Games of No Chance 6
MSRI Publications
Volume **71**, 2025

Dead-ending day-2 games under misère play

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Combinatorial games do not exhibit the same algebraic structure in misère play as they do in normal play; e.g., no nonzero game has an additive inverse under misère play. In recent years, misère research has considered "restricted" play, where games can be equal or comparable modulo a subset (universe) of games, even if they are not in general. One universe well suited for misère analysis is the set of *dead-ending games*: games with the property that if a specific player cannot move at some point in the game, then that player will never again be able to move. Dead-ending games have many nice properties: some games, including normal-play numbers, are invertible "modulo dead-ending", there is an easy test for inequality, and there are reductions that give unique reduced forms. We apply recent results for inequalities and game simplification to find the unique, reduced dead-ending games born by day 2, and we determine which of these are invertible modulo dead-ending games.

1. Introduction

A combinatorial game is a two-player game of perfect information and no elements of chance. Many well-known combinatorial games, including DOMINEERING,[1] HACKENBUSH, and others, have the *dead-end* property: if a player cannot move at some point, then that player can never move again; i.e., no move by the player's opponent can "open up" a move.

Most research in combinatorial game theory assumes *normal play*, where the first player unable to move on their turn loses; in this paper, we assume *misère play*, where that player wins. In general, misère play is much harder to analyze than normal play (see Section 1.2). Dead-ending games were introduced in [6] as a set of interest for misère analysis; in this universe, misère play has more structure and exhibits some of the familiar algebraic properties from normal play. Recently, results from *absolute* game theory [4] have been applied to

All authors supported in part by the Natural Sciences and Engineering Research Council of Canada.
MSC2020: 91A46.
Keywords: combinatorial game theory, misère games, dead-ending games.

[1] We refer to DOMINEERING several times in this paper. It is a tiling game, usually played on a rectangular grid, in which Left places vertical dominoes and Right places horizontal dominoes. Play continues until a player cannot move on their turn.

dead-ending games [2; 3] to show that these games have unique reduced forms under misère play. The primary purpose of this paper is to find the reduced forms of all dead-ending positions "born by 2" — i.e., those games that end in at most two moves. We also completely classify which of these games have an additive inverse in the dead-ending universe.

1.1. *Definitions.* The two players are called Left and Right. A game $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$ is defined by the set of *Left options*, $G^{\mathcal{L}}$, that Left can reach in one move and the set of *Right options*, $G^{\mathcal{R}}$, that Right can reach in one move. The game $\{\cdot \mid \cdot\}$ with no moves for either player is called the *zero* game, 0. The *followers* of a game G include G, its options, the options of its options, etc. The *game tree* of a game is a directed graph with each node representing a follower of the game; the root is the original position, moves available to Left are drawn to the left, and moves available to Right are drawn to the right. Each leaf is a zero position.

A game with no moves for Left is called a *Left end*. If all followers of a Left end are also Left ends, then it is a *dead* Left end. (We similarly define Right ends and dead Right ends, and note that 0 is both a dead Left end and dead Right end.) To have the dead-end property — i.e., to be *dead-ending* — all end followers of a game must be dead ends. The game tree shown in Figure 1(a) is dead-ending; the game tree in Figure 1(b) is not dead-ending, because the Right end at ∘ is not dead.

The *outcome* $\mathrm{o}(G)$ is the winner under optimal play:

$$\mathrm{o}(G) = \begin{cases} \mathcal{L} & \text{if Left wins } G \text{ whether she goes first or second,} \\ \mathcal{R} & \text{if Right wins } G \text{ whether he goes first or second,} \\ \mathcal{N} & \text{if the next player to move in } G \text{ wins,} \\ \mathcal{P} & \text{if the previous player (i.e., not the next player) wins.} \end{cases}$$

The outcome function depends on the winning convention; in this paper, unless stated otherwise, we always assume misère play. Outcomes are ordered according to preference by Left: $\mathcal{L} > \mathcal{N} > \mathcal{R}$ and $\mathcal{L} > \mathcal{P} > \mathcal{R}$, with $\mathcal{N}$ and $\mathcal{P}$ incomparable.

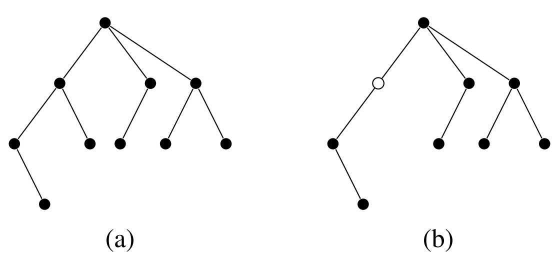

Figure 1. (a) A game that is dead-ending. (b) A game that is not dead-ending.

The *disjunctive sum* of two games G and H is the game $G+H$ in which, on a player's turn, they may play in G or in H: $G+H=\{G^{\mathcal{L}}+H, G+H^{\mathcal{L}} \mid G^{\mathcal{R}}+H, G+H^{\mathcal{R}}\}$, where $G^{\mathcal{L}}+H=\{G^L+H : G^L \in G^{\mathcal{L}}\}$, etc. Two games G and H are *equal* if they can be interchanged in any sum without affecting the outcome: that is, if $o(G+X)=o(H+X)$ for any sum of games X. A partial order of games is inherited from the ordering of outcomes: $G \geqslant H$ if $o(G+X) \geqslant o(H+X)$ for all X. Equality and inequality are dependent on the winning convention; e.g., two games could be equal in normal play but not in misère play.

In both normal and misère play, given a game G, if Left has options to G^{L1} and G^{L2} with $G^{L1} \geqslant G^{L2}$, then by definition, Left will prefer G^{L1} in every situation; thus, we can remove the *dominated* option G^{L2} and obtain a simpler position that is still equal to G. This is one of two game reductions that lead to unique canonical forms. The other reduction involves *reversible* options. A Left option G^L is *reversible* if there is a Right reply G^{LR} with $G^{LR} \leqslant G$; to be specific, we say G^L is reversible *through* G^{LR}. In normal play and in general misère play, we can *bypass* reversible options, replacing G^L with the Left options of G^{LR}, if there are any, or by removing G^L, if there are not. The result is a simpler game that is equal to the original.

In normal play, a game whose game tree is the mirror image of G is called its *negative*, $-G$, because $G+(-G)=0$ for all G. In general misère play, this is actually *never* true, unless G is exactly 0 [5]. Thus, we use the term *conjugate* and symbol $\overline{G}$ for the same game, defined recursively as $\overline{G}=\{\overline{G^{\mathcal{R}}} \mid \overline{G^{\mathcal{L}}}\}$, with $\overline{0}=0$ and with $\overline{G^{\mathcal{R}}}=\{\overline{G^R} : G^R \in G^{\mathcal{R}}\}$, etc.

The *birthday* of a game is the depth of its game tree. The zero game is "born on day 0", and any game with only options to 0 is "born on day 1". The day-1 games are

$$1=\{0 \mid \cdot\}, \quad \overline{1}=\{\cdot \mid 0\}, \quad *=\{0 \mid 0\}.$$

This paper studies games born by day 2; that is, games whose options are limited to 0, 1, $\overline{1}$, and $*$.

1.2. ***Restricted misère play.*** There are many problems with general misère play. In normal play, all games with outcome $\mathcal{P}$ are equal to zero; in general misère play, *no* game is equal to zero besides the literal game with no moves for either player. In particular, a nonzero game and its negative do not sum to zero in misère play; thus, unlike normal play, the set of all games under misère play is not a group. There is less intuition for outcomes of sums under misère play: in normal play, $\mathcal{L}+\mathcal{L}$ is always $\mathcal{L}$, but in misère play, it could be any of the four outcomes [5]. Equality and inequality are rare in misère play, and the easy test from normal play — i.e., that $G \geqslant H \iff G-H \in \mathcal{L} \cup \mathcal{P}$ — does not hold. For example, the game 0 and the game 1 are incomparable in misère play: Left wins first on 0 and loses first on 1, but Left loses first on $0+*$ and wins first on $1+*$.

To combat some of these issues, *restricted play* was introduced [8], with weaker equality and inequality relations, so that games might be equal or comparable "modulo some subset of games", even if they are not equal or comparable in general misère play. A *universe* $\mathcal{U}$ is a set of games closed under addition, negation (conjugation), and followers.[2] The restricted-play relations are

$$G \equiv_{\mathcal{U}} H \iff \mathrm{o}(G+X) = \mathrm{o}(H+X) \text{ for all } X \in \mathcal{U};$$
$$G >_{\mathcal{U}} H \iff \mathrm{o}(G+X) > \mathrm{o}(H+X) \text{ for all } X \in \mathcal{U}.$$

The universe could be taken to be all positions that occur under a particular rule set, such as DOMINEERING, or could be defined by a game property; commonly studied universes include impartial games $\mathcal{I}$, dicots $\mathcal{D}$ (where at every point of the game, either both players can move or neither player can), and dead-ending games $\mathcal{E}$. Note that $\mathcal{I} \subset \mathcal{D} \subset \mathcal{E}$.

One notable problem with restricted misère play is reversibility through ends. In general misère play, bypassing reversible options works, but is not often applicable because nontrivial inequalities are rare [9]. In restricted misère play, we are more often able to find $G^{LR} \leqslant_{\mathcal{U}} G$, and if G^{LR} has Left options, bypassing G^L works as above; but if not — if G^{LR} is a Left end — then removing G^L may not leave an equivalent game [9]. There is no general solution to the problem of *end-reversibility* for restricted play in an arbitrary universe. However, there are solutions for dicots [1] and dead-ending games [3].

Indeed, the dead-ending universe exhibits a number of nice properties under misère play: modulo $\mathcal{E}$, all ends are invertible [6], there is a recursive test for inequality [2], and the end-reversibility reductions mentioned above give unique reduced forms [3]. The number of reduced dicot games born by day 3 are enumerated in the study of the dicot subuniverse in [1]. We start a similar line of study in $\mathcal{E}$, and consider the dead-ending games born by day 2.

1.3. ***Objective and outline.*** In this paper, we analyze the set of dead-ending games born by day 2. We use the recursive comparison test and reductions from [2] and [3] to determine the unique reduced day-2 forms modulo $\mathcal{E}$, and furthermore determine which of these positions are invertible modulo $\mathcal{E}$.

Section 2 sets up the day-2 dead-ending game trees and establishes some preliminary results, including comparability with and equivalence to zero modulo $\mathcal{E}$. In Section 3, we see how end-reversibility in $\mathcal{E}$ applies to day-2 games

[2]Some authors also require a universe to have the *parental* or *dicotic* property, where $\{\mathcal{S} | \mathcal{T}\} \in \mathcal{U}$ for all nonempty subsets of games $\mathcal{S}, \mathcal{T} \subseteq \mathcal{U}$. The dead-ending universe does have this additional property.

and computationally determine the number of unique reduced positions born by day 2, modulo $\mathcal{E}$. In Section 4, we determine which of these positions are invertible modulo $\mathcal{E}$. Finally, Section 5 discusses future work and a general conjecture arising from the invertibility results for day-2 games.

2. Dead-ending day-2 games

Which positions born by day 2 are dead-ending? The four games born by day 1 (0, 1, $\bar{1}$, $*$) are all dead-ending. Thus, any game born on day 2 will have all dead-ending followers, and so will be dead-ending itself *unless* it is a Left end with an option that is not a Left end, or a Right end with an option that is not a Right end. In particular, non-ends born on day 2 are all dead-ending. There are $2^4 - 1 = 15$ nonempty subsets of $\{0, 1, \bar{1}, *\}$ to choose as nonempty option sets $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$, which gives $15 \times 15 = 225$ non-end positions. However, note that if $G^{\mathcal{L}} = G^{\mathcal{R}} = \{0\}$, then $G = *$, which we have already counted as a day-1 game.

For the ends, we cannot have a Left end with a Right option to 1 or $*$ or a Right end with a Left option to $\bar{1}$ or $*$. Thus, the dead ends born on day 2 are $\{1 | \cdot\}$, $\{0, 1 | \cdot\}$, and their conjugates. Note that $\{1 | \cdot\}$ is the game called “2” in normal play. The game $\{0, 1 | \cdot\}$ would reduce to 2 in normal play, but not in misère play — not even modulo $\mathcal{E}$, because 0 and 1 are still incomparable here. The 4×1 DOMINEERING board is an example of this game: Left placing a vertical domino can play in the middle to 0 or at the top or bottom to 1.

Including the day-2 dead ends gives a total of 232 dead-ending positions born by day 2. These are illustrated in Table 3 (page 280). We already know the day-1 games 1, $\bar{1}$, and $*$ cannot be simplified, because neither can be equivalent to zero; and it is already known that 1 and $\bar{1}$ are invertible, while $*$ is not invertible. Thus, the remaining work is focused specifically on the 228 games born on day 2 — i.e., the day-2 dead-ending games.

2.1. ***Strong outcome.*** Given a game G, the *Left outcome*, $\mathrm{o}_L(G)$, is the result (L wins or R wins) when Left plays first, and the *Right outcome*, $\mathrm{o}_R(G)$, is the result when Right plays first. The concept of *strong outcome* was introduced in [2]. The *strong Left outcome* is the worst possible Left outcome, from Left’s perspective, of G plus a Left end, and the *strong Right outcome* is the worst case scenario for Right of G plus a Right end. That is,

$$\hat{\mathrm{o}}_L(G) = \begin{cases} L & \text{if } \mathrm{o}_L(G+X) = L \text{ for every Left end } X, \\ R & \text{if there is a Left end } X \text{ such that } \mathrm{o}_L(G+X) = R; \end{cases}$$

$$\hat{\mathrm{o}}_R(G) = \begin{cases} R & \text{if } \mathrm{o}_R(G+X) = R \text{ for every Right end } X, \\ L & \text{if there is a Right end } X \text{ such that } \mathrm{o}_R(G+X) = L. \end{cases}$$

The *strong outcome* of G is then defined by the pair $(\hat{o}_L(G), \hat{o}_R(G))$:

$$\hat{o}(G) = \begin{cases} \mathcal{L} & \text{if } \hat{o}_L(G) = L \text{ and } \hat{o}_R(G) = L, \\ \mathcal{N} & \text{if } \hat{o}_L(G) = L \text{ and } \hat{o}_R(G) = R, \\ \mathcal{P} & \text{if } \hat{o}_L(G) = R \text{ and } \hat{o}_R(G) = L, \\ \mathcal{R} & \text{if } \hat{o}_L(G) = R \text{ and } \hat{o}_R(G) = R. \end{cases}$$

Recall that all ends in normal play reduce to numbers. In misère play, what would the "worst Left end" look like, for Left? It is a game over which Right has total control; after each move, Right can either end the game immediately (move to zero), or continue the game. These games were introduced in [2], where they are named *perfect murders*.[3] They appear in the reversibility reductions for $\mathcal{E}$. The perfect murder Left end born on day n is denoted by M_n, with $M_0 = 0$ and

$$M_n = \{\cdot \mid 0, M_{n-1}\}.$$

One of the conditions of the recursive comparison test for $G \geqslant_{\mathcal{E}} H$ is that the strong outcome of G be greater than or equal to the strong outcome of H. Thus, we will want a quick way to determine the strong outcome of the dead-ending games born on day 2. Recall, $\hat{o}_L(G)$ is the worst outcome (for Left) of Left playing first on $G + X$, where X ranges over all possible Left ends, including $X = 0$.

Theorem 1. *If G is a day-2 dead-ending game and is not an end, then*:

(1) $\hat{o}_L(G) = L \iff \overline{1} \in G^{\mathcal{L}}$ *or* $\{0, *\} \subseteq G^{\mathcal{L}}$.

(2) $\hat{o}_R(G) = R \iff 1 \in G^{\mathcal{R}}$ *or* $\{0, *\} \subseteq G^{\mathcal{R}}$.

Proof. We prove (1), and (2) follows by symmetry. Assume G is not an end.

($\Rightarrow$) We prove the contrapositive. Suppose G does not have a Left move to $\overline{1}$ and does not have Left moves to both 0 and $*$. If Left has no move to $\overline{1}$ and no move to $*$, then Left will lose G playing first, as the Left options are either 1 or 0. If Left has no move to $\overline{1}$ and no move to 0, then Left will lose $G + \overline{1}$ playing first, as the Left options are either $1 + \overline{1}$ or $* + \overline{1}$. In either case, the strong outcome $\hat{o}_L(G)$ is R.

($\Leftarrow$) Suppose G has a Left move to $\overline{1}$ or Left moves to both 0 and $*$. We will show Left wins $G + X$ playing first, for any Left end X. If $X = 0$, then Left wins $G + X$ playing first by moving to either $\overline{1}$ or $*$. If $X = \overline{1}$, then Left wins $G + X$ playing first by moving to either $\overline{1} + \overline{1}$ or to $0 + \overline{1}$. If X is any other Left end, Left will always win $G + X$ playing first, because after Left's turn, there is at most one more Left move and at least two more Right moves. □

[3] In later papers these have been renamed to "waiting games".

2.2. *Comparability with zero.* In Section 3, to apply reversibility reductions to the day-2 dead-ending games, we will need to know which games are comparable to the zero game. We establish this here, using the following recursive test for inequality (comparison) modulo $\mathcal{E}$, from [2]. Recall that 1 and $\overline{1}$ are incomparable with 0, even modulo $\mathcal{E}$.

Theorem 2 [2]. *Let $G, H \in \mathcal{E}$. Then $G \geqslant_{\mathcal{E}} H$ if and only if G and H satisfy*:

(1) *Proviso*: $\hat{o}(G) \geqslant \hat{o}(H)$.

(2) *Maintenance Property*:

(a) *for all $H^L \in H^{\mathcal{L}}$, there exists $G^L \in G^{\mathcal{L}}$ such that $G^L \geqslant_{\mathcal{E}} H^L$ or there exists $H^{LR} \in H^{L\mathcal{R}}$ such that $G \geqslant_{\mathcal{E}} H^{LR}$*;

(b) *for all $G^R \in G^{\mathcal{R}}$, there exists $H^R \in H^{\mathcal{R}}$ such that $G^R \geqslant_{\mathcal{E}} H^R$ or there exists $G^{RL} \in G^{R\mathcal{L}}$ such that $G^{RL} \geqslant_{\mathcal{E}} H$.*

Using Theorem 2, we can now determine which day-2 dead-ending games are comparable with zero.

Theorem 3. *If G is a day-2 dead-ending game, then*:

- *$G \geqslant_{\mathcal{E}} 0$ if and only if $G^{\mathcal{L}}$ contains $\overline{1}$ or both 0 and $*$, and $G^{\mathcal{R}}$ is a nonempty subset of $\{1, *\}$.*
- *$G \leqslant_{\mathcal{E}} 0$ if and only if $G^{\mathcal{R}}$ contains 1 or both 0 and $*$, and $G^{\mathcal{L}}$ is a nonempty subset of $\{\overline{1}, *\}$.*

Proof. We prove the first item and the second follows by symmetry.

Suppose $G \in \mathcal{E}_2$ satisfies $G \geqslant_{\mathcal{E}} 0$. The strong outcome of zero is $\mathcal{N}$, so from the Proviso we must have $\hat{o}(G) = \mathcal{L}$ or $\mathcal{N}$; that is, we must have $\hat{o}_L(G) = L$. By Theorem 1, this means the Left options of G must include $\overline{1}$ or both 0 and $*$. With $H = 0$, part (a) of the Maintenance Property is vacuously true. In (b), since there are no games H^R, for each G^R we need a G^{RL} with $G^{RL} \geqslant 0$. Since G is born on day 2, any G^{RL} is actually exactly 0. So the Maintenance Property stipulates that every G^R has a Left option to 0; thus, the only allowable Right options for G are 1 and $*$.

The converse follows similarly: if $G^{\mathcal{L}}$ contains $\overline{1}$ or both 0 and $*$, then Theorem 1 tells us that the Proviso is satisfied. If $G^{\mathcal{R}}$ is a nonempty subset of $\{1, *\}$ then the Maintenance Property is satisfied. □

Four of the 228 day-2 dead-ending games satisfy both conditions of Theorem 3 simultaneously and are therefore equivalent to zero modulo $\mathcal{E}$ (see Figure 2):

$$\{\overline{1}|1\} \equiv_{\mathcal{E}} 0, \quad \{\overline{1}, *\,|\,1\} \equiv_{\mathcal{E}} 0, \quad \{\overline{1}\,|\,1, *\} \equiv_{\mathcal{E}} 0, \quad \{\overline{1}, *\,|\,1, *\} \equiv_{\mathcal{E}} 0.$$

Note that $\{\overline{1}|1\}$ is precisely $1 + \overline{1}$, which is already known to be equivalent to zero modulo $\mathcal{E}$ [6], because all ends are invertible in the dead-ending universe.

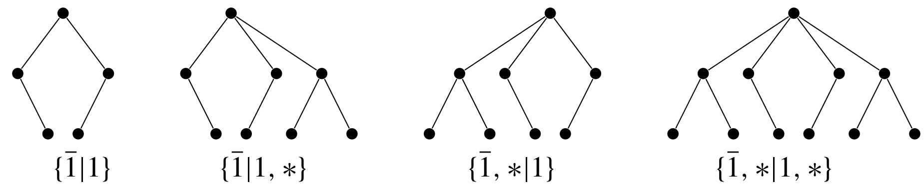

Figure 2. The four day-2 dead-ending game trees that are equivalent to 0 modulo $\mathcal{E}$.

3. Reversibility and reduced form

Recall that G^L is reversible through G^{LR} if $G^{LR} \leqslant G$. Let us consider how we could have $G^{LR} \leqslant_{\mathcal{E}} G$ for a day-2 game G.

Theorem 4. *If G is a day-2 dead-ending game, then*:

- *$G^L \in G^{\mathcal{L}}$ is reversible if and only if $G^{\mathcal{R}}$ is a nonempty subset of $\{1, *\}$, and*
  - (i) *$G^L = \bar{1}$, or*
  - (ii) *$G^L = *$, and either 0 or $\bar{1}$ is also a Left option.*
- *$G^R \in G^{\mathcal{R}}$ is reversible if and only if $G^{\mathcal{L}}$ is a nonempty subset of $\{\bar{1}, *\}$, and*
  - (i) *$G^R = 1$, or*
  - (ii) *$G^R = *$, and either 0 or 1 is also a Right option.*

Proof. If G is born on day 2, then any G^{LR} is actually the zero game, and for $G^{LR} = 0$ to exist, G^L must be $\bar{1}$ or $*$. For G^L to be reversible, we must have $G \geqslant_{\mathcal{E}} G^{LR} = 0$. Now, by Theorem 3, $G \geqslant_{\mathcal{E}} 0$ if and only if $G^{\mathcal{L}}$ contains $\bar{1}$ or both 0 and $*$, and $G^{\mathcal{R}}$ is a nonempty subset of $\{1, *\}$. This proves the theorem for Left options, and the result for Right follows by symmetry. □

Now that we know what reversible options will look like, let us see how to handle them. The reversibility reductions for $\mathcal{E}$ are presented in Theorem 5. We require one new definition: G^L is a *fundamental option* if

$$\hat{o}_L(G) = L \quad \text{and} \quad \hat{o}_L\big(\{G^{\mathcal{L}} \setminus \{G^L\} \mid G^{\mathcal{R}}\}\big) = R.$$

That is, G^L is fundamental if for some Left end X, G^L is the one and only good first move for left in $G + X$. Similarly, G^R is a fundamental option for Right if

$$\hat{o}_R(G) = R \quad \text{and} \quad \hat{o}_R\big(\{G^{\mathcal{L}} \mid G^{\mathcal{R}} \setminus \{G^R\}\}\big) = L.$$

Theorem 5 (end-reversibility in $\mathcal{E}$ [3]). *If $G^L \in G^{\mathcal{L}}$ and there is a Left end $G^{LR} \leqslant_{\mathcal{E}} G$, then G^L is reversible and one of the following simplifications can be made*:

(i) *If $G = \{G^L \mid G^R\}$, where both G^L and G^R are end-reversible, then remove both options simultaneously*: $G \equiv_{\mathcal{E}} 0$.

(ii) *If* G^L *is not fundamental, then remove it*: $G \equiv_{\mathcal{E}} \{G^{\mathcal{L}} \setminus \{G^L\} \mid G^{\mathcal{R}}\}$, *as long as this is still in* $\mathcal{E}$.

(iii) *If* G^L *is fundamental, or if its removal via* (ii) *would create a non-dead-ending game, then replace* G^L *with* $\{\cdot \mid M_n\}$, *where n is the smallest integer such that* $G \geqslant_{\mathcal{E}} M_n$. *In this case,* $G \equiv_{\mathcal{E}} \left\{G^{\mathcal{L}} \setminus \{G^L\}, \{\cdot \mid M_n\} \mid G^{\mathcal{R}}\right\}$.

Symmetric statements to (ii) *and* (iii) *hold for a reversible Right option* G^R (*i.e.,* G^R *with* $G^{RL} \geqslant G$).

How does Theorem 5 apply to day-2 games? First note that case (i) of that theorem can only occur if $G = \{\bar{1}|1\}$, and as mentioned in Section 2.2, we already know this game reduces to zero modulo $\mathcal{E}$. The other cases can similarly be simplified using our previous results for day-2 dead-ending games. The reductions specifically for day-2 games are given as Theorem 6.

Theorem 6 (end-reversibility in $\mathcal{E}$ for day-2 games). *Let G be a day-2 dead-ending game with* $\varnothing \neq G^{\mathcal{R}} \subseteq \{1, *\}$ *and* $G^L \in G^{\mathcal{L}}$.

(i) *If* $G = \{\bar{1}|1\}$, *then* $G \equiv_{\mathcal{E}} 0$.

(ii) *If* $G^L = \bar{1}$ *and* $0, * \in G^{\mathcal{L}}$, *then* G^L *is a nonfundamental reversible option, and can be removed*: $G \equiv_{\mathcal{E}} \{G^{\mathcal{L}} \setminus \{\bar{1}\} \mid G^{\mathcal{R}}\}$.

(iii) *If* $G^L = *$ *and* $\bar{1} \in G^{\mathcal{L}}$, *then* G^L *is a nonfundamental reversible option, and can be removed*: $G \equiv_{\mathcal{E}} \{G^{\mathcal{L}} \setminus \{*\} \mid G^{\mathcal{R}}\}$.

(iv) *If* $G^L = *$ *and* $0 \in G^{\mathcal{L}}$, $\bar{1} \notin G^{\mathcal{L}}$, *then* G^L *is a fundamental reversible option, and can be replaced by* $\bar{1}$: $G \equiv_{\mathcal{E}} \left\{G^{\mathcal{L}} \setminus \{*\} \cup \{\bar{1}\} \mid G^{\mathcal{R}}\right\}$.

The symmetric results hold for reversible Right options.

Proof. We have already noted that case (i) of Theorem 5 is equivalent to case (i) of Theorem 6. To apply the other cases of Theorem 5, we must determine which Left options are fundamental for day-2 games in $\mathcal{E}$. Recall, G^L is fundamental if the strong Left outcome of G is L, but the strong Left outcome of G *without* G^L is R. By Theorem 1, $\hat{o}_L(G) = L$ if and only if $\bar{1} \in G^{\mathcal{L}}$ or both $0 \in G^{\mathcal{L}}$ and $* \in G^{\mathcal{L}}$. This is already satisfied if G^L is reversible. So whether or not G^L is fundamental equates to the strong Left outcome of G with G^L removed. If $G^L = \bar{1}$, the strong Left outcome is still L if 0 and $*$ are Left options, and otherwise it is R. Thus, if $G^L = \bar{1}$, it is nonfundamental if $0, * \in G^{\mathcal{L}}$, and so by Theorem 5 case (ii), G^L can be removed.

If $G^L = *$, the strong Left outcome is still L if $\bar{1}$ is a Left option, and otherwise it is R. Thus, in this case, if $\bar{1} \in G^{\mathcal{L}}$ then $G^L = *$ is not fundamental and can be removed. If $\bar{1} \notin G^{\mathcal{L}}$ but $0 \in G^{\mathcal{L}}$ then $G^L = *$ is a fundamental reversible option and by Theorem 5, it can be replaced by $\bar{1}$. □

Note that if $\bar{1}$ is a fundamental Left option, then, by Theorem 5, we should replace it with $\{\cdot\,|M_n\}$, where n is the smallest integer such that $G \geqslant_{\mathcal{E}} M_n$. But we have $G \geqslant 0 = M_0$, so this instruction is to replace $\bar{1}$ with $\{\cdot\,|0\} = \bar{1}$; i.e., there is no simplification to be made here. Thus, Theorem 6 omits the case where $\bar{1}$ is fundamental.

We can tidy Theorem 6 slightly. In the case where $\bar{1}, 0, *$ are all Left options, reductions (ii) and (iii) could both apply; but if we apply (ii) and remove the $\bar{1}$, then we are in case (iv), which says to replace the remaining $*$ with $\bar{1}$; thus, we should just apply reduction (iii) from the start. This, along with analogous reductions for Right options, is summarized in Corollary 7, a complete guide to reducing day-2 dead-ending games.

Corollary 7. *Let G be a day-2 dead-ending game.*

(i) *If $G = \{\bar{1}|1\}$, then $G \equiv_{\mathcal{E}} 0$.*

(ii) *If $G^{\mathcal{R}}$ is a nonempty subset of $\{1, *\}$ and $\bar{1}, * \in G^{\mathcal{L}}$, then remove the Left option to $*$.*

(iii) *If $G^{\mathcal{R}}$ is a nonempty subset of $\{1, *\}$ and $0, * \in G^{\mathcal{L}}$ but $\bar{1} \notin G^{\mathcal{L}}$, then replace the Left option to $*$ with $\bar{1}$.*

(iv) *If $G^{\mathcal{L}}$ is a nonempty subset of $\{\bar{1}, *\}$ and $1, * \in G^{\mathcal{R}}$, then remove the Right option to $*$.*

(v) *If $G^{\mathcal{L}}$ is a nonempty subset of $\{\bar{1}, *\}$ and $0, * \in G^{\mathcal{R}}$ but $1 \notin G^{\mathcal{R}}$, then replace the Right option to $*$ with 1.*

Applying these reductions to 228 day-2 dead-ending game trees, we find 193 unique positions. As we know, one of these — $\{\bar{1}|1\}$ — reduces to the zero game. Including 1, $\bar{1}$, and $*$, we have the following theorem.

Theorem 8. *There are* 196 *unique, reduced dead-ending game born by day* 2.

Proof. This number was obtained computationally, applying the reductions from Corollary 7. The 196 reduced positions are highlighted (circled) in Table 3. The Python code is on GitHub (https://github.com/rmilley/day2deadend). □

Of the 192 day-2 games, 172 are not equivalent to any other day-2 game modulo $\mathcal{E}$, and 20 are equivalent to one or two other day-2 games. Table 1 lists the 20 sets of equivalent positions, with reduced forms identified in bold. For completeness, we also list the day-2 games equivalent to zero, although they are not counted in the 192.

4. Invertibility

We know that among day-1 games, modulo $\mathcal{E}$, $*$ is not invertible, while 1 and $\bar{1}$ are. For day-2 games, it is already known that the four ends — $\{1|\cdot\}$, $\{0, 1|\cdot\}$,

$\mathbf{0}$, $\{\bar{1}\mid 1\}$, $\{\bar{1}, *\mid 1\}\{\bar{1}\mid 1, *\}$, $\{\bar{1}, *\mid 1, *\}$	$\boldsymbol{\{0, \bar{1}\mid 1, *\}}$, $\{0, *\mid 1, *\}$, $\{0, \bar{1}, *\mid 1, *\}$
$\boldsymbol{\{\bar{1}\mid *\}}$, $\{\bar{1}, *\mid *\}$	$\boldsymbol{\{0, 1, \bar{1}\mid 1\}}$, $\{0, 1, *\mid 1\}$, $\{0, 1, \bar{1}, *\mid 1\}$
$\boldsymbol{\{*\mid 1\}}$, $\{*\mid 1, *\}$	$\boldsymbol{\{0, 1, \bar{1}\mid *\}}$, $\{0, 1, *\mid *\}$, $\{0, 1, \bar{1}, *\mid *\}$
$\boldsymbol{\{\bar{1}\mid 1, \bar{1}\}}$, $\{\bar{1}\mid 1, \bar{1}, *\}$	$\boldsymbol{\{0, 1, \bar{1}\mid 1, *\}}$, $\{0, 1, *\mid 1, *\}$, $\{0, 1, \bar{1}, *\mid 1, *\}$
$\boldsymbol{\{*\mid 1, \bar{1}\}}$, $\{*\mid 1, \bar{1}, *\}$	$\boldsymbol{\{\bar{1}\mid 0, 1\}}$, $\{\bar{1}\mid 0, *\}$, $\{\bar{1}\mid 0, 1, *\}$
$\boldsymbol{\{\bar{1}, *\mid 1, \bar{1}\}}$, $\{\bar{1}, *\mid 1, \bar{1}, *\}$	$\boldsymbol{\{*\mid 0, 1\}}$, $\{*\mid 0, *\}$, $\{*\mid 0, 1, *\}$
$\boldsymbol{\{1, \bar{1}\mid 1\}}$, $\{1, \bar{1}, *\mid 1\}$	$\boldsymbol{\{\bar{1}, *\mid 0, 1\}}$, $\{\bar{1}, *\mid 0, *\}$, $\{\bar{1}, *\mid 0, 1, *\}$
$\boldsymbol{\{1, \bar{1}\mid *\}}$, $\{1, \bar{1} * \mid *\}$	$\boldsymbol{\{\bar{1}\mid 0, 1, \bar{1}\}}$, $\{\bar{1}\mid 0, \bar{1}, *\}$, $\{\bar{1}\mid 0, 1, \bar{1}*\}$
$\boldsymbol{\{1, \bar{1}\mid 1, *\}}$, $\{1, \bar{1}, *\mid 1, *\}$	$\boldsymbol{\{*\mid 0, 1, \bar{1}\}}$, $\{*\mid 0, \bar{1}, *\}$, $\{*\mid 0, 1, \bar{1}*\}$
$\boldsymbol{\{0, \bar{1}\mid 1\}}$, $\{0, *\mid 1\}$, $\{0, \bar{1}, *\mid 1\}$	$\boldsymbol{\{\bar{1}, *\mid 0, 1, \bar{1}\}}$, $\{\bar{1}, *\mid 0, \bar{1}, *\}$, $\{\bar{1}, *\mid 0, 1, \bar{1}*\}$
$\boldsymbol{\{0, \bar{1}\mid *\}}$, $\{0, *\mid *\}$, $\{0, \bar{1}, *\mid *\}$	

Table 1. Reduced day-2 dead-ending positions (bold) and the day-2 games to which they are equivalent.

and their negatives — are invertible modulo $\mathcal{E}$. Which other reduced day-2 games in $\mathcal{E}$ are invertible? By the symmetry of the position $G + \overline{G}$ (for any G, not just day-2 games), it suffices to show that $G + \overline{G} \geqslant_{\mathcal{E}} 0$. We can thus apply Theorem 2 to determine which dead-ending games are invertible.

Theorem 9. *For any dead-ending game* G, $G + \overline{G} \equiv_{\mathcal{E}} 0$ *if and only if these hold*:

(1) *Proviso*: *The strong outcome of* $G + \overline{G}$ *is* $\mathcal{N}$.

(2) *Maintenance*: *Every Right move in* $G + \overline{G}$ *has a Left response that is greater than or equal to* 0.

Proof. Since $G + \overline{G}$ is self-conjugate, $G + \overline{G} \equiv_{\mathcal{E}} 0$ if and only if $G + \overline{G} \geqslant_{\mathcal{E}} 0$. By Theorem 2, $G + \overline{G} \geqslant_{\mathcal{E}} 0$ if and only if the Proviso and Maintenance Property are satisfied. For the Proviso, we need $\hat{o}(G + \overline{G}) \geqslant \hat{o}(0) = \mathcal{N}$. Again, by the symmetry of $G + \overline{G}$, we cannot have that its strong outcome is Left; thus, the Proviso is equivalent here to $\hat{o}(G + \overline{G}) = \mathcal{N}$, which is condition (1). For the Maintenance Property, part (a) is vacuously true because 0 has no options. For all $(G + \overline{G})^R$, part (b) of the Maintenance Property can only be satisfied with a $(G + \overline{G})^{RL} \geqslant 0$, because there are no Right options of 0. This is condition (2). □

We used our computer program to check which day-2 dead-ending positions satisfy $G + \overline{G} \equiv_{\mathcal{E}} 0$ via Theorem 2. We found that 43 of the 192 reduced day-2 dead-ending games are invertible modulo $\mathcal{E}$: 3 symmetric positions and 20 conjugate pairs. These are listed in Table 2. The table also includes the invertible form $\{\bar{1}|1\} \equiv_{\mathcal{E}} 0$, which we did not count in the 43. Including 0, 1, and $\bar{1}$, we have 46 invertible dead-ending positions born by day 2.

Observe that the invertible reduced day-2 games do not have outcome $\mathcal{P}$ and do not have an option to $*$. In fact, this is if and only if; we confirmed computationally that any day-2 dead-ending game that is $\mathcal{P}$-win or that has an

$\{\bar{1}\|1\}$	$\{0, \bar{1}\|0, 1\}$	$\{1, \bar{1}\|1, \bar{1}\}$	$\{0, 1, \bar{1}\|0, 1, \bar{1}\}$
$\{\cdot\|1\}$	$\{0\|0, 1, \bar{1}\}$	$\{\bar{1}\|0, 1\}$	$\{0, 1\|1, \bar{1}\}$
$\{\bar{1}\|\cdot\}$	$\{0, 1, \bar{1}\|0\}$	$\{0, \bar{1}\|1\}$	$\{1, \bar{1}\|0, \bar{1}\}$
$\{\cdot\|0, \bar{1}\}$	$\{1\|1\}$	$\{\bar{1}\|0, \bar{1}\}$	$\{0, 1\|0, 1, \bar{1}\}$
$\{0, 1\|\cdot\}$	$\{\bar{1}\|\bar{1}\}$	$\{0, 1\|1\}$	$\{0, 1, \bar{1}\|0, \bar{1}\}$
$\{0\|1\}$	$\{1\|0, 1\}$	$\{\bar{1}\|1, \bar{1}\}$	$\{0, \bar{1}\|1, \bar{1}\}$
$\{\bar{1}\|0\}$	$\{0, \bar{1}\|\bar{1}\}$	$\{1, \bar{1}\|1\}$	$\{1, \bar{1}\|0, 1\}$
$\{0\|0, 1\}$	$\{1\|1, \bar{1}\}$	$\{\bar{1}\|0, 1, \bar{1}\}$	$\{0, \bar{1}\|0, 1, \bar{1}\}$
$\{0, \bar{1}\|0\}$	$\{1, \bar{1}\|\bar{1}\}$	$\{0, 1, \bar{1}\|1\}$	$\{0, 1, \bar{1}\|0, 1\}$
$\{0\|1, \bar{1}\}$	$\{1\|0, 1, \bar{1}\}$	$\{0, 1\|0, 1\}$	$\{1, \bar{1}\|0, 1, \bar{1}\}$
$\{1, \bar{1}\|0\}$	$\{0, 1, \bar{1}\|\bar{1}\}$	$\{0, \bar{1}\|0, \bar{1}\}$	$\{0, 1, \bar{1}\|1, \bar{1}\}$

Table 2. The invertible reduced day-2 dead-ending positions.

option to $*$ (after reduction) is not invertible. To explore this relationship further, we prove exactly which day-2 dead-ending games are previous-win.

Lemma 10. *If $G \in \mathcal{E}$ is born on day 2 then $o(G) = \mathcal{P}$ if and only if $G^{\mathcal{L}}$ is a nonempty subset of $\{0, 1\}$ and $G^{\mathcal{R}}$ is a nonempty subset of $\{0, \bar{1}\}$.*

Proof. If G is born on day 2 and $o(G) = \mathcal{P}$, then the second player must win immediately after the first turn; i.e., there must be no alternating reply to any opening move. This means the only Left options are 0 and 1, and the only Right options are 0 and $\bar{1}$. The converse is clearly also true. □

Here are the previous-win day-2 games:

$$\{0|\bar{1}\},\quad \{0|0, \bar{1}\},\quad \{1|0\},\quad \{1|\bar{1}\},\quad \{1|0, \bar{1}\},\quad \{0, 1|0\},\quad \{0, 1|\bar{1}\},\quad \{0, 1|0, \bar{1}\}.$$

With this we can prove our observation about the categorization of invertible positions and confirm theoretically the results in Table 2.

Theorem 11. *If $G \in \mathcal{E}$ is born by day 2 and G is in reduced form, then G is invertible if and only if*

$$* \notin G^{\mathcal{L}}, G^{\mathcal{R}} \quad \textit{and} \quad o(G) \neq \mathcal{P},$$

i.e., if and only if

$$* \notin G^{\mathcal{L}}, G^{\mathcal{R}} \quad \textit{and} \quad \textit{either } G^{\mathcal{L}} \nsubseteq \{0, 1\} \textit{ or } G^{\mathcal{R}} \nsubseteq \{0, \bar{1}\}.$$

Proof. We have confirmed this computationally and will spare the reader the tedious proof that these conditions on $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ are equivalent to the two conditions of Theorem 9; i.e., to (1) the strong outcome of $G + \overline{G}$ is $\mathcal{N}$ and (2) every Right move in $G + \overline{G}$ has a Left response $\geqslant 0$. The code can be found on GitHub. □

5. Summary and future work

In this paper, we identified all dead-ending games born by 2, determined the reduced forms modulo $\mathcal{E}$, and classified each as invertible or noninvertible modulo $\mathcal{E}$. To do this, we determined how to calculate the strong outcome from Left and Right options (Theorem 1), and similarly determined how to apply the recursive comparison test from [2], based on the options of the given day-2 games. We used end-reversibility reductions for $\mathcal{E}$ from [3] to determine the number of reduced day-2 games, and finally, we determined algebraically and computationally the invertible day-2 positions. Our results are summarized below.

- There are 228 dead-ending game trees of depth 2.
- Of these, 4 are equivalent to 0 modulo $\mathcal{E}$.
- There are 192 unique, reduced day-2 dead-ending games, 172 of which are not equivalent to any other day-2 game modulo $\mathcal{E}$.
- Of the 192 reduced games, 43 are invertible modulo $\mathcal{E}$.
- Including 0, 1, $\bar{1}$, and $*$, there are 196 reduced dead-ending games born by day 2, and 46 of them are invertible.

Table 3 shows all game positions born by day 2, with possible Left option sets $G^{\mathcal{L}}$ down the left and possible Right option sets $G^{\mathcal{R}}$ across the top. Gray cells are not dead-ending. A checkmark ✓ indicates that a position is invertible, and a cross × indicates that it is noninvertible. Reduced positions are highlighted with a circle ◌. When a game is equivalent to 0, we indicate so, but note that these cells could also be marked as ✓ for a nonreduced, invertible position.

Future work could consider the partial order of the dead-ending games born by day 2. We could also begin to look at reductions and invertibility for day-3 dead-ending games.

It is very interesting that the invertible day-1 and day-2 games are precisely those with outcome not $\mathcal{P}$ and with no $\mathcal{P}$ followers. We used our computer program to create random day-3 dead-ending games, and found that this pattern continues. Our code is available on GitHub. We end this paper with the following conjecture for all dead-ending games.

Conjecture 12. A dead-ending game G is invertible if and only if $o(G') \neq \mathcal{P}$ for all followers G' of G (including G itself).

Addendum

The research in this paper was completed during 2019–2021 and submitted for publication in 2021. Since that time, some of the results have been generalized beyond day 2; in particular, Conjecture 12 was proven and published in [7].

	·	0	1	$\bar{1}$	$*$	0, 1	0, $\bar{1}$	0, $*$	1, $\bar{1}$	1, $*$	$\bar{1}$, $*$	0, 1, $\bar{1}$	0, 1, $*$	0, $\bar{1}$, $*$	1, $\bar{1}$, $*$	0, 1, $\bar{1}$, $*$
·	(✓)	(✓)		(✓)			(✓)									
0	(✓)	(×)	(✓)	(×)	(×)	(✓)	(×)	(×)	(✓)	(×)	(×)	(✓)	(×)	(×)	(×)	(×)
$\bar{1}$		(✓)	0	(✓)	(×)	(✓)	(✓)	✓	(✓)	0	(×)	(✓)	✓	✓	✓	✓
1	(✓)	(×)	(✓)	(×)	(×)	(✓)	(×)	(×)	(✓)	(×)	(×)	(✓)	(×)	(×)	(×)	(×)
$*$		(×)	(×)	(×)	(×)	(×)	(×)	×	(×)	×	(×)	(×)	×	×	×	×
0, $\bar{1}$		(✓)	(✓)	(✓)	(×)	(✓)	(✓)	(×)	(✓)	(×)	(×)	(✓)	(×)	(×)	(×)	(×)
0, 1	(✓)	(×)	(✓)	(×)	(×)	(✓)	(×)	(×)	(✓)	(×)	(×)	(✓)	(×)	(×)	(×)	(×)
0, $*$		(×)	✓	(×)	×	(×)	(×)	(×)	(×)	×	(×)	(×)	(×)	(×)	(×)	(×)
1, $\bar{1}$		(✓)	(✓)	(✓)	(×)	(✓)	(✓)	(×)	(✓)	(×)	(×)	(✓)	(×)	(×)	(×)	(×)
$\bar{1}$, $*$		(×)	0	(×)	×	(×)	(×)	×	(×)	0	(×)	(×)	×	×	×	×
1, $*$		(×)	(×)	(×)	(×)	(×)	(×)	(×)	(×)	(×)	(×)	(×)	(×)	(×)	(×)	(×)
0, 1, $\bar{1}$		(✓)	(✓)	(✓)	(×)	(✓)	(✓)	(×)	(✓)	(×)	(×)	(✓)	(×)	(×)	(×)	(×)
0, $\bar{1}$, $*$		(×)	✓	(×)	×	(×)	(×)	(×)	(×)	×	(×)	(×)	(×)	(×)	(×)	(×)
0, 1, $*$		(×)	✓	(×)	×	(×)	(×)	(×)	(×)	×	(×)	(×)	(×)	(×)	(×)	(×)
1, $\bar{1}$, $*$		(×)	✓	(×)	×	(×)	(×)	(×)	(×)	×	(×)	(×)	(×)	(×)	(×)	(×)
0, 1, $\bar{1}$, $*$		(×)	✓	(×)	×	(×)	(×)	(×)	(×)	×	(×)	(×)	(×)	(×)	(×)	(×)

Table 3. All dead-ending games born by day 2 (white cells), with ✓ for invertible positions and × for noninvertible, and with a circle ( ) indicating that a position is in reduced form.

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