1}{2},\{1\mid *\}\mid \downarrow *,\{0,*\mid -1\}\}, \{1*\Vert *\mid -1\},$ $\{1*\mid \downarrow,\downarrow *\}, \{1,1*\mid \downarrow,\downarrow *,\{0,*\mid -1\}\}, \{1,\{1\mid 0\}\Vert *\mid -1\}, \{1,\{1\mid 0\},\{1\mid *\}\mid -\frac{1}{2}\}, \{1,\{1\mid 0,*\}\mid 0,\downarrow *\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid *,\downarrow\},$ $\{1*\mid 0,\downarrow *,\{0,*\mid -1\}\}, \{1,\{1\mid 0,*\}\mid *,\downarrow\}, \{1,\{1\mid 0,*\}\mid 0,*,\pm 1,*2\}, \{1,1*\mid *,\{*\mid -1\}\}, \{1,1*\mid \pm 1,-\frac{1}{2}\}, \{1,1*\mid -1\}, \{1,\{1\mid 0\}\mid *,\downarrow,\pm 1\},$ $\{1*\Vert 0\mid -1\}, \{1,\{1\mid 0\}\mid \downarrow,\downarrow *\}, \{2\mid \downarrow *,\{*\mid -1\}\}, \{2\mid -\frac{1}{2},\{0,*\mid -1\}\}, \{2\mid -1,\pm 1\}$
U_{19}	$\{\uparrow *,\{1\mid *\}\mid 0,*,*2\}, \{0,*\mid 0,*2\}, \{\frac{1}{2},\{1\mid *\}\mid 0,\downarrow *\}, \{\uparrow,\uparrow *,\{1\mid 0,*\}\mid 0,*\}, \{\{1\mid 0\},\{1\mid *\}\mid 0,\downarrow *\}, \{1\mid *,\downarrow\}, \{0,\uparrow *\mid \downarrow\}, \{1,\{1\mid 0\}\mid 0,\downarrow *,\{0,*\mid -1\}\},$ $\{\uparrow,\{1\mid 0\}\mid 0,*,*2\}, \{1\Vert 0\mid -1\}, \{*,\{1\mid *\}\mid 0,*\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid *,\downarrow,\{0,*\mid -1\}\}, \{0,\uparrow *\mid 0,\{0,*\mid -1\}\}, \{1\mid 0,*,\pm 1,*2\},$ $\{\frac{1}{2},\{1\mid 0\}\mid 0,*,\{0,*\mid -1\}\}, \{0\mid *,*2\}, \{\frac{1}{2},\{1\mid 0,*\}\mid 0,*,*2\}, \{1,\{1\mid 0,*\}\mid 0,*,\{0,*\mid -1\}\}, \{\frac{1}{2}\mid 0,*\}, \{\frac{1}{2},\{1\mid 0\}\mid 0,\downarrow *\}, \{\frac{1}{2},\{1\mid *\}\mid *,\downarrow\},$ $\{\frac{1}{2},\{1\mid *\}\mid 0,*,\{0,*\mid -1\}\}, \{0,\{1\mid 0\}\mid \downarrow,\{0,*\mid -1\}\}, \{\uparrow\mid *,\{0,*\mid -1\}\}, \{\frac{1}{2}\mid \downarrow *,\{0,*\mid -1\}\}, \{1,\{1\mid *\}\mid 0,\downarrow *,\{0,*\mid -1\}\},$ $\{\{1\mid 0\},\{1\mid *\}\mid *,\downarrow\}, \{0,\{1\mid 0\}\mid 0,*\}, \{1\Vert *\mid -1\}, \{\uparrow\mid \downarrow *\}, \{\frac{1}{2}\mid \downarrow,\{0,*\mid -1\}\}, \{1,\{1\mid *\}\mid \downarrow,\downarrow *,\pm 1\}, \{\{1\mid 0\},\{1\mid *\}\mid 0,*,\{0,*\mid -1\}\},$ $\{1,\{1\mid 0\},\{1\mid *\}\mid 0,\{0\mid -1\}\}, \{1,1*\mid \downarrow,\{0\mid -1\}\}, \{2\mid \{0\mid -1\},\{*\mid -1\}\}, \{1,\{1\mid *\}\mid *,\downarrow,\{0,*\mid -1\}\}, \{1,\{1\mid 0\}\mid \downarrow,\downarrow *,\pm 1\},$ $\{1\mid *\Vert \downarrow *,\{0,*\mid -1\}\}, \{1*\mid *,\{*\mid -1\}\}, \{1*\mid \downarrow,\downarrow *,\{0,*\mid -1\}\}, \{1,\{1\mid 0\},\{1\mid *\}\mid \downarrow,\downarrow *,\{0,*\mid -1\}\}, \{\frac{1}{2},\{1\mid 0\}\Vert *\mid -1\}, \{1,\{1\mid *\}\mid -\frac{1}{2}\},$ $\{1\mid 0,\downarrow *\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid \downarrow,\downarrow *\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid 0,\downarrow *,\{0,*\mid -1\}\}, \{1,\{1\mid 0,*\}\mid *,\downarrow,\pm 1\}, \{1,\{1\mid 0,*\}\mid 0,\downarrow *,\pm 1\},$ $\{1,\{1\mid 0\},\{1\mid *\}\mid *,\{*\mid -1\}\}, \{1,\{1\mid 0\},\{1\mid *\}\mid \pm 1,-\frac{1}{2}\}, \{1,\{1\mid 0\},\{1\mid *\}\mid -1\}, \{\frac{1}{2},\{1\mid 0\}\mid *,\downarrow\}, \{\frac{1}{2},\{1\mid *\}\Vert 0\mid -1\}, \{1,\{1\mid 0,*\}\mid \downarrow,\downarrow *\},$ $\{1,1*\mid \downarrow *,\{*\mid -1\}\}, \{1,1*\mid -\frac{1}{2},\{0,*\mid -1\}\}, \{1,1*\mid -1,\pm 1\}, \{1*\mid 0,\{0\mid -1\}\}, \{1,\{1\mid 0\}\mid -\frac{1}{2}\}, \{1*\mid -\frac{1}{2}\}, \{2\mid -\frac{1}{2},\{*\mid -1\}\},$ $\{1,\{1\mid 0\}\mid *,\downarrow,\{0,*\mid -1\}\}, \{2\mid -1,\{0,*\mid -1\}\}, \{2\mid \{0\mid -1\},-\frac{1}{2}\}$
U_{20}	$\{\uparrow *,\{1\mid *\}\mid 0,*,\{0,*\mid -1\}\}, \{0,*\mid \downarrow\}, \{\frac{1}{2},\{1\mid *\}\mid 0,\downarrow *,\{0,*\mid -1\}\}, \{\uparrow,\uparrow *\mid 0,*\}, \{\uparrow *,\{1\mid *\}\mid 0,\downarrow *\}, \{1\mid \downarrow,\downarrow *\}, \{\uparrow *,\{1\mid *\}\mid *,\downarrow\},$ $\{1,\{1\mid 0\}\mid \downarrow,\downarrow *,\{0,*\mid -1\}\}, \{0,\{1\mid 0\}\mid 0,*,*2\}, \{1,\{1\mid 0\}\mid 0,\{0\mid -1\}\}, \{*,\uparrow,\{1\mid 0,*\}\mid 0,*\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid *,\{*\mid -1\}\}, \{0,*\mid 0,\{0,*\mid -1\}\},$ $\{1\mid 0,*,\{0,*\mid -1\}\}, \{\uparrow,\{1\mid 0\}\mid 0,*,\{0,*\mid -1\}\}, \uparrow^2, \{\uparrow,\uparrow *,\{1\mid 0,*\}\mid 0,*,*2\}, \{\frac{1}{2},\{1\mid 0,*\}\mid 0,*,\{0,*\mid -1\}\}, \{\frac{1}{2}\mid 0,*,*2\}, \{\frac{1}{2},\{1\mid 0,*\}\mid 0,\downarrow *\},$ $\{\frac{1}{2},\{1\mid 0,*\}\mid *,\downarrow\}, \{\frac{1}{2},\{1\mid *\}\mid *,\downarrow,\{0,*\mid -1\}\}, \{\{1\mid 0\},\{1\mid *\}\mid *,\downarrow,\{0,*\mid -1\}\}, \{0\mid *,\{0,*\mid -1\}\}, \{\uparrow\mid \downarrow *,\{0,*\mid -1\}\}, \{1,\{1\mid *\}\mid 0,\{0\mid -1\}\},$ $\{\uparrow,\{1\mid 0\}\mid *,\downarrow\}, \{0,\uparrow *,\{1\mid 0,*\}\mid 0,*\}, \{\frac{1}{2}\Vert *\mid -1\}, \{\uparrow,\{1\mid 0\}\mid 0,\downarrow *\}, \{0,\uparrow *\mid \downarrow,\{0,*\mid -1\}\}, \{1,\{1\mid 0,*\}\mid \downarrow,\downarrow *,\pm 1\},$ $\{\{1\mid 0\},\{1\mid *\}\mid 0,\downarrow *,\{0,*\mid -1\}\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid 0,\{0\mid -1\}\}, \{1,1*\mid \{0\mid -1\},\{*\mid -1\}\}, \{2\mid \{0\mid -1\},-\frac{1}{2},\{*\mid -1\}\},$ $\{1,\{1\mid *\}\mid \downarrow,\downarrow *,\{0,*\mid -1\}\}, \{\frac{1}{2},\{1\mid 0\}\mid \downarrow,\downarrow *\}, \{1\mid *\Vert 0\mid -1\}, \{1*\mid \downarrow *,\{*\mid -1\}\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid \downarrow,\downarrow *,\{0,*\mid -1\}\},$ $\{1,\{1\mid 0\},\{1\mid *\}\mid \downarrow,\{0\mid -1\}\}, \{0,\{1\mid 0\}\Vert *\mid -1\}, \{1,\{1\mid *\}\mid \pm 1,-\frac{1}{2}\}, \{1\mid 0,\downarrow *,\pm 1\}, \{\frac{1}{2},\{1\mid *\}\mid \downarrow,\downarrow *\}, \{\frac{1}{2},\{1\mid 0\}\mid 0,\downarrow *,\{0,*\mid -1\}\},$ $\{1\mid *,\downarrow,\pm 1\}, \{1,\{1\mid 0,*\}\mid 0,\downarrow *,\{0,*\mid -1\}\}, \{1,\{1\mid *\}\mid *,\{*\mid -1\}\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid -\frac{1}{2}\}, \{1,\{1\mid *\}\mid -1\}, \{\frac{1}{2},\{1\mid 0\}\mid *,\downarrow,\{0,*\mid -1\}\},$ $\{\frac{1}{2}\Vert 0\mid -1\}, \{1,\{1\mid 0,*\}\mid -\frac{1}{2}\}, \{1,\{1\mid 0\},\{1\mid *\}\mid \downarrow *,\{*\mid -1\}\}, \{1,\{1\mid 0\},\{1\mid *\}\mid -\frac{1}{2},\{0,*\mid -1\}\}, \{1,\{1\mid 0\},\{1\mid *\}\mid -1,\pm 1\}, \{1*\mid \downarrow,\{0\mid -1\}\},$ $\{1,\{1\mid 0\}\mid -1\}, \{1*\mid -\frac{1}{2},\{0,*\mid -1\}\}, \{1,1*\mid -\frac{1}{2},\{*\mid -1\}\}, \{1,\{1\mid 0,*\}\mid *,\downarrow,\{0,*\mid -1\}\}, \{1,1*\mid -1,\{0,*\mid -1\}\}, \{1,1*\mid \{0\mid -1\},-\frac{1}{2}\},$ $\{*,\{1\mid *\}\mid 0,*,*2\}, \{\{1\mid 0\},\{1\mid *\}\mid \downarrow,\downarrow *\}, \{1,\{1\mid 0\}\mid \pm 1,-\frac{1}{2}\}, \{1*\mid -1\}, \{1,\{1\mid 0\}\mid *,\{*\mid -1\}\}, \{2\mid -1,\{*\mid -1\}\}, \{2\mid -1,\{0\mid -1\}\}$
U_{21}	$\{\uparrow *,\{1\mid *\}\mid *,\downarrow,\{0,*\mid -1\}\}, \{*,\{1\mid *\}\mid *,\downarrow\}, \{\uparrow *,\{1\mid *\}\mid 0,\downarrow *,\{0,*\mid -1\}\}, \{\uparrow,\uparrow *\mid 0,*,*2\}, \{*,\{1\mid *\}\mid 0,\downarrow *\}, \{1\mid -\frac{1}{2}\}, \{\uparrow,\uparrow *,\{1\mid 0,*\}\mid *,\downarrow\},$ $\{\frac{1}{2},\{1\mid 0\}\mid \downarrow,\downarrow *,\{0,*\mid -1\}\}, \{0,\{1\mid 0\}\mid 0,*,\{0,*\mid -1\}\}, \{1,\{1\mid 0\}\mid \downarrow,\{0\mid -1\}\}, \{0,*,\{1\mid 0,*\}\mid 0,*\}, \{\frac{1}{2},\{1\mid 0\}\mid *,\{*\mid -1\}\},$ $\{*,\{1\mid *\}\mid 0,*,\{0,*\mid -1\}\}, \{1\mid 0,\downarrow *,\{0,*\mid -1\}\}, \{\uparrow,\{1\mid 0\}\mid 0,\downarrow *,\{0,*\mid -1\}\}, \{0,\{1\mid 0\}\mid 0,\downarrow *\}, \{0,\uparrow *,\{1\mid 0,*\}\mid 0,*,*2\},$ $\{\uparrow,\uparrow *,\{1\mid 0,*\}\mid 0,*,\{0,*\mid -1\}\}, \{\frac{1}{2}\mid 0,*,\{0,*\mid -1\}\}, \{\frac{1}{2}\mid 0,\downarrow *\}, \{\frac{1}{2}\mid *,\downarrow\}, \{\frac{1}{2},\{1\mid *\}\mid *,\{*\mid -1\}\}, \{\uparrow,\{1\mid 0\}\mid *,\downarrow,\{0,*\mid -1\}\},$ $\{0\mid \downarrow *,\{0,*\mid -1\}\}, \{\uparrow\Vert 0\mid -1\}, \{\frac{1}{2},\{1\mid *\}\mid 0,\{0\mid -1\}\}, \{0,\{1\mid 0\}\mid *,\downarrow\}, \{0,\uparrow *\mid 0,*\}, \{0,\uparrow *\Vert *\mid -1\}, \{\uparrow,\uparrow *,\{1\mid 0,*\}\mid 0,\downarrow *\},$ $\{0,*\mid \downarrow,\{0,*\mid -1\}\}, \{\frac{1}{2},\{1\mid 0,*\}\mid \downarrow,\downarrow *\}, \{\{1\mid 0\},\{1\mid *\}\mid \downarrow,\downarrow *,\{0,*\mid -1\}\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid \downarrow,\{0\mid -1\}\}, \{1,1*\mid \{0\mid -1\},-\frac{1}{2},\{*\mid -1\}\},$ $\{2\mid -1,\{0\mid -1\},\{*\mid -1\}\}, \{1,\{1\mid *\}\mid \downarrow,\{0\mid -1\}\}, \{\uparrow,\{1\mid 0\}\mid \downarrow,\downarrow *\}, \{\{1\mid 0\},\{1\mid *\}\mid 0,\{0\mid -1\}\}, \{1*\mid \{0\mid -1\},\{*\mid -1\}\},$ $\{\frac{1}{2},\{1\mid *\}\mid \downarrow,\downarrow *,\{0,*\mid -1\}\}, \{1,\{1\mid 0\},\{1\mid *\}\mid \{0\mid -1\},\{*\mid -1\}\}, \{\{1\mid 0\},\{1\mid *\}\mid *,\{*\mid -1\}\}, \{\frac{1}{2},\{1\mid *\}\mid -\frac{1}{2}\}, \{1\mid \downarrow,\downarrow *,\pm 1\}, \{\uparrow *,\{1\mid *\}\mid \downarrow,\downarrow *\},$ $\{\frac{1}{2},\{1\mid 0,*\}\mid 0,\downarrow *,\{0,*\mid -1\}\}, \{1\mid *,\downarrow,\{0,*\mid -1\}\}, \{1,\{1\mid 0,*\}\mid 0,\{0\mid -1\}\}, \{1,\{1\mid *\}\mid \downarrow *,\{*\mid -1\}\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid -1\},$ $\{1,\{1\mid 0,*\}\mid -1\}, \{\frac{1}{2},\{1\mid 0,*\}\mid *,\downarrow,\{0,*\mid -1\}\}, \{\frac{1}{2},\{1\mid 0\}\mid 0,\{0\mid -1\}\}, \{1,\{1\mid 0,*\}\mid \pm 1,-\frac{1}{2}\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid \downarrow *,\{*\mid -1\}\},$ $\{1,\{1\mid *\}\mid -\frac{1}{2},\{0,*\mid -1\}\}, \{1,\{1\mid *\}\mid -1,\pm 1\}, \{1*\mid \{0\mid -1\},-\frac{1}{2}\}, \{1,\{1\mid 0\}\mid -1,\pm 1\}, \{\frac{1}{2},\{1\mid 0\},\{1\mid *\}\mid -\frac{1}{2},\{0,*\mid -1\}\},$ $\{1,\{1\mid 0\},\{1\mid *\}\mid -\frac{1}{2},\{*\mid -1\}\}, \{1,\{1\mid 0,*\}\mid \downarrow,\downarrow *,\{0,*\mid -1\}\}, \{1,\{1\mid 0\},\{1\mid *\}\mid -1,\{0,*\mid -1\}\}, \{1,\{1\mid 0\},\{1\mid *\}\mid \{0\mid -1\},-\frac{1}{2}\},$ $\{*,\uparrow,\{1\mid 0,*\}\mid 0,*,*2\}, \{\{1\mid 0\},\{1\mid *\}\mid -\frac{1}{2}\}, \{\frac{1}{2},\{1\mid 0\}\mid -\frac{1}{2}\}, \{1*\mid -1,\{0,*\mid -1\}\}, \{1,\{1\mid 0\}\mid \downarrow *,\{*\mid -1\}\}, \{1,1*\mid -1,\{*\mid -1\}\},$ $\{1,1*\mid -1,\{0\mid -1\}\}, \{1*\mid -\frac{1}{2},\{*\mid -1\}\}, \{1,\{1\mid 0\}\mid -\frac{1}{2},\{0,*\mid -1\}\}, \{1,\{1\mid 0,*\}\mid *,\{*\mid -1\}\}, \{*,\uparrow\mid 0,*\}, \{2\mid -1*\}$

Table 3. Elements in U_i $(14 \leq i \leq 21)$.

<table>
<tr><td>U_{22}</td><td>
$\{\uparrow *, \{1|*\}|\downarrow, \downarrow *, \{0, *|-1\}\}, \{*, \{1|*\}|*, \downarrow, \{0, *|-1\}\}, \{\uparrow *, \{1|*\}|0, \{0|-1\}\}, \{\uparrow, \uparrow *|0, *, \{0, *|-1\}\}, \{*, \{1|*\}|\downarrow, \downarrow *\}, \{1|\pm 1, -\frac{1}{2}\},$
$\{*, \uparrow, \{1|0, *\}|*, \downarrow\}, \{\frac{1}{2}, \{1|0, *\}|\downarrow, \downarrow *, \{0, *|-1\}\}, \{0, \uparrow *, \{1|0, *\}|0, *, \{0, *|-1\}\}, \{1, \{1|0\}|\{0|-1\}, \{*|-1\}\}, \{0, *, \{1|0, *\}|0, *, *2\},$
$\{\frac{1}{2}, \{1|0\}|\downarrow *, \{*|-1\}\}, \{*, \{1|*\}|0, \downarrow *, \{0, *|-1\}\}, \{1|0, \{0|-1\}\}, \{0, \{1|0\}|0, \downarrow *, \{0, *|-1\}\}, \{0, \{1|0\}|\downarrow, \downarrow *\}, \{0, \uparrow *, \{1|0, *\}|0, \downarrow *\},$
$\{*, \uparrow, \{1|0, *\}|0, *, \{0, *|-1\}\}, \{\frac{1}{2}|0, \downarrow *, \{0, *|-1\}\}, \{\frac{1}{2}|\downarrow, \downarrow *\}, \{\uparrow, \uparrow *|*, \downarrow\}, \{\frac{1}{2}, \{1|0, *\}|*, \{*|-1\}\}, \{0, \{1|0\}|*, \downarrow, \{0, *|-1\}\}, \mathrm{Tiny}_1,$
$\{\uparrow, \{1|0\}|0, \{0|-1\}\}, \{\frac{1}{2}, \{1|*\}|\downarrow, \{0|-1\}\}, \{0, \uparrow *, \{1|0, *\}|*, \downarrow\}, \{0, \uparrow *|0, *, *2\}, \{\uparrow *, \{1|*\}|*, \{*|-1\}\}, \{\uparrow, \uparrow *|0, \downarrow *\}, \{0, *||*|-1\},$
$\{\frac{1}{2}, \{1|0, *\}|-\frac{1}{2}\}, \{\{1|0\}, \{1|*\}|\downarrow *, \{*|-1\}\}, \{\frac{1}{2}, \{1|0\}, \{1|*\}|\{0|-1\}, -\frac{1}{2}\}, \{1, 1*|-1, \{0|-1\}, \{*|-1\}\}, \{2|-1, -1*\},$
$\{1, \{1|*\}|\{0|-1\}, -\frac{1}{2}\}, \{\uparrow, \{1|0\}|\downarrow, \downarrow *, \{0, *|-1\}\}, \{\{1|0\}, \{1|*\}|\downarrow, \{0|-1\}\}, \{1*|\{0|-1\}, -\frac{1}{2}, \{*|-1\}\}, \{\frac{1}{2}, \{1|*\}|\downarrow *, \{*|-1\}\},$
$\{1, \{1|*\}|\{0|-1\}, \{*|-1\}\}, \{\uparrow, \{1|0\}|*, \{*|-1\}\}, \{\frac{1}{2}, \{1|*\}|-\frac{1}{2}, \{0, *|-1\}\}, \{1|\downarrow, \downarrow *, \{0, *|-1\}\}, \{\uparrow, \uparrow *, \{1|0, *\}|\downarrow, \downarrow *\},$
$\{\uparrow, \uparrow *, \{1|0, *\}|0, \downarrow *, \{0, *|-1\}\}, \{\frac{1}{2}|*, \downarrow, \{0, *|-1\}\}, \{\frac{1}{2}, \{1|0, *\}|0, \{0|-1\}\}, \{1, \{1|*\}|-\frac{1}{2}, \{*|-1\}\}, \{\frac{1}{2}, \{1|*\}|-1\}, \{1, \pm 1|-1\},$
$\{\uparrow, \uparrow *, \{1|0, *\}|*, \downarrow, \{0, *|-1\}\}, \{\frac{1}{2}, \{1|0\}|\downarrow, \{0|-1\}\}, \{1, \{1|0, *\}|-\frac{1}{2}, \{0, *|-1\}\}, \{\frac{1}{2}, \{1|0\}, \{1|*\}|\{0|-1\}, \{*|-1\}\},$
$\{1, \{1|*\}|-1, \{0, *|-1\}\}, \{1, \{1|0, *\}|-1, \pm 1\}, \{1*|-1, \{0|-1\}\}, \{\frac{1}{2}, \{1|0\}|-1\}, \{\{1|0\}, \{1|*\}|-\frac{1}{2}, \{0, *|-1\}\},$
$\{\frac{1}{2}, \{1|0\}, \{1|*\}|-\frac{1}{2}, \{*|-1\}\}, \{1, \{1|0, *\}|\downarrow, \{0|-1\}\}, \{1, \{1|0\}|-1, \{0, *|-1\}\}, \{1, \{1|0\}, \{1|*\}|\{0|-1\}, -\frac{1}{2}, \{*|-1\}\},$
$\{*, \uparrow, \{1|0, *\}|0, \downarrow *\}, \{\uparrow *, \{1|*\}|-\frac{1}{2}\}, \{\uparrow, \{1|0\}|-\frac{1}{2}\}, \{\frac{1}{2}, \{1|0\}, \{1|*\}|-1, \{0, *|-1\}\}, \{1, \{1|0, *\}|\downarrow *, \{*|-1\}\},$
$\{1, \{1|0\}, \{1|*\}|-1, \{*|-1\}\}, \{1, \{1|0\}, \{1|*\}|-1, \{0|-1\}\}, \{1*|-1, \{*|-1\}\}, \{\frac{1}{2}, \{1|0\}|-\frac{1}{2}, \{0, *|-1\}\}, \{1|*, \{*|-1\}\},$
$\{0, *, *2|0, *\}, \{1, 1*|-1*\}, \{\{1|0\}, \{1|*\}|-1\}, \{1, \{1|0\}|\{0|-1\}, -\frac{1}{2}\}, \{*, \uparrow|0, *, *2\}, \{1, \{1|0\}|-\frac{1}{2}, \{*|-1\}\}$
</td></tr>
<tr><td>U_{23}</td><td>
$\{\uparrow *, \{1|*\}|-\frac{1}{2}, \{0, *|-1\}\}, \{*, \{1|*\}|\downarrow, \downarrow *, \{0, *|-1\}\}, \{\uparrow *, \{1|*\}|\downarrow, \{0|-1\}\}, \{0, \uparrow *|0, *, \{0, *|-1\}\}, \{*, \uparrow, \{1|0, *\}|\downarrow, \downarrow *\}, \pm\frac{1}{2},$
$\{0, *, \{1|0, *\}|*, \downarrow\}, \{\frac{1}{2}|\downarrow, \downarrow *, \{0, *|-1\}\}, \pm(0, \uparrow *, \{1|0, *\}), \{\frac{1}{2}, \{1|0\}|\{0|-1\}, \{*|-1\}\}, \pm(0, *, \{1|0, *\}), \{\uparrow, \{1|0\}|\downarrow *, \{*|-1\}\},$
$\{*, \{1|*\}|0, \{0|-1\}\}, \{1|\downarrow, \{0|-1\}\}, \pm(0, \{1|0\}), \{0, \{1|0\}|-\frac{1}{2}\}, \pm(0, \uparrow *), \pm(*, \uparrow, \{1|0, *\}), \{\uparrow, \uparrow *|0, \downarrow *, \{0, *|-1\}\}, \pm(\uparrow, \uparrow *), \pm(*, \uparrow),$
$\{\frac{1}{2}|*, \{*|-1\}\}, \{0, \{1|0\}|\downarrow, \downarrow *, \{0, *|-1\}\}, 0, \{\uparrow, \uparrow *, \{1|0, *\}|0, \{0|-1\}\}, \{\frac{1}{2}, \{1|*\}|\{0|-1\}, -\frac{1}{2}\}, \{0, \uparrow *, \{1|0, *\}|*, \downarrow, \{0, *|-1\}\},$
$\{0, \uparrow *|*, \downarrow\}, \{\uparrow, \uparrow *, \{1|0, *\}|*, \{*|-1\}\}, \{*, \uparrow|0, \downarrow *\}, \pm(*, \{1|*\}), \{\uparrow, \uparrow *, \{1|0, *\}|-\frac{1}{2}\}, \pm(\uparrow *, \{1|*\}), \pm(\frac{1}{2}, \{1|0\}), \pm(1, \{1|0\}, \{1|*\}),$
$\pm(1, 1*), \{1, \{1|*\}|-1, \{0|-1\}\}, \pm(\uparrow, \{1|0\}), \pm(\{1|0\}, \{1|*\}), \pm(\frac{1}{2}, \{1|0\}, \{1|*\}), \{\frac{1}{2}, \{1|0, *\}|\downarrow *, \{*|-1\}\},$
$\{1, \{1|*\}|\{0|-1\}, -\frac{1}{2}, \{*|-1\}\}, \{0, \{1|0\}|*, \{*|-1\}\}, \{\frac{1}{2}, \{1|*\}|-1, \{0, *|-1\}\}, \{1|-\frac{1}{2}, \{0, *|-1\}\}, \{0, \uparrow *, \{1|0, *\}|\downarrow, \downarrow *\},$
$\{*, \uparrow, \{1|0, *\}|0, \downarrow *, \{0, *|-1\}\}, \{\uparrow, \uparrow *|*, \downarrow, \{0, *|-1\}\}, \{\frac{1}{2}|0, \{0|-1\}\}, \pm(\frac{1}{2}, \{1|*\}), \{\uparrow *, \{1|*\}|-1\}, \pm(1, \pm 1), \pm(\uparrow, \uparrow *, \{1|0, *\}),$
$\{\frac{1}{2}, \{1|0, *\}|\downarrow, \{0|-1\}\}, \pm(\frac{1}{2}, \{1|0, *\}), \{\frac{1}{2}, \{1|*\}|\{0|-1\}, \{*|-1\}\}, \pm(1, \{1|*\}), \{\frac{1}{2}, \{1|0, *\}|-1\}, \{\frac{1}{2}, \{1|0\}, \{1|*\}|-1, \{0|-1\}\},$
$\{\uparrow, \{1|0\}|-1\}, \{\{1|0\}, \{1|*\}|\{0|-1\}, -\frac{1}{2}\}, \{\{1|0\}, \{1|*\}|-\frac{1}{2}, \{*|-1\}\}, \{1, \{1|0, *\}|\{0|-1\}, \{*|-1\}\}, \pm(1, \{1|0, *\}),$
$\{1, \{1|0\}|\{0|-1\}, -\frac{1}{2}, \{*|-1\}\}, \{0, *, \{1|0, *\}|0, \downarrow *\}, \{*, \{1|*\}|-\frac{1}{2}\}, \{\uparrow, \{1|0\}|-\frac{1}{2}, \{0, *|-1\}\}, \{\frac{1}{2}, \{1|0\}, \{1|*\}|-1, \{*|-1\}\},$
$\{1, \{1|0, *\}|-\frac{1}{2}, \{*|-1\}\}, \{1, \{1|0\}|-1, \{*|-1\}\}, \pm(1, \{1|0\}), \{1*|-1, \{0|-1\}, \{*|-1\}\}, \{\frac{1}{2}, \{1|0\}|-1, \{0, *|-1\}\},$
$\{1|\downarrow *, \{*|-1\}\}, *2, \{1, \{1|0\}, \{1|*\}|-1*\}, \{\{1|0\}, \{1|*\}|-1, \{0, *|-1\}\}, \{1, \{1|0, *\}|\{0|-1\}, -\frac{1}{2}\}, \{*, \uparrow|0, *, \{0, *|-1\}\},$
$\{\frac{1}{2}, \{1|0\}|-\frac{1}{2}, \{*|-1\}\}, \pm 1, *, *3, \pm(1*), \pm 2$
</td></tr>
</table>

Table 4. Elements in U_i $(22 \leq i \leq 23)$.

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