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Values of generic impartial combinatorial games

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We introduce the study of “typical” combinatorial games, where a specific game is chosen at random from a large set of games. We study one such class of games, those arising from a k-regular multigraph, and show that the limiting distribution of (Sprague–Grundy) values converges to a stationary distribution which only depends on k. We provide an iterative procedure for computing this distribution and prove several high probability results for finite plays. Our work provides some initial steps towards formalizing the “renormalization approach” to combinatorial games which has proven effective at describing the properties of several classic combinatorial games but as yet is nonrigorous in most applications. In addition, our results may provide insights into properties of complex combinatorial games that have so far resisted formal analyses.

1. Introduction

Combinatorial game theory primarily focuses on the detailed analysis and techniques for specific games [1; 2]. We propose a complementary approach in which we study statistical regularities across large classes of games. Our main result shows that for a specific class of games, there exist strong regularities among the distribution of Sprague–Grundy values (henceforth referred to simply as “values”, capitalized to distinguish from other uses of the word). We show that almost all k-regular multigraph games converge to a specific distribution that only depends on k. We also prove other nonasymptotic properties of the distribution to understand the convergence process.

Our analysis suggests a path to the formalization of renormalization analyses, which have been effective in understanding games which appear to be unsolvable, such a Wythoff’s game, Chomp, and perturbed versions of Nim [4] as well as many other games and variants [5], but are mostly nonrigorous, with the notable exception of [6].

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Also, as discussed in the final section of this paper, this model may also be useful for understanding deterministic, but complex, games where the distributions of values seem to behave randomly.

2. Definitions

In order to compute the value of a game one considers the game's directed graph, G, where nodes are positions and edges from a position point to other positions via legal moves (their options) [2]. We will assume that this graph has a unique sink (where the game ends) and is acyclic. One can then compute the values of the positions iteratively, from the sink, by setting the value of the sink, $V(r) = 0$, and then defining the value of node w to be the mex( ) of all its options, where mex( ) is the minimum excluded value of its arguments, i.e., $\text{mex}(0, 1, 3) = 2$.

To simplify the presentation we will allow the digraph to be a directed multigraph, with potentially multiple edges from one node to another. This allows us to define k-regular games as those in which every node in the directed multigraph, except the sink, has exactly k outgoing edges.

Note that one could consider a directed graph that is not a multigraph but is approximately k-regular asymptotically without changing the asymptotic behavior of the values, since for large n the probability of two edges from a node pointing to the same node vanishes. However, the assumption of strict k-regularity chosen in the paper simplifies the presentation and analysis.

Since there always exists a topological ordering for a directed graph, to simplify the presentation, we will assume that all nodes are numbered, $0, 1, \ldots$ and nodes only point to nodes with lower numeric values, i.e., the game proceeds from higher numbered nodes with legal moves to lower numbered nodes. Then we can view the value function as $V(n) : N_0 \to N_0$, where $N_0 = \{0, 1, 2, \ldots\}$.

In this paper we will study the set of games which consists of all k-regular multigraphs with labeled edges. In order to construct a (uniformly) randomly chosen k-regular multigraph, we can take each node and then sequentially for each of the k edges chose a random node with lower numeric label. For example, if k=2 then node 6 could chose nodes 1 and 3, or node 4 twice.

We will focus on the dependent empirical probabilities,

$$p_j^k(n) = |\{0 \le i < n \mid V(i) = j\}|/n. \tag{1}$$

We will also consider their asymptotic values

$$p_j^k = \lim_{n\to\infty} p_j^k(n) \tag{2}$$

which we will show are well defined.

Note that both $V(n)$ and $p^k_j(n)$ depend deterministically on the exact game so will be viewed as random variables when a game is chosen at random.

2.1. ***Examples.*** Consider the classic game of one pile Nim [3], which is neither random nor k-regular but provides a simple example to demonstrate our notation. In this game there is an initial number of tokens and players alternately remove any number of tokens and the player who takes the last token wins. Clearly this is a simple game as the optimal strategy is to take all the tokens, but it will illustrate our basic notation. In this case node j corresponds to the case with j tokens and since all moves move to lower number of tokens, node j has exactly j edges, one to each node i for $0 \le i < j$. Since $V(0) = 0$, by definition, $v(1) = \text{mex}(V(0)) = \text{mex}(0) = 1$ and inductively, $V(j) = j$ so $p^{NIM}_j(n)$ does not converge for any j.

If we modify the game so that a player is only allowed to take 1 token at a turn (the she loves me she loves me not game), then we see that every node $j > 1$ has one edge to the node directly before it and $V(j) = 0$ if j is even and $V(j) = 1$ if j is odd. In this case $p_j(n)$ converges to $p_j = \frac{1}{2}$ for $j \in \{0, 1\}$.

Now consider a randomly chosen k-regular game where $k = 1$. Note that in this case node 1 has 1 edges pointing to node 0, while node 2 has 1 edge which is chosen randomly to point at either node 1 or node 0. Since $V(0) = 0$ we must have $V(1) = 1$, thus $p^1_0(1) = p^1_1(1) = \frac{1}{2}$. The next value, $V(2)$ is equally likely to be 0 or 1 and thus $p^1_0(2) = \frac{2}{3}$ or $p^1_0(2) = \frac{1}{3}$. It seems intuitively obvious that this $p^1_j(N)$ converges to $\frac{1}{2}$ for $j \in \{0, 1\}$ and 0 otherwise, since if $p^1_0(n) > \frac{1}{2}$ for some n then it is more likely for $V(n) = 1$ which will move it towards the stationary distribution. In fact, this convergence is quite strong and we will show that both $p^1_0(n) = p^1_1(n)$ converge to $\frac{1}{2}$ for large n for *almost all games chosen at random from the* 1*-regular games.*

For $k = 2$ we similarly see that $V(0) = 0$ and V(1) = 1, but $V(2) = 0$ with probability $\frac{1}{4}$, when both edges from node 2 point to node 1. Similarly, $V(2) = 1$ with probability $\frac{1}{4}$ so $V(2) = 2$ with probability $\frac{1}{2}$. To compute the stationary distribution, we note that $V(n) = 0$ whenever none of the options have value 0 which occurs with probability $(1 - p^2_0(n))^2$. Therefore the distribution will be stationary if $(1 - p^2_0(n))^2 = p^2_0(n)$ which implies that

$$p^2_0(n) = (3 - 5^{1/2})/2 \approx 0.38$$

As we show below, this is the asymptotic distribution and we will show that $p^2_0(n)$ will converge to a fixed value p^2_0 for *almost all games chosen at random from the* 2*-regular games.*

Now consider $p^5_j(n)$ in as shown in Figure 1. Note that the numerical values converge rapidly and that p^5_5 is quite small.

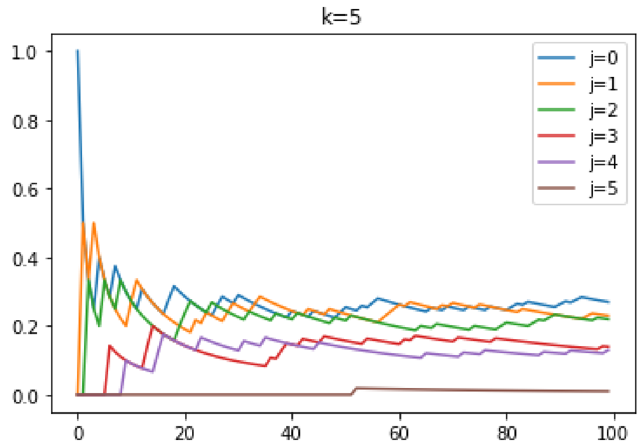

Figure 1. Numerical values of $p_j^5(n)$ for a randomly chosen 5-regular multigraph. (Curves for higher j generally sit below those with lower j. A color version of this and subsequent graphs is available from https://library.slmath.org and also on the publisher's web site (see copyright page for URL.)

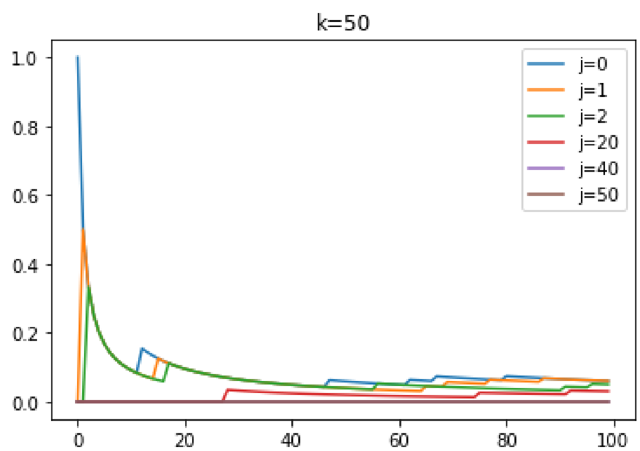

Figure 2. Numerical values of $p_j^{50}(n)$ for a randomly chosen 50-regular multigraph.

Now compare to $k = 50$ in Figure 2. Note that the convergence is rapid for $p_j^{50}(n)$ when $j = 0$ then a slower for each increase in j. In fact, as we show later, the first appearance of a value of $j = 50$ is almost always around $n = 2^{50}$ which is not shown on this plot for obvious reasons.

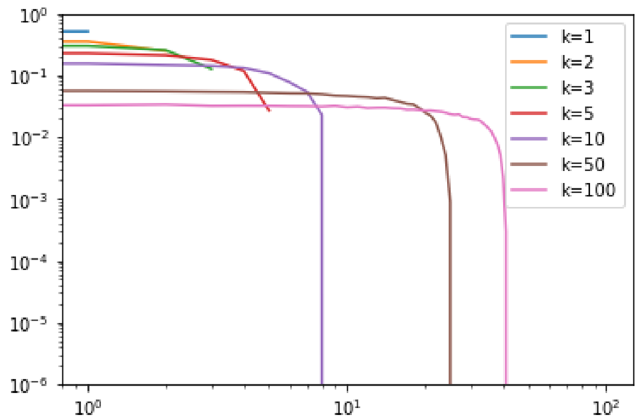

Figure 3. Numerical approximation of p_j^k for various values of k with j on the x-axis.

Consider Figure 3 with the asymptotic numerical values p_j^k. Notice that the values for small j's are fairly constant and then drop off rapidly. Similarly to the late appearance of values for j close to k, those values are extremely rare and $p_k^k \le 2^{-k}$ as proven later. Numerically, note that for $k = 50$, $p_0^k \approx 0.0560 \approx W(50)/50 = 0.057$ where $W(\,)$ is the Lambert W function as discussed later. Similarly $p_1^k \approx 0.0552$ and $p_2^k \approx 0.0547$.

Also note that p_j^k is decreasing in j, as we will prove later.

3. Stationary distributions

We first focus on the stationary distribution. Consider an arbitrary distribution q^k. Define $f^k(q^k)$ where $f_j^k(q^k)$ is the probability that j is the value chosen under probabilities q^k.

Thus, We say that a distribution is stationary if it maps to itself and define $p^k(*) = f^k(p^k(*))$ to be a stationary distribution.

Now note that $f_j^k(q^k)$ can be computed using only the numerical values q_i^k for $i \le j$ since this probability is unaffected by the precise numerical values greater than j, only their sum which can be computed from the numerical values q_i^k for $i \le j$. Thus we can write $g_j^k(q_0^k, \ldots, q_j^k) = f_j^k(q^k)$.

We first show that $p_j^k(*)$ is well defined by the following lemma.

Lemma 1. *$f_j^k(q^k)$ is decreasing in q_j^k.*

Proof. Notice that increasing q_j^k while holding the other q_i^k fixed increases the probability of it being chosen by a random edge thus decreasing its probability of being chosen. □

Theorem 2. *The stationary distribution $p^k(*)$ exists and is unique for all k.*

Proof. Solving iteratively for each j, $p_j^k(*)$ is the solution of

$$p_j^k(*) = g_j^k(p_0^k(*), \ldots, p_j^k(*)).$$

Since the right-hand side is strictly decreasing in $p_j^k(*)$, by the previous lemma. Since the left-hand side is strictly increasing in $p_j^k(*)$ the stationarity equation must have a unique solution. □

Next we show monotonicity in j.

Theorem 3. *The stationary distribution $p^k(*)$ satisfies $p_i^k(*) < p_j^k(*)$ for $i < j$.*

Proof. For all y we have $g_j^k(p_1^k(*), \ldots, p_{j-1}^k(*), y) > g_{j+1}^k(p_1^k(*), \ldots, p_j^k(*), y)]$, since the left side of the equation adds an additional constraint to be satisfied, lowering the probability. Thus, $p_j^k(*) > p_{j+1}^k(*)$ must also decrease as it is the intersection of the identity line and that function. □

4. Asymptotics

We now show that the empirical distribution of the value function will converge to the stationary distribution almost surely for all j, k.

Our analysis will be iterative; assuming that $p_j^k(n)$ have already converged for all $j < t$ to show the convergence of $p_t^k(n)$.

Consider

$$W(n) = |p_j^k(n) - p_j^k(*)|$$

and note that

$$p_j^k(n) = (1 - 1/n)p_j^k(n-1) + (1/n)X(n)$$

where $X(n) = 1$ if j is chose at step n and $X(n) = 0$ otherwise. Also note that $E[x(n)] = f_j^k(p_j^k(n-1))$.

Now consider

$$\frac{W(n)}{W(n-1)} = \frac{|p_j^k(n) - p_j^k(*)|}{|p_j^k(n-1) - p_j^k(*)|}$$

and apply the previous equation to get

$$E\left[\frac{W(n)}{W(n-1)}\right] = \frac{|(1-1/n)p_j^k(n-1) + (1/n)f_j^k(p_j^k(n-1)) - p_j^k(*)|}{|p_j^k(n-1) - p_j^k(*)|}.$$

Let $b_j^k = f_j^k(p_j^k(n-1) - p_j^k(*)$ to get

$$E\left[\frac{W(n)}{W(n-1)}\right] = \frac{|(1-1/n)(p_j^k(n-1) - p_j^k(*)) + (1/n)b_j^k|}{|p_j^k(n-1) - p_j^k(*)|}.$$

Since $f(\cdot)$ is a decreasing function, if $p_j^k(n-1) < p_j^k(*)$ (resp. $p_j^k(n-1) > p_j^k(*)$) then $1 > b_j^k \geq 0$ (resp. $-1 < b_j^k \leq 0$) define $c_j^k = 1 - |b|$ to get

$$E\left[\frac{W(n)}{W(n-1)}\right] \leq (1 - c_j^k/n).$$

Since $f(\cdot)$ is a continuous function on a bounded interval, we can let c be the maximum over the interval to prove the following:

Lemma 4. $$E[W(n)] \leq W(n-1)(1 - c/n).$$

Now we apply the Robbins–Siegmund theorem [9].

Theorem 5 (Robbins–Siegmund). *If* $Z_n \geq 0$ *and*

$$E[Z_{n+1}] \leq (1 - a_n + b_n)Z_n + c_n$$

for positive adaptive random variables Z_n, a_n, b_n, c_n *such that with probability* 1,

$$\sum_n a_n = \infty \sum_n b_n < \infty \sum_n c_n < \infty$$

then $\lim_{n\to\infty} Z_n = 0$.

This allows us to prove our main theorem.

Theorem 6. *For a randomly chosen k-regular multigraph* j, k, $p_j^k(n)$ *converges to* $p_j^k(*)$ *a.s. as* $n \to \infty$.

Proof. Since the harmonic series diverges, the Robinson–Siegmund theorem implies that $W(n) \to p_j^k(*)$ almost surely. (Note that we set $b_n = c_n = 0$.) We can then apply this inductively, first allowing $p_0(n)$ to converge then $p_1(n)$ to converge up to $p_k(n)$, yielding the theorem. □

5. Convergence properties

We now consider some convergence properties. We note that initially, the value function grows rapidly.

Theorem 7. *Given a randomly chosen k-regular multigraph game*,

$$V(n) = n$$

with probability greater than $1 - n^2 e^{-k/n}$, *for* $k > n$.

Proof. Our proof is a variant of a standard analysis of the coupon collector problem [8]. First we compute the union bound for the probability that $V(i) \neq i$ conditional on $V(j) = j$ for all $j < i$. This will occur if any of the nodes $j < i$ are pointed at by edges from node i. Since all previous nodes are equally likely, we can bound this by a union bound $j(1-1/j)^k < je^{-k/j}$. Now combine these with a union bound for all $j \leq n$ to show that the unconditional probability that $V(n) \neq n$ is at most $\sum_{j<n} je^{-k/j} < n^2 e^{-k/n}$ proving the theorem. □

For example, if $n = k/(2\log(k))$ then the probability that $V(n) \neq n$ is less than $(\log(k))^{-2}$, which approaches 0 as $k \to \infty$.

However, once we get past this initial period of linear growth, growth of $V(n)$ will slow significantly. For example one can get exponential lower bounds for the convergence to p_j^k for large j.

We first consider the first time the event that $V(n) = k$ occurs.

Theorem 8. *Given a randomly chosen k-regular multigraph game, let m be the smallest n such that* $V(n) = k$. *Then* $E[m] \geq 2^k$.

Proof. The probability of getting $V(n) = k$ is

$$k!\Pi_j p_j^k(n) \leq (k/2)^k/k^k = 2^{-k} \tag{3}$$

since $k! \leq (k/2)^k$ and $\Pi_j p_j^k(n) \leq 1/k^k$, which is attained with the uniform distribution. Thus the expected number of steps is at least 2^k. □

We now generalize this to other large values of i for $V(n) = i$.

Theorem 9. *Given a randomly chosen k-regular multigraph game, let m be the smallest n such that* $V(n) = i$ *for* $0 < i < k$. *Then* $E[m] \geq (1 - e^{-k/i})^{-i}/2$.

Proof. In order to attain $V(n) = k$ there must be edges from node n to nodes with values less than i and no edges to nodes with value i. The relevant probabilities are defined by the $p_j^k(n)$. A direct calculation is cumbersome so we bound this by considering the uniform distribution given by $q_j^k = 1/i$ for $j < i$ and $q_j^k = 0$ otherwise. To compute this probability we can approximate this event by assuming that number of edges attaining each value $j < i$ are independent Poisson distributions. Then the probability would be given by $(1 - e^{-k/i})^i$. Theorem 5.10 from [8] shows that the true probability must be less than twice this amount implying the stated result. (That theorem shows that the effects of the correlations between events can be precisely bounded. We refer the reader to Chapter 5 of [8] for a detailed discussion of this methodology.) □

For example if $i = k/2$ then $E[m] > 1.15^i$.

The proof also showed that $p_i^k(n)$ is small for large i

Corollary 10. $p_i^k(n) \leq 2(1 - e^{-k/i})^i$ *for all* $n > 0$.

Notice that this implies that for large k, $p_j^k(n)$ is exponentially small for most values of j. As above if $i = k/2$ then $p_j^k(n) < 0.87^i$. More generally if $i = \alpha k$ with $0 < \alpha < 1$, $p_j^k(n) \leq 2(1 - e^{-1/\alpha})^i$ since $(1 - e^{-1/\alpha}) < 1$.

6. Stationary distributions for large k

When k is large, we can approximate some of the numerical values of $p_j^k(*)$. We begin with values of j which are much smaller than k.

Theorem 11. *For fixed j as $k \to \infty$,*

$$p_j^k(*) \to W(k)/k$$

where $W(k)$ is the Lambert W function.

Proof. Recall that $p_j^k(*) = g_j^k(p_1^{k+1}(*), \ldots, p_{j-1}^{k+1}(*))(1 - p_j^k(*))^k$. Define $p_j^k(*) = a_j^k/k$ then

$$a_j^k = k g_k^j(p_1^{k+1}(*), \ldots, p_{j-1}^{k+1}(*))(1 - a_j^k/k)^k \to k e^{-a_j^k}$$

since

$$g_j^k(p_1^{k+1}(*), \ldots, p_{j-1}^{k+1}(*)) \to 1$$

as $k \to \infty$. Thus we get

$$k = a_j^k e^{a_j^k}$$

which yields

$$a_j^k = W(k)$$

proving the theorem. □

We now bound the probabilities for values of j close to k using Theorem 9.

Corollary 12. *For fixed j as $k \to \infty$, $p_{k-j}^k(*) \le 2^{-(k-j)}$*

7. Conclusions

While interesting in its own right, the study of generic games may provide insights into specific games with complicated values. For example, the game of Chomp is extremely complex but numerical analyses (with analytical components) suggest that the p-positions satisfy certain probabilistic properties [4]. Similarly the values of various arithmetic games [7] appear to display statistical patterns that can be (partially) understood by analogies to random games by extending our analysis. However, in these games the degree is not typically bounded and grows with n. Numerical results suggest that generic games capture several key attributes of these arithmetic games (unpublished); however, a formal analysis appears quite challenging.

References

[1] M. H. Albert, R. J. Nowakowski, and D. Wolfe, *Lessons in play*: *An introduction to combinatorial game theory*, 2nd ed., CRC Press, Boca Raton, FL, 2019. Zbl

[2] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning ways for your mathematical plays, IV*, 2nd ed., Peters, Wellesley, MA, 2004. MR Zbl

[3] C. L. Bouton, "Nim, a game with a complete mathematical theory", *Ann. of Math.* (2) **3**:1-4 (1901/02), 35–39. MR Zbl

[4] E. J. Friedman and A. S. Landsberg, "On the geometry of combinatorial games: a renormalization approach", pp. 349–376 in *Games of no chance* 3: *Papers from the workshop on combinatorial game theory* (Banff, Canada, June 2005), Cambridge University Press, 2009. Zbl

[5] E. Friedman, S. M. Garrabrant, I. K. Phipps-Morgan, A. S. Landsberg, and U. Larsson, "Geometric analysis of a generalized Wythoff game", pp. 343–372 in *Games of no chance* 5, edited by U. Larsson, Math. Sci. Res. Inst. Publ. **70**, Cambridge University Press, 2019. MR Zbl

[6] S. M. Garrabrant, E. J. Friedman, and A. S. Landsberg, "Cofinite induced subgraphs of impartial combinatorial games: an analysis of CIS-Nim", *Integers* **13** (2013), Paper No. G2, 27. MR Zbl

[7] D. E. Iannucci and U. Larsson, "Game values of arithmetic functions", pp. 245–279 in *Combinatorial game theory—a special collection in honor of Elwyn Berlekamp*, *John H. Conway and Richard K. Guy*, edited by R. J. Nowakowski et al., De Gruyter, Berlin, [2022] ©2022. MR Zbl

[8] M. Mitzenmacher and E. Upfal, *Probability and computing*, Cambridge University Press, 2005. MR Zbl

[9] H. Robbins and D. Siegmund, "A convergence theorem for non negative almost supermartingales and some applications", pp. 233–257 in *Optimizing methods in statistics* (Proc. Sympos., Ohio State Univ., Columbus, Ohio, 1971), edited by J. S. Rustagi, Academic Press, New York, 1971. MR Zbl

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