Games of No Chance 6
MSRI Publications
Volume **71**, 2025

# A family of Nim-like arrays: stabilization

LOWELL ABRAMS AND DENA S. COWEN-MORTON

In previous work, the authors and others constructed a family of Nim-like arrays using the operation of Nim addition composed with the operation of sequential compound. Every row of these arrays is a permutation of the natural numbers that, for large enough values, is arithmetically periodic. In this work, we regularize, or "stabilize", these arrays row by row, so that each row becomes both doubly infinite and everywhere arithmetically periodic. We study basic properties of these stabilized arrays, in particular showing that various interesting properties of the original arrays continue to hold for their stabilized counterparts. We then show that the row-permutations of the stabilized arrays are in fact affine permutations of $\mathbb{Z}$, and thus the groups generated by these permutations are themselves groups of affine permutations. To give a taste of the complexity of these groups, we analyze an illustrative example highlighting the structure of a subgroup of the multiplication group for one particular array of the family, and examine its Cayley graph.

## 1. Introduction

We continue our study of the family $\mathcal{A}_*$ of recursively generated arrays we call *Nim-like arrays*. These arise from sequential compounds of the game of Nim.

**1.1.** ***Nim, sequential compound, and the arrays $\mathcal{A}_s$.*** The game of Nim is a two-person combinatorial game in which the players alternate turns removing any number of stones they wish from a single pile of stones; the winner is the player who takes the last stone. The direct sum $G_1 \oplus G_2$ of two combinatorial games $G_1, G_2$ is the game in which a player, on their turn, has the option of making a move in exactly one of the games $G_1$ or $G_2$ as long as it is not yet exhausted (in Nim this simply means having several independent piles of stones). Again, the winner is the last player to make a move. The importance of

This work was completed while Abrams was on the faculty of the University Writing Program and the Department of Mathematics of The George Washington University in Washington, D.C. Abrams is the corresponding author.
*MSC2020:* 20N05, 68R15, 91A46.
*Keywords:* mex, Nim, sequential compound, multiplication group, stabilization, affine permutations.

Nim was established by the Sprague–Grundy theorem [11; 18] (also developed in [9, Chapter 11]), which essentially asserts that Nim is universal among finite, impartial two-player combinatorial games in which the winner is the player to move last. Briefly, that is to say that every such game $G$ is, vis-à-vis direct sum, equivalent to a single-pile Nim game.

The Sprague–Grundy theorem implies that direct sum of Nim piles yields an operation, called Nim addition, and it is well known that Nim addition may be represented as a recursively generated array [7].

In [19], Stromquist and Ullman define an operation on games called "sequential compound". Essentially, the sequential compound $G \to H$ of games $G$ and $H$ is the game in which play proceeds in $G$ until it is exhausted, at which point play switches to $H$. In this paper we continue our exploration of combinatorial games whose structure is $(G_1 \oplus G_2) \to H$, where $G_1$ and $G_2$ are individual combinatorial games, and $H$ is equivalent to the Nim pile with $s$ stones. This more complicated operation gives rise to the family $\mathcal{A}_*$ of recursively generated arrays $\mathcal{A}_s$ (the formal definition is reviewed in Section 2).

The connections of the arrays $\mathcal{A}_s$ to the game of Nim and combinatorial games in general are elucidated further in our previous works [1; 2; 3]. In particular, in [1] we discuss the algebraic structure of the arrays $\mathcal{A}_s$, and in [2] we describe periodicity properties which hold in the rows and along the diagonals of $\mathcal{A}_s$. Most recently, in [3] we show that for certain values of $s$, the entries of $\mathcal{A}_s$ satisfy a property we call the "locator property", which links the location in the array $\mathcal{A}_s$ of the entry $j$ in row $i$ to the entry in column $j$ of row $i$; in fact, this is equivalent to the permutation corresponding to that row being an involution.

**1.2.** ***Aim and content of this work.*** Interest in the periodic behavior of the Sprague–Grundy function of a game, which returns for each instance of the game the size of the Nim pile equivalent to that instance, is not new. One motivation for this, aside from the joys of pattern hunting, is that periodicity of the Sprague–Grundy values implies the existence of a polynomially computable strategy for the game. Of particular interest in our work here is arithmetic periodicity, which is to say that the Sprague–Grundy function $\mathcal{G}$ satisfies $\mathcal{G}(n+p) = \mathcal{G}(n) + s$ for some fixed $p$ and $s$. There are many well-known periodicity results. For example, Horrocks and Nowakowski [12] show that some octal games exhibit arithmetic periodicity when a pass move is allowed, and Howse and Nowakowski [13] point out that hexadecimal games can have a variety of interesting kinds of periodicity, including arithmetic periodicity.

As will be explained below, our Sprague–Grundy functions arise from a kind of generalization of misère play. Periodicity results abound here as well. Allen [5] proves periodicity in the context of misère play using the traditional analysis tool of genus sequences. Periodicity results for misère games are placed in a very contemporary context by Weimerskirch [20], who uses them for the calculation of quotient monoids. In fact, it may be that "misère-like" situations are more likely to exhibit periodicity. Fairly recently, Sopena [17] presented a subtraction/division game for which the Sprague–Grundy function is aperiodic under normal play, but periodic under misère play.

Nevertheless, much remains less than perfectly understood about periodicity. Althöfer and Bültermann [6] showed that periods for Sprague–Grundy functions can be arbitrarily large, but for specific situations it may not be clear what the precise behavior of the periods is. For instance, Albert and Nowakowski [4] demonstrate the existence of periodic behavior in the $k$-th diagonals $\mathcal{G}(a, a+k)$ for two-pile greedy Nim, but not the form of the period. Indeed, this is the situation with the Sprague–Grundy functions we are studying, for which we conjecture, but cannot yet prove, that there are infinitely many different periods. More generally, after observing periodic behavior for the Sprague–Grundy function of arc Kayles for various classes of graphs, Huggan and Stevens [14, p. 13] conclude that "a general theorem on periodic behavior of games would help this analysis", and that is certainly the case here as well.

The original aim of this work was to develop a new approach to studying periodicity, with the hope that this would shed light on the periodicity properties of $\mathcal{A}_s$ first described in [2]. In that work, we found that each row of every array $\mathcal{A}_s$ is eventually arithmetically periodic. In this paper, we utilize the arrays $\mathcal{A}_s$ to construct the stabilization arrays $\overleftrightarrow{\mathcal{A}}_s$. In effect, the stabilization arrays arise from the original arrays in the following manner: Within each array $\mathcal{A}_s$, we continue down each individual row until we reach the arithmetically periodic part of that row, and then extend the arithmetic periodicity for that row backwards. The resulting arrays become doubly infinite and everywhere periodic.

As is true of the original arrays $\mathcal{A}_s$, the stabilization arrays $\overleftrightarrow{\mathcal{A}}_s$ exhibit interesting properties, some of which we highlight in Sections 3 and 4. Notably, the same mex rule that was used to construct the original arrays $\mathcal{A}_s$ is still satisfied in the stabilized arrays $\overleftrightarrow{\mathcal{A}}_s$, and if in each row of $\mathcal{A}_s$, the "locator property" eventually holds, then it holds for every entry in $\overleftrightarrow{\mathcal{A}}_s$. We conclude Section 4 with the conjecture that for $s \neq t$ and $\{s, t\} \neq \{0, 1\}$ we have $\overleftrightarrow{\mathcal{A}}_s \neq \overleftrightarrow{\mathcal{A}}_t$, and provide some evidence for this conjecture. In Section 5, we build, from the stabilized arrays $\overleftrightarrow{\mathcal{A}}_s$, the multiplication groups $\overleftrightarrow{\mathcal{M}}_s$, which are generated by the regular actions of $\overleftrightarrow{\mathcal{A}}_s$ on itself. The groups $\overleftrightarrow{\mathcal{M}}_s$ are the second focus of this article, and its highlight. As the reader will see, these groups are surprisingly intricate and yield some beautiful results and conjectures. After first studying, in Section 6.1, the simplest examples, $\overleftrightarrow{\mathcal{M}}_0$ and $\overleftrightarrow{\mathcal{M}}_1$, our discussion in Section 6.2 of a specific example of a small subgroup of $\overleftrightarrow{\mathcal{M}}_2$ further helps detail the complexity of the groups $\overleftrightarrow{\mathcal{M}}_s$.

We end with a "mex–maxx conjecture", which, in the stabilization, connects the usual mex operations to a new operation, denoted by maxx. This property does not hold in the original arrays $\mathcal{A}_s$ for $s \geq 2$. Additional applications of stabilization to periodicity questions will be discussed elsewhere.

## 2. Mex and the arrays $\mathcal{A}_s$

In the following, we closely follow [1; 2]. We begin by constructing a family of infinite arrays using the mex operation:

**Definition 1.** For a set $X \subset \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ we define mex $X$ to be the smallest nonnegative integer not contained in $X$. Here, *mex* stands for *m*inimal *ex*cluded value.

**Definition 2.** For any 2-dimensional array $\mathcal{A}$ indexed by $\mathbb{N}_0 \times \mathbb{N}_0$, let $a_{i,j}$ denote the entry in row $i$, column $j$, where $i, j \geq 0$. The *principal subarray* $\mathcal{A}(p, q)$ is the subarray of $\mathcal{A}$ consisting of entries $a_{i,j}$ with indices $(i, j) \in \{0, \ldots, p\} \times \{0, \ldots, q\}$. For $i \geq 0$ define $\mathrm{Up}(i, j) = \{a_{p,j} : p < i\}$, and for $j \geq 0$ define $\mathrm{Left}(i, j) = \{a_{i,q} : q < j\}$ and $\mathrm{Right}(i, j) = \{a_{i,q} : q > j\}$.

Observe that Definition 2 gives $\mathrm{Left}(i, 0) = \mathrm{Up}(0, j) = \varnothing$.

**Definition 3.** The infinite $\mathbb{N}_0 \times \mathbb{N}_0$ array $\mathcal{A}_s$, for $s \in \mathbb{N}_0$, is constructed recursively: the entry $a_{0,0}$ is set to the *seed* $s$, and for $(i, j) \neq (0, 0)$,

$$a_{i,j} := \mathrm{mex}\big(\mathrm{Left}(i, j) \cup \mathrm{Up}(i, j)\big).$$

We sometimes write $\mathcal{A}_*$ for $\{\mathcal{A}_s\}_{s \in \mathbb{N}_0}$.

Definition 3 yields the Grundy values of Nim when $s = 0$, of misère Nim when $s = 1$, and of the sequential compound with a Nim pile with $s$ stones for $s > 1$.

To motivate the terminology in the definitions, see, for example, Figure 1, in which we are concerned with the principal subarrays $\mathcal{A}_0(7, 7)$ and $\mathcal{A}_2(7, 7)$, with row indices $i = 0, 1, 2, \ldots, 7$ and column indices $j = 0, 1, 2, \ldots, 7$.

The reader may easily verify that a change of seed from 0 to 1 has a minimal effect; other than the top left $2 \times 2$ block (i.e., $\mathcal{A}_0(1, 1)$), the entries in the array $\mathcal{A}_1$ are exactly the same as those of $\mathcal{A}_0$. As seed $s = 1$ corresponds to misère play, this highlights the subtle difference between regular play and misère play.

The following simple, yet fundamental, property is an immediate consequence of the recursive construction:

**Proposition 4** [1, Proposition 2.4]. *For each $s$, the array $\mathcal{A}_s$ is symmetric, and each nonnegative integer appears exactly once in each row* (*and, by symmetry, each column*).

As we first demonstrated in [1], there are many patterns that occur in the rows, columns, and diagonals of the arrays $\mathcal{A}_s$, some of which will be useful

$$\begin{bmatrix} 0&1&2&3&4&5&6&7\\ 1&0&3&2&5&4&7&6\\ 2&3&0&1&6&7&4&5\\ 3&2&1&0&7&6&5&4\\ 4&5&6&7&0&1&2&3\\ 5&4&7&6&1&0&3&2\\ 6&7&4&5&2&3&0&1\\ 7&6&5&4&3&2&1&0 \end{bmatrix} \qquad \begin{bmatrix} 2&0&1&3&4&5&6&7\\ 0&1&2&4&3&6&5&8\\ 1&2&0&5&6&3&4&9\\ 3&4&5&0&1&2&7&6\\ 4&3&6&1&0&7&2&5\\ 5&6&3&2&7&0&1&4\\ 6&5&4&7&2&1&0&3\\ 7&8&9&6&5&4&3&0 \end{bmatrix}$$

**Figure 1.** $\mathcal{A}_0(7,7)$ (left) and $\mathcal{A}_2(7,7)$ (right).

throughout the discussion below; we include the relevant patterns here. The following lemma showcases the patterns in rows 0–3 of the arrays $\mathcal{A}_s$.[1]

**Lemma 5** [1, Propositions 3.1–3.4]. *For seed $s$ in $\mathcal{A}_s$:*

(1) *For all $j > s$, $a_{0,j} = j$.*

(2) *If $j > s > 0$ then*

$$a_{1,j} = \begin{cases} j-1 & \text{if } j-s \equiv 0 \bmod 2, \\ j+1 & \text{if } j-s \equiv 1 \bmod 2. \end{cases}$$

(3) *If $j > s \geq 2$ then*

$$a_{2,j} = \begin{cases} j+1 & \text{if } s \equiv 0,1 \bmod 3 \text{ and } j > s+1 \text{ and } j-s \equiv 0 \bmod 2, \\ j-1 & \text{if } s \equiv 0,1 \bmod 3 \text{ and } j > s+1 \text{ and } j-s \equiv 1 \bmod 2, \\ j-2 & \text{if } s \equiv 2 \bmod 3 \text{ and } j-s \equiv 0,3 \bmod 4, \\ j+2 & \text{if } s \equiv 2 \bmod 3 \text{ and } j-s \equiv 1,2 \bmod 4. \end{cases}$$

(4) *If $j > s \geq 5$ then*

$$a_{3,j} = \begin{cases} j-2 & \text{if } s \equiv 0,4 \bmod 9 \text{ and } j-s \equiv 0,3 \bmod 4, \\ j+2 & \text{if } s \equiv 0,4 \bmod 9 \text{ and } j-s \equiv 1,2 \bmod 4, \\ j+2 & \text{if } s \equiv 1,6 \bmod 9 \text{ and } j > s+1 \text{ and } j-s \equiv 0 \bmod 2, \\ j-2 & \text{if } s \equiv 1,6 \bmod 9 \text{ and } j > s+1 \text{ and } j-s \equiv 1 \bmod 2, \\ j+2 & \text{if } s \equiv 2 \bmod 9 \text{ and } j-s \equiv 0,3 \bmod 4, \\ j-2 & \text{if } s \equiv 2 \bmod 9 \text{ and } j-s \equiv 1,2 \bmod 4, \\ j-2 & \text{if } s \equiv 3,7 \bmod 9 \text{ and } j > s+1 \text{ and } j-s \equiv 0,1 \bmod 4, \\ j+2 & \text{if } s \equiv 3,7 \bmod 9 \text{ and } j > s+1 \text{ and } j-s \equiv 2,3 \bmod 4, \\ j+1 & \text{if } s \equiv 5,8 \bmod 9 \text{ and } j > s+1 \text{ and } j-s \equiv 0 \bmod 2, \\ j-1 & \text{if } s \equiv 5,8 \bmod 9 \text{ and } j > s+1 \text{ and } j-s \equiv 1 \bmod 2. \end{cases}$$

[1] For the reader consulting reference [1], which expresses the "pattern properties" in terms of an operation $*$, we note that while $a_{i,j} \neq i * j$ for small values of $i$, $j$, we do have $a_{i,j} = i * j$ for $i$, $j > s$.

## 3. The stabilization arrays $\vec{\mathcal{A}}_s$

### 3.1. *Construction of* $\vec{\mathcal{A}}_s$.

**Definition 6.** For fixed seed $s$, associated to each index pair $(i, j)$ in $\mathcal{A}_s$ is the *offset* $d_i(j) := a_{i,j} - j$.

**Definition 7.** We say a sequence $\{x_i\}_{i=1}^{\infty}$ is *eventually periodic* if there is an index $N$ such that $\{x_i\}_{i=N}^{\infty}$ is periodic. In this case, we say that $\{x_i\}_{i=1}^{\infty}$ is *periodic from* $N$.

The offset is simply the difference between an entry and its column index. It satisfies the following important property which sets the groundwork for much of this paper:

**Theorem 8** [2, Theorem 4.1] . *For fixed seed $s$, and for each row $i$ in $\mathcal{A}_s$, the sequence $\{d_i(j)\}_{j=0}^{\infty}$ is eventually periodic.*

In [2], the authors prove this theorem using the finite state machine approach of Landman [15]. Another possible approach is that of Dress et al. [10].

**Definition 9.** We say a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ is *arithmetically periodic* if the sequence $\{f(n) - n\}_{n=0}^{\infty}$ is periodic. We sometimes then say that $f$ is "fully arithmetically periodic". The function $f$ is *arithmetically $p$-periodic* if $f(a + p) = f(a) + p$ for all $a \in \mathbb{N}_0$. In this case, we sometimes say that $f$ is "fully arithmetically $p$-periodic". We extend these terms analogously to the case when $f$ is $\mathbb{Z}$.

Note that in the terminology of [12], for example, our definition of arithmetic periodicity is equivalent to having saltus equal to the period $p$. Combining Definitions 9 and 7 yields:

**Definition 10.** We say a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ is *arithmetically $p$-periodic from* $N \in \mathbb{N}$ if the sequence $\{f(n) - n\}_{n=N}^{\infty}$ is $p$-periodic, i.e., $f(n + p) = f(n) + p$ for all $n \geq N$. We sometimes then say that $f$ is "eventually $p$-periodic". We extend these terms analogously to the case when $f$ is $\mathbb{Z}$.

**Definition 11.** For a given seed $s$, let $N_i$ denote the smallest integer greater than $s$ such that row $i$ of $\mathcal{A}_s$ is arithmetically $p_i$-periodic from $N_i$. The *offset period in row $i$*, denoted by $p_i$, is the period of the offsets $\{d_i(j)\}_{j=N_i}^{\infty}$.

**Example 12.** Using Figure 1 as a reference, in $\mathcal{A}_0$, we have

$$\{d_0(j)\}_{j=0}^{\infty} = \{0, 0, 0, \dots\}, \qquad \text{so } p_0 = 1 \text{ and } N_0 = 0,$$
$$\{d_1(j)\}_{j=0}^{\infty} = \{1, -1, 1, -1, \dots\}, \qquad \text{so } p_1 = 2 \text{ and } N_1 = 0,$$
$$\{d_2(j)\}_{j=0}^{\infty} = \{2, 2, -2, -2, 2, 2, \dots\}, \qquad \text{so } p_2 = 4 \text{ and } N_2 = 0.$$

Similarly, in $\mathcal{A}_2$, we have

$$\{d_0(j)\}_{j=0}^{\infty} = \{2, -1, -1, 0, 0, 0, \dots\}, \qquad \text{so } p_0 = 1 \text{ and } N_0 = 3,$$
$$\{d_1(j)\}_{j=0}^{\infty} = \{0, 0, 0, 1, -1, 1, -1, \dots\}, \qquad \text{so } p_1 = 2 \text{ and } N_1 = 3,$$
$$\{d_2(j)\}_{j=0}^{\infty} = \{1, 1, -2, 2, 2, -2, -2, 2, 2, -2, \dots\}, \qquad \text{so } p_2 = 4 \text{ and } N_2 = 2.$$

With this notation, Theorem 8 tells us that for each row $i$ we have $a_{i,j+p_i} = a_{i,j} + p_i$ for sufficiently large $j$, so for each row $i$, the sequence $\{d_i(j)\}_{j=0}^{\infty}$ is eventually arithmetically $p_i$-periodic. We will use Theorem 8 to construct, from the family $\mathcal{A}_*$, a family $\overleftrightarrow{\mathcal{A}}_*$ of "stabilized arrays", where we use the term "stabilized" to connote the idea that each row in the resulting array will be arithmetically periodic, rather than just eventually arithmetically periodic.

We begin with the central definition of this paper. Note that $\lceil x \rceil$ denotes the usual ceiling function.

**Definition 13.** We define the *stabilization of* $\mathcal{A}_s$, denoted by $\overleftrightarrow{\mathcal{A}}_s$, to be the $\mathbb{N}_0 \times \mathbb{Z}$ array arising as follows: set $r_{i,j} = \lceil (N_i - j)/p_i \rceil$ for each $j \in \mathbb{Z}$ and define the $(i, j)$ entry of $\overleftrightarrow{\mathcal{A}}_s$ to be $\overleftrightarrow{a}_{i,j} = a_{i,(j+r_{i,j} \cdot p_i)} - r_{i,j} p_i$.

Expressed verbally, to define $\overleftrightarrow{\mathcal{A}}_s$ we do the following: In each row $i$ of the original array $\mathcal{A}_s$ we move (in jumps of length $p_i$) until we reach the arithmetically periodic part of that row; thus to define $\overleftrightarrow{a}_{i,j}$ we examine $a_{i,j} = a_{i,(j+r_{i,j} \cdot p_i)}$. Then we extend the arithmetic periodicity for that row backwards, which corresponds to $a_{i,(j+r_{i,j} \cdot p_i)} - r_{i,j} p_i$.

The values $r_{i,j}$ tell how many jumps of size $p_i$ to the right (if $r_{i,j} > 0$) or to the left (if $r_{i,j} \le 0$, although in that case, strictly speaking, no jumps are needed) must be performed in row $i$ to reach from column $j$ to the periodic part of $\mathcal{A}_s$. Once a given row in $\mathcal{A}_s$ has reached its arithmetically periodic part, the entries in $\overleftrightarrow{\mathcal{A}}_s$ agree with the entries in $\mathcal{A}_s$. We note that, as a direct result of the construction, row $i$ in $\overleftrightarrow{\mathcal{A}}_s$ is arithmetically $p_i$-periodic everywhere.

Within the stabilized arrays, we define the column indexing by declaring that entry 0 in row $i = 0$ appears in column $j = 0$, i.e., so that $\overleftrightarrow{a}_{0,0} = 0$. Since the entry 0 occurs exactly once in row $i = 0$, this indexing is well defined.

**3.2.** ***Examples of construction.*** To illustrate Definition 13, we now look at the construction of $\overleftrightarrow{\mathcal{A}}_2$ based on $\mathcal{A}_2(7, 7)$ from Figure 1. A portion of $\overleftrightarrow{\mathcal{A}}_2$ can be found in Figure 2, so that the reader can verify that the calculations do, indeed, work. We refer the reader to Example 12 for calculations of $p_i$ and $N_i$.

**Example 14.** In this example, we construct the values of row $i = 0$ in $\overleftrightarrow{\mathcal{A}}_2$. Here, $N_0 = 3$ and $p_0 = 1$.

(1) To find $\overleftrightarrow{a}_{0,0}$: We have $r_{0,0} = \lceil (N_0 - 0)/p_0 \rceil = 3$, and therefore $\overleftrightarrow{a}_{0,0} = a_{0,(0+r_{0,0} \cdot p_0)} - r_{0,0} \cdot p_0 = a_{0,0+3\cdot 1} - 3 \cdot 1 = a_{0,3} - 3 = 0$.

(2) To find $\overleftrightarrow{a}_{0,1}$: We have $r_{0,1} = \lceil (N_0 - 1)/p_0 \rceil = 2$, and therefore $\overleftrightarrow{a}_{0,1} = a_{0,(1+r_{0,1}\cdot p_0)} - r_{0,1} \cdot p_0 = a_{0,1+2\cdot 1} - 2 \cdot 1 = a_{0,3} - 2 = 1$.

(3) To find $\overleftrightarrow{a}_{0,2}$: We have $r_{0,2} = \lceil (N_0 - 2)/p_0 \rceil = 1$, and therefore $\overleftrightarrow{a}_{0,2} = a_{0,(2+r_{0,2}\cdot p_0)} - r_{0,2} \cdot p_0 = a_{0,2+1\cdot 1} - 1 \cdot 1 = a_{0,3} - 1 = 2$.

(4) To find $\overleftrightarrow{a}_{0,3}$: We have $r_{0,3} = \lceil (N_0 - 3)/p_0 \rceil = 0$, and therefore $\overleftrightarrow{a}_{0,3} = a_{0,(3+r_{0,3}\cdot p_0)} - r_{0,3} \cdot p_0 = a_{0,3+0\cdot 1} - 0 \cdot 1 = a_{0,3} - 0 = 3$. Note as $a_{0,3}$ is in the periodic part of the row, we expected $\overleftrightarrow{a}_{0,3} = a_{0,3}$.

(5) Skipping ahead, to $\overleftrightarrow{a}_{0,7}$: We have $r_{0,7} = \lceil (N_0 - 7)/p_0 \rceil = -4$, and therefore $\overleftrightarrow{a}_{0,7} = a_{0,(7+r_{0,7}\cdot p_0)} - r_{0,7} \cdot p_0 = a_{0,7+(-4)\cdot 1} - (-4) \cdot 1 = a_{0,3} + 4 = 7$. Note as $a_{0,7}$ is in the periodic part of the row, we expected $\overleftrightarrow{a}_{0,7} = a_{0,7}$.

(6) Moving backwards, to $\overleftrightarrow{a}_{0,-1}$: We have $r_{0,-1} = \lceil (N_0 + 1)/p_0 \rceil = 4$, and therefore $\overleftrightarrow{a}_{0,-1} = a_{0,(-1+r_{0,-1}\cdot p_0)} - (r_{0,-1}) \cdot p_0 = a_{0,-1+(4)\cdot 1} - (4) \cdot 1 = a_{0,3} - 4 = -1$.

(7) Likewise, for $\overleftrightarrow{a}_{0,-2}$: We have $r_{0,-2} = \lceil (N_0 + 2)/p_0 \rceil = 5$, and therefore $\overleftrightarrow{a}_{0,-2} = a_{0,(-2+r_{0,-2}\cdot p_0)} - (r_{0,-2}) \cdot p_0 = a_{0,-2+(5)\cdot 1} - (5) \cdot 1 = a_{0,3} - 5 = -2$.

**Example 15.** In this example, we construct the values of row $i = 1$ in $\overleftrightarrow{\mathcal{A}}_2$. Here $N_1 = 3$ and $p_1 = 2$.

(1) To find $\overleftrightarrow{a}_{1,0}$: We have $r_{1,0} = \lceil (N_1 - 0)/p_1 \rceil = 2$, and therefore $\overleftrightarrow{a}_{1,0} = a_{1,(0+r_{1,0}\cdot p_1)} - r_{1,0} \cdot p_1 = a_{1,0+2\cdot 2} - 2 \cdot 2 = a_{1,4} - 4 = -1$.

(2) To find $\overleftrightarrow{a}_{1,1}$: We have $r_{1,1} = \lceil (N_1 - 1)/p_1 \rceil = 1$, and therefore $\overleftrightarrow{a}_{1,1} = a_{1,(1+r_{1,1}\cdot p_1)} - r_{1,1} \cdot p_1 = a_{1,1+1\cdot 2} - 1 \cdot 2 = a_{1,3} - 2 = 2$.

(3) To find $\overleftrightarrow{a}_{1,2}$: We have $r_{1,2} = \lceil (N_1 - 2)/p_1 \rceil = 1$, and therefore $\overleftrightarrow{a}_{1,2} = a_{1,(2+r_{1,2}\cdot p_1)} - r_{1,2} \cdot p_1 = a_{1,2+1\cdot 2} - 1 \cdot 2 = a_{1,4} - 2 = 1$.

(4) To find $\overleftrightarrow{a}_{1,7}$: We have $r_{1,7} = \lceil (N_1 - 7)/p_1 \rceil = -2$, and therefore $\overleftrightarrow{a}_{1,7} = a_{1,(7+r_{1,7}\cdot p_1)} - r_{1,7} \cdot p_1 = a_{1,7+-2\cdot 2} - (-2) \cdot 2 = a_{1,3} + 4 = 8$.

(5) To find $\overleftrightarrow{a}_{1,-1}$: We have $r_{1,-1} = \lceil (N_1 + 1)/p_1 \rceil = 2$, and therefore $\overleftrightarrow{a}_{1,-1} = a_{1,(-1+r_{1,-1}\cdot p_1)} - (r_{1,-1}) \cdot p_1 = a_{1,-1+2\cdot 2} - 2 \cdot 2 = a_{1,3} - 4 = 0$.

(6) To find $\overleftrightarrow{a}_{1,-2}$: We have $r_{1,-2} = \lceil (N_1 + 2)/p_1 \rceil = 3$, and therefore $\overleftrightarrow{a}_{1,-2} = a_{1,(-2+r_{1,-2}\cdot p_1)} - (r_{1,-2}) \cdot p_1 = a_{1,-2+3\cdot 2} - 3 \cdot 2 = a_{1,4} - 6 = -3$.

Figure 2 shows a portion of the stabilized array for $\overleftrightarrow{\mathcal{A}}_2$. We index the rows with $i = 0, 1, 2, \ldots$ increasing down the page and we index the columns by the entries in row 0. By Lemma 16, this indexing is well defined. It is interesting to note that in the original array $\mathcal{A}_2$ the $i$-th row and $i$-th column give the same infinite sequence, whereas in $\overleftrightarrow{\mathcal{A}}_2$ the $i$-th column seems to no longer give the same infinite sequence as (even a portion of) the $i$-th row.

$$\begin{bmatrix} \cdots & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \cdots & -7 & -4 & -5 & -2 & -3 & 0 & -1 & 2 & 1 & 4 & 3 & 6 & 5 & \cdots \\ \cdots & -8 & -3 & -2 & -5 & -4 & 1 & 2 & -1 & 0 & 5 & 6 & 3 & 4 & \cdots \\ \cdots & -5 & -6 & -3 & -4 & -1 & -2 & 1 & 0 & 3 & 2 & 5 & 4 & 7 & \cdots \\ \cdots & -3 & -8 & -7 & -6 & 0 & 2 & -2 & 3 & -1 & 1 & 7 & 8 & 9 & \cdots \\ \cdots & -4 & -2 & -6 & -1 & -5 & -3 & 3 & 4 & 5 & 0 & 1 & 2 & 8 & \cdots \\ \cdots & -9 & -7 & -1 & 0 & 1 & -4 & -3 & -2 & 4 & 6 & 2 & 7 & 3 & \cdots \\ \cdots & -2 & -1 & -8 & -7 & -6 & -5 & 4 & 5 & 6 & 7 & 0 & 1 & 2 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

**Figure 2.** A portion of $\overleftrightarrow{\mathcal{A}}_2$: the row indexing is $0, 1, 2, \ldots$, and the column indexing is defined to be the values in row $i = 0$.

**Lemma 16.**
$$\overleftrightarrow{a}_{0,j} = j \quad \textit{in } \overleftrightarrow{\mathcal{A}}_s.$$

*Proof.* Lemma 5(1) shows that for every $\mathcal{A}_s$, if $j > s$, then $a_{0,j} = j$, so row 0 is always arithmetically periodic in the original arrays $\mathcal{A}_s$ from column $s + 1$, with period $p_0 = 1$. Thus $\overleftrightarrow{a}_{0,j} = a_{0,j} = j$ for all $j \geq s + 1$, and since the rows of $\overleftrightarrow{\mathcal{A}}_s$ are arithmetically periodic we have $\overleftrightarrow{a}_{0,j} = j$ for all $j$. □

## 4. Properties of the stabilization and the stabilized mex rule

In this section, we highlight some of the stabilization properties of the arrays $\overleftrightarrow{\mathcal{A}}_s$. Recall the definitions of $\overleftrightarrow{\mathcal{A}}_s$ and $N_i$ from Definitions 11 and 13, which will be heavily used throughout this section.

### 4.1. *Basic properties of the stabilization.*

**Proposition 17.** *For a fixed seed $s$ that is in $\mathcal{A}_s$ for all $j \geq N_i$ and in $\overleftrightarrow{\mathcal{A}}_s$ for all $j \in \mathbb{Z}$, the entries in row $i$ from columns $j$ to $j + p_i - 1$ form a complete set of representatives* mod $p_i$.

*Proof.* We use the surjectivity of rows in the arrays $\mathcal{A}_s$ from Proposition 4. In $\mathcal{A}_s$, fix row $i$ and recall that $d_i(j)$ is periodic with offset period $p_i$ for $j \geq N_i$, i.e., for $j \geq N_i$ we have $a_{i,j} \equiv a_{i,j+p_i} \bmod p_i$. Thus all $a_{i,j}$ with $j \geq N_i$ belong to one of the congruence classes mod $p_i$ of $a_{i,N_i}, \ldots, a_{i,(N_i+p_i-1)}$. Since row $i$ contains all natural numbers, this must be a complete set of representatives mod $p_i$.

The full result follows by arithmetic periodicity and the construction of $\overleftrightarrow{\mathcal{A}}_s$. □

**Definition 18.** For each $i \in \mathbb{N}_0$ we define the arithmetically $p_i$-periodic function $\sigma_i : \mathbb{Z} \to \mathbb{Z}$ by $\sigma_i(j) := \overleftrightarrow{a}_{i,j}$.

We speak interchangeably of row $i$ in $\overleftrightarrow{\mathcal{A}}_s$ being arithmetically $p_i$-periodic and $\sigma_i$ being arithmetically $p_i$-periodic.

**Theorem 19.** *For each $i \in \mathbb{N}_0$, the function $\sigma_i : \mathbb{Z} \to \mathbb{Z}$ is an arithmetically $p_i$-periodic bijection. Furthermore, for any $i \neq i'$ and any $j \in \mathbb{Z}$ we have $\sigma_i(j) \neq \sigma_{i'}(j)$.*

*Proof.* Arithmetic periodicity holds by Theorem 8. The remainder of the proof makes frequent use of Proposition 4.

Fix row $i$ and define $S = \{\overleftrightarrow{a}_{i,N_i}, \overleftrightarrow{a}_{i,N_i+1}, \overleftrightarrow{a}_{i,N_i+2}, \dots\} = \{a_{i,N_i}, a_{i,N_i+1}, \dots\}$. By Proposition 17, $S$ contains representatives of all congruence classes mod $p_i$, and arithmetic periodicity implies that in each such congruence class all but finitely many positive representatives are contained in $S$. It follows that there is a well-defined maximal element $m$ of $\mathbb{Z} \setminus S$ and so $S = \{m+1, m+2, \dots\} \cup T$ for some finite subset $T$ of $\mathbb{N}_0$.

For any $x \in \mathbb{Z}$ there is $l \in \mathbb{Z}$ such that $x + lp_i > \max\{m, i + N_i\}$. By definition of $m$ this means there is a $j$ such that $a_{i,j} = x + lp_i$. By [2, Proposition 4.2] (which states that for all $i, j > 0$, $a_{i,j} \leq i + j$) and our choice of $l$ we have $i + N_i < a_{i,j} \leq i + j$, so $j > N_i$. Thus, by definition of $N_i$ we have $\sigma_i(j) = \overleftrightarrow{a}_{i,j} = a_{i,j} = x + lp_i$, which shows that $\sigma_i(j - lp_i) = x$. It follows that $\sigma_i$ is surjective.

If $\sigma_i(j) = \sigma_{i'}(j')$ then

$$\sigma_i(j + lp_i p_{i'}) = \sigma_i(j) + lp_i p_{i'} = \sigma_{i'}(j') + lp_i p_{i'} = \sigma_{i'}(j' + lp_i p_{i'})$$

for any integer $l$. Taking $lp_i p_{i'} \geq \max\{N_i, N_{i'}\}$ we get

$$a_{i,(j+lp_i p_{i'})} = \sigma_i(j + lp_i p_{i'}) = \sigma_{i'}(j' + lp_i p_{i'}) = a_{i',(j'+lp_i p_{i'})}.$$

If $i = i'$, then by the injectivity of the rows in $\mathcal{A}_s$ we have $j = j'$, and thus the rows of $\overleftrightarrow{\mathcal{A}}_s$ are one-to-one. Last, if $j = j'$, then $a_{i,(j+lp_i p_{i'})} = a_{i',(j+lp_i p_{i'})}$, and we obtain the same element twice in column $j + lp_i p_{i'}$ of $\mathcal{A}_s$, which is a contradiction unless $i = i'$. □

**4.2.** ***The stabilized mex rule.*** We now introduce definitions and lemmas leading up to the proof of Theorem 25, which asserts that the mex rule from Definition 3 applies, with suitable alteration, to the stabilization $\overleftrightarrow{\mathcal{A}}_s$.

**Definition 20.** Working in $\overleftrightarrow{\mathcal{A}}_s$ instead of $\mathcal{A}_s$, define $\overleftrightarrow{\mathrm{Up}}(i, j)$, $\overleftrightarrow{\mathrm{Left}}(i, j)$, and $\overleftrightarrow{\mathrm{Right}}(i, j)$ by analogy with Definition 2, but allowing for any $j \in \mathbb{Z}$.

For any set $S \subset \mathbb{Z}$ and any $j \in \mathbb{Z}$, we denote by $S + j$ the set $\{x + j \mid x \in S\}$.

**Lemma 21.** *For fixed $s$ and for each row $i$ and each $j, m \in \mathbb{Z}$ we have*

$$\overleftrightarrow{\mathrm{Left}}(i, j+mp_i) = \overleftrightarrow{\mathrm{Left}}(i, j) + mp_i \quad \textit{and} \quad \overleftrightarrow{\mathrm{Right}}(i, j+mp_i) = \overleftrightarrow{\mathrm{Right}}(i, j) + mp_i.$$

*If $p := \mathrm{lcm}\{p_0, \dots, p_i\}$ and $j, m \in \mathbb{Z}$, then*

$$\overleftrightarrow{\mathrm{Up}}(i, j + mp) = \overleftrightarrow{\mathrm{Up}}(i, j) + mp.$$

*Proof.* We repeatedly use the arithmetic periodicity in row $i$:

$$\begin{aligned}\overleftrightarrow{\text{Left}}(i, j+mp_i) &= \{\ldots, \overleftrightarrow{a}_{i,(j+mp_i-2)}, \overleftrightarrow{a}_{i,(j+mp_i-1)}\}\\ &= \{\ldots, \overleftrightarrow{a}_{i,(j-2)}+mp_i, \overleftrightarrow{a}_{i,(j-1)}+mp_i\}\\ &= \{\ldots, \overleftrightarrow{a}_{i,(j-2)}, \overleftrightarrow{a}_{i,(j-1)}\}+mp_i\\ &= \overleftrightarrow{\text{Left}}(i, j)+mp_i.\end{aligned}$$

The proof for $\overleftrightarrow{\text{Right}}$ is similar.

As for $\overleftrightarrow{\text{Up}}$, we have

$$\begin{aligned}\overleftrightarrow{\text{Up}}(i, j+mp) &= \{\overleftrightarrow{a}_{0,(j+mp)}, \overleftrightarrow{a}_{1,(j+mp)}, \ldots, \overleftrightarrow{a}_{i-1,(j+mp)}\}\\ &= \{\overleftrightarrow{a}_{0,j}+mp, \overleftrightarrow{a}_{1,j}+mp, \ldots, \overleftrightarrow{a}_{(i-1),j}+mp\}\\ &= \{\overleftrightarrow{a}_{0,j}, \overleftrightarrow{a}_{1,j}, \ldots, \overleftrightarrow{a}_{(i-1),j}\}+mp\\ &= \overleftrightarrow{\text{Up}}(i, j)+mp.\end{aligned}$$

□

**Definition 22.** For any subset $S \subsetneq \mathbb{Z}$ such that $\mathbb{Z} \setminus S$ contains only finitely many negative integers, we define $\text{mex}_{\mathbb{Z}} S$ to be the smallest integer not contained in $S$.

We note that for each seed $s$ and pair $(i, j)$, Proposition 17 and Theorem 19 ensure that the set $\mathbb{Z} \setminus \overleftrightarrow{\text{Left}}(i, j)$ contains only finitely many negative integers.

**Lemma 23.** *If $S \subsetneq \mathbb{Z}$ is such that $\mathbb{Z} \setminus S$ contains only finitely many negative integers, then* $\text{mex}_{\mathbb{Z}}(S+j) = \text{mex}_{\mathbb{Z}}(S)+j$ *for all* $j \in \mathbb{Z}$.

*Proof.* Let $\chi_S : \mathbb{Z} \to \{0, 1\}$ denote the characteristic function of $S$. Since $\mathbb{Z} \setminus S$ contains only finitely many negative integers the same is true for $\mathbb{Z} \setminus (S+j)$, and thus $\text{mex}_{\mathbb{Z}}(S+j)$ is well defined. We see that, for each $j$,

$$\begin{aligned}\text{mex}_{\mathbb{Z}}(S+j) &= \min\{z \in \mathbb{Z} \mid \chi_{(S+j)}(z)=0\}\\ &= \min\{z \in \mathbb{Z} \mid \chi_S(z-j)=0\}\\ &= \min\{z'+j \in \mathbb{Z} \mid \chi_S(z')=0\}\\ &= \text{mex}_{\mathbb{Z}}(S)+j.\end{aligned}$$

□

The next lemma follows immediately from the definitions.

**Lemma 24.** *Let $S \subsetneq \mathbb{N}_0$. Then* $\text{mex}(S) = \text{mex}_{\mathbb{Z}}(S \cup (\mathbb{Z} \setminus \mathbb{N}_0))$.

Recalling our definition of $N_i$ (Definition 11), we may now confirm that the mex rule still holds in the stabilized array.

**Theorem 25.** *For fixed $s$ and for all* $(i, j) \in \mathbb{N}_0 \times \mathbb{Z}$,

$$\overleftrightarrow{a}_{i,j} = \text{mex}_{\mathbb{Z}}\big(\overleftrightarrow{\text{Left}}(i, j) \cup \overleftrightarrow{\text{Up}}(i, j)\big).$$

*Proof.* Fix $i$ and let $N = \max\{N_0, N_1, \ldots, N_i\}$; this allows $N$ to be sufficiently large that rows 0 through $i$ have stabilized. Throughout this proof we let $i'$ denote a row index with $i' \le i$ and $j'$ a column index with $j' \ge N$. In each of the rows $i'$ and columns $j'$, we are in the part of the original array $\mathcal{A}_s$ which is arithmetically periodic, and so $\overleftrightarrow{a}_{i',j'} = a_{i',j'} \in \mathbb{N}_0$. Moreover, we have $\overleftrightarrow{\mathrm{Up}}(i', j') = \mathrm{Up}(i', j')$ and $\overleftrightarrow{\mathrm{Right}}(i', j') = \mathrm{Right}(i', j')$. Since $\sigma_{i'}$ is bijective, we therefore have

$$\begin{aligned}\overleftrightarrow{\mathrm{Left}}(i', j') &= \mathbb{Z} \setminus \big(\{\overleftrightarrow{a}_{i',j'}\} \cup \overleftrightarrow{\mathrm{Right}}(i', j')\big) = \mathbb{Z} \setminus \big(\{a_{i',j'}\} \cup \mathrm{Right}(i', j')\big) \\ &= \mathbb{N}_0 \setminus \big(\{a_{i',j'}\} \cup \mathrm{Right}(i', j')\big) \cup (\mathbb{Z} \setminus \mathbb{N}_0) \\ &= \mathrm{Left}(i', j') \cup (\mathbb{Z} \setminus \mathbb{N}_0).\end{aligned}$$

By Lemma 24,

$$\begin{aligned}a_{i',j'} &= \mathrm{mex}\big(\mathrm{Left}(i', j') \cup \mathrm{Up}(i', j')\big) \\ &= \mathrm{mex}_{\mathbb{Z}}\big(\mathrm{Left}(i', j') \cup (\mathbb{Z} \setminus \mathbb{N}_0) \cup \mathrm{Up}(i', j')\big),\end{aligned}$$

proving that for $i' \le i$ and $j' \ge N$, we have

$$\overleftrightarrow{a}_{i',j'} = a_{i',j'} = \mathrm{mex}_{\mathbb{Z}}\big(\overleftrightarrow{\mathrm{Left}}(i', j') \cup \overleftrightarrow{\mathrm{Up}}(i', j')\big). \tag{4.2.1}$$

We now specialize to row $i$ and consider an arbitrary column $j$. Let $p = \mathrm{lcm}\{p_0, \ldots, p_i\}$ and choose $k \in \mathbb{Z}$ so $j + kp \ge N$. Then, by the analysis in (4.2.1) and the periodic behavior of $\overleftrightarrow{\mathcal{A}}_s$,

$$\overleftrightarrow{a}_{i,j} + kp = \overleftrightarrow{a}_{i,j+kp} = \mathrm{mex}_{\mathbb{Z}}\big(\overleftrightarrow{\mathrm{Left}}(i, j + kp) \cup \overleftrightarrow{\mathrm{Up}}(i, j + kp)\big).$$

By Lemma 21 we obtain

$$\begin{aligned}\overleftrightarrow{a}_{i,j} + kp &= \mathrm{mex}_{\mathbb{Z}}\big(\big(\overleftrightarrow{\mathrm{Left}}(i, j) + kp\big) \cup \big(\overleftrightarrow{\mathrm{Up}}(i, j) + kp\big)\big) \\ &= \mathrm{mex}_{\mathbb{Z}}\big(\big(\overleftrightarrow{\mathrm{Left}}(i, j) \cup \overleftrightarrow{\mathrm{Up}}(i, j)\big) + kp\big),\end{aligned}$$

and, finally, by Lemma 23 we have $\overleftrightarrow{a}_{i,j} = \mathrm{mex}_{\mathbb{Z}}\big(\overleftrightarrow{\mathrm{Left}}(i, j) \cup \overleftrightarrow{\mathrm{Up}}(i, j)\big)$. □

**4.3.** ***The locator property.*** The focus of our paper [3] was the "locator property", which establishes a connection between the row index, column index, and value of an entry. This property may also be understood in terms of cycles of a permutation, as the locator property holds whenever the row permutations are involutions.

**Definition 26.** Let seed $s$ be fixed. For each $i$, define the bijective function $m_i : \mathbb{N}_0 \to \mathbb{N}_0$ by $m_i(j) := a_{i,j}$.

We will at times refer to the function $m_i$ as "row $i$".

**Definition 27** [3]. For a fixed seed $s$ we say that the *locator property holds in $\mathcal{A}_s$ at* $(i, j)$ if there exists $k \in \mathbb{Z}$ such that $m_i(j) = k$ and $m_i(k) = j$. We say that the *locator property eventually holds in row $i$ of $\mathcal{A}_s$* if there is some $J_i$ such that for $j \ge J_i$ the locator property holds at $(i, j)$.

Note that this property yields $m_i(m_i(j)) = j$. Another way to state the locator property is that the entry $j$ in row $i$ appears in column $a_{i,j}$, i.e., $a_{i,a_{i,j}} = j$.

**Theorem 28** [3, Proposition 3.2, Theorem 3.13]. *For seeds $s = 0, 1$ the locator property holds in $\mathcal{A}_s$ for all $(i, j)$. For seeds $s = 2, 3, 5, 7$, the locator property holds in $\mathcal{A}_s$ for all $(i, j) \in L_s$, where*

$$L_2 = \{(i, j) : i > 2 \text{ or } j > 2\},$$

$$L_3 = \{(i, j) : (i > 3 \text{ or } j > 3) \text{ and } j \neq 1, 2 \text{ and } |i - j| > 1\},$$

$$L_5 = \{(i, j) : (i > 6 \text{ or } j > 6) \text{ and } j \neq 1, 2, 3, 4 \text{ and } |i - j| > 2\},$$

$$L_7 = \{(i, j) : (i > 9 \text{ or } j > 9) \text{ and } j \neq 1, 2, 3, 4, 5 \text{ and } |i - j| > 3\}.$$

Computational evidence given in [3] suggests that the locator property does not hold for seeds other than those mentioned in Theorem 28 above.

We now bring the locator property into the context of the stabilization arrays. Recall our definition of the function $\sigma_i$ (Definition 18).

**Definition 29.** For a fixed seed $s$ we say that the *locator property holds in $\overleftrightarrow{\mathcal{A}}_s$ at $(i, j)$* if there exists $k \in \mathbb{Z}$ such that $\sigma_i(j) = k$ and $\sigma_i(k) = j$.

As with Definition 27, this means that $\sigma_i(\sigma_i(j)) = j$.

As might be expected, the locator property holding in the original array yields the locator property in the stabilized array as well:

**Proposition 30.** *If $\mathcal{A}_s$ is such that for each $i \in \mathbb{N}_0$ the locator property eventually holds in row $i$, then in $\overleftrightarrow{\mathcal{A}}_s$ the locator property holds for all $i$ and $j$.*

*Proof.* Fix $i, j$ and suppose that the locator property holds in $\mathcal{A}_s$ in row $i$ for all columns $j \geq J_i$. Choose $k \in \mathbb{N}_0$ large enough that both $\overleftrightarrow{a}_{i,j} + kp_i = a_{i,j+kp_i} \geq N_i$ and $j + kp_i \geq \max\{J_i, N_i\}$. Then

$$\begin{aligned}
\sigma_i(\sigma_i(j)) &= \sigma_i(\sigma_i(j) + kp_i) - kp_i && \text{(by periodicity)}\\
&= m_i(\sigma_i(j) + kp_i) - kp_i && \text{(because } \overleftrightarrow{a}_{i,j} + kp_i \geq N_i\text{)}\\
&= m_i(\sigma_i(j + kp_i)) - kp_i && \text{(by periodicity)}\\
&= m_i(m_i(j + kp_i)) - kp_i && \text{(since } j + kp_i \geq N_i\text{)}\\
&= (j + kp_i) - kp_i && \text{(since } j + kp_i \geq J_i\text{)}\\
&= j.
\end{aligned}$$

Thus the locator property holds for all $i$ and $j$. ☐

**4.4.** ***Contrasting distinct arrays in $\overleftrightarrow{\mathcal{A}}_*$.*** Building on Lemma 16, we end this section by examining the differences between $\overleftrightarrow{\mathcal{A}}_s$ and $\overleftrightarrow{\mathcal{A}}_t$ for $s \neq t$. Lemma 16 tells us that *all* stabilization arrays agree on row 0 entries. The smallest changes to the overall original arrays occur in the stabilizations for seeds 0 and 1. In particular, for columns $j \geq 0$ the stabilization of $\mathcal{A}_0$ agrees with $\mathcal{A}_0$ itself; i.e., for all $j \geq 0$ in $\mathcal{A}_0$, $\overleftrightarrow{a}_{i,j} = a_{i,j}$, as $\mathcal{A}_0$ is always arithmetically periodic. Since $\mathcal{A}_0$ and $\mathcal{A}_1$ differ only in their respective $2 \times 2$ principal subarrays, it follows immediately from Definition 13 that $\overleftrightarrow{\mathcal{A}}_0 = \overleftrightarrow{\mathcal{A}}_1$.

A natural question arising from Lemma 16 is whether the stabilizations arising from different seeds could possibly turn out to be equal. We conjecture that, except for the case $\overleftrightarrow{\mathcal{A}}_0 = \overleftrightarrow{\mathcal{A}}_1$ above, $\overleftrightarrow{\mathcal{A}}_s \neq \overleftrightarrow{\mathcal{A}}_t$ if $s \neq t$. The following proposition gives strong evidence for the veracity of this conjecture, and proves that, once again, seeds 0 and 1 behave differently than the rest:

**Proposition 31.** *There is a partition $S_1, S_2, \ldots, S_6$ of $\mathbb{N}_0$ such that if $s \in S_i$ and $t \in S_j$ for $i \neq j$ we have $\overleftrightarrow{\mathcal{A}}_s \neq \overleftrightarrow{\mathcal{A}}_t$.*

*Proof.* Define the partition $S_1, S_2, \ldots, S_6$ as follows:

$$S_1 = \{0, 1\},$$
$$S_2 = \{s \in \mathbb{N}_0 : s \equiv 0 \pmod 2,\ s \neq 0\},$$
$$S_3 = \{s \in \mathbb{N}_0 : s \equiv 0, 1 \pmod 3 \text{ and } s \equiv 1, 3 \pmod 4,\ s \neq 1\},$$
$$S_4 = \{s \in \mathbb{N}_0 : s \equiv 2 \pmod 3 \text{ and } s \equiv 1 \pmod 4\},$$
$$S_5 = \{s \in \mathbb{N}_0 : s \equiv 5, 8 \pmod 9 \text{ and } s \equiv 3 \pmod 4\},$$
$$S_6 = \{s \in \mathbb{N}_0 : s \equiv 2 \pmod 9 \text{ and } s \equiv 3 \pmod 4\}.$$

The crux of the proof is that the entries in column $j = 0$ vary from stabilized array to stabilized array. We make heavy use of Lemma 5, as it identifies patterns for rows 0 through 3 for every one of the original arrays $\mathcal{A}_s$.

The array $\mathcal{A}_0$ is fully arithmetically periodic, and hence, in its stabilization $\overleftrightarrow{\mathcal{A}}_0$, we have $\overleftrightarrow{a}_{i,j} \geq 0$ for $j \geq 0$, whereas we have $\overleftrightarrow{a}_{i,j} < 0$ for $j < 0$. As we previously noted, $\overleftrightarrow{\mathcal{A}}_0 = \overleftrightarrow{\mathcal{A}}_1$.

Suppose $s > 1$. If $s$ is even then $\overleftrightarrow{a}_{1,0} = -1$, whereas if $s$ is odd then $\overleftrightarrow{a}_{1,0} = 1$. If $s$ is odd and $s \equiv 0, 1 \pmod 3$, then $\overleftrightarrow{a}_{2,0} = -1$. If $s$ is odd and $s \equiv 2 \pmod 3$, then we need two cases: if $s \equiv 1 \pmod 4$, then $\overleftrightarrow{a}_{2,0} = -2$, whereas if $s \equiv 3 \pmod 4$, then $\overleftrightarrow{a}_{2,0} = 2$. Last, if $s \equiv 3 \pmod 4$, and $s \equiv 5, 8 \pmod 9$ then $\overleftrightarrow{a}_{3,0} = -1$ but if $s \equiv 3 \pmod 4$ and $s \equiv 2 \pmod 9$, then we get $\overleftrightarrow{a}_{3,0} = -2$.

Thus, for $s > 1$, at least one of $a_{1,0}$, $a_{2,0}$, $a_{3,0}$ is negative, contrary to the situation for $s = 0, 1$. More generally, we have shown that seeds from distinct blocks of the partition yield distinct triples $(a_{1,0}, a_{2,0}, a_{3,0})$, proving the result. □

**Conjecture 32.** For $s \neq t$ and $\{s, t\} \neq \{0, 1\}$ we have $\overleftrightarrow{\mathcal{A}}_s \neq \overleftrightarrow{\mathcal{A}}_t$.

In the proof of Proposition 31, considering columns other than column 0 or larger values of the row index $i$ would readily yield an even finer partition, adding to the evidence for our conjecture.

## 5. The multiplication group of the stabilization: general properties

For each fixed seed $s$, Definition 18 and Theorem 19 tell us that row $i$ of $\overleftrightarrow{\mathcal{A}}_s$ defines the arithmetically $p_i$-periodic permutation $\sigma_i : \mathbb{Z} \to \mathbb{Z}$. In this section, we focus on the observation that the functions $\sigma_i$ are affine permutations.

Write $[p] = \{1, \ldots, p\}$ and let $S_p$ denote the symmetric group on $[p]$. Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is an arithmetically $p$-periodic permutation of $\mathbb{Z}$. For each $i \in [p]$ and $k \in \mathbb{Z}$, there are $r_i \in \mathbb{Z}$ and $n_i \in [p]$ such that $f(i + kp) = r_i p + n_i + kp$, so, adopting notation suggested by [8], we may write $f = [r_1, \ldots, r_p \mid \bar{\tau}]$, where $\bar{\tau} \in S_p$ is given by $\bar{\tau}(i) = n_i$. Of course, this notation for $f$ is valid; since $f$ is arithmetically $p$-periodic, it is determined by its values on $[p]$.

If, with this notation, we have $f = [r_1, \ldots, r_p \mid \bar{\tau}]$ and $g = [s_1, \ldots, s_p \mid \bar{\sigma}]$, then a straightforward calculation gives

$$fg = [s_1 + r_{\bar{\sigma}(1)}, \ldots, s_p + r_{\bar{\sigma}(p)} \mid \bar{\tau}\bar{\sigma}]. \tag{5.0.1}$$

This shows that the group of arithmetically $p$-periodic permutations of $\mathbb{Z}$ is isomorphic to the semidirect product $\mathbb{Z}^p \ltimes S_p$. The semidirect product of groups is a standard construction; see [10, §5.5] for further details.

**Definition 33.** [8; 16] The group of *affine $p$-periodic permutations* of $\mathbb{Z}$ is the subgroup

$$\left\{ [r_1, \ldots, r_p \mid \bar{\tau}] \in \mathbb{Z}^p \ltimes S_p \;\middle|\; \sum r_i = 0 \right\}.$$

Proposition 34 is the key observation of this section.

**Proposition 34.** *For fixed seed $s$ and each $i$, the map $\sigma_i$ is an affine $p_i$-periodic permutation of $\mathbb{Z}$.*

*Proof.* By Proposition 17, $a_{i,(N_i+1)}, \ldots, a_{i,(N_i+p_i)}$ is a complete set of representatives mod $p_i$. For each $k$, Lemma 21 (which we may apply in this context since we are considering column indices greater than $N_i$) tells us that all entries congruent to and larger than $a_{i,(N_i+k)}$ appear to the right of column $N_i + k$. By Proposition 4 all other elements of $\mathbb{N}_0$ congruent to, and necessarily smaller than, $a_{i,(N_i+k)}$ must appear to the left, hence among $a_{i,0}, \ldots, a_{i,N_i}$. There are $\lfloor a_{i,(N_i+k)}/p_i \rfloor$ such entries (where $\lfloor x \rfloor$ denotes the usual floor function) and so

$$N_i + 1 = |\{a_{i,0}, \ldots, a_{i,N_i}\}| = \sum_{k=1}^{p_i} \left\lfloor \frac{a_{i,(N_i+k)}}{p_i} \right\rfloor. \tag{5.0.2}$$

Because (5.0.2) depends only on the fact that arithmetic periodicity in row $i$ holds for column indices at least $N_i$, but not on $N_i$ being the smallest such possible column index, we can, without loss of generality, choose any integer $l > 0$ such that $lp_i > N_i$ and replace $N_i$ in (5.0.2) with $lp_i$. Then

$$lp_i + 1 = \sum_{k=1}^{p_i} \left\lfloor \frac{a_{i,(lp_i+k)}}{p_i} \right\rfloor = \sum_{k=1}^{p_i} \left\lfloor \frac{\overleftrightarrow{a}_{i,(lp_i+k)}}{p_i} \right\rfloor,$$

by definition of $N_i$. But then, using the periodicity of $\overleftrightarrow{\mathcal{A}}_s$, we have

$$lp_i + 1 = \sum_{k=1}^{p_i} \left\lfloor \frac{\overleftrightarrow{a}_{i,(lp_i+k)}}{p_i} \right\rfloor = \sum_{k=1}^{p_i} \left\lfloor \frac{(\overleftrightarrow{a}_{i,k}) + lp_i}{p_i} \right\rfloor = lp_i + \sum_{k=1}^{p_i} \left\lfloor \frac{\overleftrightarrow{a}_{i,k}}{p_i} \right\rfloor,$$

which yields

$$\sum_{k=1}^{p_i} \left\lfloor \frac{\overleftrightarrow{a}_{i,k}}{p_i} \right\rfloor = 1. \tag{5.0.3}$$

By Proposition 17 and Theorem 19 we know that for $k = 1, \dots, p_i$ there are integers $r_1, \dots, r_{p_i}$ and a permutation $(n_1, \dots, n_{p_i})$ of $1, \dots, p_i$ such that $\overleftrightarrow{a}_{i,k} = r_k p_i + n_k$, and we have

$$\left\lfloor \frac{\overleftrightarrow{a}_{i,k}}{p_i} \right\rfloor = \begin{cases} r_k + 1 & \text{if } n_k = p_i, \\ r_k & \text{otherwise.} \end{cases}$$

Using this to rewrite (5.0.3), the result follows. □

**Definition 35.** For fixed seed $s$, we define $\overleftrightarrow{\mathcal{M}}_s$ to be the subgroup generated by $\{\sigma_i \mid i \in \mathbb{N}_0\}$ of the symmetric group on $\mathbb{Z}$, with the group operation defined to be composition of the permutations. We refer to $\overleftrightarrow{\mathcal{M}}_s$ as the *multiplication group* for $\overleftrightarrow{\mathcal{A}}_s$.

Since a composition of a $p$-periodic affine permutation and a $q$-periodic affine permutation is also a periodic permutation of period at most $\operatorname{lcm}(p, q)$, by Proposition 34, $\overleftrightarrow{\mathcal{M}}_s$ is a group of affine permutations, although not necessarily all of the same period. By Proposition 30, when $s$ is such that the locator property holds in $\overleftrightarrow{\mathcal{A}}_s$, we have $\sigma_i(\sigma_i(j)) = j$ for each $i$, $j$, and thus $\overleftrightarrow{\mathcal{M}}_s$ is generated by involutory affine permutations. This is interesting because the affine permutation groups $\tilde{A}_n$ (not to be confused with $\overleftrightarrow{\mathcal{A}}_s$) are known to be generated by involutions.

## 6. The multiplication group of the stabilization: some structure

**6.1.** ***The structure of $\overleftrightarrow{\mathcal{M}}_0$ and $\overleftrightarrow{\mathcal{M}}_1$.*** For any seed $s$, periodicity in each row implies that any element of $\overleftrightarrow{\mathcal{M}}_s$ is determined by its values on $\mathbb{N}_0$. In $\overleftrightarrow{\mathcal{M}}_0$, though, we have the additional property that, for every row $i$, the permutation $\sigma_i$ restricts to a bijection $\sigma_i : \mathbb{N}_0 \to \mathbb{N}_0$. It follows that the structure of $\overleftrightarrow{\mathcal{M}}_0$ is exactly the structure of the group $\Gamma$ generated by the restrictions $\sigma_i|_{\mathbb{N}_0}$.

Now, as is well known, $\mathcal{A}_0$ can itself be viewed as the multiplication table for a group structure on $\mathbb{N}_0$, namely, the Cartesian product of countably many copies of $\mathbb{Z}/2\mathbb{Z}$, each corresponding to a digit in the binary representation of the natural numbers [7]. It follows that $\Gamma$, and hence $\overleftrightarrow{\mathcal{M}}_0$, is isomorphic to this same Cartesian product. Since $\overleftrightarrow{\mathcal{A}}_0 = \overleftrightarrow{\mathcal{A}}_1$, we have $\overleftrightarrow{\mathcal{M}}_0 = \overleftrightarrow{\mathcal{M}}_1$ as well.

For comparison with Section 6.2, consider the following four elements of $\overleftrightarrow{\mathcal{M}}_0$; it is not difficulty to verify that, as expected, these comprise a finite subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

$$\begin{aligned}\sigma_0 &= e,\\ \sigma_1 &= [-1, 0, 0, 1 \mid (14)(23)],\\ \sigma_2 &= [0, -1, 0, 1 \mid (13)(24)],\\ \sigma_3 &= [0, 0, -1, 1 \mid (12)(34)].\end{aligned}$$

For convenience we have written $\sigma_1$ with period 4, even though as a map it has period 2. Proposition 38 addresses in more detail the issue of multiple representations of an affine permutation.

### 6.2. *An interesting subgroup of $\overleftrightarrow{\mathcal{M}}_2$.*

In this section we study a small subgroup of the multiplication group $\overleftrightarrow{\mathcal{M}}_2$, one of the highlights of this paper; even this initial analysis of $\overleftrightarrow{\mathcal{M}}_2$ demonstrates its complexity and beauty. As noted above more generally, $\overleftrightarrow{\mathcal{M}}_2$ is a group of affine permutations and each $\sigma_i$ in $\overleftrightarrow{\mathcal{M}}_2$ is an involution. This is of course also true in $\overleftrightarrow{\mathcal{M}}_0$ (and $\overleftrightarrow{\mathcal{M}}_1$), but as we will now see, the similarity does not go much beyond that observation.

We write $\bar{e}$ for the identity element of $S_n$ regardless of the value of $n$ and write $e$ for the identity element of $\overleftrightarrow{\mathcal{M}}_2$. Reading off rows 0, 1, 2, 3 from $\overleftrightarrow{\mathcal{A}}_2$ (see Figure 2), consider the following four elements of $\overleftrightarrow{\mathcal{M}}_2$:

$$\begin{aligned}\sigma_0 &= e,\\ \sigma_1 &= [0, 0, 0, 0 \mid (12)(34)],\\ \sigma_2 &= [-1, -1, 1, 1 \mid (13)(24)],\\ \sigma_3 &= [-1, 0, 0, 1 \mid (14)(23)].\end{aligned}$$

Again for convenience, we have written $\sigma_1$ and $\sigma_3$ with period 4, even though as maps they have period 2.

**Proposition 36.** *The permutations $\sigma_1$, $\sigma_2$, $\sigma_3$ satisfy the following relations for all $k \in \mathbb{Z}$* (we write $\boldsymbol{a}$ for $(a, b, c)$):

(1) $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = e$.

(2) $(\sigma_1\sigma_2)^2 = e$.

(3) $(\sigma_1\sigma_3\sigma_2\sigma_3)^2 = e$.

$$
(4)\quad \sigma_1^a(\sigma_3\sigma_1)^{2k+b}\sigma_3^c=\begin{cases}
[k,-k,k,-k\,|\,\bar{e}] & \textit{if } \boldsymbol{a}=(0,0,0),\\
[k,-k-1,k+1,-k\,|\,(13)(24)] & \textit{if } \boldsymbol{a}=(0,1,0),\\
[k,-k,k,-k\,|\,(12)(34)] & \textit{if } \boldsymbol{a}=(1,0,0),\\
[k,-k-1,k+1,-k\,|\,(14)(23)] & \textit{if } \boldsymbol{a}=(1,1,0),\\
[-k-1,k,-k,k+1\,|\,(14)(23)] & \textit{if } \boldsymbol{a}=(0,0,1),\\
[-k-1,k+1,-k-1,k+1\,|\,(12)(34)] & \textit{if } \boldsymbol{a}=(0,1,1),\\
[-k-1,k,-k,k+1\,|\,(13)(24)] & \textit{if } \boldsymbol{a}=(1,0,1),\\
[-k-1,k+1,-k-1,k+1\,|\,\bar{e}] & \textit{if } \boldsymbol{a}=(1,1,1).
\end{cases}
$$

$$
(5)\quad \sigma_2^a(\sigma_3\sigma_2)^{2k+b}\sigma_3^c=\begin{cases}
[-k,-k,k,k\,|\,\bar{e}] & \textit{if } \boldsymbol{a}=(0,0,0),\\
[-k-1,-k,k,k+1\,|\,(12)(34)] & \textit{if } \boldsymbol{a}=(0,1,0),\\
[-k-1,-k-1,k+1,k+1\,|\,(13)(24)] & \textit{if } \boldsymbol{a}=(1,0,0),\\
[-k-2,-k-1,k+1,k+2\,|\,(14)(23)] & \textit{if } \boldsymbol{a}=(1,1,0),\\
[k-1,k,-k,-k+1\,|\,(14)(23)] & \textit{if } \boldsymbol{a}=(0,0,1),\\
[k,k,-k,-k\,|\,(13)(24)] & \textit{if } \boldsymbol{a}=(0,1,1),\\
[k,k+1,-k-1,-k\,|\,(12)(34)] & \textit{if } \boldsymbol{a}=(1,0,1),\\
[k+1,k+1,-k-1,-k-1\,|\,\bar{e}] & \textit{if } \boldsymbol{a}=(1,1,1).
\end{cases}
$$

*Proof.* Relation (1) follows from Proposition 30. The proofs of the other relations are all straightforward calculations, some involving induction. To illustrate the group operation in $\overleftrightarrow{\mathcal{M}}_2$, we verify the case $(a, b, c) = (0, 1, 0)$ of relation (4).

First, using (5.0.1) we have

$$
\begin{aligned}
\sigma_3\sigma_1 &= [-1, 0, 0, 1 \mid (14)(23)] \cdot [0, 0, 0, 0 \mid (12)(34)]\\
&= [0+0, 0+(-1), 0+1, 0+0 \mid (13)(24)] = [0, -1, 1, 0 \mid (13)(24)],
\end{aligned}
$$

and therefore also

$$
(\sigma_3\sigma_1)^2 = [0+1, -1+0, 1+0, 0+(-1) \mid \bar{e}] = [1, -1, 1, -1 \mid \bar{e}].
$$

Proceeding by induction on $k$, suppose that

$$
(\sigma_3\sigma_1)^{2k+1} = [k, -k-1, k+1, -k \mid (13)(24)].
$$

We then have

$$
\begin{aligned}
(\sigma_3\sigma_1)^{2(k+1)+1} &= [k, -k-1, k+1, -k \mid (13)(24)] \cdot [1, -1, 1, -1 \mid \bar{e}]\\
&= [1+k, -1+(-k-1), 1+(k+1), -1+(-k) \mid (13)(24)]\\
&= [k+1, -(k+1)-1, (k+1)+1, -(k+1) \mid (13)(24)],
\end{aligned}
$$

as desired, verifying the relation for positive $k$. To see that the relation holds for negative $k$, note first that $(\sigma_3\sigma_1)^{-2} = [-1, 1, -1, 1 \mid \bar{e}]$, and then use this in a second proof by induction. □

We now use the results of Proposition 36 to verify that the subgroup of $\overleftrightarrow{\mathcal{M}}_2$ generated by $\sigma_1$, $\sigma_2$, $\sigma_3$ is given up to isomorphism by the presentation

$$\langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = (\sigma_1\sigma_2)^2 = (\sigma_1\sigma_3\sigma_2\sigma_3)^2 = e \rangle$$

or, more picturesquely, that the graph depicted in Figure 3 is the Cayley graph $G(\overleftrightarrow{\mathcal{M}}_2, \{\sigma_1, \sigma_2, \sigma_3\})$ of that subgroup, using the right action of $\overleftrightarrow{\mathcal{M}}_2$ on itself. We will write $G_{123}$ for $G(\overleftrightarrow{\mathcal{M}}_2, \{\sigma_1, \sigma_2, \sigma_3\})$. We encourage the reader to refer to Figure 3 while reading through the following discussion verifying its validity.

First, we note that by relation (1) of Proposition 36, all edges in $G_{123}$ may be treated as bidirectional, hence undirected. Relations (2) and (3) validate the $\sigma_1, \sigma_2$-squares and $\sigma_1, \sigma_2, \sigma_3$-octagons in $G_{123}$, respectively.

Now consider the subgraph $G_{13}$ of $G_{123}$ obtained by deleting all $\sigma_2$ edges. All vertices of $G_{13}$ have degree 2, and hence $G_{13}$ is a disjoint union of infinite paths and finite cycles, each alternating between $\sigma_1$-edges and $\sigma_3$-edges. However, relation (4) of Proposition 36 shows that the only possible solutions to $\sigma_1^a(\sigma_3\sigma_1)^{2k+b}\sigma_3^c = e$ fail to correspond to any finite cycle in $G_{123}$. Thus, the $\sigma_1$-edges and $\sigma_3$-edges form a disjoint union of infinite paths. We refer to each such infinite path as a "1,3-path".

A similar analysis, using relation (5) of Proposition 36, shows that the subgraph $G_{23}$ of $G_{123}$ obtained by deleting all $\sigma_1$ edges also consists of a disjoint union of infinite paths, which we refer to as "2,3-paths".

**Proposition 37.** *If $P_1$ is a 1,3-path and $P_2$ is a 2,3-path, then $P_1$ and $P_2$ share exactly two vertices, and these are the ends of a single $\sigma_3$ edge.*

*Proof.* We first show that $P_1$ and $P_2$ have a vertex $X$ in common. Consider the group $\widehat{\mathcal{M}}$ given by the presentation

$$\widehat{\mathcal{M}} = \langle x_1, x_2, x_3 \mid x_1^2, x_2^2, x_3^2 \rangle.$$

We view the elements of $\widehat{\mathcal{M}}$ as reduced words in $x_1$, $x_2$, $x_3$, i.e., we presume that all possible cancellations have been performed. Let $\pi : \widehat{\mathcal{M}} \to \overleftrightarrow{\mathcal{M}}_2$ be the map given by $\pi : x_i \mapsto \sigma_i$ for $i = 1, 2, 3$, and let $\mathrm{inv}_{12} : \widehat{\mathcal{M}} \to \mathbb{N}_0$ be defined on a word $q_1q_2\cdots q_n$ by

$$\mathrm{inv}_{12}(q_1q_2\cdots q_n) := \big|\{(i, j) \mid q_i = x_2,\ q_j = x_1 \text{ and } i < j\}\big|.$$

(Note that $\mathrm{inv}_{12}(q_1q_2\cdots q_n)$ counts inversions of the order $x_1, x_2$.) We define two kinds of operations on a word $w \in \widehat{\mathcal{M}}$. If there are words $w_1, w_2 \in \widehat{\mathcal{M}}$ such that $w = w_1x_2x_1w_2$, then we refer to $w_1x_1x_2w_2$ as a 2,1-swap of $w$.

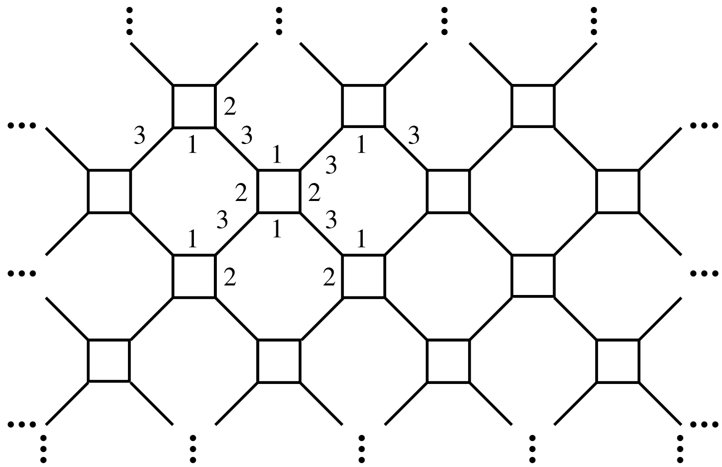


**Figure 3.** The Cayley graph for the subgroup generated by $\sigma_1$, $\sigma_2$, and $\sigma_3$. Horizontal edges correspond to $\sigma_1$, vertical edges correspond to $\sigma_2$, and diagonal edges correspond to $\sigma_3$.

If there are words $w_1, w_2 \in \widehat{\mathcal{M}}$ such that $w = w_1 x_2 x_3 x_1 w_2$ then we refer to $w_1 x_3 x_1 x_3 x_2 x_3 w_2$ as a 2,3,1-swap of $w$. These swaps correspond to taking alternate paths around the squares and octagons, respectively. Note that if a word $w'$ is obtained from a word $w$ via a sequence of one or more swaps of either kind, then $\mathrm{inv}_{1,2}(w') < \mathrm{inv}_{1,2}(w)$, and by relations (2) and (3) of Proposition 36 we have $\pi(w) = \pi(w')$.

Suppose $w \in \widehat{\mathcal{M}}$ has $\mathrm{inv}_{1,2}(x) > 0$. Then in the word $w$ there are subwords $w_1$, $w_2$, $w_3$, possibly empty, such that $w = w_1 x_2 w_2 x_1 w_3$. Without loss of generality, we may assume that $w_1$ is as large as possible; this implies that $w_2$ contains no instance of $x_2$. Given $w_1$, we may also suppose that $w_3$ is as large as possible; this implies that $w_2$ contains no instance of $x_1$. If follows that either $w_2$ is empty or $w_2 = x_3$. In the former case we may apply a 2,1-swap to $w$ and in the latter case we may apply a 2,3,1-swap. Thus, we have proven that if $\mathrm{inv}_{1,2}(x) > 0$ then we can always use a sequence of 2,1-swaps and 2,3,1-swaps to obtain a word $w'$ with $\mathrm{inv}_{1,2}(w') = 0$ and $\pi(w') = \pi(w)$.

Now let $\alpha$ be a vertex on $P_1$ and let $\beta$ be a vertex on $P_2$; since $G_{123}$ is connected, there is a path $P$ from $\alpha$ to $\beta$. By viewing vertices as paths from the "origin", i.e., the vertex corresponding to the group identity, we can interpret $\alpha$, $\beta$ and $P$ as elements of $\overleftrightarrow{\mathcal{M}}_2$ such that $\beta = \alpha P$, and that $P = \pi(\hat{P})$ for some word $\hat{P} \in \widehat{\mathcal{M}}$. Using 2,1-swaps and 2,3,1-swaps as necessary, we may assume $\mathrm{inv}_{1,2}(\hat{P}) = 0$. In that case there are words $\hat{P}_1$, $\hat{P}_2$ with $\hat{P} = \hat{P}_1 \hat{P}_2$ such that $\hat{P}_1$ has no instance of $x_2$ and $\hat{P}_2$ has no instance of $x_1$. Because $\pi(\hat{P}_1)$ has no instance of $\sigma_2$, $\alpha\pi(\hat{P}_1)$ corresponds to a vertex on $P_1$. Similarly, because

$\pi(\hat{P}_2)$ has no instance of $\sigma_1$, $\beta\pi(\hat{P}_2)^{-1}$ corresponds to a vertex on $P_2$. Now we have $\beta = \alpha P = \alpha\pi(\hat{P}_1)\pi(\hat{P}_2)$, and thus $\beta\pi(\hat{P}_2)^{-1} = \alpha\pi(\hat{P}_1)$, which shows that we have a vertex $\gamma = \alpha\pi(\hat{P}_1)$ at which $P_1$ and $P_2$ intersect. Of course, each of $P_1$ and $P_2$ has two edges incident to $\gamma$. Since every vertex in $G_{123}$ has exactly one $\sigma_1$ edge, one $\sigma_2$ edge, and one $\sigma_3$ edge, and $P_1$ and $P_2$ can share neither a $\sigma_1$ edge nor a $\sigma_2$ edge, they necessarily share a $\sigma_3$ edge $d$ for which $\gamma$ is an endpoint.

Suppose now, for the sake of contradiction, that $P_1$ and $P_2$ also share a vertex $\delta$ which is not an endpoint of $d$. Let $P_1'$, $P_2'$ be the paths along $P_1$, $P_2$, respectively, from $\gamma$ to $\delta$. Then $P_1'$ necessarily corresponds to an element $g_1$ of the form $\sigma_1^a(\sigma_3\sigma_1)^{2k+b}\sigma_3^c$ for some $a, b, c \in \{0, 1\}$ and $k \in \mathbb{Z}$, and $P_2'$ necessarily corresponds to an element $g_2$ of the form $\sigma_2^q(\sigma_3\sigma_2)^{2l+r}\sigma_3^s$ for some $q, r, s \in \{0, 1\}$ and $l \in \mathbb{Z}$. Since $P_1'$ and $P_2'$ both start at $\gamma$ and end at $\delta$, interpreting $\gamma$ and $\delta$ as elements of $\overleftrightarrow{\mathcal{M}}_2$ we have $\gamma g_1 = \delta = \gamma g_2$, and thus $g_1 = g_2$.

A careful analysis of the cases in relations (4) and (5), though, shows that $g_1 \neq g_2$. To see this, suppose that $g_1 = [r_1, r_2, r_3, r_4 \mid \bar{\tau}]$ and $g_2 = [s_1, s_2, s_3, s_4 \mid \bar{\sigma}]$. If $\bar{\tau} = \bar{\sigma} = \bar{e}$ or $\bar{\tau} = \bar{\sigma} = (14)(23)$, then $[r_1, r_2, r_3, r_4]$ and $[s_1, s_2, s_3, s_4]$ cannot be equal since in both cases they necessarily have different sign patterns. If $\bar{\tau} = \bar{\sigma} = (12)(34)$ or $\bar{\tau} = \bar{\sigma} = (13)(24)$, then $[r_1, r_2, r_3, r_4]$ and $[s_1, s_2, s_3, s_4]$ cannot be equal since in both cases the entries of one will all have the same absolute value and the entries of the other will not. □

This confirms that Figure 3 does indeed correctly depict $G_{123}$.

We note that the inclusion of even the single additional generator $\sigma_4$, given by

$$\sigma_4 = [0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 \mid (1\,3)(2\ 11)(4\,7)(5\,8)(6\,9)(10\ 12)],$$

complicates the situation significantly. Indeed, we have the following relations:

(1) $\sigma_4^2 = e$.

(2) $(\sigma_3\sigma_4)^3 = e$.

(3) $\sigma_1\sigma_4$ and $\sigma_2\sigma_4$ have infinite order.

(4) $(\sigma_1\sigma_3\sigma_4)^8 = (\sigma_3\sigma_1\sigma_4)^8 = (\sigma_2\sigma_3\sigma_4)^8 = (\sigma_3\sigma_2\sigma_4)^8 = e$.

(5) $\sigma_1\sigma_2\sigma_4 = \sigma_2\sigma_1\sigma_4$ has infinite order.

(6) $$(\sigma_3\sigma_2\sigma_1\sigma_4)^{10} = (\sigma_1\sigma_3\sigma_1\sigma_4)^4 = (\sigma_1\sigma_3\sigma_2\sigma_1\sigma_4)^6 = (\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_4)^{12} = (\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3\sigma_4)^5 = e.$$

To verify these, it is helpful to make use of a simple lemma describing how our notation can be used to represent the same permutation with different periods.

**Proposition 38.** *Suppose that* $f = [r_1, r_2, \ldots, r_p \mid \tau]$, *where* $\tau \in S_p$. *Then, for each positive integer* $l$, *the permutation* $f$ *can also be expressed in the* *form* $f = [R_1, R_2, \dots, R_{lp} \mid T]$, *where* $T \in S_{lp}$ *and for each* $0 \le k < l$ *and* $i \in [p]$,

$$R_{i+kp} = \left\lfloor \frac{f(i) + kp - 1}{lp} \right\rfloor \quad \textit{and} \quad T(i+kp) = f(i) + (k - R_{i+kp} l)p.$$

*Proof.* If we write $n_i = \tau(i)$ for $i \in [p]$ and $\nu_i = T(i)$ for $i \in [lp]$, then for any $0 \le k < l$ and $i \in [p]$ we want to construct $R_{i+kp} \in \mathbb{Z}$ and $\nu_{i+kp} \in [lp]$ such that

$$r_i p + n_i + kp = f(i) + kp = f(i+kp) = R_{i+kp} lp + \nu_{i+kp}. \qquad (*)$$

Solving for $\nu_{i+kp}$, we require

$$0 < r_i p + n_i + kp - R_{i+kp} lp \le lp,$$

and now solving for $R_{i+kp}$ gives

$$\frac{r_i p + n_i + kp}{lp} > R_{i+kp} \ge -1 + \frac{r_i p + n_i + kp}{lp}.$$

There exists a unique integer $R_{i+kp}$ satisfying this condition, and it is given by the formula in the statement of the lemma. From $(*)$, we easily get $\nu_{i+kp} = f(i) + kp - R_{i+kp} lp$. □

Using Proposition 38, we can express $\sigma_1$, $\sigma_2$, and $\sigma_3$ in period 12, to match the period of $\sigma_4$. This gives

$$\sigma_1 = [0, 0, \dots, 0 \mid (1\,2)(3\,4)(5\,6)(7\,8)(9\,10)(11\,12)],$$
$$\sigma_2 = [-1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1 \mid (1\,11)(2\,12)(3\,5)(4\,6)(7\,9)(8\,10)],$$
$$\sigma_3 = [-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 \mid (1\,12)(2\,3)(4\,5)(6\,7)(8\,9)(10\,11)].$$

Verification of the various relations now becomes primarily a matter of direct computation; we sketch here some key points and examples. For instance,

$$\sigma_3\sigma_4 = [0, -1, -1, 0, 0, 0, 0, 0, 0, 1, 1, 0 \mid (1\,2\,10)(3\,12\,11)(4\,6\,8)(5\,9\,7)],$$
$$(\sigma_3\sigma_4)^2 = [-1, 0, -1, 0, 0, 0, 0, 0, 0, 1, 0, 1 \mid (1\,10\,2)(3\,11\,12)(4\,8\,6)(5\,7\,9)],$$

from which item (2) readily follows.

To verify claims of infinite order of a permutation $f = [r_1, r_2, \dots, r_p \mid \tau]$, as in items (3) and (5), it is necessary and sufficient to show that if $q$ is the order of $\tau$ and we write $f^q = [R_1, R_2, \dots, R_p \mid \bar{e}]$, then at least one of $R_1, \dots, R_p$ is not 0. This is because $f^{aq} = [aR_1, aR_2, \dots, aR_p \mid \bar{e}]$ for all $a \in \mathbb{Z}$.

For example, we have

$$\sigma_2\sigma_4 = [0, 0, -1, 0, 0, 0, 0, 0, 0, 1, 0, 0 \mid (1\,5\,10\,2)(3\,11\,12\,8)(4\,9)(6\,7)],$$
$$(\sigma_2\sigma_4)^2 = [0, 0, -1, 0, 1, 0, 0, -1, 0, 1, 0, 0 \mid (1\,10)(2\,5)(3\,12)(8\,11)],$$
$$(\sigma_2\sigma_4)^4 = [1, 1, -1, 0, 1, 0, 0, -1, 0, 1, -1, -1 \mid \bar{e}],$$

which confirms the second assertion in item (3).

The other assertions are verified similarly. Note that our list of relations implicitly conveys all relations involving $\sigma_4$ which are of length 3 or less. However, we have omitted the vast majority of relations, both of finite and infinite exponent and even among those of length at most 6.

## 7. The mex–maxx conjecture

We end this paper with a definition and a conjecture.

**Definition 39.** Let $S \subsetneq \mathbb{Z}$ be any set such that $\mathbb{Z} \setminus S$ contains only finitely many positive integers. Define $\mathrm{maxx}_{\mathbb{Z}}(S)$ to be the *max*imal e*x*cluded element of $S$, i.e., the largest integer not in $S$.

**Conjecture 40** (mex–maxx conjecture). For each seed $s$ and for every $(i, j) \in \mathbb{N}_0 \times \mathbb{Z}$ we have

$$\mathrm{mex}_{\mathbb{Z}}\big(\overleftrightarrow{\mathrm{Left}}(i, j) \cup \overleftrightarrow{\mathrm{Up}}(i, j)\big) = \overleftrightarrow{a}_{i,j} = \mathrm{maxx}_{\mathbb{Z}}\big(\overleftrightarrow{\mathrm{Right}}(i, j) \cup \overleftrightarrow{\mathrm{Up}}(i, j)\big).$$

We note that this property does not hold in the original arrays $\mathcal{A}_s$ for $s \geq 2$.

## 8. Acknowledgements

Michael Cowen and Thomas Zaslavsky offered helpful suggestions during the writing stage. We also thank the referees for helpful comments which led to improvements in this paper.

## References

[1] L. Abrams and D. S. Cowen-Morton, "Algebraic structure in a family of Nim-like arrays", *J. Pure Appl. Algebra* **214**:2 (2010), 165–176. MR Zbl

[2] L. Abrams and D. S. Cowen-Morton, "Periodicity and other structure in a colorful family of Nim-like arrays", *Electron. J. Combin.* **17**:1 (2010), art. id. 103. MR Zbl

[3] L. Abrams and D. S. Cowen-Morton, "A family of Nim-like arrays: the locator theorem", *Theoret. Comput. Sci.* **535** (2014), 31–37. MR Zbl

[4] M. H. Albert and R. J. Nowakowski, "NIM restrictions", *Integers* **4** (2004), art. id. G01. MR Zbl

[5] M. R. Allen, "On the periodicity of genus sequences of quaternary games", *Integers* **7** (2007), art. id. G04. MR Zbl

[6] I. Althöfer and J. Bültermann, "Superlinear period lengths in some subtraction games", *Theoret. Comput. Sci.* **148**:1 (1995), 111–119. MR Zbl

[7] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning ways for your mathematical plays*, *I*, 2nd ed., A K Peters, 2001. Zbl

[8] A. Björner and F. Brenti, "Affine permutations of type *A*", *Electron. J. Combin.* **3**:2 (1996), art. id. 18. MR Zbl

[9] J. H. Conway, *On numbers and games*, 2nd ed., A K Peters, 2001. MR Zbl

[10] A. Dress, A. Flammenkamp, and N. Pink, "Additive periodicity of the Sprague–Grundy function of certain Nim games", *Adv. in Appl. Math.* **22**:2 (1999), 249–270. MR Zbl

[11] P. M. Grundy, "Mathematics and games", *Eureka* **1939**:2 (1939), 6–8.

[12] D. G. Horrocks and R. J. Nowakowski, "Regularity in the $\mathcal{G}$-sequences of octal games with a pass", *Integers* **3** (2003), art. id. G01. MR Zbl

[13] S. Howse and R. J. Nowakowski, "Periodicity and arithmetic-periodicity in hexadecimal games", *Theoret. Comput. Sci.* **313**:3 (2004), 463–472. MR Zbl

[14] M. Huggan and B. Stevens, "Polynomial time graph families for Arc Kayles", *Integers* **16** (2016), art. id. A86. MR Zbl

[15] H. A. Landman, "A simple FSM-based proof of the additive periodicity of the Sprague–Grundy function of Wythoff's game", pp. 383–386 in *More games of no chance* (Berkeley, CA, 2000), edited by R. J. Nowakowski, Math. Sci. Res. Inst. Publ. **42**, Cambridge Univ. Press, 2002. MR Zbl

[16] G. Lusztig, "Some examples of square integrable representations of semisimple $p$-adic groups", *Trans. Amer. Math. Soc.* **277**:2 (1983), 623–653. MR Zbl

[17] E. Sopena, "$i$-Mark: a new subtraction division game", *Theoret. Comput. Sci.* **627** (2016), 90–101. MR Zbl

[18] R. P. Sprague, "Über mathematische Kampfspiele", *Tôhoku Math. J.* **41** (1935), 438–444. Zbl

[19] W. Stromquist and D. Ullman, "Sequential compounds of combinatorial games", *Theoret. Comput. Sci.* **119**:2 (1993), 311–321. MR Zbl

[20] M. Weimerskirch, "An algorithm for computing indistinguishability quotients in misère impartial combinatorial games", pp. 267–277 in *Games of no chance* 4, edited by R. J. Nowakowski, Math. Sci. Res. Inst. Publ. **63**, Cambridge Univ. Press, 2015. MR Zbl

abrams.lowell@gmail.com *Cincinnati, OH, United States*

morton@xavier.edu *Department of Mathematics, Xavier University, Cincinnati, OH, United States*