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$\mathcal{P}$ play in CANDY NIM

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CANDY NIM is a variant of NIM in which both players aim to take the last candy in a game of NIM, with the added simultaneous secondary goal of taking as many candies as possible. We give bounds on the number of candies the first and second players obtain in 3-pile $\mathcal{P}$ positions as well as strategies that are provably optimal for some families of such games. We also show how to construct a game with N candies such that the loser takes the largest possible number of candies and how to bound the number of candies the winner can take in an arbitrary $\mathcal{P}$ position with N total candies.

1. Introduction

One of the first serious results in the study of combinatorial games was Bouton's solution to the game of NIM in [2]. NIM is a two-player game played with several piles of stones. In a turn, a player removes some number of stones from one pile. The player taking the last stone wins.

Beyond its historical interest, the game of NIM is interesting because a wide family of games, the so-called *finite normal-play impartial combinatorial games*, can all be reduced to the game of NIM, thanks to the celebrated *Sprague–Grundy theory*, as first described in [7] and [3]. In this paper, we describe and study a slight modification of the game of NIM, known as CANDY NIM, which is interesting in its own right as being a blend of an impartial combinatorial game and a scoring game. While impartial combinatorial games have been widely studied ever since the time of Bouton, the study of scoring games has only recently attracted interest, for instance, in [4; 5; 6; 8].

In any NIM game, either the first player to move or the second player to move must have a winning strategy, but not both. We classify the NIM positions based on which player wins with optimal play. Games in which the first player wins with optimal play are called $\mathcal{N}$ positions. Similarly, games in which the second player wins with optimal play are called $\mathcal{P}$ positions. We call $\mathcal{N}$ and $\mathcal{P}$ the

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outcome classes of the associated games. In the two-player game of NIM, we refer to the losing player as Luca and the winning player as Windsor.

It is easy to compute the outcome classes and winning strategy for NIM games, based on the $\oplus$ function of $a, b \in \mathbb{Z}$, called the *nim-sum*, defined as follows: $a \oplus b$ is given by the XOR of a and b (obtained by writing both a and b in binary and adding without carrying).

Theorem 1.1 (Bouton [2]). *The* NIM *game with piles of size* $a_1, a_2, \ldots, a_p$ *is a* $\mathcal{P}$ *position if and only if* $n := a_1 \oplus a_2 \oplus \cdots \oplus a_p = 0$. *If* $n \neq 0$, *winning moves take the total nim-sum to zero.*

This quantity $a_1 \oplus a_2 \oplus \cdots \oplus a_p$ is of general importance in NIM, and we will need to use it again later, so we introduce some notation for it. Given a NIM game G, with piles of size $a_1, \ldots, a_p$, we define its *Grundy value* $\mathcal{G}(G)$ to be $a_1 \oplus a_2 \oplus \cdots \oplus a_p$.

The above famous theorem of Bouton gives an easily computable winning strategy for NIM and more generally has made outcomes of impartial combinatorial games more straightforward to understand. As in [4; 5; 6; 8], scoring variants of combinatorial games are much more challenging to understand and are a topic of much recent interest. Given NIM's position as the canonical impartial combinatorial game, we are interested in studying the natural scoring variant we obtain when we change the objective of the game slightly.

Definition 1.2. CANDY NIM is a two-player combinatorial game with the same setup and game play as NIM. However, in addition to the primary goal of making the last move (as in NIM), players have a secondary goal of collecting as many stones, or *candies*, as possible.

Remark 1.3. In CANDY NIM, winning always takes priority over collecting candies. No number of candies can fully compensate for the embarrassment of losing the game. Alternatively, at the end of the game, the winner gets a prize that is more desirable than all the candies put together.

CANDY NIM was first introduced by Michael Albert [1]. In traditional NIM, the role of the losing player, Luca, and her choices are irrelevant to the outcome of the game, but by giving Luca a natural, secondary goal, game play becomes much more interesting and challenging to analyze.

Albert observed, among other things, that it is not always optimal for Luca to remove candies from the largest pile and also provided some results on values of CANDY NIM games (see Section 2 for a definition of the value of a game). This surprising finding along with several other counterintuitive observations regarding optimal CANDY NIM play (some of which are examined later in this article) makes the game interesting to study, especially as an indication of the tools that may be deployed in a broader analysis of scoring variants of other impartial normal-play combinatorial games.

Remark 1.4. In this paper, we focus on the games whose outcome class is $\mathcal{P}$. This is because Luca has many more options to play with than does the winning player, Windsor. At every turn, in optimal play, Windsor must bring the nim-sum of all of the pile sizes down to zero. This severely limits the options of the winning player. In many of the positions we will study, Windsor will only have a single move available on each turn. On the other hand, Luca loses no matter what her move is, giving room for optimizing her move with respect to the number of candies she collects. Consequently, their turns are more interesting to consider. Throughout, we will assume that Windsor is *forced* to play winning moves in the underlying NIM games, so that losing moves are illegal.

In this article, we focus on two broad classes of problems: understanding optimal play and the resulting scores in 3-pile games and describing extremal allocations of N candies with respect to the value of the game.

Our primary objective is to obtain bounds on the maximal difference between the number of candies Luca and Windsor take in a game of k-pile CANDY NIM (focusing on $\mathcal{P}$ positions). Our main results are Theorems 3.3 and 3.5, where we obtain upper and lower bounds for 3-pile CANDY NIM games.

We begin with some notation and definitions in Section 2, defining, notably, the value of a game G (which we work to bound for the remainder of the article). In Section 3, we state our main results concerning 3-pile games, deferring the proofs of these theorems to Section 6. In Section 4, we subsequently present a simple strategy for the 3-pile game (the flip-flop strategy) and an iterative variant (the fractal strategy) that yields the lower bound of our main result, Theorem 3.3. Some explicit examples of these strategies are worked out in Section 5.

We investigate several related questions beyond the scope of 3-pile CANDY NIM. In Section 7, we consider optimal allocations of N candies from Luca's perspective and give extremal bounds in Theorems 7.1, 7.2, and 7.3. We defer the proofs of Theorem 7.1 to Section 8 and of Theorem 7.3 to Section 9, as both require some involved casework. We conclude with some potential generalizations and remaining open questions in Section 10.

Throughout this work, we provide some worked examples of optimal play for specific CANDY NIM games. We encourage readers to generate their own examples using our program, which is available for download.[1]

2. Preliminaries

We begin with some definitions and notation that will be helpful for the analysis

[1] https://github.com/nmani2/candynim

of the 3- and n-pile CANDY NIM games. Unless otherwise specified, G will always refer to CANDY NIM games.

Definition 2.1. Given a CANDY NIM game G, let $N(G)$ be the total number of candies in the game. Let $N_W(G)$ be the number of candies collected by winning player Windsor, and let $N_L(G)$ be the number of candies collected by losing player Luca, assuming optimal play.

Our primary goal is to bound the number of candies Luca (the losing player) can collect relative to Windsor in a $\mathcal{P}$ game, assuming optimal play. This difference in candies will be called the *value* of the associated CANDY NIM game.

Definition 2.2. The *value* $V(G)$ of a game G is given by

$$V(G) = N_L(G) - N_W(G).$$

Definition 2.3. A *turn* is a triple of games $T = (G, G', G'')$ where Luca moves from G to G' and Windsor moves from G' to G''. We call each move made by a player from G to G' a *ply* $P = (G, G')$.

Many of the bounds, such as those in Theorems 3.3 and 3.5, arise from analyzing specific strategies or sequences of moves by Luca and Windsor. To simplify these analyses, we introduce some notation:

Definition 2.4. The *single-turn value* $V_T(G)$ of a turn $T = (G, G', G'')$ is

$$V_T(G) = (N(G) - N(G')) - (N(G') - N(G'')) = N(G) + N(G'') - 2N(G').$$

Definition 2.5. A *strategy* S of a game G is a sequence of turns $T_i = (G_i, G'_i, G''_i)$ for $1 \leq i \leq n$ such that $G_1 = G$ and for each $j < n$, we have $G''_j = G_{j+1}$. Furthermore, $G''_n = \varnothing$. We call the *strategic value* of G with strategy S the difference between the number of candies collected by Luca and Windsor under strategy S, i.e., $V_S(G) = \sum V_{T_i}(G)$. An *optimal strategy* is a strategy S from which neither player has an opportunity to vary and guarantee strictly more candies than offered from S.

Remark 2.6. If S is an optimal strategy, then $V_S(G) = V(G)$.

Definition 2.7. We write $G = [a_1, a_2, a_3, a_4, \ldots, a_p]$ if G is the game with p piles where pile i has $a_i \geq 0$ candies.

To gain some basic intuition about CANDY NIM we note that if Luca follows the strategy where she takes all of the candies from the largest pile in game G_i for the i-th turn T_i, then $V_{T_i}(G_i) \geq 0$ for all i. This implies the following:

Lemma 2.8. *For any game G, $V(G) \geq 0$.*

3. 3-pile CANDY NIM

In our main results, Theorems 3.3 and 3.5, we bound the value of 3-pile CANDY NIM games. In giving such results, it is helpful to use an alternative characterization of such games.

Definition 3.1. Let $\mathfrak{G}(a, m, x)$ be the 3-pile CANDY NIM game

$$\mathfrak{G}(a, m, x) = [a, 2^{k+1} \cdot m + x, 2^{k+1} \cdot m + a \oplus x],$$

where $k = \lfloor \log_2 a \rfloor$, $m \geq 1$, and $0 \leq x < 2^k$.

Note that, as per Definition 3.1, 2^{k+1} is the smallest power of 2 strictly greater than a. We first show results for the following subset of the class of 3-pile CANDY NIM games.

Definition 3.2. A game G is a *standard-form game* if for some k and m,

$$G = \mathfrak{G}(2^{k+1} - 1, m, 0) = [2^{k+1} - 1, 2^{k+1} \cdot m, 2^{k+1}(m+1) - 1].$$

We pay special attention to standard-form games because of the existence of simple strategies for Luca that are likely optimal, and that we show are close to optimal in a precise sense. In particular, we have the following bounds on $V(G)$ when G is a standard-form game:

Theorem 3.3. *Given a standard-form game* $G = \mathfrak{G}(2^{k+1} - 1, m, 0)$, *we have*

$$V(\mathfrak{G}(2^{k+1} - 1, m, 0)) \leq (2^{k+2} - 2)m + (2^{k+2} - 2) - 2 + \delta_{0k},$$

where δ_{0k} *is the Kronecker delta function, which is* 1 *if* $k = 0$ *and* 0 *otherwise. Furthermore*,

$$V(\mathfrak{G}(2^{k+1} - 1, m, 0)) \geq 2(2^{k+1} - 1)m - 2(2^{\lceil k/2 \rceil} - 1) + V(\mathfrak{G}(2^{\lceil k/2 \rceil} - 1, 2^{\lfloor k/2 \rfloor + 1} - 1, 0)).$$

Thus,

$$V(\mathfrak{G}(2^{k+1} - 1, m, 0)) \geq 2(2^{k+1} - 1)(m - 1) + b(k),$$

where

$$3(2^{k+1} - 1) \leq b(k) \leq 4(2^{k+1} - 1) - 2.$$

We obtain our lower bound on $V(G)$ for standard-form games by constructing two strategies that yield large values of $N_L(G)$, the *flip-flop* and more general *fractal strategies*, described in more detail in Section 4. We conjecture that an instance of the fractal strategy is optimal for standard-form games. We obtain the upper bound by greedily bounding the maximum number of candies Luca can take relatively to Windsor each turn, using the *semiratio* of a turn, defined in Section 6.

We can also leverage our standard-form game bounds to prove looser bounds for more general 3-pile CANDY NIM games.

Corollary 3.4. $V(\mathfrak{G}(a, m, x)) \geq 2a(m-1) + x \oplus a + a - x.$

Theorem 3.5. *If* $G = \mathfrak{G}(2^{k+1}-1, m, x)$, *then*

$$V(\mathfrak{G}(2^{k+1}-1, m-1, 0)) + 2(2^{k+1}-1) - 2x \leq V(G) \leq V(\mathfrak{G}(2^{k+1}-1, m+1, 0)) - 2(2^{k+1}-1) + 2x.$$

We defer the proofs of Theorems 3.3 and 3.5 to Section 6.

4. Strategies for the 3-pile game

In order to give the lower bounds of Theorems 3.3 and 3.5, we construct an strategy for a 3-pile game whose value we can explicitly compute. We present two strategies for certain families of the 3-pile game, which we call the *flip-flop strategy* and the *fractal strategy*.

The flip-flop strategy is a simple strategy that, until the last turn, allows Luca to take as many candies as possible subject to allowing Windsor only to take one candy on that turn. The fractal strategy is an iterative variant of the flip-flop strategy. We compute the value of this strategy, but obtain a more involved result. We conjecture that a certain specific version of the fractal strategy is optimal for games in standard form, and obtain our lower bound on $V(G)$ from the fractal strategy for 3-pile standard-form games.

The flip-flop strategy. Consider the class of games $[1, 2m, 2m+1]$. This class of games has a simple inductive optimal strategy. If Luca removes three candies from the largest pile, we obtain the game $[1, 2m, 2(m-1)]$. Windsor is then forced to remove a single candy from the middle pile to obtain the game $[1, 2(m-1)+1, 2(m-1)]$. The process may then be repeated with the middle and rightmost piles swapped. This strategy is optimal and gives $V([1, 2m, 2m+1]) = 2m$, proved and illustrated in Proposition 5.1. This motivating example suggests the following strategy for games of the form $G = [2^k-1, 2^k \cdot m, 2^k \cdot (m+1) - 1]$, which we will term the *flip-flop* strategy:

Definition 4.1. Given a game $G = \mathfrak{G}(2^k-1, m, 0) = [2^k-1, 2^k \cdot m, 2^k \cdot (m+1)-1]$, the *flip-flop strategy* $\mathrm{FF}(G)$ is given as follows:

(1) If $m \geq 1$, Luca removes $2^{k+1}-1$ candies from the third pile, and then Windsor removes one candy from the second pile. The resulting game is $\mathfrak{G}(2^k-1, m-1, 0)$. Then continue with $\mathrm{FF}(\mathfrak{G}(2^k-1, m-1, 0))$.

(2) If $m = 0$, then we have $G = [2^k-1, 2^k-1]$. Luca removes one pile, then Windsor removes the other one.

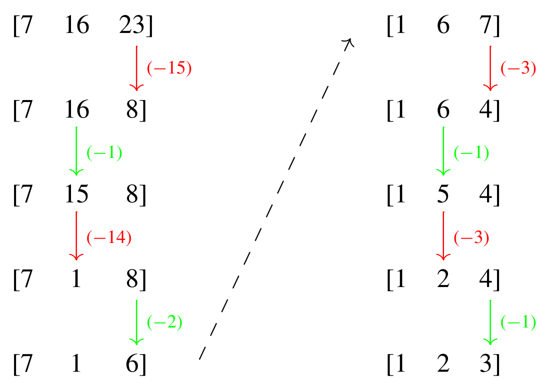

Figure 1. In the fractal strategy with $G = [7, 16, 23]$, we begin by applying the flip-flop strategy until the game reaches $H = [7, 15, 8]$. If Luca continued via the flip-flop strategy, the next turn would be $T = (H, [7, 8], [7, 7])$, giving Luca 22 of 30 candies in H. If Luca instead reduced the smallest pile from size 7 to 1, the single-turn value of that reduction would be 12. This yields the game $[1, 6, 7]$, which has value 6, as shown in Proposition 5.1. With this sequence of moves, Luca does better, obtaining 24 of 30 candies of H.

Proposition 4.2. *For* $G = [2^k - 1, 2^k \cdot m, 2^k \cdot (m+1) - 1]$, *we have*

$$V(G) \geq V_{\mathrm{FF}(G)}(G) = (m-1) \cdot (2^{k+1} - 2).$$

Proof. Consider the strategy $\mathrm{FF}(G)$ with initial turn $T_1 = \big(G = G^{(0)}, G^{(1)}, G^{(2)}\big)$, where Luca takes $2^{k+1} - 1$ candies from the largest pile and Windsor takes one candy from the middle pile. Therefore, $V_{T_1}(G) = 2^{k+1} - 2$ with $G^{(2)} = [2^k - 1, 2^k \cdot (m-1), 2^k \cdot m - 1]$. Repeat for turns $T_2, \ldots, T_{m-1}$, where

$$T_i = (G^{(2i-2)}, G^{(2i-1)}, G^{(2i)}).$$

For $i = 1, \ldots, m-1$,

$$V_{T_i}(G^{(2i-2)}) = 2^{k+1} - 2.$$

When $m = 0$, the resulting game is $G^{(2m-2)} = [2^k - 1, 2^k - 1]$ with $V(G^{(2m-2)}) = 0$. Thus, $V_{\mathrm{FF}(G)}(G) = (m-1) \cdot (2^{k+1} - 2)$. □

The fractal strategy. We can improve the above strategy to one that exhibits a curious fractal-like behavior, as in Figure 1 and Proposition 5.2.

Definition 4.3. Say that a function $f : \mathbb{N} \to \mathbb{N}$ is *contractive* if for all $a \in \mathbb{N}$, $f(a) \leq a$. Let $\mathscr{F}$ denote the family of contractive functions.

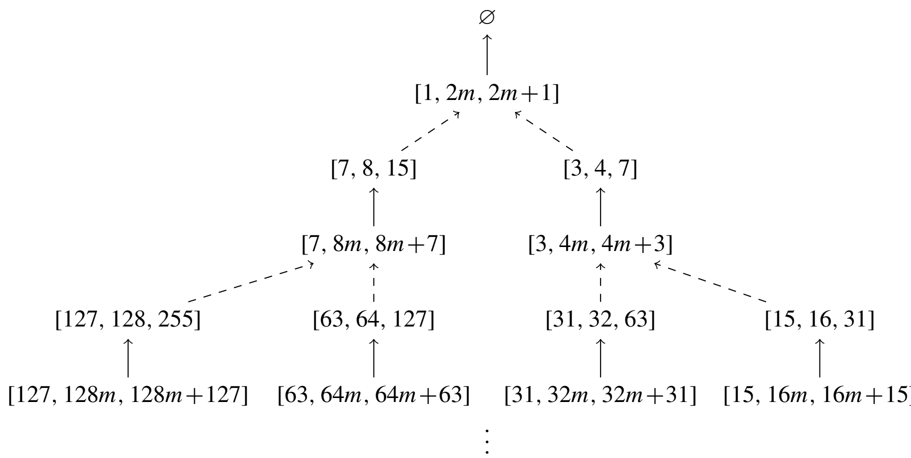

Figure 2. The fractal strategy: the solid arrows indicate the flip-flop strategy of Section 4, and the dashed arrows indicate a change of smallest pile size.

Definition 4.4. Consider a game G of the form

$$G = \mathfrak{G}(2^k - 1, m, 0) = [2^k - 1, 2^k \cdot m, 2^k(m+1) - 1] \quad \text{for } m, k \geq 1.$$

Let $f \in \mathscr{F}$. We define the *fractal strategy* $\mathrm{Fractal}_f(G)$ based on f as follows:

(1) If $m > 1$, then Luca plays as in $\mathrm{FF}(G)$ by removing $2^{k+1} - 1$ candies from the third pile, and then Windsor moves to $\mathfrak{G}(2^k - 1, m-1, 0)$. Then play $\mathrm{Fractal}_f(\mathfrak{G}(2^k - 1, m-1, 0))$.

(2) If $m = 1$ and $f(a) = a$, then play as in the flip-flop strategy.

(3) If $m = 1$ and $f(a) < a$, then Luca moves the smallest pile to $2^{f(a)} - 1$, and Windsor moves to $\mathfrak{G}(2^{f(a)} - 1, 2^{a-f(a)}, 0)$. Then play $\mathrm{Fractal}_f$ from there.

We conjecture that with an appropriate choice of contractive function $f \in \mathscr{F}$, the fractal strategy $\mathrm{Fractal}_f$ is an optimal strategy for standard-form games. In the following theorem, we identify a best possible contractive function for the fractal strategy, namely $f(a) = \lfloor a/2 \rfloor$. We prove that this function is at least as good as any other function and compute the value of G under the corresponding strategy.

Theorem 4.5. *Given* $G = \mathfrak{G}(2^k - 1, m, 0)$ *with* $k, m \geq 1$,

$$\sup_{f \in \mathscr{F}} V_{\mathrm{Fractal}_f}(G) = (m-2)\cdot(2^{k+1} - 2) + \sum_{i=0}^{\lceil \log_2 k \rceil} \bigl(2^{\lfloor k/2^i \rfloor + 1} - 2^{\lfloor k/2^{i+1} \rfloor + 1} + (2^{\lfloor k/2^{i+1} \rfloor + 1} - 1)(2^{\lfloor k/2^i \rfloor - \lfloor k/2^{i+1} \rfloor} - 2)\bigr),$$

with the supremum achieved by taking $f : a \mapsto \lfloor a/2 \rfloor$.

Proof. We first show that the function $f : a \mapsto \lfloor a/2 \rfloor$ achieves the stated bound. Consider the strategy Fractal_f, where $f : a \mapsto \lfloor a/2 \rfloor$. We show

$$V_{\text{Fractal}_f}(G) = (m-2)\cdot(2^{k+1}-2) + \sum_{i=0}^{\lceil \log_2 k \rceil} 2^{\lfloor k/2^i \rfloor + 1} - 2^{\lfloor k/2^{i+1} \rfloor + 1} + (2^{\lfloor k/2^{i+1} \rfloor + 1} - 1)(2^{\lfloor k/2^i \rfloor - \lfloor k/2^{i+1} \rfloor} - 2).$$

It suffices to show that for the game $H = [2^k - 1, 2^k, 2^{k+1} - 1]$,

$$V_{\text{Fractal}_f}(H) = \sum_{i=0}^{\lceil \log_2 k \rceil} 2^{\lfloor k/2^i \rfloor + 1} - 2^{\lfloor k/2^{i+1} \rfloor + 1} + (2^{\lfloor k/2^{i+1} \rfloor + 1} - 1)(2^{\lfloor k/2^i \rfloor - \lfloor k/2^{i+1} \rfloor} - 2).$$

Under Fractal_f, the first turn is

$$T = (H, H', H'') = ([2^k - 1, 2^k, 2^{k+1} - 1], [2^k - 1, 2^k, 2^{\lfloor k/2 \rfloor} - 1], [2^k - 1, 2^k - 2^{\lfloor k/2 \rfloor}, 2^{\lfloor k/2 \rfloor} - 1]).$$

Then we perform $\text{FF}(G)$ until we reach the game $[2^{\lfloor k/2 \rfloor} - 1, 2^{\lfloor k/2 \rfloor}, 2^{\lfloor k/2 \rfloor - 1}]$. This involves repeating the following sequence of moves $2^{k - \lfloor k/2 \rfloor} - 2$ times:

$$[2^{\lfloor k/2 \rfloor} - 1, a\cdot 2^{\lfloor k/2 \rfloor}, (a+1)\cdot 2^{\lfloor k/2 \rfloor} - 1] \mapsto [2^{\lfloor k/2 \rfloor} - 1, a\cdot 2^{\lfloor k/2 \rfloor}, (a-1)\cdot 2^{\lfloor k/2 \rfloor}] \mapsto [2^{\lfloor k/2 \rfloor} - 1, a\cdot 2^{\lfloor k/2 \rfloor} - 1, (a-1)\cdot 2^{\lfloor k/2 \rfloor}].$$

Since for any $g \in \mathscr{F}$ and $a \neq 1$, we have that

$$V_{\text{Fractal}_g}([2^{\lfloor k/2 \rfloor} - 1, a\cdot 2^{\lfloor k/2 \rfloor}, (a+1)\cdot 2^{\lfloor k/2 \rfloor} - 1]) = V_{\text{Fractal}_g}([2^{\lfloor k/2 \rfloor} - 1, a\cdot 2^{\lfloor k/2 \rfloor} - 1, (a-1)\cdot 2^{\lfloor k/2 \rfloor}]) + 2^{\lfloor k/2 \rfloor + 1} - 1,$$

we obtain that

$$V_{\text{Fractal}_f}(H) = V_{\text{Fractal}_f}([2^{\lfloor k/2 \rfloor} - 1, 2^{\lfloor k/2 \rfloor}, 2^{\lfloor k/2 \rfloor + 1} - 1]) + 2^{k+1} - 2^{\lfloor k/2 \rfloor} + (2^{\lfloor k/2 \rfloor + 1} - 1)(2^{k - \lfloor k/2 \rfloor} - 2).$$

Via the inductive hypothesis we obtain the desired result:

$$V_{\text{Fractal}_f}(G) = (m-2)\cdot(2^{k+1}-2) + \sum_{i=0}^{\lceil \log_2 k \rceil} 2^{\lfloor k/2^i \rfloor + 1} - 2^{\lfloor k/2^{i+1} \rfloor + 1} + (2^{\lfloor k/2^{i+1} \rfloor + 1} - 1)(2^{\lfloor k/2^i \rfloor - \lfloor k/2^{i+1} \rfloor} - 2).$$

We now show that the contractive function $f : a \mapsto \lfloor a/2 \rfloor$ is optimal over all of $\mathscr{F}$. It suffices to show that for $g \in \mathscr{F}$,

$$V_{\text{Fractal}_g}([2^k - 1, 2^k, 2^{k+1} - 1]) \le V_{\text{Fractal}_f}([2^k - 1, 2^k, 2^{k+1} - 1]).$$

We consider two cases based on whether $g(k) < \lfloor k/2 \rfloor$ or $g(k) > \lfloor k/2 \rfloor$ First, if $g(k) = i < \lfloor k/2 \rfloor$, then

$$V_{\text{Fractal}_g}([2^k - 1, 2^k, 2^{k+1} - 1]) \le V_{\text{Fractal}_f}([2^k - 1, 2^k, 2^{k+1} - 1]). \qquad (4\text{-}1)$$

Equivalently, we wish to show that the left side of (4-1) minus the right side is less than or equal to zero. By the definition of the fractal strategy, we have

$$\begin{aligned} &V_{\text{Fractal}_g}([2^k - 1, 2^k, 2^{k+1} - 1]) - V_{\text{Fractal}_f}([2^k - 1, 2^k, 2^{k+1}]) \\ &= V_{\text{Fractal}_g}([2^i - 1, 2^i, 2^{i+1} - 1]) - V_{\text{Fractal}_f}([2^{\lfloor k/2 \rfloor} - 1, 2^{\lfloor k/2 \rfloor}, 2^{\lfloor k/2 \rfloor + 1} - 1]) \\ &\qquad - 2^{k-i} + 2^{i+1} + 2^{k - \lfloor k/2 \rfloor} + 2^{\lfloor k/2 \rfloor - i} - 2 \\ &\le 2^{i+1} + 2^{k - \lfloor k/2 \rfloor} + 2^{\lfloor k/2 \rfloor - i} - 2^{k-i} - 2. \end{aligned}$$

Now, suppose $i \ne \lfloor k/2 \rfloor - 1$. Then we have

$$\begin{aligned} &V_{\text{Fractal}_g}([2^k - 1, 2^k, 2^{k+1} - 1]) - V_{\text{Fractal}_f}([2^k - 1, 2^k, 2^{k+1} - 1]) \\ &\qquad \le 2^{i+1} + 2^{k - \lfloor k/2 \rfloor} + 2^{\lfloor k/2 \rfloor - i} - 2^{k-i} - 2 < 0. \end{aligned}$$

On the other hand, if $i = \lfloor k/2 \rfloor - 1$, we get

$$\begin{aligned} &V_{\text{Fractal}_g}([2^k - 1, 2^k, 2^{k+1} - 1]) - V_{\text{Fractal}_f}([2^k - 1, 2^k, 2^{k+1} - 1]) \\ &\qquad \le 2^{i+1} + 2^{k - \lfloor k/2 \rfloor} + 2^{\lfloor k/2 \rfloor - i} - 2^{k-i} - 2 \le 0. \end{aligned}$$

Next suppose that $g(k) = i > \lfloor k/2 \rfloor$. First, recall the notation from Definition 3.1, $\mathfrak{G}(a, m, x) = [a, 2^k \cdot m + x, 2^k \cdot m + a \oplus x]$, with $2^k > a \ge 2^{k-1}$. Let $f', g' \in \mathscr{F}$ be defined by $f'(n) = f(n)$ if $n \ne \lfloor k/2 \rfloor$ and $f'(\lfloor k/2 \rfloor) = \lfloor i/2 \rfloor$, and $g'(n) = f(n)$ if $n \ne k$ and $g'(k) = i$. We will show that $V_{\text{Fractal}_{f'}}(\mathfrak{G}(2^k - 1, 1, 0)) \ge V_{\text{Fractal}_{g'}}(\mathfrak{G}(2^k - 1, 1, 0))$. By induction, this implies $V_{\text{Fractal}_f}(\mathfrak{G}(2^k - 1, 1, 0)) \ge V_{\text{Fractal}_g}(\mathfrak{G}(2^k - 1, 1, 0))$. To this end, we have

$$\begin{aligned} &V_{\text{Fractal}_{f'}}(\mathfrak{G}(2^k - 1, 1, 0)) - V_{\text{Fractal}_{g'}}(\mathfrak{G}(2^k - 1, 1, 0)) \\ &= V_{\text{Fractal}_{f'}}(\mathfrak{G}(2^{\lfloor k/2 \rfloor} - 1, 1, 0)) - V_{\text{Fractal}_{g'}}(\mathfrak{G}(2^i - 1, 1, 0)) - 2^{\lfloor k/2 \rfloor} \\ &\qquad + (2^{\lfloor k/2 \rfloor + 1} - 1)(2^{k - \lfloor k/2 \rfloor} - 2) + 2^i - (2^{i+1} - 1)(2^{k-i} - 2) \\ &= 2^{k-i} + 3 \cdot 2^i + 2^{i - \lfloor i/2 \rfloor} - 2^{\lfloor k/2 \rfloor - \lfloor i/2 \rfloor} - 2^{\lfloor k/2 \rfloor + 1} - 2^{k - \lfloor k/2 \rfloor}. \end{aligned}$$

Now, we can see that the trio of inequalities $i \ge k - \lfloor k/2 \rfloor$ and $i \ge \lfloor k/2 \rfloor + 1$ and $i \ge \lfloor k/2 \rfloor - \lfloor i/2 \rfloor$ are each true, and so that allows us to simplify to get

$$V_{\text{Fractal}_f}(\mathfrak{G}(2^k - 1, 1, 0)) - V_{\text{Fractal}_g}(\mathfrak{G}(2^k - 1, 1, 0)) \ge 2^{k-i} + 2^{i - \lfloor i/2 \rfloor},$$

which is positive. This resolves the last case, yielding the desired result. □

5. 3-pile strategy examples

Below are worked examples of 3-pile strategies and bounding techniques. We first highlight a family of 3-pile games for which the flip-flop strategy of Definition 4.1 achieves the maximal $V(G)$:

Proposition 5.1. $$V([1, 2m, 2m+1]) = 2m.$$

Proof. We show optimality and give the strategy inductively. If $m = 1$, then $G = [1, 2, 3]$. Optimal play occurs when the first move is $T_1 = (G, [1, 2, 0], [1, 1, 0])$ with $V(G) = 2$ and $N_L(G) = 4$. Now assume that $V([1, 2m, 2m+1]) = 2m$ is true for all $1 \le m \le m'$. We first show that for $G = [1, 2(m'+1), 2(m'+1)+1]$, $V(G) \ge 2m'+2$. Consider the strategy S consisting of initial turn

$$T_1 = \big(G, G' = [1, 2(m'+1), 2m'], G'' = [1, 2m'+1, 2m']\big)$$

followed by optimal play as per the inductive hypothesis for the resulting game $G'' = [1, 2m', 2m'+1]$. $V_{T_1}(G) = 2$ and $V(G'') = 2m'$ by the inductive hypothesis, giving $V(G) \ge 2m'+2$.

To show this strategy is optimal, we prove $V(G) \le 2m'+2$. Consider the four possible cases for Luca's first move.

<u>Case 1</u>: Consider strategy S_1, where Luca takes from the smallest pile. Then the first turn $T_1 = (G, G', G'')$ satisfies $G'' = [0, 2(m'+1), 2(m'+1)]$. Then

$$V_{S_1}(G) = V_{T_1}(G) + V(G'') = 0 + 0 < 2m'+2.$$

<u>Case 2</u>: Consider strategy S_2, where Luca takes $2k$ candies from the largest pile such that the first turn is

$$T_1 = \big(G, G' = [1, 2(m'+1), 2j+1], G'' = [1, 2j, 2j+1]\big),$$

where $j = m'+1-k$. Note that $V_{T_1}(G) = 0$. By induction, $V(G'') = 2j$. So

$$V_{S_2}(G) = V_{T_1}(G) + V(G'') = 2j < 2m'+2.$$

<u>Case 3</u>: Consider strategy S_3, where Luca takes $2k+1$ candies from the largest pile such that the first turn is

$$T_1 = \big(G, G' = [1, 2(m'+1), 2j], G'' = [1, 2j+1, 2j]\big),$$

where $j = m'+1-k$. This time, $V_{T_1}(G) = 2$. By induction, $V(G'') = 2j$. So

$$V_{S_3}(G) = V_{T_1}(G) + V(G'') = 2 + 2j \le 2m'+2.$$

<u>Case 4</u>: Consider strategy S_4, where Luca takes k candies from the medium pile such that the first turn is

$$T_1 = \big(G, G' = [1, 2j, 2(m'+1)+1], G'' = [1, 2j, 2j+1]\big)$$

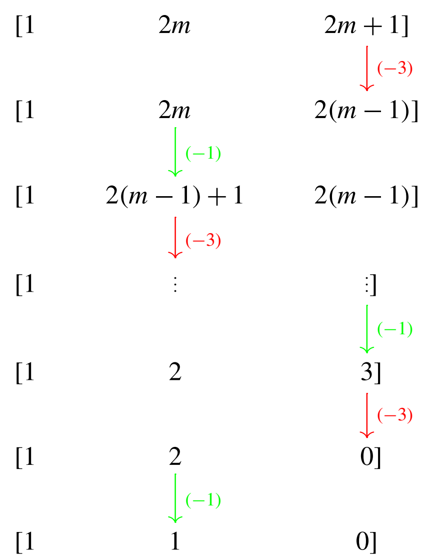

Figure 3. The sequence of moves that occurs when using the flip-flop strategy in the game $G = [1, 2m, 2m+1]$.

or

$$T_1 = \big(G, G' = [1, 2j+1, 2(m'+1)+1], G'' = [1, 2j, 2j+1]\big).$$

In this case, $V_{T_1} \le 0$. By induction, $V(G'') = 2j$. Therefore,

$$V_{S_4}(G) = V_{T_1}(G) + V(G'') \le 2j < 2m' + 2.$$

Since $\max\big(V_{S_1}(G), V_{S_2}(G), V_{S_3}(G), V_{S_4}(G)\big) \le 2m'+2$, we obtain the desired equality $V(G) = 2m' + 2$. □

The inductive optimal strategy in the proof above is pictorially represented by the sequence of moves in Figure 3.

The improvement obtained by the more involved fractal strategy is exhibited by the following family of games, providing an example of the methods used to prove the lower and upper bounds on standard-form 3-pile games:

Proposition 5.2. $62m + 60 \ge V([31, 32m, 32m+31]) \ge 62(m-1) + 98.$

Proof. We observe that in any turn, Luca can take at most 63 times as many stones as Windsor will be able to subsequently take (for a more formal proof, see Lemma 6.2). This implies that

$$V([31, 32m, 32m+31]) \le \tfrac{63-1}{63+1} \cdot (31 + 32m + 32m + 31) = 60.0625 + 62m.$$

We use a fractal strategy to give a lower bound. Consider the following strategy, broken down into $m > 1$ and $m = 1$.

(1) While $m > 1$, Luca recursively removes 63 candies from the largest pile, requiring Windsor to respond by removing one candy from the middle pile, creating the turn

$$T = (G, [31, 32m, 32(m-1)], [31, 32(m-1)+31, 32(m-1)]).$$

The accumulated value is $\sum_{i=2}^{m}(63-1) = 62(m-1)$.

(2) When $m = 1$, the turn is

$$T_4 = ([31, 32, 63], [31, 32, 3], [31, 28, 3]),$$

with a single-turn value of $V_{T_4}(G_4) = 60 - 4 = 56$.

(a) Then, for all games $G_2 = [3, 4m', 4m'+3]$ with $m' > 1$, the turn would be

$$T = (G_2, [3, 4m', 4(m'-1)], [3, 4(m'-1)+3, 4(m'-1)]).$$

The accumulated value for all G_2 is $\sum_{m'>1}(7-1) = 6(7-1) = 36$.

(b) When $m' = 1$, the turn is $T_1 = ([3, 4, 7], [3, 4, 1], [3, 2, 1])$. Finally, the last two turns are $T_0 = ([1, 2, 3], [1, 2], [1, 1])$ and $T_0' = ([1, 1], [1], \varnothing)$. In total, we get an overall value of $62(m-1)+56+36+4+2+0 = 62(m-1)+98$. □

Remark 5.3. We have verified numerically that $V([31, 32m, 32m+31]) = 62(m-1)+98$ for $m < 12$. We conjecture that equality holds for all $m \in \mathbb{N}$.

6. Proofs of Theorems 3.3 and 3.5

The strategies of Section 4 enable us to give lower bounds on $V(G)$ for standard-form and general 3-pile games. To give upper bounds, we will consider maximal candy allocations turn-by-turn via the *semiratio*:

Definition 6.1. The *semiratio* $r_T(G)$ of a turn $T = (G, G', G'')$ is defined to be

$$r_T(G) = \frac{N(G)-N(G')}{N(G')-N(G'')}.$$

Lemma 6.2. *Given $G = \mathfrak{G}(a, m, x)$ and any turn $T = (G, G', G'')$ the semiratio $r_T(G)$ is at most $2a+1$.*

Proof. Consider a turn $T = (G, G', G'')$. We show $r_T(G) \le 2a+1$. If Luca's ply (G, G') is in the smallest pile, she would take at most a candies, yielding a semiratio at most $a < 2a+1$. Suppose Luca moves in either the middle pile or the largest pile such that $G'' = \mathfrak{G}(a', m', x')$, where $a' \neq a$. There are several cases to consider.

<u>Case 1</u>: Suppose $G' = [a, 2^{k+1}m + x, a']$ and $G'' = [a, a \oplus a', a']$. Then

$$V_T(G) = (2^{k+1}m + x \oplus a - a') - (2^{k+1}m + x - a' \oplus a) \le 2a.$$

Therefore, under this strategy,

$$r_T(G) = \frac{N(G) - N(G')}{N(G') - N(G'')} \leq 2a + 1.$$

Case 2: Suppose $G' = [a, a', 2^{k+1}m + x \oplus a]$ and $G'' = [a, a \oplus a', a']$. Then

$$V_T(G) = (2^{k+1}m + x - a') - (2^{k+1}m + x \oplus a - a' \oplus a) \leq 2x.$$

Therefore, $r_T(G) \leq 2x + 1 \leq 2a + 1$ under this strategy.

Case 3: Suppose

$$G' = [a, 2^{k+1}m + x, 2^{k+1}m + x \oplus a'],$$
$$G'' = [a', 2^{k+1}m + x, 2^{k+1}m + x \oplus a'].$$

Then

$$V_T(G) = (x \oplus a - x \oplus a') - (a - a') \leq a' + x - |a' - x|.$$

This single-turn value is either $2x$ or $2a'$, so $r_T(G) \leq 2\max(x, a') + 1 \leq 2a + 1$ under this strategy.

Case 4: Suppose

$$G' = [a, 2^{k+1}m + x \oplus a \oplus a', 2^{k+1}m + x \oplus a],$$
$$G'' = [a', 2^{k+1}m + x \oplus a \oplus a', 2^{k+1}m + x \oplus a].$$

Then

$$V_T(G) = x - x \oplus a \oplus a' - (a - a') \leq x - a + a'.$$

Therefore, $r_T(G) \leq 2a + 1$ under this strategy.

The last possible situation is when $G'' = \mathfrak{G}(a, m', x')$. Here, there are two cases to consider.

Case 1: $m' = m$ and $x' < x$. Then

$$N_L(G) - N_L(G'') < 2a + 1.$$

Case 2: $m < m'$. Then $V_T(G)$ is maximized when

$$G' = [a, 2^{k+1}m + x, 2^{k+1}m' + x'].$$

Here,

$$\begin{aligned} V_T(G) &= ((2^{k+1}m + x \oplus a) - (2^{k+1}m' + x')) - ((2^{k+1}m + x) - (2^{k+1}m' + x' \oplus a)) \\ &\leq 2a. \end{aligned}$$

In both cases, $r_T(G) = (N(G) - N(G'))/(N(G') - N(G'')) \leq 2a + 1$. □

From the above, we conclude our main results concerning 3-pile CANDY NIM games:

Proof of Theorem 3.3. Upper bound: We will first show that

$$V(G) = V(\mathfrak{G}(2^{k+1}-1, m, 0)) \le (2^{k+1}-2)m + (2^{k+1}-2) - 2 + \delta_{0k}.$$

By Lemma 6.2, $r_T(G) \le s = 2^{k+2}-1$. Then

$$V_L(G) \le \frac{s-1}{s+1} N(G) = (2^{k+2}-2)m + (2^{k+2}-2) - 2 + \frac{1}{2^k}.$$

Lower bound: Given the game $G = \mathfrak{G}(2^{k+1}-1, m, 0)$, consider the strategy where Luca removes $2^{k+2}-1$ candies from the largest pile when $m > 1$ and $2^{k+2} - 2^{\lfloor (k+1)/2 \rfloor}$ from the largest pile when $m = 1$. Then

$$V_L(G) \ge 2(2^{k+1}-1)m - 2(2^{\lceil k/2 \rceil}-1) + V(\mathfrak{G}(2^{\lceil k/2 \rceil}-1, 2^{\lfloor k/2 \rfloor + 1}-1, 0)).$$

This is an example of the fractal strategy as in Definition 4.4 with $f(k) = \lfloor (k+1)/2 \rfloor$. □

Proof of Corollary 3.4. Let

$$G = \mathfrak{G}(a, m, x) = [a, 2^{\lfloor \log_2 a \rfloor + 1} m + x, 2^{\lfloor \log_2 a \rfloor + 1} m + x \oplus a].$$

Consider the first turn $T_0 = (G, G', G'')$ such that

$$G' = [a, 2^{\lfloor \log_2 a \rfloor + 1} m + x, 2^{\lfloor \log_2 a \rfloor + 1}(m-1)],$$
$$G'' = [a, 2^{\lfloor \log_2 a \rfloor + 1}(m-1) + a, 2^{\lfloor \log_2 a \rfloor + 1}(m-1)].$$

Then $V_{T_0}(G) = x \oplus a + a - x$. For $0 < i < m$, let the i-th turn be $T_i = (G_i, G_i', G_i'')$, where (similar to the flip-flop strategy of Definition 4.1)

$$G_i = [a, 2^{\lfloor \log_2 a \rfloor + 1}(m-i), 2^{\lfloor \log_2 a \rfloor + 1}(m-i) + a],$$
$$G_i' = [a, 2^{\lfloor \log_2 a \rfloor + 1}(m-i), 2^{\lfloor \log_2 a \rfloor + 1}(m-i-1)],$$
$$G_i'' = [a, 2^{\lfloor \log_2 a \rfloor + 1}(m-i-1) + a, 2^{\lfloor \log_2 a \rfloor + 1}(m-i-1)],$$

with $V_{T_i}(G) = 2a$. After turn T_{m-1}, $G''_{m-1} = [a, a]$. Thus, the game concludes after m turns, and in total, Luca takes $2a(m-1) + x \oplus a + a - x$ candies. □

Proof of Theorem 3.5. Let $G = [2^{k+1}-1, 2^{k+1}m + x, 2^{k+1}m + 2^{k+1} - 1 - x]$.

Upper bound: We construct a strategy S that achieves a value of

$$V_S(G) = V(\mathfrak{G}(2^{k+1}-1, m-1, 0)) + 2(2^{k+1}-1) - 2x.$$

Let the first turn $T_1 = (G, G', G'')$ consist of

$$G' = [2^{k+1}-1, 2^{k+1}m + x, 2^{k+1}(m-1)],$$
$$G'' = [2^{k+1}-1, 2^{k+1}(m-1) + 2^{k+1} - 1, 2^{k+1}(m-1)].$$

Then the single-turn value is $V_{T_1}(G) = 2(2^{k+1} - 1) - 2x$, and we have that $G'' = \mathfrak{G}(2^{k+1} - 1, m - 1, 0)$, yielding an overall value of

$$V_S(G) = V(\mathfrak{G}(2^{k+1} - 1, m - 1, 0)) + 2(2^{k+1} - 1) - 2x,$$

which gives a lower bound on $V(G)$.

Lower bound: We prove

$$V(\mathfrak{G}(2^{k+1} - 1, m + 1, 0)) - 2(2^{k+1} - 1) + 2x \geq V(G).$$

Given game $G_0 = \mathfrak{G}(2^{k+1} - 1, m + 1, 0)$, under any strategy S,

$$V_S(G_0) \leq V(G_0).$$

Suppose the first turn in S is $T_1 = (G_0, G_0', G_0'')$, where

$$\begin{aligned} G_0' &= [2^{k+1} - 1, 2^{k+1}(m + 1), 2^{k+1}m + x], \\ G_0'' &= [2^{k+1} - 1, 2^{k+1}m + x, 2^{k+1}m + 2^{k+1} - 1 - x]. \end{aligned}$$

Note that G_0'' is the only move to a $\mathcal{P}$ position from G_0'. The resulting game is $G_0'' = G$, implying that $V_S(\mathfrak{G}(2^{k+1} - 1, m+1, 0)) = 2(2^{k+1} - 1) - 2x + V(G)$. Thus

$$\begin{aligned} 2(2^{k+1} - 1) - 2x + V(G) &= V_S(\mathfrak{G}(2^{k+1} - 1, m + 1, 0)) \\ &\leq V(\mathfrak{G}(2^{k+1} - 1, m + 1, 0)). \end{aligned}$$ □

7. Optimal allocation of N candies

Thus far, we have only considered CANDY NIM positions with three piles. We have seen that, in these games, Luca can take a substantial majority of the candies, and indeed there are 3-pile $\mathcal{P}$ positions in which Luca takes a proportion of at least $1 - \varepsilon$ of the candies, for any fixed $\varepsilon > 0$. It is natural, then, to consider the problem of Luca allocating N candies, in a $\mathcal{P}$ position, so that she maximizes the number of candies that she can take with optimal play. This problem is the motivating question for this section.

Windsor always obtains a logarithmic number of candies no matter how Luca plays, reaching this bound only for a small family of values of N and a unique arrangement for each such N.

Theorem 7.1. *Given a game G, $N_W(G) \geq \lfloor \log_2(N(G)) \rfloor$. Equality is achieved only when $N(G) = 2^n$, $2^n - 2$, or $2^n - 2^k - 2$, for $n, k \in \mathbb{Z}^+$, $n > k + 1$, $n > 2$ in the following arrangements*:

(1) $N(G) = 2^n$ *and* $G = [1, 1, 1, 2, 4, 8, \dots, 2^{n-2}, 2^{n-1} - 1]$.

(2) $N(G) = 2^n - 2$ *and* $G = [1, 2, 4, 8, \dots, 2^{n-2}, 2^{n-1} - 1]$.

(3) $N(G) = 2^n - 2^k - 2$ *and* $G = [1, 2, 4, 8, \dots, 2^{k-2}, 2^k, \dots, 2^{n-2}, 2^{n-1} - 1 - 2^{k-1}]$.

The proof of the above result, Theorem 7.1, is rather tedious; as a result, we defer the details to Section 8. For small N, we can use the above result to identify the games G that minimize $N_W(G)$.

Example. For $N = 10$, 12, 14, 16, we find the unique games G that minimize $N_W(G)$ via Theorem 7.1.

- If $N = 10$, then $G = [1, 4, 5]$ minimizes N_W, with $N_W(G) = 3$.
- If $N = 12$, then $G = [2, 4, 6]$ minimizes N_W, with $N_W(G) = 3$.
- If $N = 14$, then $G = [1, 2, 4, 7]$ minimizes N_W, with $N_W(G) = 3$.
- If $N = 16$, then $G = [1, 1, 1, 2, 4, 7]$ minimizes N_W, with $N_W(G) = 4$.

We can guarantee that Windsor only obtains $\Theta(\sqrt{N})$ candies for all even values of N by judicious arrangement of N candies into at most 5 piles. This constant-pile example represents a contrast to the above result, which required $\Theta(\log N)$ piles to meet the bound:

Theorem 7.2. *For all $N \in 2\mathbb{Z}^+$, there exists a 5-pile game G with $N(G) = N$ and $N_W(G) \le \frac{3}{2}\sqrt{2N} - 2$.*

Proof. We can write N in binary as

$$N = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_n} + 2^{k_{n+1}} + 2^{k_{n+2}} + \cdots + 2^{k_{n+m}},$$

where $k_1 > k_2 > \cdots > k_{n+p}$, where n is defined so that $k_n \ge \lfloor k_1/2 \rfloor$ but $k_{n+1} < \lfloor k_1/2 \rfloor$. Therefore n is the minimal i such that $2^{k_{i+1}} < \sqrt{N}$. Consider the game $G_1 = \mathfrak{G}(a, m, 0)$, where

$$m = 2^{k_1 - \lfloor k_1/2 \rfloor} + 2^{k_2 - \lfloor k_1/2 \rfloor} + 2^{k_3 - \lfloor k_1/2 \rfloor} + \cdots + 2^{k_n - \lfloor k_1/2 \rfloor} - 1,$$
$$a = 2^{\lfloor k_1/2 \rfloor - 1} - 1.$$

By construction, $N(G_1) < N$. From this, we can construct the game

$$G = [2^{\lfloor k_1/2 \rfloor - 1} - 1, 2^{k_1 - 1} + 2^{k_2 - 1} + \cdots + 2^{k_n - 1} - 2^{\lfloor k_1/2 \rfloor - 1}, 2^{k_1 - 1} + 2^{k_2 - 1} + \cdots + 2^{k_n - 1} - 1, 2^{k_{n+1} - 1} + 2^{k_{n+2} - 1} + \cdots + 2^{k_{n+m} - 1} - 1, 2^{k_{n+1} - 1} + 2^{k_{n+2} - 1} + \cdots + 2^{k_{n+m} - 1} - 1],$$

where $N(G) = N$. Note the last two piles of G are identical. Corollary 3.4 gives

$$N_W(G_1) \le 2^{\lfloor k_1/2 \rfloor} - 2 + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_n} - 2^{\lfloor k_1/2 \rfloor} - (2^{k_1} + 2^{k_2} + \cdots + 2^{k_n}) + r_N, \quad r_N \le \sqrt{2N},$$

and therefore

$$N_W(G) \le \tfrac{3}{2}\sqrt{2N} - 2,$$

because

$$N_W(G) = 2^{k_{n+1} - 1} + 2^{k_{n+2} - 1} + \cdots + 2^{k_{n+p} - 1} + r_N - 2. \qquad \square$$

Via induction, we can also give a lower bound on $N_W(G)$ as a function of the number of piles of a game G (this reflects the $\log_2(N)$ bound on $N_W(G)$ shown with the arrangement of Theorem 7.1). We defer the details of this proof to Section 9:

Theorem 7.3. *If G is a game containing p piles with no duplicate piles, then $N_W(G) \geq p-1$.*

8. Proof of Theorem 7.1

We work to show the upper bound $N_W(G) \geq \lfloor \log_2(N(G)) \rfloor$, and explicitly enumerate the games G for which this inequality is an equality.

Lemma 8.1. *If $G \in \mathcal{P}$, for any ply (G, G'), we have $N(G) - N(G') \leq \frac{1}{2} N(G)$.*

Proof. Let $G = [a_1, a_2, \ldots, a_p]$, where $a_1 \geq a_2 \geq \cdots \geq a_p$. For any ply (G, G'), we have $N(G) - N(G') \leq a_1$. Therefore, it suffices to show that $a_1 \leq \frac{1}{2} N(G)$. Since $G \in \mathcal{P}$, we have $a_1 = a_2 \oplus a_3 \oplus \cdots \oplus a_p$. For any $x_1, \ldots, x_k$, we have $x_1 \oplus \cdots \oplus x_k \leq x_1 + \cdots + x_k$, so $a_1 \leq a_2 + a_3 + \cdots + a_p$. Since $N(G) = a_1 + a_2 + a_3 + \cdots + a_p$, this implies that $a_1 \leq N(G) - a_1$. Thus $a_1 \leq \frac{1}{2} N(G)$. $\square$

Lemma 8.2. *For any game G,*

$$N_W(G) \geq \lfloor \log_2 N(G) \rfloor. \tag{8-1}$$

Proof. We induct on $N(G)$. As our base case, we consider the position where $N(G) = 1$, when $N_W(G) = 1 > \log_2 N(G) = 0$. Now, we perform the inductive step. Fix a game G and suppose that the result holds for any game H such that $N(H) < N(G)$. Let $n = \lfloor \log_2 N(G) \rfloor$. We divide our analysis into two cases:

(1) If G is a $\mathcal{P}$ position, consider a ply (G, G'). Then $N(G') \geq \frac{1}{2} N(G) \geq 2^{n-1}$ by Lemma 8.1. Suppose first that Windsor only removes a single candy when going from G' to G'', i.e., $N(G'') = N(G') - 1$. If $N(G') > 2^{n-1}$, then

$$N_W(G'') \geq \lfloor \log_2 N(G'') \rfloor \geq n-1,$$

so

$$N_W(G) \geq 1 + N_W(G'') \geq n = \lfloor \log_2 N(G) \rfloor,$$

proving the desired result. On the other hand, if $N(G') = 2^{n-1}$, then $N(G'') = 2^{n-1} - 1$ has an odd number of candies and is thus an $\mathcal{N}$ position, meaning that Windsor's last ply was invalid.

The only case left to consider is if Windsor removes at least two candies, i.e., if $N(G'') \leq N(G') - 2$. Since $N(G') - N(G'') > 1$, we have

$$N(G') - N(G'') > \lfloor \log_2(N(G')) \rfloor - \lfloor \log_2(N(G'')) \rfloor.$$

If $N(G'') = 0$, then $G = [a, a]$ where (8-1) holds, and if $N(G'') \neq 0$, then

$$\begin{aligned} N_W(G) - N_W(G'') &= N(G') - N(G'') \\ &> \lfloor \log_2(N(G')) \rfloor - \lfloor \log_2(N(G'')) \rfloor \\ &= n - 1 - \lfloor \log_2(N(G'')) \rfloor, \end{aligned}$$

which implies that

$$N_W(G) > n - 1 - \lfloor \log_2(N(G'')) \rfloor + N_W(G'').$$

Since $N_W(G'') \geq \lfloor \log_2(N(G'')) \rfloor$, this gives $N_W(G) \geq n$, as desired.

(2) Now suppose G is an $\mathcal{N}$ position. Consider a ply (G, G'). Since Windsor moves $G \mapsto G'$,

$$N_W(G) - N_W(G') = N(G) - N(G') \geq \lfloor \log_2(N(G)) \rfloor - \lfloor \log_2(N(G')) \rfloor$$

whenever $N(G') > 0$. By the inductive hypothesis, $N_W(G') \geq \lfloor \log_2(N(G')) \rfloor$, so $N_W(G) \geq \lfloor \log_2(N(G)) \rfloor$, as desired. □

For convenience, we will represent the concatenation of two games by addition:

Definition 8.3. Games $G = [g_1, g_2, g_3, \dots, g_p]$ and $H = [h_1, h_2, h_3, \dots, h_q]$ have sum $G + H$ defined by concatenation as follows:

$$G + H = [g_1, g_2, g_3, \dots, g_p, h_1, h_2, h_3, \dots, h_q].$$

Lemma 8.4. *If $K = [a_1, a_1, a_2, a_2, \dots, a_p, a_p]$, then for all games G, $V(G) = V(G + K)$.*

Proof. We induct on $N(G) + N(K)$. First, the base case $G = K = \varnothing$ is trivial. Now we consider the inductive step. Consider a turn $T = \big(H := G + K, H', H''\big)$. If the optimal move is in G, then $H' = G' + K$, with $N(G') < N(G)$. Thus, by the inductive hypothesis, $V(H') = V(G' + K) = V(G')$, so

$$V(G + K) = N(G) - N(G') + V(G' + K) = N(G) - N(G') + V(G') = V(G).$$

If (G, G') is the optimal ply in G, by the same argument we have

$$V(G) = N(G) - N(G') + V(G').$$

On the other hand, for any ply (K, K'), the opponent can mimic in K, and hence move to $H'' = G + K''$, where K'' consists of only equal piles. It follows that

$$V(G + K') \leq N(K) - N(K') + V(G + K'').$$

Thus no move in K can be strictly better than the optimal move in G, so we have $V(G + K) = V(G)$, completing the inductive step. □

Lemma 8.5. *For every positive integer a, there exists a positive integer k such that $a \oplus (a - 1) = 2^k - 1$.*

Proof. If a is odd, then $a \oplus (a-1)$ is 1, or $2^1 - 1$. If a is even, then we write

$$a = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_\ell}, \quad k_1 > k_2 > \cdots > k_\ell > 0.$$

Then we have

$$a - 1 = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_{\ell-1}} + 2^{k_\ell - 1} + 2^{k_\ell - 2} + \cdots + 2^3 + 2^2 + 2^1 + 1.$$

This gives the desired result:

$$a \oplus (a-1) = 2^{k_\ell} + 2^{k_\ell - 1} + \cdots + 2^3 + 2^2 + 2^1 + 1 = 2^{k_l + 1} - 1. \qquad \square$$

Lemma 8.6. *The game $G = [1, 2, 4, 8, 16, \ldots, 2^{n-2}, 2^{n-1} - 1]$ maximizes $N_L(G)$ subject to the constraint that $N(G) = 2^n - 2$. In this case, we have $N_W(G) = n - 1$.*

Proof. First, let us compute $N_W([1, 2, 4, 8, 16, \ldots, 2^{n-2}, 2^{n-1} - 1])$. If Luca removes the entire largest pile, then Windsor is forced to remove a single candy, leaving the game $G' = [1, 2, 4, 8, 16, \ldots, 2^{n-3}, 2^{n-2} - 1]$. When $n = 2$ we get $N_W = 1$. By induction, $N_W = n - 1$. By Lemma 8.2, $N_W(G) \geq n - 1$, for an arbitrary $\mathcal{P}$ position G with $N(G) = 2^n - 2$. Since $G = [1, 2, 4, 8, \ldots, 2^{n-2}, 2^{n-1} - 1]$ achieves equality, it minimizes N_L subject to $N(G) = 2^n - 2$. $\square$

Lemma 8.7. *Given*

$$G = [1, 2, 4, 8, \ldots, 2^{k-2}, 2^k, \ldots, 2^{n-2}, 2^{n-1} - 1 - 2^{k-1}],$$

the ply $P = (G, G')$ with

$$G' = [1, 2, 4, 8, \ldots, 2^{k-2}, 2^k, \ldots, 2^{n-2}, 2^{k-1}]$$

is an optimal move. Then

$$N_W(G) = N_W(G') = n - 1.$$

Proof. Lemma 8.2 shows that it is impossible for Luca to concede fewer than $n-1$ candies to Windsor. Therefore, to show optimality, it suffices to show that $N_W(G) = n - 1$. If Luca moves the pile of size $2^{n-1} - 1 - 2^{k-1}$ to a pile of size 2^{k-1}, the remaining game G' has $2^n - 2$ candies, with $N_W(G') = n - 1$ by Lemma 8.6 assuming optimal play by Luca. Thus, G minimizes N_W with $N_W(G) = n - 1$, as desired. $\square$

Proof of Theorem 7.1. First, note that it is sufficient to prove the result when G is a $\mathcal{P}$ position. To see this, suppose that we have proven the result for all $\mathcal{P}$ positions, and G is an $\mathcal{N}$ position with a ply (G, G'), where G' is a $\mathcal{P}$ position. Then

$$\begin{aligned} N_W(G) &\geq N(G) - N(G') + N_W(G') \\ &\geq N(G) - N(G') + \lfloor \log_2(N(G')) \rfloor \geq \lfloor \log_2(N(G)) \rfloor. \end{aligned}$$

Thus from now on, we shall always assume that G is a $\mathcal{P}$ position.

We prove the result by induction on $N(G)$. Via a finite check, this is true whenever $N(G) \leq 16$. For the inductive step, suppose that equality is achieved only in the above positions for all positions with $N(G) < M$. We want to show that if $N(G) = M$, this theorem holds.

First we show that $N_W(G) = \lfloor \log_2(N(G)) \rfloor$ implies $N_W(G'') = \lfloor \log_2(N(G'')) \rfloor$. Let $M = 2^n + x$, where $n = \lfloor \log_2 M \rfloor$. Then

$$2^{n-1} \leq 2^{n-1} + \tfrac{1}{2}x \leq N(G') \leq 2^n + x - 1 \leq 2^{n+1}.$$

If $N(G') - N(G'') = 1$, then

$$2^{n-1} - 1 \leq N(G'') < 2^{n+1} - 1.$$

Since G is a $\mathcal{P}$ position, $N(G)$ is even and thus $N(G'') \neq 2^{n-1} - 1$. Thus $2^{n-1} \leq N(G'') < 2^{n+1} - 1$, so by Lemma 8.2, $N_W(G'') \geq n - 1$. If $N_W(G'') \geq n$, then $N_W(G) \geq n+1$, so in any potential equality case, we must have $N_W(G'') = n-1$. If $N(G') - N(G'') \geq 2$, then whenever $N(G'') > 0$ and $N(G') - N(G'') > 1$, we have

$$N(G') - N(G'') > \lfloor \log_2(N(G')) \rfloor - \lfloor \log_2(N(G'')) \rfloor.$$

Since $N(G'') \geq \lfloor \log_2(N(G'')) \rfloor$, if $N_W(G) = \lfloor \log_2(N(G)) \rfloor$, then $N_W(G'') = \lfloor \log_2(N(G'')) \rfloor$. If $N(G'') = 0$, then $N(G') = N(G)/2$. Therefore, $N_W(G) = \lfloor \log_2(N(G)) \rfloor$, which implies $N_W(G'') = \lfloor \log_2(N(G'')) \rfloor$. Thus, by the inductive hypothesis, G'' must be one of the three positions above.

Now we show that if G'' is any one of the above three positions, then so is G, thereby completing the induction.

(1) Suppose that

$$G'' = [1, 1, 1, 2, 4, 8, \ldots, 2^{n-3}, 2^{n-2} - 1].$$

In order to have

$$N_W(G') - N_W(G'') = \lfloor \log_2(N(G)) \rfloor - \lfloor \log_2(N(G'')) \rfloor,$$

we must have $N(G) \in \{2^n, 2^n + 2\}$. If $N(G) = 2^n + 2$, then $N(G) - N(G') = 2^{n-1} + 1$. However, this implies that

$$G = [2^{n-1} + 1, 2^{n-1} + 1] \quad \text{or} \quad G = [1, 2^{n-1}, 2^{n-1} + 1],$$

since those are the only two $\mathcal{P}$ positions with $N(G) = 2^n + 2$. Neither of those can produce a G'' of the specified form. Therefore, $N(G) = 2^n$ and $N(G) - N(G') = 2^{n-1} - 1$. and there must have been a pile of size at least $2^{n-1} - 1$ in G. If there was a pile of size at least 2^{n-1}, we have the same issue as above with $2^n + 2$. Consequently, there must be a pile of size exactly $2^{n-1} - 1$. If $N(G) = 2^n$, Windsor removed one candy on the first term, giving the Grundy value of G', $\mathcal{G}(G') \in \{1, 3, 2^{n-1}-1\}$. In the first two cases, there is no way to achieve $N(G) - N(G') = 2^{n-1} - 1$. Therefore,

$$G = [1, 1, 1, 2, 4, 8, \dots, 2^{n-2}, 2^{n-1} - 1].$$

(2) Now suppose that

$$G'' = [1, 2, 4, 8, \dots, 2^{n-3}, 2^{n-2} - 1].$$

As Windsor removed one candy, $\mathcal{G}(G') \in \{1, 3, 2^{n-1}-1\}$. If $\mathcal{G}(G') = 1$, then

$$G = [1, 2, 4, 8, \dots, 2^l + 1, \dots, 2^m + 1, \dots, 2^{n-3}, 2^{n-2} - 1],$$

which allows Windsor to remove one candy from a different pile to increase his winnings, contradicting optimal play. If $\mathcal{G}(G') = 3$, then

$$G = [2, 2, 4, 8, \dots, 2^l + 3, \dots, 2^{n-2} - 1] \quad \text{or} \quad G = [2, 2, 3, 4, \dots, 2^{n-2} - 1].$$

Windsor could have removed three from the $2^{n-2} - 1$ and received more candies while still winning, again contradicting optimal play. If $\mathcal{G}(G') = 2^{n-1} - 1$, then

$$G' = [1, 2, 4, 8, \dots, 2^{n-3}, 2^{n-2}].$$

So, either Luca moved from $2^{n-1} - 2^k - 1$ to 2^k or from $2^{n-1} - 1$ to 0. The first case gives the third game above, and the second gives the second game above.

(3) Finally, suppose that

$$G'' = [1, 2, 4, 8, \dots, 2^{k-2}, 2^k, \dots, 2^{n-3}, 2^{n-2} - 2^{k-1} - 1],$$

with $\mathcal{G}(G') \in \{1, 3, 2^k - 1\}$. If $k \geq 2$, then as $G - G'' \geq 2^k + 1$, it would be impossible for Windsor to both remove one, and have $\lfloor \log_2(G'') \rfloor < \lfloor \log_2(G) \rfloor$. Otherwise $k = 1$, $\mathcal{G}(G') = 1$, and thus $G = 2 + G''$ so $\lfloor \log_2(G'') \rfloor = \lfloor \log_2(G) \rfloor$, a contradiction. □

9. Proof of Theorem 7.3

Via induction and some careful casework, we show that Windsor can always take at least one candy fewer than the number of piles in a game G.

Proof of Theorem 7.3. We prove this by induction on $N(G)$. When $N(G) < 2$, the result is trivial. When $N(G) = 2$, then G is either [2] or [1, 1]. The first one gives $2 \geq 0$, and the second $1 \geq 1$, as desired. Suppose the claim is true for all G with $N(G) < n$. We show it holds when $N(G) = n$.

($\mathcal{N}$) Let G be an $\mathcal{N}$ position. Then the number of piles in G' is at most one fewer than that of G. So, either $N_W(G') \geq p - 2$ or Windsor made a move to make two piles equal sizes. In the first case, Windsor must have removed at least one candy, so $N_W(G) \geq p-1$, as desired. If Windsor moved to create a duplicate pile,

$$G' = [a, a, g_1, g_2, g_3, \dots, g_{p-2}],$$

where the g_i's are all distinct, then by Lemma 8.4,

$$N_W(G') = a + N_W([g_1, g_2, \dots, g_{p-2}]).$$

By induction, $N_W([g_1, g_2, \dots, g_{p-2}]) \geq p-3$. Because $a \geq 1$, we get that $N_W(G') \geq p-2$, so $N_W(G) \geq p-1$, as desired

($\mathcal{P}$) Suppose G is a $\mathcal{P}$ position.

(1) If Luca doesn't remove a full pile, then G' has the same number of piles as G. We consider cases:

(a) If there are no duplicates in G', by the inductive hypothesis, $N_W(G) = N_W(G') \geq p-1$, as desired.

(b) Suppose Luca creates a duplicate pile, so

$$G' = [a, a, g_3, \dots, g_p].$$

Then we have $N_W(G') = a + N_W([g_3, \dots, g_p])$. If $a \neq 1$, $N_W(G) = N_W(G') \geq 2 + p - 3 = p - 1$, via inductive hypothesis. Suppose

$$G' = [1, 1, g_3, \dots, g_p].$$

In that case, we must have had $G = [g_1, g_2, g_3, \dots, g_p]$ with $g_1 = 1$. Windsor cannot move in a 1-pile, or else Luca would have been able to move to $G'' = [g_2, g_3, \dots, g_p]$, contradicting the assumption that $G \in \mathcal{P}$. So, his winning move must be in one of the piles $g_3, \dots, g_p$. If Windsor doesn't remove a pile, we get

$$N_W(G) = N_W(G') = 2 + N_W([z, g_4, \dots, g_p]) \geq p - 1, \tag{9-1}$$

where the first equality holds because it is currently Luca's move, and the second equality follows from Lemma 8.4 and because Windsor removed one candy. The inequality follows from the inductive hypothesis.

(c) If Luca first creates a 1, 1 duplicate (i.e., moves a pile g_2 to size 1 with an existing pile g_1 of size 1) to obtain G', Windsor removes a pile in G'. We have

$$N_W(G) = N_W(G') = 1 + g_3 + N_W([g_4, \dots, g_p]) \geq 1 + g_3 + p - 4,$$

where g_3 is the pile Windsor removes. If $g_3 \neq 1$, we have

$$N_W(G) \geq 1 + 2 + p - 4 = p - 1,$$

as desired. But if $g_3 = 1$, then G had a 1, 1 duplicate already, contrary to hypothesis.

(2) Suppose Luca removes a pile. We have $G' = [0, g_2, g_3, \dots, g_p]$. We further subdivide into cases:

(a) If Windsor removes a pile g_2, then it is Luca's turn, so $g_1 \oplus g_2 = 0$ and $g_1 = g_2$, giving an initial duplicate pile.

(b) Suppose Windsor doesn't remove a pile and creates no duplicate piles when he moves G' to G''. Via the inductive hypothesis $N_W(G'') \geq p-2$. Since Windsor removed at least one candy, $N_W(G) \geq p-1$, as desired.

(c) Suppose Windsor removes no entire pile, but creates some duplicate pile of size $a \geq 2$, so $G'' = [a, a, g_4, g_5, \dots g_p]$ with

$$N_W(G'') = a + N_W([g_4, g_5, \dots, g_p]) \geq a + p - 4.$$

Since $a \geq 2$, and Windsor removed at least one candy,

$$N_W(G) \geq 1 + a + p - 4 \geq 1 + 2 + p - 3 = p - 1,$$

as desired.

(d) Finally, suppose that Windsor removes some candies to create a duplicate pile of size 1 with $G'' = [1, 1, g_4, g_5, \dots, g_p]$. This would give

$$G' = [1, 2, g_4, g_5, \dots, g_p], \quad G = [1, 2, 3, g_4, g_5, \dots, g_p],$$

as Luca removed a pile (so no other pile had size 2). Since $G'' \in \mathcal{P}$, $\mathcal{G}(G') = 3$. It suffices to show that if $H = [g_4, g_5, \dots, g_p]$, then $N_W(H) \geq p-3$. If $H = \varnothing$, we are done. Thus suppose H has at least one pile. Note that for all $i \geq 4$, $g_i > 1$ and $g_i \equiv 0, 1 \pmod 4$, and all these piles of H are distinct. We can consider the possible moves in Luca's ply (H, H') as we did above.

• Any duplicate pile created has size at least 4, so creating a duplicate pile would yield the desired bound:

$$N_W(H) = N_W(H') \geq 4 + N_W([g_6, g_7, \dots, g_p]) \geq 4 + p - 6 = p - 2.$$

• If Luca neither removes a pile nor creates a duplicate, Windsor must move in a distinct pile from Luca. If Windsor removed a pile, he removed at least four candies, so $N_W(G) \geq n - 1$. Since H is duplicate-free, Windsor cannot create a duplicate. Thus, if Windsor didn't remove a pile, we thus obtain $H'' = [a, b, g_6, g_7, \dots, g_p]$, with

$$\begin{aligned} N_W(G) = N_W(G') &\geq 1 + N_W(G'') \\ &= 2 + N_W(H') = 3 + N_W([a, b, g_6, g_7, \dots, g_p]) \geq n - 1. \end{aligned}$$

• Suppose Luca removes a pile. Since $g_i \equiv 0, 1 \pmod 4$ for all piles in H, Windsor must have removed at least three candies since $H'' \in \mathcal{P}$; furthermore, because H contains no duplicates, Windsor cannot have removed an entire pile in moving from H' to H''. Thus H'' consists of $p-4$ piles. If H'' has no duplicate piles, then by induction, $N_W(H'') \geq p-5$, so

$$N_W(H) \geq 3 + (p-5) = p-2,$$

which is greater than the required $p-3$. On the other hand, if H'' has a duplicate pile, say with $H'' = [g_6, g_6, g_7, g_8, \ldots, g_p]$, then

$$\begin{aligned} N_W(H) \geq 3 + N_W(H'') &= 3 + g_6 + N_W([g_7, g_8, \ldots, g_p]) \\ &\geq 3 + g_6 + (p-6) \\ &\geq p-3. \end{aligned}$$ □

10. Conjectures and concluding remarks

While our primary focus of this study was the 3-pile CANDY NIM game, there are a huge number of interesting open questions that remain.

4-*pile* CANDY NIM. Most of our attention with respect to strategies and bounds on $V(G)$ has been focused on the case when G is a 3-pile game. We include a brief analysis and several conjectures regarding $V(G)$ and optimal play for 4-pile games.

In the 4-pile game, Luca does not always have an optimal move in the largest pile.

Example. Let $G = [1, 5, 16, 20]$. We have $V([1, 5, 16, 20]) = 28$, where Luca's optimal move is to remove three candies from the pile of size 5. By checking, we have the following optimal game play:

$$\begin{aligned} &[1, \overline{5}, 16, 20] \xrightarrow{\overline{L}} [1, \overline{2}, 16, \underline{20}] \xrightarrow{\underline{W}} [1, 2, 16, \underline{1}\overline{9}] \xrightarrow{\overline{L}} [1, 2, \overline{12}, \underline{16}] \xrightarrow{\underline{W}} [1, 2, 12, \underline{1}\overline{5}] \\ &\quad \xrightarrow{\overline{L}} [1, 2, \overline{8}, \underline{12}] \xrightarrow{\underline{W}} [1, 2, 8, \underline{1}\overline{1}] \xrightarrow{\overline{L}} [1, 2, \underline{8}, \overline{4}] \xrightarrow{\underline{W}} [1, 2, \underline{7}, 4] = [1, 2, 4, 7]. \end{aligned}$$

By Theorem 7.1, $V([1, 2, 4, 7]) = 8$. Thus,

$$V(G) = 20 + 8 = 28.$$

We can obtain lower bounds on some families of 4-pile games G using related 3-pile games. We first consider the 4-pile games G with smallest two piles of size 1, 2, and show that their values $V(G)$ are bounded by the "corresponding" 3-pile game with smallest pile size 3.

Proposition 10.1. *Let m be a positive integer. Then both of the following hold*:

$$V([1, 2, 4m, 4m+3]) \geq V([3, 4m, 4m+3]),$$
$$V([1, 2, 4m+1, 4m+2]) \geq V([3, 4m+1, 4m+2]).$$

Proof. Let

$$G_1 = [3, 4m, 4m+3], \qquad G_2 = [3, 4m+1, 4m+2],$$
$$H_1 = [1, 2, 4m, 4m+3], \quad H_2 = [1, 2, 4m+1, 4m+2].$$

We will show the desired result by induction on m. Let $m = 1$ be our base case. We can check $6 = V([3, 4, 7]) \le V([1, 2, 4, 7]) = 8$ and $V([3, 5, 6]) = V([1, 2, 5, 6]) = 6$. For every possible optimal turn $T_{G_i} = (G_i, G_i', G_i'')$ $(i = 1, 2)$, we show there exists a turn $T_{H_i} = (H_i, H_i', H_i'')$ such that

$$V_{T_{G_i}}(G_i) + V(G_i'') \le V_{T_{H_i}}(H_i) + V(H_i'').$$

Suppose that $V(G_i) \le V(H_i)$ for $m < w$. Let $m = w$. Suppose $G_i' = [3, a, b]$ and $G_i'' = [3, a, c]$. Then we set $H_i' = [1, 2, a, b]$ and $H_i'' = [1, 2, a, c]$, so that $V_{T_{G_i}}(G_i) = V_{T_{H_i}}(H_i)$ and $V(G_i'') \le V(H_i'')$, by the inductive hypothesis. If $G_i' = [2, a, b]$ or $G_i' = [1, a, b]$, then we set $H_i' = [0, 2, a, b]$ and $H_i' = [1, 0, a, b]$, respectively. This yields

$$V_{T_{G_i}}(G_i) + V(G_i'') = V_{T_{H_i}}(H_i) + V(H_i'').$$

Now suppose that $G_i'' = [0, a, b]$. If $i = 1$, then $V_{T_{G_1}}(G_1) + V(G_1'') = 0$ and $V_{T_{H_1}}(H_1) + V(H_1'') \ge 0$, by Lemma 2.8. If $i = 2$, then $V_{T_{G_2}}(G_2) + V(G_2'') \le 2$, which implies that it is not an optimal move since Luca could instead remove the largest pile in G_2 and obtain an overall value of 4. Thus, by induction, we have

$$V([1, 2, 4m, 4m+3]) \ge V([3, 4m, 4m+3]),$$
$$V([1, 2, 4m+1, 4m+2]) \ge V([3, 4m+1, 4m+2]). \qquad \square$$

General play. We can hope to make even more general inferences from the 3-pile game to multipile CANDY NIM games. Notably, we wonder if a similar result to Proposition 10.1 holds for a broader family of CANDY NIM games.

Question 10.2. *Suppose $G = [a, b, c]$ with $a < b < c$. Then for some $j > 1$, do there exist $a_1, \dots, a_j$ with*

$$a = a_1 + \cdots + a_j = a_1 \oplus \cdots \oplus a_j$$

such that the game

$$H = [a_1, a_2, \dots, a_j, b, c]$$

satisfies $V(H) \ge V(G)$?

Remark 10.3. It is *not true* that for any game $G = [a, b, c]$ and any such decomposition $a = a_1 + \cdots + a_j = a_1 \oplus \cdots \oplus a_j$ the resulting game H in Question 10.2 satisfies $V(H) \ge V(G)$. As a counterexample, consider the game $G = [31, 42, 53]$ with $a = 31$. Using the decomposition $a_1 = 1$, $a_2 = 2$, $a_3 = 4$,

$a_4 = 8$, $a_5 = 16$, we obtain the game $H = [1, 2, 4, 8, 16, 42, 53]$. However, $V(G) = 96$ while $V(H) = 94$.

We can also hope to extend the analysis of Section 7. Observationally, for a fixed number of candies, the games G that optimize $N_W(G)$ have specific structural properties that we conjecture hold in general:

Conjecture 10.4. *For all fixed $N > 0$, there exist (not necessarily distinct) games G_1, G_2 with $N(G_1) = N(G_2) = N$ such that*

$$N_W(G_1) = N_W(G_2) = \max_{H:N(H)=N} N_W(H),$$

where G_1 has a pile with at least $N/4$ candies and G_2 has at most $c \log N$ piles for some absolute constant $c > 0$.

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