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Reversibility, canonical form, and invertibility in dead-ending misère play

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In normal play combinatorial game theory, there is a slick reduction of game forms via domination and reversibility, which yields a unique reduced game form, dubbed the canonical form or simply the game value. In misère play, the situation is much more varied and complex. In restricted misère play (Plambeck and Siegel 2008), where the definition of inequality is weakened, domination is in analogy with normal play, but reversibility is not: in particular, if a Left option is reversible through a position with no Left option, then the reversible Left option cannot always be removed (Siegel 2015). Dorbec et al. (2015) found a modified reversibility reduction to give unique reduced forms for dicot games. We present a set of reductions for reversible options in *dead-ending* games (Milley et al. 2013). We prove that the reduced forms are unique with respect to our choice of reduction. We use uniqueness of reduced forms to prove that dead-ending, dicotic, and impartial restrictions have the conjugate property: if a game has an inverse, then it is the conjugate, i.e., the game where players have swapped roles.

1. Introduction

In combinatorial game theory there are two ways to simplify a game: remove dominated options and bypass reversible options. These simplifications give unique reduced game forms in both normal play, where the first player unable to move loses, and in misère play, where the first player unable to move wins [17]. Generic misère play has been much less studied than normal play, because of a loss of algebraic structure, including fewer instances of simplification.

In recent years, study of misère games has focused on *restricted* or *modular*

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play [15], where the relations of equality and inequality are weakened by restricting to a given subset, or *universe*, of games, say $\mathcal{U}$.[1] The idea is that games may be equal or comparable, or have an inverse, "modulo $\mathcal{U}$", even if they are not equal/comparable/invertible in full misère play. Modular misère play has facilitated successful analysis of many types of games, as we will review in Section 2.1.

However, the theory of reductions opens up new research problems in the study of modular misère play. Specifically, domination can be applied as usual, but reversibility cannot. In particular, if a Left option is reversible (modulo $\mathcal{U}$) through a position with no Left options, i.e., through a *Left-end*, then removing the reversible Left option may not leave an equivalent game [17]; this constitutes the problem of so-called *end-reversibility*.

Dorbec et al. [4] introduced a modified reversibility reduction to give unique reduced forms for the universe of *dicots*, $\mathcal{D}$, where at every position, either both players can move or neither can; i.e., where the only *end* is zero. Dicot games appear in rulesets such as FLOWER GARDENS [3], CLOBBER [1] and BIPASS [6].

The universe of dicots is a subset of another well-studied misère universe. A *dead-ending* game has the property that if, at some stage of play, a player cannot move, then they cannot move after any sequence of moves by the other player. HACKENBUSH and DOMINEERING [3] are examples of dead-ending rulesets.

In this paper we present a solution for *end-reversibility* in the universe $\mathcal{E}$ of dead-ending games, and we prove that the resulting reduced form is unique modulo $\mathcal{E}$. As a consequence, we also prove that $\mathcal{D}$, $\mathcal{E}$, and the subuniverse of impartial games satisfy the *conjugate property*: if a game has an inverse (which is not guaranteed even in modular misère play), then the inverse is the *conjugate*, where the roles of the players are swapped.

Standard definitions, including outcome, sum, conjugation and partial order are reviewed in Section 2. In Section 2.1 we define game comparison in the dicot and dead-ending universes. In Section 2.2 we define the concept of "waiting games", which leads to the version of absolute game comparison relevant to this work. In Section 3 we discuss the standard reductions, domination and reversibility. In Section 3.1 we discuss the case of open-reversibility, and in Section 3.2 we discuss the case of end-reversibility. In Section 4 we discuss our choice to arrive at a canonical form. In Section 5 we round up with proofs that the conjugate property holds.

2. Some standard definitions

The players are called Left (she) and Right (he). A game $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$ is

[1] A set of misère games is a *universe* if it satisfies standard closure properties: it has a neutral element and is closed under sums, taking options and conjugates. See Definition 1.

defined recursively by the set of *Left options* $G^{\mathcal{L}}$ and the set of *Right options* $G^{\mathcal{R}}$. The *zero game*, $\mathbf{0} = \{\varnothing \mid \varnothing\}$, is a position with no options for either player. The *rank* (or birthday) of a game is the depth of its (literal form) game tree; the rank of $\mathbf{0}$ is 0 and the rank of any nonzero game is one more than the maximum rank of its options. A *follower* of a game G is any position that can be reached from play in G, including G.

A game is called a *Left-end* if $G^{\mathcal{L}} = \varnothing$ and a *Right-end* if $G^{\mathcal{R}} = \varnothing$; thus, the zero game is both a Left-end and a Right-end. A game is an *end* if it is a Left-end or a Right-end. In normal play, all ends reduce to an integer game. This is not the case in general misère play, but we will use the same notation to mean a game with the same game tree as a normal-play canonical form integer: for an integer $n \geqslant 1$, $\boldsymbol{n} = \{\boldsymbol{n-1} \mid \varnothing\}$ and $\overline{\boldsymbol{n}} = \{\varnothing \mid \overline{\boldsymbol{n-1}}\}$.

Under normal play, the first player unable to move on their turn loses. Under misère play, this player wins. Under a specified winning condition, with a specified starting player, there are two possible results of a game: Left wins, denoted by L, or Right wins, denoted by R. The *Left outcome* $\mathrm{o}_L(G)$ is the optimal result of G when Left plays first and the *Right outcome* $\mathrm{o}_R(G)$ is the optimal result of G when Right plays first. The *outcome* of G is

$$\mathrm{o}(G) = \begin{cases} \mathscr{L} & \text{if } (\mathrm{o}_L(G), \mathrm{o}_R(G)) = (\mathrm{L}, \mathrm{L}); \\ \mathscr{N} & \text{if } (\mathrm{o}_L(G), \mathrm{o}_R(G)) = (\mathrm{L}, \mathrm{R}); \\ \mathscr{P} & \text{if } (\mathrm{o}_L(G), \mathrm{o}_R(G)) = (\mathrm{R}, \mathrm{L}); \\ \mathscr{R} & \text{if } (\mathrm{o}_L(G), \mathrm{o}_R(G)) = (\mathrm{R}, \mathrm{R}). \end{cases}$$

The results are ordered with $\mathrm{L} > \mathrm{R}$, so that the outcomes are partially ordered with $\mathscr{L} > \mathscr{N} > \mathscr{R}$ and $\mathscr{L} > \mathscr{P} > \mathscr{R}$, while $\mathscr{N}$ and $\mathscr{P}$ are incomparable.

The *disjunctive sum* of two games G and H is the game in which, on a player's turn, they choose to play in exactly one of G or H:

$$G + H = \{G^{\mathcal{L}} + H, G + H^{\mathcal{L}} \mid G^{\mathcal{R}} + H, G + H^{\mathcal{R}}\},$$

where $G^{\mathcal{L}} + H$ is the set of all positions of the form $G^L + H$ with $G^L \in G^{\mathcal{L}}$. Games are compared by the inequality relation

$$G \geqslant H \quad \text{if } \mathrm{o}(G + X) \geqslant \mathrm{o}(H + X) \text{ for all } X,$$

that is, $G \geqslant H$ means G is always at least as good as H for Left.

Two games G, H are *equivalent* if $G \geqslant H$ and $H \geqslant G$; i.e.,

$$G \equiv H \quad \text{if } \mathrm{o}(G + X) = \mathrm{o}(H + X) \text{ for all } X.$$

Thus, games are equivalent if they can be interchanged in any sum without affecting the outcome. Like outcome, note that equivalence and inequality are dependent upon the winning condition. In this paper, we always assume misère play, unless otherwise specified.

The *conjugate* of G is the game $\overline{G}$ with the roles of Left and Right swapped: the hereditary definition is $\overline{G} = \{\overline{G^{\mathcal{R}}} \mid \overline{G^{\mathcal{L}}}\}$, where $\overline{\mathcal{X}} = \{\overline{X} \mid X \in \mathcal{X}\}$.

2.1. ***Misère play modulo $\mathcal{D}$ and $\mathcal{E}$.*** Analysis of games in full misère play is intricate, for example, the only invertible game is **0**, and most pairs of games are incomparable.[2] To combat this, we consider restricted or modular misère play [15]. When we restrict attention to a subsets of games we must verify that the set satisfies standard closure properties.

Definition 1 (Universe of Games). A set of misère games $\mathcal{U}$ is a universe if

- $\mathbf{0} \in \mathcal{U}$;
- for all $G, H \in \mathcal{U}$, $G + H \in \mathcal{U}$;
- for all $G \in \mathcal{U}$, $\overline{G} \in \mathcal{U}$;
- for all $G \in \mathcal{U}$, for all options G', $G' \in \mathcal{U}$.

Given a universe $\mathcal{U}$, and two games G and H (not necessarily in $\mathcal{U}$),[3] inequality modulo $\mathcal{U}$ is defined as

$$G \geqslant_{\mathcal{U}} H \quad \text{if } \; \mathrm{o}(G + X) \geqslant \mathrm{o}(H + X) \; \text{ for all } \; X \in \mathcal{U}.$$

Consequently, equivalence is defined by $G \equiv_{\mathcal{U}} H$, if $G \geqslant_{\mathcal{U}} H$ and $H \geqslant_{\mathcal{U}} G$. In fact, inequality is a *congruence relation* [9], so that if $G \geqslant_{\mathcal{U}} H$ then, for all $J \in \mathcal{U}$,

$$G + J \geqslant_{\mathcal{U}} H + J. \tag{1}$$

Moreover, $G \in \mathcal{U}$ is *invertible modulo* $\mathcal{U}$ if there exists an $H \in \mathcal{U}$ such that $G + H \equiv_{\mathcal{U}} 0$. As a consequence of the congruence relation (1), if J is invertible, then $G + J \geqslant_{\mathcal{U}} H + J$ if and only if $G \geqslant_{\mathcal{U}} H$.

When the surrounding context is clear, we may omit the subscript $\mathcal{U}$; for example, when we explicitly announce $G, H \in \mathcal{U}$, then we usually prefer the more slick notation"$G \geqslant H$" instead of "$G \geqslant_{\mathcal{U}} H$"; if $G, H \in \mathcal{U}$ then, unless otherwise stated, comparison is modulo $\mathcal{U}$.

Let us mention a central property that every well structured universe should satisfy. (See Section 5.)

Definition 2 (Conjugate Property). A universe $\mathcal{U}$ has the *conjugate property* if, for all $G \in \mathcal{U}$, $G + H \equiv \mathbf{0}$ implies $H \equiv \overline{G}$.

[2] See [14] for a full survey of partizan misère play.

[3] In this study G and H mostly belongs to the same universe as X. The general definition is used in Corollary 26 and in another paper in this volume [10].

Two well studied universes of games are those of the dicots and the dead-ends.

Definition 3 (Dicots and Dead-ends). A *dicot* is a game in which the only end is zero: either both players can move, or neither can. The set of all dicots is $\mathcal{D}$. Each follower of a *dead Left-end* is a Left-end, and similar for Right. A dead end is a dead Left-end or a dead Right-end. A *dead-ending* game is a game in which each end follower is a dead end. The set of all dead-ending games is $\mathcal{E}$.

Notice that $\mathcal{D} \subset \mathcal{E}$.

Proposition 4. *The sets $\mathcal{D}$ and $\mathcal{E}$ are universes of games.*

Proof. Obvious. □

Critical results for the dicot and dead-ending universes are the recursive comparison tests introduced in [9; 8], stated in Theorem 8 below. For the dead-ending variant, we will require the definition of *waiting protected outcome* (Definition 7).[4]

2.2. ***Misère waiting games.*** Suppose that Right receives the following offer just when he is about to start a game: for $G \in \mathcal{E}$, an arbitrary game of rank $k > 0$, he may, if he wishes, *design* a Left-end $E \in \mathcal{E}$ to be played in disjunctive sum with G. Note that the outcome of a nontrivial Left-end is Left-win: so does the challenge really make sense? In fact, there is a class of games, with a modest number of nodes, about twice the rank of G, which makes the offer very attractive. The *waiting game* of dead-ending misère play, introduced in [8], is a good tool for any player adventurous enough to take on a challenge, where in general "moves are good", except at the very end. In a sense, Right wants to use a waiting game to maximize the chance that Left plays last in an arbitrary game G of specified rank.[5]

Definition 5 [8]. The (Right) waiting game of rank n, $W_n \in \mathcal{E}$, is

$$W_n = \begin{cases} \mathbf{0} & \text{if } n = 0; \\ \{\varnothing \mid \mathbf{0}, W_{n-1}\} & \text{if } n > 0. \end{cases}$$

Thus, $W_0 = \mathbf{0}$, $W_1 = \{\varnothing \mid \mathbf{0}\}$, $W_2 = \{\varnothing \mid \mathbf{0}, \{\varnothing \mid \mathbf{0}\}\}$, and so on. Right Waiting games of rank up to 4 are displayed in Figure 1.

The importance of waiting game ends in the dead-ending universe is established in [8]; a "waiting game Left-end" is a worst Left-end, from Left's perspective, because it maximizes Right's movability in case he needs it, and it minimizes his risk of extra unwanted moves.

Theorem 6 [8]. *If G is a Left-end with* $\operatorname{rank}(G) = k > 0$, *then* $G \geqslant W_n$, *for all* $n \geqslant k$.

[4] This is sometimes abbreviated "strong outcome".

[5] The concept of "waiting" is central in absolute theory, and it leads to concepts of strong outcomes and more (see the Introduction in [9]).

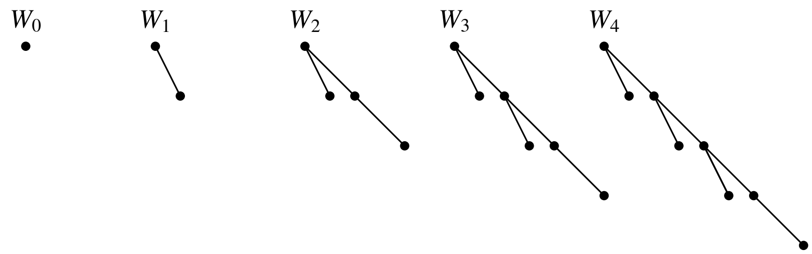


Figure 1. Waiting games of ranks 0 to 4.

Waiting games will appear in our reversibility reductions in Section 3.1.

A major tool is the waiting protected outcome.

Definition 7 [8]. The waiting protected Left-outcome and Right-outcome of $G \in \mathcal{E}$ are

$$\hat{o}_L(G) = \min\{o_L(G), o_L(G + W_{n-1})\},$$

$$\hat{o}_R(G) = \max\{o_R(G), o_R(G + \overline{W}_{n-1})\},$$

respectively, where $n = \operatorname{rank}(G)$. The waiting protected outcome of $G \in \mathcal{E}$ is

$$\hat{o}(G) = \begin{cases} \mathscr{L} & \text{if } (\hat{o}_L(G), \hat{o}_R(G)) = (\mathrm{L}, \mathrm{L}); \\ \mathscr{N} & \text{if } (\hat{o}_L(G), \hat{o}_R(G)) = (\mathrm{L}, \mathrm{R}); \\ \mathscr{P} & \text{if } (\hat{o}_L(G), \hat{o}_R(G)) = (\mathrm{R}, \mathrm{L}); \\ \mathscr{R} & \text{if } (\hat{o}_L(G), \hat{o}_R(G)) = (\mathrm{R}, \mathrm{R}). \end{cases}$$

A fundamental use of this protected outcome follows.

Theorem 8 (Recursive Comparison Test [8]). *Let $\mathcal{U} \in \{\mathcal{D}, \mathcal{E}\}$ and let $G, H \in \mathcal{U}$. Then $G \geqslant_{\mathcal{U}} H$ if and only if G, H satisfy*:

(1) Proviso: *if $\mathcal{U} = \mathcal{D}$, $o(G) \geqslant o(H)$; if $\mathcal{U} = \mathcal{E}$, $\hat{o}(G) \geqslant \hat{o}(H)$.*

(2) Maintenance Property:

(a) *for all $H^L \in H^{\mathcal{L}}$, there exists $G^L \in G^{\mathcal{L}}$ such that $G^L \geqslant_{\mathcal{U}} H^L$ or there exists $H^{LR} \in H^{L\mathcal{R}}$ such that $G \geqslant_{\mathcal{U}} H^{LR}$;*

(b) *for all $G^R \in G^{\mathcal{R}}$, there exists $H^R \in H^{\mathcal{R}}$ such that $G^R \geqslant_{\mathcal{U}} H^R$ or there exists $G^{RL} \in G^{R\mathcal{L}}$ such that $G^{RL} \geqslant_{\mathcal{U}} H$.*

Some proofs seem to require these simplifying constructive techniques, while other proofs will apply the standard "by definition and induction on the rank of game trees". One advantage of being able to use the standard induction proofs is that the "absolute" criterion is not required, and techniques may generalize in a broader sense. We will take a note of these aspects as we go along.

We make use of the *hand-tying principle* in modular misère play. Notice that this applies in any universe, not just the dead-ending universe.

Lemma 9 (Misère Hand-tying Principle [12]). *Consider a misère play universe* $\mathcal{U}$, *and let* $G \in \mathcal{U}$. *If* $|G^{\mathcal{L}}| \geqslant 1$ *then for any* $H \in \mathcal{U}$, $\{G^{\mathcal{L}}, H \mid G^{\mathcal{R}}\} \geqslant_{\mathcal{U}} G$.

Proof. Since $G^{\mathcal{L}}$ is nonempty, Left may ignore the option H. □

3. Domination and reversibility

Let us give some intuition for Theorem 10. Consider a game G and a winning condition, normal, or misère. If Left has two options G^{L_1}, G^{L_2} with $G^{L_1} \geqslant G^{L_2}$, then by definition, Left prefers G^{L_1} in every possible situation, and we may as well remove G^{L_2} from the set of Left options. In this case we say G^{L_2} is *dominated* by G^{L_1}, and

$$G \equiv \{G^{\mathcal{L}} \backslash \{G^{L_2}\} \mid G^{\mathcal{R}}\},$$

in both normal and misère play, with equivalence defined by the respective winning condition. Similarly, if $G^{R_1} \leqslant G^{R_2}$, then we can reduce G by removing G^{R_2} from the set of Right options.

Removing dominated options also applies in modular misère play. This is demonstrated in [4] for the dicot case, but generalizes to dead-ending, where the reduced game is trivially also dead-ending: if $G \in \mathcal{D}$ (or $\mathcal{E}$), a reduced game $\{G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\}$, as in Theorem 10, is still in $\mathcal{D}$ ($\mathcal{E}$).

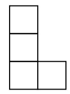
Theorem 10 (Domination [4]). *Let* $\mathcal{U} \in \{\mathcal{D}, \mathcal{E}\}$, *with* $G \in \mathcal{U}$. *If* $A, B \in G^{\mathcal{L}}$ *with* $A \leqslant B$, *then* $G \equiv \{G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\}$.

Let us give some intuition for Definition 11 in the next section. Removal of dominated options is one of two type of game reductions. The second reduction technique applies when a Left option of G, say $A = G^L$, has a Right option A^R that is at least as good for Right as the original game G; i.e., $A^R \leqslant G$. In this case, A is *reversible* and can be *bypassed*. The intuition is that Left can expect a Right reply to A^R, and so Left should immediately consider the Left options of A^R. Bypassing the reversible Left option A gives the following reduced game:

$$\{G^{\mathcal{L}} \backslash \{A\}, A^{R\mathcal{L}} \mid G^{\mathcal{R}}\}. \tag{2}$$

In normal play, reversible options can be bypassed in this way, while maintaining equivalence, even if A^R is a Left-end; in this case the reduction is simply to remove A from the left options of G. In full misère, the same situation cannot happen: if X is a Left-end and Y is not, then $X \not\leqslant Y$ [17].

This brings us to study reversibility under modular misère play. With the weakening of the inequality relation, it is now possible to have $X \leqslant_{\mathcal{U}} Y$ for a Left-end X and a non-Left-end Y. For example, in the universe of dead-ending games, the position $G = \{\mathbf{0}, \bar{\mathbf{1}} \mid \mathbf{1}\}$ satisfies $G \geqslant_{\mathcal{E}} \mathbf{0}$, by Theorem 8. Note that the game $\{\mathbf{0}, \bar{\mathbf{1}} \mid \mathbf{1}\}$ is the DOMINEERING position



In full misère play, this position does not reduce.

Thus, in modular misère play, we may find $G^{LR} \leqslant_{\mathcal{U}} G$ when G^{LR} is a Left-end; however removing G^{L} may not leave a position that is equivalent modulo $\mathcal{U}$. For example, in the same $G = \{\mathbf{0}, \bar{\mathbf{1}} \mid \mathbf{1}\}$ given above, the Left option $G^{L} = \bar{\mathbf{1}}$ has a Right option $G^{LR} = \mathbf{0} \leqslant_{\mathcal{E}} G$. However, it is not the case that $G \equiv_{\mathcal{E}} \{\mathbf{0} \mid \mathbf{1}\}$: they do not even have the same outcome under misère play, as

$$\mathrm{o}(G) = \mathscr{N}, \quad \mathrm{o}(\{\mathbf{0} \mid \mathbf{1}\}) = \mathscr{R}.$$

A primary goal of this paper is to determine what to do with end-reversible options modulo dead-ending games (Section 3.2). First, we settle the simpler case of open reversibility.

3.1. ***Reversibility in modular misère play.*** We collect the reversibility discussion from Section 3 by adapting some definitions from [7]. In that paper, ends are called "atomic games", and hence atomic-reversible is synonymous with end-reversible. We give the definitions from Left's perspective, but the analogous definitions for Right are also used.

Definition 11 [7]. Consider a universe $\mathcal{U}$ under misère play, and let $G \in \mathcal{U}$. Suppose that there are followers $A \in G^{\mathcal{L}}$ and $A^{R} \in A^{\mathcal{R}}$ with $A^{R} \leqslant_{\mathcal{U}} G$. Then the Left option A is *reversible through* its Right option A^{R}. This A^{R} is called the *reversing option*. If $A^{R\mathcal{L}}$ is nonempty then it is a *replacement set* for A, and in this case A is *open-reversible*. If the reversing option A^{R} is a Left-end (i.e., $A^{R\mathcal{L}} = \varnothing$), then A is *end-reversible*.

Observe that no "replacement set" is indicated whenever A is end-reversible. A special treatment is required to analyze such situations, and indeed this analysis is a main contribution of this work. We devote Section 3.2 to this study.

We begin by showing that in the case of dead-ending, the definition is justified by replacing open-reversible options by its replacement set: when A is replaced by the set $A^{R\mathcal{L}}$, one does obtain an equivalent game. Indeed, this is in analogy with classical normal-play theory, and it is a straightforward generalization of the result for dicots [4]. In the next subsection, we consider the more intricate case of end-reversible options.

Theorem 12 (Open Reversibility). *Consider $\mathcal{U} \in \{\mathcal{D}, \mathcal{E}\}$, and $G \in \mathcal{U}$. If $G^{L} = A$ is open-reversible, with replacement set $A^{R\mathcal{L}} \neq \varnothing$, then*

$$G \equiv_{\mathcal{U}} \{G^{\mathcal{L}} \setminus \{A\}, A^{R\mathcal{L}} \mid G^{\mathcal{R}}\}.$$

Proof. This result holds in the universe of dicots [4].

Consider $\mathcal{U} = \mathcal{E}$, with games as in the statement of the theorem. Since A is open reversible, the replacement set $A^{R\mathcal{L}}$ is nonempty. Hence, let

$$H = \{G^{\mathcal{L}} \setminus \{A\}, A^{R\mathcal{L}} \mid G^{\mathcal{R}}\}.$$

Note that $G^{\mathcal{R}}$ cannot be empty since G is dead-ending and $A \in G^{\mathcal{L}}$ has a Right option. Thus, $H \in \mathcal{E}$. We prove that $H \equiv G$.

We will prove that Left wins $G+X$ whenever she wins $H+X$, and vice versa. We proceed by induction on the rank of X.

First assume that Left wins $H+X$ moving first. If the winning move is to some G^L, then clearly Left wins $G+X$ with the same move. If it is to an X^L, then Left wins $G+X^L$ by induction. Finally, if the winning move is to some A^{RL}, then since $G \geqslant A^R$, there is a move in G that is at least as good as A^{RL}. So Left wins $G+X$ playing first.

Next assume Left wins $H+X$ moving second. So Left wins from any $H+X^R$, which by induction means Left wins any $G+X^R$, and also Left wins from any H^R+X; but the Right options of G are identical to those of H, so Left will win from any G^R+X, as well. By symmetry, this also proves Left wins $H+X$ playing second whenever she wins $G+X$ playing second.

Finally, assume Left wins $G+X$ moving first. If the good move is to X^L, then Left also wins $H+X^L$ by induction. If the good move is to any G^L besides A, then the same move is available in H. The remaining case is if the good move is to $A+X$. Then Left must be able to win from A^R+X; i.e., Left wins first on A^R+X. But every Left move in A^R+X is available to Left in $H+X$, so Left also wins $H+X$ moving first. Fix X. Note that $B^{\mathcal{L}}$, $G^{\mathcal{L}}$ and $H^{\mathcal{L}}$ are nonempty, and so A^R+X, $G+X$ and $H+X$ all have Left options. Also, $A+X$ has Right options, because A has at least one Right option, namely A^R.

Claim. $$H \geqslant A^R.$$

Proof of Claim. Suppose that Left starts in the game A^R+X. If $C \in A^{\mathcal{RL}}$ then $C \in H^{\mathcal{L}}$ and thus, if Left wins in A^R+X with $C+X$, Left also wins in $H+X$ with $C+X$. For the base case $X^{\mathcal{L}} = \varnothing$, this is enough. If $X^{\mathcal{L}} \neq \varnothing$, if Left wins in A^R+X with A^R+X^L, by induction, Left also wins in $H+X$ with $H+X^L$.

Suppose that Right starts in the game $H+X$. If Right wins in $H+X$ with $H+X^R$ then, by induction, Right wins in A^R+X with A^R+X^R. For the Right moves H^R+X (this includes the base case $X=0$), we observe that H^R+X is also a Right option of $G+X$. Since $G \geqslant A^R$, if Right wins with H^R+X, we must have a winning move for Right in A^R+X. This concludes the proof of the claim. □

To prove that $G \equiv H$, we have to prove that $G \geqslant H$ and $G \leqslant H$. Besides induction, we only need to use two arguments.

For $G \geqslant H$, if $A^{RL} + X$ is a winning move for Left in $H + X$, then because $G \geqslant A^R$, there is a winning move for Left in $G + X$.

For $G \leqslant H$, if $A + X$ is a winning move for Left in $G + X$, after an automatic Right reply to $A^R + X$, Left must have a winning move again. But, by the claim, the existence of that winning move implies that Left has a winning move in $H + X$.

Thus $o(H + X) = o(G + X)$, for all X, and so $G \equiv H$. □

3.2. ***End-reversibility.*** The idea for simplifying games with end-reversible options in specified modular misère play is as follows: played alone end-reversible options are winning options. Thus we attempt to replace them with an appropriate simpler winning option. However, if an end-reversible option is not the only winning option, we rather attempt a brute force removal. The simplification has to be justified in each specified universe. A simplification for dicot misère is proven in [4].

Theorem 13 (End-reversibility in $\mathcal{D}$ [4]). *Let $G \in \mathcal{D}$ and suppose that G has an end-reversible Left option, say A. Then $G \equiv \{*, G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\}$. The Left option $*$ is omitted if Left has a winning option in $G^{\mathcal{L}} \setminus \{A\}$.*

In the special case where G has only one Left and one Right option, both end-reversible, then the following substitution can be made for G. Theorem 13 is applied to both end-reversible options to obtain $G \equiv_{\mathcal{D}} \{* \mid *\}$, and then we use the known result that $\{* \mid *\}$ (or $* + *$) is equivalent to $\mathbf{0}$ modulo dicots [2; 11].

Corollary 14 (Substitution Theorem for $\mathcal{D}$ [4]). *If $G = \{A \mid B\} \in \mathcal{D}$, where A and B are end-reversible, then $G \equiv \mathbf{0}$.*

We will now extend the technique of [4] to develop a similar solution for end-reversibility in $\mathcal{E}$, and we introduce the concept of a *revocable option*.

Definition 15 (Nonessential Left Option). Consider $G \in \mathcal{E}$ such that $\hat{o}_L(G) = \mathrm{L}$. A Left option $A \in G^{\mathcal{L}}$ is *nonessential* if $\hat{o}_L(\{G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\}) = \mathrm{L}$, and otherwise A is *essential*.

Notice that if G has only one Left option, it is nonessential, because its removal leaves a Left-end, and Left would win playing this in a sum with a waiting Left-end.

We are now ready for our first end-reversible results for $\mathcal{E}$. There are more cases to consider than there were in $\mathcal{D}$. We start by proving that if an end-reversible option is nonessential, then it can be removed to leave an equivalent game. In particular, if there is exactly one end-reversible Left option and one end-reversible Right option, then these are nonessential and both can be removed, so that $G \equiv \mathbf{0}$. It is important to note that we will only remove a lone Left option if the result is still in $\mathcal{E}$. If the Right options of G are not all Left-ends, then changing $G = \{A \mid G^{\mathcal{R}}\}$ to $\{\varnothing \mid G^{\mathcal{R}}\}$ would create a game that is not dead-ending. In this case, we will instead settle for simplifying G using a substitution (Theorem 19).

Suppose that $G \in \mathcal{E}$ has an end-reversible Left option, say A, with reversing option, say A^R. Then, since A^R is a Left-end, $G \geqslant A^R$ implies that

$$\hat{o}_L(G) = \hat{o}_L(A^R) = \mathrm{L}. \tag{3}$$

Thus, the notion of nonessential Left option is well defined in the following result.

Theorem 16 (Nonessential End-reversibility in $\mathcal{E}$). *Suppose that $G \in \mathcal{E}$ has a nonessential end-reversible option $A \in G^{\mathcal{L}}$.*

(1) *If $\{G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\} \in \mathcal{E}$, then $G \equiv \{G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\}$.*

(2) *If $G = \{A \mid B\}$, where B is also end-reversible, then $G \equiv \mathbf{0}$.*

Proof. Case 1: Consider the end-reversible Left option $A \in G^{\mathcal{L}}$, and let $A^R \in A^{\mathcal{R}}$, with $A^{R\mathcal{L}} = \varnothing$ and $G \geqslant A^R$.

Let us begin by proving the inequality $G \geqslant H$. Suppose Left wins $H + X$, for some $X \in \mathcal{E}$. In the case $X^{\mathcal{L}} = \varnothing$, Left wins $G + X$ by (3), so suppose X has a Left option. If $|G^{\mathcal{L}}| = 1$, then Left has no move in H, so the winning move must be to $H + X^L$. By induction, $G + X^L$ is also a winning move. If $|G^{\mathcal{L}}| > 1$, then $G \geqslant H$ by the Hand-tying Principle (Lemma 9). Thus $G \geqslant H$.

Next, we prove the inequality $G \leqslant H$. Suppose Left wins $G + X$ for some $X \in \mathcal{E}$. If Left's winning move in $G + X$ is $G^L + X$ where $G^L \neq A$, then this is also a Left winning move in $H + X$. If Left's winning move in $G + X$ is $G + X^L$, then by induction $H + X^L$ is a Left winning move in $H + X$. The remaining case is if $A + X$ is the only winning Left move in $G + X$. In this case, X must be a Left-end; else by Lemma 17, there would exist a winning move $G + X^L$, a contradiction.

If X is a Left-end and $|G^{\mathcal{L}}| = 1$, then H is also a Left-end, and so Left wins $H + X$.

If X is a Left-end and $|G^{\mathcal{L}}| > 1$, since $A^R + X$ is a Left-end, $o_L(A^R + X) = \mathrm{L}$, which by $G \geqslant A^R$ implies $o_L(G + X) = \mathrm{L}$. But then, since A is nonessential, $o_L(H + X) = \mathrm{L}$. Thus, Left wins $H + X$.

To conclude this case, $G \equiv H$.

Case 2: If $G = \{A \mid B\}$, with both options end-reversible, then the above argument shows $G \equiv \{\varnothing \mid B\} \equiv \{\varnothing \mid \varnothing\}$ (and the test if $\{\varnothing \mid B\} \notin \mathcal{E}$ becomes obsolete). □

Let us prove a strategic fact for Left playing in a game with an end-reversible Left option.

Lemma 17 (Weak Avoidance Property). *Consider a misère universe $\mathcal{U}$ and let $G \in \mathcal{U}$. Suppose that A is an end-reversible Left option of G. If Left wins $G + X$ with $A + X$, where X is not a Left-end, then Left also wins $G + X$ with some $G + X^L$.*

Proof. With notation as in the statement, let A^R be a reversing option for A. By assumption, $G \geqslant A^R$ and $A^{R\mathcal{L}} = \varnothing$. If Right, playing first, loses $A + X$, then Left, playing first, wins $A^R + X$. Of course, because $A^{R\mathcal{L}} = \varnothing$ and $X^{\mathcal{L}} \neq \varnothing$, a Left winning move must be some $A^R + X^L$. But $G \geqslant A^R$, so $G + X^L$ must be a winning move for Left, from $G + X$. □

Next, we have a general structural simplification for any end-reversible Left option A; in particular this allows us to simplify end-reversible options that do not satisfy Theorem 16, either because they are essential or because they are lone Left options whose removal bumps the game out of $\mathcal{E}$. If A is end-reversible through a Left-end A^R, the simplification is to trim A so that it has no Left options and its only Right option is A^R.

Proposition 18. *Let $G \in \mathcal{E}$. If $A \in G^{\mathcal{L}}$ is reversible through a Left-end A^R, then $G \equiv \big\{\{\varnothing \mid A^R\}, G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\big\}$.*

Proof. Let $H = \big\{\{\varnothing \mid A^R\}, G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\big\}$.

Consider $X \in \mathcal{E}$ such that $o_L(H + X) = \mathrm{L}$. If $\{\varnothing \mid A^R\} + X$ is a winning move for Left in $H + X$, then Left has a good response to Right's move to $A^R + X$, and this means that Left can also win in the game $G + X$, because, by assumption, $G \geqslant A^R$. Any other winning option for Left in $H + X$ is also available in $G + X$, so $o_L(G + X) = \mathrm{L}$. Observe that any winning option for Right in $G + X$ is also available in $H + X$, so trivially $o_R(H + X) = o_R(G + X)$. Altogether, this shows that $G \geqslant H$.

For the other direction, we prove by induction on the rank of X that, for all X, $o(H + X) \geqslant o(G + X)$.

Suppose Left wins $G + X$, playing first. If Left's winning move in $G + X$ is $G + X^L$, then by induction $o(H + X^L) \geqslant o(G + X^L)$ and so $H + X^L$ is a left winning move in $H + X$. If Left's winning move in $G + X$ is $G^L + X$, where $G^L \in G^{\mathcal{L}} \setminus \{A\}$, then this is also a Left winning move in $H + X$. The remaining case is if $A + X$ is the only winning move in $G + X$. In this case, X must be a Left-end; else by Lemma 17, there would exist a winning move $G + X^L$, a contradiction. But if X is a Left-end, then $\{\varnothing \mid A^R\} + X$ is a winning move for Left, because it is a Left-end. Thus Left wins $H + X$. □

In fact, we can do better than Proposition 18, and simplify instead to a game of the same form but with A^R replaced by a position with a weakly decreased rank. This is shown next in the Substitution Theorem (Theorem 19). To get there, we use the waiting games — specifically the fact that they are worse than any other Left-end — to construct another substitution for end-reversible options. Like Proposition 18, this substitution applies to all end-reversible options, but is only required for those options that are not removed by Theorem 16: that is, when A is essential (and therefore $|G^{\mathcal{L}}| > 1$), or when A is a lone option and $\{\varnothing \mid G^{\mathcal{R}}\} \notin \mathcal{E}$.

Theorem 19 (Substitution Theorem for $\mathcal{E}$). *Let $G \in \mathcal{E}$. If $A \in G^{\mathcal{L}}$ is end-reversible, then there exists a smallest nonnegative integer n such that $G \geqslant W_n$ and $G \equiv \big\{\{\varnothing \mid W_n\}, G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\big\}$.*

Proof. Suppose that $A \in G^{\mathcal{L}}$ is end-reversible through the Left-end A^R. Let $k = \mathrm{rank}(A^R)$. By assumption, $G \geqslant A^R$ and thus, by Theorem 6, $G \geqslant A^R \geqslant W_k$. Since k is a nonnegative integer, the existence part is clear. Let n be the minimum nonnegative integer such that $G \geqslant W_n$.

Let $H = \big\{\{\varnothing \mid W_n\}, G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\big\}$, and let $G' = \big\{\{\varnothing \mid W_n\}, G^{\mathcal{L}} \mid G^{\mathcal{R}}\big\}$. By Lemma 9, we have $G' \geqslant G$. Then $G \geqslant W_n$ and $G \geqslant A^R$ imply that both $\{\varnothing \mid W_n\}$ and A are end-reversible options of G'.

Now, $\hat{o}_L(H) = \mathrm{L}$, because Left wins moving first in any $H + W_m$, by moving to the Left-end $\{\varnothing \mid W_n\} + W_m$. Likewise, $\hat{o}_L(G) = \mathrm{L}$ because $G \geqslant A^R$ and $\hat{o}_L(A^R) = \mathrm{L}$. Since H and G are the games G' with A and $\{\varnothing \mid W_n\}$ removed respectively, this shows that both A and $\{\varnothing \mid W_n\}$ are nonessential in G'. Thus, by Theorem 16, $G' \equiv H$ and $G' \equiv G$. This gives $G \equiv H$, as required. □

The rank of $\{\varnothing \mid A^R\}$ is weakly greater than the rank of $\{\varnothing \mid W_n\}$, because n is chosen to be minimum. So Theorem 19 will usually give a substitution that is simpler than the substitution in Proposition 18.

Example 20. Suppose we apply the Substitution Theorem to the form described in Theorem 16(2). That is, $G = \{A \mid B\} \in \mathcal{E}$ with both A and B end-reversible. Then, by Theorem 19,

$$G \equiv \big\{\{\varnothing \mid W_n\} \mid \{\overline{W}_k \mid \varnothing\}\big\}.$$

Since the options are reversible, we have $\overline{W}_k \geqslant G \geqslant W_n$; but $\overline{W}_k \geqslant W_n$ is only possible if both are zero, because otherwise the former is Right-win and the latter is Left-win. So $k = n = 0$ and $G \equiv \big\{\{\varnothing \mid \mathbf{0}\} \mid \{\mathbf{0} \mid \varnothing\}\big\} = \{-\mathbf{1} \mid \mathbf{1}\}$. This game would then reduce to $\mathbf{0}$ by Theorem 16.

These are all of the reductions for the dead-ending universe. To summarize, there are four types of reductions in $\mathcal{D}$ [4] and five types in $\mathcal{E}$; the first two are common:

(1) Remove dominated options (Theorem 10).

(2) Reverse open-reversible options (Theorem 12).

In $\mathcal{D}$ we have two additional reductions, for end-reversible options [4]:

(3) Replace end-reversible options by $*$ (Theorem 13).

(4) Replace $\{* \mid *\}$ by $\mathbf{0}$ (Corollary 14).

And in $\mathcal{E}$ we instead have three additional reductions, for end-reversible options:

(3) Remove nonessential end-reversible options, including removal of lone options as long as the result is in $\mathcal{E}$ (Theorem 16(1)).

(4) Simultaneously remove lone Left and lone Right-end-reversible options; i.e., replace $\{A \mid B\}$ with $\mathbf{0}$ if both A and B are end-reversible (Theorem 16(2)).

(5) Replace other end-reversible options of rank $n > 0$ (i.e., those that are essential and those whose removal give a game not in $\mathcal{E}$) by $\{\varnothing \mid W_n\}$ for Left options, or by $\{\overline{W}_n \mid \varnothing\}$ for Right options (Theorem 19).

We call a game *reduced* if the above reductions have been applied, in any order, until application stabilizes.

Definition 21 (Reduction). A game $G \in \mathcal{E}$ is *reduced* if none of Theorems 10, 12, 16, or 19 can be applied to give a game with differing options. A game $G \in \mathcal{D}$ is *reduced* if none of Theorems 10, 12, 13, or 14 can be applied to give a game with differing options.

Note that our results for end-reversible reductions guarantee that reduced dead-ending games remain in $\mathcal{E}$.

In Section 4 we show that, given our choice for end-reversible substitution, there is a simplest unique reduced form of any game in $\mathcal{E}$.

4. Uniqueness and simplicity of reduced forms

We are now able to prove the existence of a simplest reduced form for a congruence class of games in the dead-ending universe. Moreover, this form is *unique* up to choice of end-reversibility substitution. (A similar result for the dicot universe is proved in [4].)

Let $\cong$ indicate that two games are identical; that is, $G \cong H$ if G and H have the same literal form.

Recall from Definition 21 that a game in $\mathcal{E}$ is *reduced* if

(i) dominated options have been removed;

(ii) open-reversible options have been reversed;

(iii) nonessential end-reversible options have been removed, unless removal gives a non-dead-ending game; and

(iv) other end-reversible options have been replaced by $\{\varnothing \mid W_n\}$ (Left) or $\{\overline{W}_n \mid \varnothing\}$ (Right), for a minimal choice of n respectively.

Let us demonstrate that any two equivalent reduced games are in fact identical. First, we note the following result from [13], which says that all ends are invertible modulo $\mathcal{E}$.

Theorem 22 [13]. *If $G \in \mathcal{E}$ is an end, then G is invertible and the inverse is the conjugate, i.e.,*

$$G + \overline{G} \equiv \mathbf{0}.$$

Thus, if $G \in \mathcal{E}$ is an end, we may make the identification $\overline{G} = -G$.

Theorem 23 (Uniqueness of Reduced Form). *Let $G, H \in \mathcal{E}$. If $G \equiv H$ and both are reduced games, then $G \cong H$.*

Proof. Assume $G \equiv H$ and both are reduced as in Definition 21. We will proceed by induction on the sum of the ranks of G and H. We will exhibit a correspondence $G^{L_i} \equiv H^{L_i}$ and $G^{R_j} \equiv H^{R_j}$ between the options of G and H. By induction, it will follow that $G^{L_i} \cong H^{L_i}$, for all i, and $G^{R_j} \cong H^{R_j}$, for all j, and consequently $G \cong H$.

For the base case, if $G \cong \mathbf{0}$ and $H \cong \mathbf{0}$, then $G \cong H$. If G and H are not both zero, then without loss of generality, assume that there is a Left option H^L. We will break the proof into two cases based on whether or not H^L is an end-reversible option. Since the games are reduced, note that there can be no open reversible options. Moreover, if a reduced game is end-reversible it must be of the form (iv) above.

<u>Case 1</u>: H^L is not end-reversible.

This means that H^L is not reversible at all, since H cannot have open-reversible options. Since $G \equiv H$, of course $G \geqslant H$. Then from Theorem 8, there exists a G^L with $G^L \geqslant H^L$ or there exists a $H^{LR} \leqslant G$. Now $H^{LR} \leqslant G \equiv H$ would contradict that H^L is not reversible. Thus, there is some G^L with $G^L \geqslant H^L$.

We claim that this G^L cannot be end-reversible. If it were, then since G is in reduced form, by Definition 21, $G^L \cong \{\varnothing \mid W_n\}$ for some nonnegative integer n. Also, from the reversibility of G^L, we would have $G \geqslant G^{LR} = W_n$. Since $G \equiv H$, this gives

$$H \geqslant W_n. \tag{4}$$

Now, from $G^L \geqslant H^L$ and $G^L \cong \{\varnothing \mid W_n\}$, we have $\{\varnothing \mid W_n\} \geqslant H^L$. But $\{\varnothing \mid W_n\}$ is invertible (Theorem 22), and so

$$0 \geqslant H^L + \{\overline{W}_n \mid \varnothing\}. \tag{5}$$

From the Maintenance Property of Theorem 8, inequality (5) implies that from every Left move in $H^L + \{\overline{W}_n \mid \varnothing\}$, there exists a Right move that is less than or equal to $\mathbf{0}$. If Left moves to $H^L + \overline{W}_n$, there is no Right response in $\overline{W}_n$, so the Right reply must look like $H^{LR} + \overline{W}_n \leqslant 0$. By the invertibility of W_n, this means $H^{LR} \leqslant W_n$. Combining this with inequality (4) gives $H \geqslant H^{LR}$, which contradicts that H^L is not reversible. Thus, G^L is not end-reversible, as claimed.

A similar argument for G^L gives a Left option $H^{L'}$ such that $H^{L'} \geqslant G^L$. Therefore, $H^{L'} \geqslant G^L \geqslant H^L$. Since there are no dominated options in the reduced game H, it must be that $H^{L'} \equiv H^L \equiv G^L$. By induction, $H^L \cong G^L$.

The symmetric argument gives that each nonreversible option H^R is identical to some G^R. In conclusion, we have a pairwise correspondence between options of G and H that are not end-reversible.

Case 2: H^L is end-reversible.

Since H is reduced, this means that $H^L \cong \{\varnothing \mid W_n\}$, for some nonnegative integer n, and $H \geqslant W_n$. We have two subcases.

Case 2a: $|H^{\mathcal{L}}| > 1$.

In this case, since H is reduced and Theorem 16 cannot be further applied, it must be that H^L is an essential Left option of H.

This means that H^L is the only good Left option in the following sense: $\hat{o}(\{H^{\mathcal{L}} \setminus \{H^L\} | H^{\mathcal{R}}\}) = \mathrm{R}$. Since $G \equiv H$, there must also be a good Left move in G, say G^L. We claim that G^L is end-reversible. If not, then by Case 1, there is a corresponding nonreversible option $H^{L'}$ in H such that $G^L \cong H^{L'}$. Then $H^{L'}$ is also a good move in H, a contradiction.

Therefore, G^L is end-reversible in G, and since G is reduced this means $G^L \cong \{\varnothing \mid W_{n'}\}$, for some nonnegative integer n'. To see $G^L \equiv H^L \cong \{\varnothing \mid W_n\}$, we need only show $n = n'$. But this follows from Theorem 19, because n, n' are minimal such that $G \geqslant W_n, W_{n'}$. Thus, $H^L \cong G^L$.

Case 2b: $|H^{\mathcal{L}}| = 1$.

We need to show that G has a Left option $G^L \cong H^L$; we claim that it suffices to show $G^{\mathcal{L}} \neq \varnothing$. To see that this is sufficient, first note that any Left option of G must be end-reversible, since otherwise the pairwise correspondence from Case 1 would mean H has a nonreversible Left option, contradicting our assumption that there is only one Left option and it is end-reversible. Since G is reduced, it has at most one end-reversible Left option, as the rest would be removed by Theorem 16, and so if $G^{\mathcal{L}} \neq \varnothing$ then in fact $|G^{\mathcal{L}}| = 1$. By Theorem 19, this one end-reversible Left option must be $G^L = \{\varnothing \mid W_{n'}\}$, for some nonnegative integer n', and then $n = n'$ follows as above, and we get $G^L \cong H^L$, as required.

So suppose by way of contradiction that $G^{\mathcal{L}} = \varnothing$. Since $H = \{\{\varnothing \mid W_n\} \mid H^{\mathcal{R}}\}$ cannot be reduced further, Theorem 16(1) cannot be applied, and it follows that $\{\varnothing \mid H^{\mathcal{R}}\} \notin \mathcal{E}$. Thus, there must exist some Right option H^R of H that is not a Left-end. If this H^R is not end-reversible then by Case 1 there is a corresponding

Right option $G^R \cong H^R$ in G; but this is a contradiction because $G \in \mathcal{E}$ would be a Left-end with a follower that is not a Left-end.

So H^R is end-reversible. If H^R is the only Right option then H is of the form in Theorem 16(2), which contradicts the fact that H is reduced. So there are other Right options, say $H^{R_2}, H^{R_3}, \ldots$, and then we know two things: first, H^R must be essential, else it would have been removed, and second, these other Right options cannot be end-reversible, else as nonessential they would have been removed.

By Case 1, this means there are corresponding nonreversible options in $G^{\mathcal{R}}$, $G^{R_2} \cong H^{R_2}$, $G^{R_3} \cong H^{R_3}, \ldots$. Since H^R is essential, it is the only good move, i.e., $\hat{o}_R(\{H^{\mathcal{L}} | H^{\mathcal{R}} \setminus \{H^R\}) = \mathrm{L}$. In particular, none of $H^{R_2}, H^{R_3}, \ldots$ is a good move. Since $G \equiv H$, G must have a good move, say G^R. This G^R must be end-reversible; else by Case 1 it would be identical to one of the H_i^R. But by Theorem 19, since G is reduced, G^R must be $\{\overline{W}_m \mid \varnothing\}$, for some nonnegative integer m, and this is a contradiction because all options of the Left-end $G \in \mathcal{E}$ must be Left-ends.

This shows that $G^{\mathcal{L}} = \varnothing$ is impossible, and so then by the argument above we get $G^{\mathcal{L}} = \{G^L\}$ with $G^L \cong H^L$.

In all cases, we have shown that $H^{\mathcal{L}}$ is identical to $G^{\mathcal{L}}$. The proof for $H^{\mathcal{R}}$ and $G^{\mathcal{R}}$ is similar. Consequently, $G \cong H$. □

Thus two equivalent reduced games have the same literal form, which means that the reduced form does not depend on the order of reductions; it is unique with respect to our choice of substitution for end-reversible options.

We can further say that the reduced form of a game in $\mathcal{E}$ is a simplest equivalent form, modulo $\mathcal{E}$.

Theorem 24 (Simplicity). *Let $G \in \mathcal{E}$ be a reduced form of a game. If $G' \equiv G$, then* $\operatorname{rank}(G') \geqslant \operatorname{rank}(G)$.

Proof. If G' is also reduced, then $G' \cong G$ and so clearly the ranks are equal. Otherwise, reduce G' to G''; then $G'' \equiv G' \equiv G$ implies $G'' \cong G$ by Theorem 23. Since all reductions either maintain or reduce the rank of G'', this means $\operatorname{rank}(G') \geqslant \operatorname{rank}(G'') = \operatorname{rank}(G)$. □

These results let us talk about *a canonical form* (with the same meaning as the defined reduced form) of a game in the dead-ending universe, $\mathcal{E}$, provided we emphasize the choice made of substitution for end-reversibility. Analogous results were shown for $\mathcal{D}$ in [4]. With this, we have completed our second major goal of the paper, while the first goal was to establish the Substitution Theorem. In the next and final section, we use these defined canonical forms to establish the third major goal, that both $\mathcal{D}$ and $\mathcal{E}$ satisfy the conjugate property.

5. Conjugates and inverses

We end the paper with a proof that the conjugate property, Definition 2, holds modulo $\mathcal{D}$ or $\mathcal{E}$. We will find good use of the defined canonical forms from the previous section.

The proofs use a technique originally employed by Ettinger in his work on dicot scoring games [5] (see specifically Claim (ii) in Case 1 of the proof of Theorem 27). Let us begin by proving the conjugate property for the simpler case of the dicots.

Theorem 25. *The universe of dicot games has the conjugate property.*

Proof. Consider the canonical forms of $G, H \in \mathcal{D}$, and suppose $G + H \equiv \mathbf{0}$. We prove, by induction on the rank of $G + H$, that $H \equiv \overline{G}$. If the conjugate is an additive inverse, we make the identification $-G = \overline{G}$. For the base case, if G and H both reduce to $\mathbf{0}$, then trivially $H \equiv \overline{G}$. Otherwise, the game $G + H \in \mathcal{D}$ has at least one Left option and at least one Right option. Consider without loss of generality a Left option of G, G^L. We break the proof into two cases depending on whether G^L is reversible. Note that, since G is in reduced form, if G^L is reversible then it must be end-reversible with $G^L = *$.

<u>Case 1</u>: G^L is not reversible.

By Theorem 8, for the Left option $G^L + H$, there exists either $G^{LR} + H \leqslant \mathbf{0}$ or $G^L + H^R \leqslant \mathbf{0}$. The former inequality is impossible, since adding G to both sides gives $G^{LR} \leqslant G$, and hence G^L is reversible, a contradiction. Therefore, there exists H^R such that $G^L + H^R \leqslant \mathbf{0}$. The proof that $G^L + H^R \equiv \mathbf{0}$ is adapted from Claim (ii) in the proof of Theorem 27 (except here the only end-reversible option H^R is $*$). By induction then $H^R \equiv -G^L$.

<u>Case 2</u>: $G^L = *$.

Since $*$ is reversible in G, then $G \geqslant \mathbf{0}$. Adding H to each side gives $G+H \geqslant H$, and therefore by assumption $\mathbf{0} \equiv G + H \geqslant H$.

Assume, by way of contradiction that $* \notin H^{\mathcal{R}}$. By Case 1, the other Right options of H are the inverses of the Left options of G (by using also induction). Since G is reduced and $*$ is an end-reversible option of G, Left does not have any other winning move in G (see also Theorem 13). By symmetry, then, H has no winning Right move. But this contradicts $\mathbf{0} \geqslant H$, as Right wins playing first on $\mathbf{0}$. Thus, if G^L is end-reversible, then there is a corresponding end-reversible H^R in H, and both are $*$. Since $-* = *$, we have $H^R \equiv -G^L$.

Thus, we have established a pairwise equivalence of the options of $\overline{G}$ and H. This gives $\overline{G} \equiv H$. □

The analogous result holds for the dicot subuniverse of impartial games, $\mathcal{I}$. Here we get use for the modular definition of inequality.

Corollary 26. *The universe of impartial games has the conjugate property.*

Proof. A result of [16] establishes that, for $G, H \in \mathcal{I}$, $G \equiv_{\mathcal{D}} H$ if and only if $G \equiv_{\mathcal{I}} H$. Apply Theorem 25. ☐

Theorem 27. *The universe of dead-ending games has the conjugate property.*

Proof. Suppose that $G, H \in \mathcal{E}$ satisfy $G + H \equiv \mathbf{0}$, and consider G, H in their reduced forms. We will prove, by induction on the rank of $G + H$, that we must have $H \equiv \overline{G}$. Because of numerous algebraic manipulations, we will revert to the short-hand notation $-G = \overline{G}$, if the existence of a negative is given by induction.

For the base case, if G and H both reduce to 0, then trivially $H \equiv \overline{G}$. Otherwise, the game $G + H$ has at least one option; without loss of generality, assume G has at least one Left option, G^L. We break the proof into two cases based on whether or not G^L is reversible.

Case 1: G^L is not reversible.

We prove two claims.

Claim (i): There exists H^R such that $G^L + H^R \leqslant \mathbf{0}$.

Let $J = G + H \equiv 0$. From Theorem 8, for all Left moves J^L, there exists J^{LR} such that $J^{LR} \leqslant 0$. In particular, for the Left option $G^L + H$, there exists either $G^{LR} + H \leqslant 0$ or $G^L + H^R \leqslant 0$. The former inequality is impossible, since adding G to both sides gives $G^{LR} \leqslant G$, which contradicts that G^L is not a reversible option. Therefore, the claim holds.

Claim (ii): With H^R as in Claim (i), $G^L + H^R \equiv 0$.

We have that $G^L + H^R \leqslant 0$, and so suppose by way of contradiction that the inequality is strict. Let us index the options starting with $G^L = G^{L_1}$ and $H^R = H^{R_1}$, so that our assumption is $G^{L_1} + H^{R_1} < 0$.

Consider the Right move in $G + H$ to $G + H^{R_1}$. Since $G + H \geqslant 0$, there exists a Left option such that $(G + H^{R_1})^L \geqslant 0$. This option is either of the form $G + H^{R_1 L} \geqslant 0$ or $G^{L_2} + H^{R_1} \geqslant 0$ for some option G^{L_2} that cannot be G^{L_1}, since $G^{L_1} + H^{R_1} < \mathbf{0}$.

If we have the first inequality, $G + H^{R_1 L} \geqslant \mathbf{0}$, then, by adding H to both sides, we get $H^{R_1 L} \geqslant H$. Therefore, since H is in canonical form, H^{R_1} is an end-reversible option and, by Theorem 19, $H^{R_1} = \{-W_n \mid \varnothing\}$ for some nonnegative integer n. The two inequalities $G^{L_1} + H^{R_1} \leqslant \mathbf{0}$ and $G + H^{R_1 L} \geqslant \mathbf{0}$ become $G^{L_1} + \{-W_n \mid \varnothing\} \leqslant \mathbf{0}$ and $G - W_n \geqslant \mathbf{0}$, respectively. In $G^{L_1} + \{-W_n \mid \varnothing\}$, by Theorem 8, if Left moves to $G^{L_1} - W_n$, then Right has a response of the form $G^{L_1 R} - W_n \leqslant \mathbf{0}$. Then $G^{L_1 R} \leqslant W_n$ and $G \geqslant W_n$ leads to $G^{L_1 R} \leqslant G$, which is a contradiction because G^{L_1} is not reversible.

If we have the second inequality, $G^{L_2} + H^{R_1} \geqslant \mathbf{0}$, then we can assume G^{L_2} is not end-reversible; if it were end-reversible, then as above, H^{R_1} would be end-reversible and so G^{L_1} would also be end-reversible, contradicting our assumption. By induction, $G^{L_2} + H^{R_1} \equiv \mathbf{0}$ implies that $H^{R_1} = -G^{L_2}$, since G and H are in reduced form. But now we have $0 > G^{L_1} + H^{R_1} = G^{L_1} - G^{L_2}$, or $G^{L_2} > G^{L_1}$, which is a contradiction because $G^{\mathcal{L}}$ should have no dominated options. Therefore, $G^{L_2} + H^{R_1} > \mathbf{0}$. By Claim (i) and the above argument, there must exist a Right option H^{R_2} in H and a Left option G^{L_3} in G, such that $G^{L_2} + H^{R_2} \leqslant \mathbf{0}$ and $G^{L_3} + H^{R_2} > \mathbf{0}$, and from there a H^{R_3} and G^{L_4}, and so on. We get the following chain of inequalities:

$$\begin{aligned} G^{L_1} + H^{R_1} &\leqslant \mathbf{0}, & G^{L_2} + H^{R_1} &> \mathbf{0},\\ G^{L_2} + H^{R_2} &\leqslant \mathbf{0}, & G^{L_3} + H^{R_2} &> \mathbf{0},\\ &\vdots & &\vdots \end{aligned}$$

But the number of Left options of G is finite; at some point, we will get an inequality like $G^{L_1} + H^{R_m} > \mathbf{0}$ (re-indexing if necessary).

Because the inequalities are strict, summing the left-hand and the right-hand inequalities gives, respectively,

$$\sum_{i=1}^{m} G^{L_i} + \sum_{i=1}^{m} H^{R_i} \leqslant \mathbf{0} \quad \text{and} \quad \sum_{i=1}^{m} G^{L_i} + \sum_{i=1}^{m} H^{R_i} > \mathbf{0},$$

which is a contradiction.

Therefore, we conclude that $G^L + H^R \equiv \mathbf{0}$, which proves the claim, and then by induction $H^R \equiv -G^L$, which solves this case.

Case 2: G^L is reversible.

That is, since G is reduced, G^L is end-reversible: there is a smallest n such that $G^L = \{\varnothing \mid W_n\}$.

We will argue that G^L must be paired with a symmetric end-reversible option in $H^{\mathcal{R}}$. If not, by Case 1, $G + H$ would be

$$\{\{\varnothing \mid W_n\}, G^{L_2}, G^{L_3}, \ldots \mid \ldots\} + \{\ldots \mid -G^{L_2}, -G^{L_3}, \ldots\}.$$

Since G^L is end-reversible, we know $G \geqslant W_n$. Adding H to both sides and using $G + H \equiv 0$, we get $0 \geqslant H + W_n$.

Let us first assume that $G^{\mathcal{L}}$ has more than one option, so the end-reversible option G^L must be essential. This means that there is a Left-end X such that $\{\varnothing \mid W_n\} + X$ is the only winning move for left in $G + X$. As ends, both W_n and X are invertible, and so we can add $\overline{W}_n + \overline{X}$ to both sides of $0 \geqslant H + W_n$ to see $\overline{W}_n + \overline{X} \geqslant H + \overline{X}$. Now, Right wins first on $\overline{W}_n + \overline{X}$ (Right-end), so Right must have a good first move on $H + \overline{X}$. But then Left would have a good first move on $\overline{H}+X$, which contradicts that G^L is essential in G. Thus, H must have an end-reversible Right option after all, say $\{\varnothing \mid W_{n'}\}$, and since G and H are reduced it must be that $n = n'$.

All that remains is the case where $G^L = \{\varnothing \mid W_n\} \in G^{\mathcal{L}}$ is the only Left option. If so, if $H^{\mathcal{R}}$ is not empty, the proof follows as above. If $H^{\mathcal{R}} = \varnothing$, by Theorem 22, $G^{\mathcal{L}}$ should be empty, which is a contradiction.

We have seen that each G^L has a corresponding $-G^L$ in the set of Right options of H. This finishes the proof. □

6. Related/future work

In a related paper in this volume, we prove that there are infinitely many (misère) absolute universes [10]. Problem: establish infinitely many reduction theorems, specifically, explain how end-reversibility generalizes to those settings.

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