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Infinitely many absolute universes

URBAN LARSSON, RICHARD J. NOWAKOWSKI
AND CARLOS P. SANTOS

Absolute combinatorial game theory was recently developed as a unifying tool for constructive/local game comparison. The theory concerns *parental universes* (a.k.a. dicot closure) of combinatorial games; standard closure properties are satisfied and each pair of nonempty sets of forms of the universe makes a form of the universe. Here we prove that there are an infinite number of absolute misère universes, by recursively expanding the dicot misère universe and the dead-ending universe. On the other hand, we prove that normal-play has exactly two absolute universes, namely the full space, and the universe of all-small games.

1. Introduction

Absolute combinatorial game theory was recently developed as a unifying tool for constructive/local game comparison [4; 7]. A small but significant number of instances from the literature motivated the theory. Here, we demonstrate that the absolute theory has infinitely many applications.

Recall the definition of partial order for short combinatorial games: $G \geqslant H$ if, for all games X, if Left wins $H+X$, then she wins $G+X$, as first and second player respectively. One of the most celebrated theorems in combinatorial game theory states that, under the normal-play convention, $G \geqslant H$ if and only if player Left wins $G-H$ playing second.

Under misère play, due to loss of group structure, comparison results are more intricate. But recent works find constructive methods in the sense that only the games G and H, together with some proviso, are required. Since equivalence classes are small in full misère,[1] one often turns towards restricted/modular misère, to find more order and algebraic structure. This naturally requires closure properties of standard game operators, such as taking options, addition and conjugation. Any set of games satisfying the closure properties is a *universe* of games.

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[1] To begin with, the zero-class is trivial, and then the standard reduction techniques lose most of their power.

In a recent breakthrough the authors were able to prove a general constructive comparison result, given another natural closure property, called *parentality*, that encompasses not only normal- and misère- play, but also scoring play. A universe is parental if, for any pair of nonempty subsets of games A and B in the universe, the game $\{A \mid B\}$ belongs to the universe. Universes that are parental and "outcome saturated" are called *absolute*, and the absolute theory applies. It turns out that outcome saturation is a consequence of parentality. Therefore the single relevant property to test whether a universe fits under the absolute umbrella is parentality. Parantality is sometimes called *dicot closure*.

Typically, a restriction of a convention concerns only the *game form*, and not the winning/ending convention. The most famous restriction is "impartial" which means that, at each stage of play, both players have the same options. However, the impartial universe is not parental. Another standard restriction is *dicot*, which means that, at each stage of play, either both players or neither player can move. The universe of all dicots is parental (irrespective of winning/ending convention). Another restriction, with wide applications, is the universe of *dead-ending* games: if, at some stage of play, player Left cannot move, then, after any sequence of Right moves, Left still cannot move. It is not possible that Right opens up any more Left options. And vice versa. All placement rulesets have this property. The dead-ending universe contains the dicotic universe, and it is parental.

Siegel [10; 11] studies full misère, Dorbec et al. [1] study dicotic misère and Milley et al. study misère dead-ending [5; 8; 6]. Milley et al. [9] give a brilliant survey on this development. Analogous parental scoring play universes, which fit under the absolute umbrella, have been studied by Stewart (full scoring play) [12; 13], Ettinger (dicot scoring play) [2], and the authors (guaranteed scoring play) [3; 4].

Thus, we already know that absolute theory embraces several common situations. However, for a theory to be really useful, one would hope that it applies to an infinite set of instances. In this paper we prove that there are an infinite number of parental universes, with respect to distinguishability. We do this by recursively expanding the misère dicots and dead-ending games respectively. We prove that, in contrast, normal-play has only two parental universes.

In Section 2 we recall some absolute terminology and basic combinatorial games' definitions. In Section 2.1 we define the notion of parentality. In Section 3 we discuss the dicot kernel of game forms and we define the (parental) closure of a given set of game forms. Section 4 is devoted to the classic conventions. We start with normal-play in Section 4.1, and we finish off in Section 4.2 with the infinitely many misère absolute universes. In Section 4.2.1 we study universes between the dicots and the dead-ending universes, and in Section 4.2.2 we study universes that enclose the dead-ending universe. In Section 5, we mention some open problems.

2. Some absolute terminology

The theory is built through a totally ordered, additive abelian, group of *adorns* $\mathcal{A}$. A terminal position is of the form $\{\varnothing^{\ell} \mid \varnothing^{r}\}$ where $\ell, r \in \mathcal{A}$. In general, if G is a game with no Left options then we write $G^{\mathcal{L}} = \varnothing^{\ell}$ for some $\ell \in \mathcal{A}$ and if Right has no options, we write $G^{\mathcal{R}} = \varnothing^{r}$ for some $r \in \mathcal{A}$.

We refer to $\varnothing^{a}$ as an *atom* and $a \in \mathcal{A}$ as the *adorn*. A position in which Left (Right) does not have a move is called a *Left-atom* (*Right-atom*). A *purely atomic* position is both Left- and Right-atomic. Let $\mathbf{0} = \{\varnothing^{0} \mid \varnothing^{0}\}$, where 0 is the identity of $\mathcal{A}$. In this paper, whenever $\mathcal{A} = \{0\}$, we will omit the adorn 0, and write simply $\varnothing = \varnothing^{0}$.

A *free space* recursively builds all possible game forms, given a group of adorns. Let $\mathcal{A}$ be a group of adorns and let $\Omega_0 = \{\{\varnothing^{\ell} \mid \varnothing^{r}\} \mid \ell, r \in \mathcal{A}\}$. For $n > 0$, Ω_n is the set of all game forms with finite sets of options in Ω_{n-1}, including game forms that are Left- and/or Right-atomic, and the set of game forms of *birthday* n is $\Omega_n \setminus \Omega_{n-1}$. Let $\Omega = \bigcup_{n \geqslant 0} \Omega_n$. Then $\Omega = (\Omega, \mathcal{A})$ is a *free space* of game forms.

By some abuse of terminology, we sometimes refer to the game form $G \in \Omega$ simply as a game, although there is not yet any incentive to play, or compare G with other games in the same free space. An evaluation function $\nu : \mathcal{A} \times \mathcal{A} \to V$ takes care of the first part; here we are concerned exclusively with $A = 0$ and $V = \{-1, +1\}$. The function ν distinguishes normal-play from misère: $\nu_L(0) = -1,\ \nu_R(0) = +1$ in normal-play, whereas in misère we have $\nu_L(0) = +1$, $\nu_R(0) = -1$.

The notion of a *game space* $(\Omega, \mathcal{A}, V, \nu)$ includes the free space together with E and ν. Usually, we write $G \in \Omega$, while assuming an underlying given game space quadruple.

In a *disjunctive sum* of games, the current player picks exactly one game component and plays in it, by leaving any remaining components intact. Here, and elsewhere, an expression of the type $G^{\mathcal{L}} + H$ denotes the set of games of the form $G^{L} + H,\ G^{L} \in G^{\mathcal{L}}$, and this notion is only defined when $G^{\mathcal{L}}$ is nonatomic.

Consider a free space $(\Omega, \mathcal{A})$, and a pair of game forms $G, H \in (\Omega, \mathcal{A})$. The disjunctive sum of G and H is given by

$$G+H = \begin{cases} \{\varnothing^{\ell_1+\ell_2} \mid \varnothing^{r_1+r_2}\} & \text{if } G=\{\varnothing^{\ell_1} \mid \varnothing^{r_1}\} \text{ and } H=\{\varnothing^{\ell_2} \mid \varnothing^{r_2}\}, \\ \{\varnothing^{\ell_1+\ell_2} \mid G^{\mathcal{R}}+H, G+H^{\mathcal{R}}\} & \text{if } G=\{\varnothing^{\ell_1} \mid G^{\mathcal{R}}\}, H=\{\varnothing^{\ell_2} \mid H^{\mathcal{R}}\}, \\ & \quad \text{and at least one of } G^{\mathcal{R}} \text{ and } H^{\mathcal{R}} \text{ nonatomic}, \\ \{G^{\mathcal{L}}+H, G+H^{\mathcal{L}} \mid \varnothing^{r_1+r_2}\} & \text{if } G=\{G^{\mathcal{L}} \mid \varnothing^{r_1}\}, H=\{H^{\mathcal{L}} \mid \varnothing^{r_2}\}, \\ & \quad \text{and at least one of } G^{\mathcal{L}} \text{ and } H^{\mathcal{L}} \text{ nonatomic}, \\ \{G^{\mathcal{L}}+H, G+H^{\mathcal{L}} \mid G^{\mathcal{R}}+H, G+H^{\mathcal{R}}\} & \text{otherwise.} \end{cases}$$

The *conjugate* of a given game switches roles of the players. The conjugate of $G \in \Omega$ is

$$\overleftrightarrow{G} = \begin{cases} \{\varnothing^{-r} \mid \varnothing^{-\ell}\} & \text{if } G = \{\varnothing^{\ell} \mid \varnothing^{r}\} \text{ for some } \ell, r \in \mathcal{A}, \\ \{\overleftrightarrow{G^{\mathcal{R}}} \mid \varnothing^{-\ell}\} & \text{if } G = \{\varnothing^{\ell} \mid G^{\mathcal{R}}\} \text{ for some } \ell \in \mathcal{A}, \\ \{\varnothing^{-r} \mid \overleftrightarrow{G^{\mathcal{L}}}\} & \text{if } G = \{G^{\mathcal{L}} \mid \varnothing^{r}\} \text{ for some } r \in \mathcal{A}, \\ \{\overleftrightarrow{G^{\mathcal{R}}} \mid \overleftrightarrow{G^{\mathcal{L}}}\} & \text{otherwise,} \end{cases}$$

where $\overleftrightarrow{G^{\mathcal{L}}}$ denotes the set of game forms $\overleftrightarrow{G^{L}}$, for $G^{L} \in G^{\mathcal{L}}$, and similarly for $G^{\mathcal{R}}$.

2.1. ***Parental universes of game forms.*** A set of game forms $\mathbb{U} \subseteq (\Omega, \mathcal{A})$ is well behaved, and forms a *universe* of games, if it satisfies the following closure properties on the basic definitions.

(a) $\Omega_0 \subseteq \mathbb{U}$.

(b) $\mathbb{U}$ is closed for disjunctive sum: if $G, H \in \mathbb{U}$, then $G + H \in \mathbb{U}$.

(c) $\mathbb{U}$ is closed for conjugation: if $G \in \mathbb{U}$, then $\overleftrightarrow{G} \in \mathbb{U}$.

(d) $\mathbb{U}$ is hereditary closed: if $G \in \mathbb{U}$ and G' is an option of G, then $G' \in \mathbb{U}$.

A set $S \subset (\Omega, \mathcal{A})$ of game forms is *parental* if, for any pair of finite nonempty sets of game forms $\mathcal{G}, \mathcal{H} \subset S$, $\{\mathcal{G} \mid \mathcal{H}\} \in S$. If a parental set $S = \mathbb{U}$ is a universe, then $\mathbb{U}$ is a *parental universe*.

There are some classical parental universes based on the following definitions.

Definition 1 (Dicot). A game form G is a *dicot* if $G^{\mathcal{L}} = \varnothing \iff G^{\mathcal{R}} = \varnothing$, and all options are dicots. The parental universe of dicotic forms is designated by $\mathbb{D}$.

In case of a nonscoring universe, for simplicity, we may drop the adorns and we often call empty sets of options "ends" instead of atoms. The existing dead-ending universes are typically nonscoring.

Definition 2 (Dead-ending). A *dead-ending* form has the property that if, at some stage of play, a player cannot move, then they cannot move after any sequence of moves by the other player. A dead-ending form G is called a *Left-end* if $G^{\mathcal{L}} = \varnothing$ and a *Right-end* if $G^{\mathcal{R}} = \varnothing$; thus, the zero game $\{\varnothing \mid \varnothing\}$ is both a Left-end and a Right-end. A game is an *end* if it is a Left-end or a Right-end. The parental universe of dead-ending forms is designated by $\mathbb{E}$.

3. Closure of parental game forms

Problem 3. For a given free space Ω, what is the "smallest" parental universe?

The notion of "smallest" requires a definition.

Definition 4 (Kernel). Consider universes of games $\mathbb{U}' \subseteq \mathbb{U} \subset (\Omega, \mathcal{A})$, where $\mathbb{U}$ is parental. Then $\mathbb{U}$ is the kernel of Ω if it satisfies: if $\mathbb{U}'$ is a parental universe, then $\mathbb{U}' = \mathbb{U}$.

It turns out that the answer to Problem 3 does not depend on the enclosing game space. For any given game space, the smallest parental universe is the dicot universe.

Theorem 5 (Dicot Forms). *Consider a free space* $(\Omega, \mathcal{A})$, *and a universe* $\mathbb{U} \subseteq (\Omega, \mathcal{A})$. *If* $\mathbb{U}$ *is parental then all dicot forms of* Ω *belong to* $\mathbb{U}$.

Proof. Suppose that there is a dicot form $G \notin \mathbb{U}$. Assume that G is a simplest form in such conditions. We cannot have $G \in \Omega_0$ since, by definition of universe, $\Omega_0 \subseteq \mathbb{U}$, contradicting the fact $G \notin \mathbb{U}$. Therefore, we have $G^{\mathcal{L}} \neq \varnothing$ and $G^{\mathcal{R}} \neq \varnothing$. But, by definition of dicotic form, the elements of $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ are dicotic and, by the smallest rank assumption, those elements belong to $\mathbb{U}$. Therefore, by parentality, $G \in \mathbb{U}$; that is again a contradiction. All dicot forms of Ω belong to $\mathbb{U}$. □

Corollary 6 (Dicot Kernel). *The kernel of a free space is its dicotic universe.*

Proof. Combine Definitions 4 and 1 with Theorem 5. □

Let us define three set operators on game forms.

Definition 7 (Set Operators). Consider the set operators on a set of game forms $S \subseteq (\Omega, \mathcal{A})$:

$$\mathcal{D}(S) = \{G + H : G, H \in S\} \quad \text{(disjunctive sum operator);}$$
$$\mathcal{C}(S) = \{\overleftrightarrow{G} : G \in S\} \quad \text{(conjugation operator);}$$
$$\mathcal{P}(S) = \{\{X \mid Y\} : X, Y \subseteq S, |X| > 0, |Y| > 0\} \quad \text{(parental operator).}$$

Definition 8 (Closure of Game Forms). Let $S \subseteq (\Omega, \mathcal{A})$ be a hereditary closed set of game forms, and consider the following recursion:

$$S_0 = S \cup \Omega_0 \quad \text{(day 0);}$$
$$S_n = \mathcal{D}\big(\textstyle\bigcup_{i=0}^{n-1} S_i\big) \cup \mathcal{C}\big(\bigcup_{i=0}^{n-1} S_i\big) \cup \mathcal{P}\big(\bigcup_{i=0}^{n-1} S_i\big) \quad \text{(day } n > 0\text{).}$$

The closure of S is the set of game forms $\overline{S} = \bigcup_{i=0}^{\infty} S_i$.

Theorem 9. *Let* $S \subseteq (\Omega, \mathcal{A})$ *be a hereditary closed set of game forms. The set* $\overline{S}$ *is a parental universe.*

Proof. The hereditary closure follows by expanding day 0, taking into account that S is hereditary closed. The closure for disjunctive sum, closure for conjugation, and parentality follows by Definition 8. □

4. The classic winning conventions

Problem 10. How many parental universes are there in the classical conventions, normal- and misère-play respectively?

We use the standard distinguishing operator for games.

Fix a convention, normal- or misère-play. The perfect play outcome of a game G is:

- $o(G) = \mathscr{L}$ if Left wins independently of who starts.
- $o(G) = \mathscr{R}$ if Right wins independently of who starts.
- $o(G) = \mathscr{N}$ if the next player wins independently of who starts.
- $o(G) = \mathscr{P}$ if the previous player wins independently of who starts.

In the classic conventions the set of adorns is $\mathcal{A} = \{0\}$, and we may make the identification $\Omega = (\Omega, \mathcal{A})$, and omit adorning empty sets of games.

Definition 11 (Distinguishability). Let $\mathbb{U} \in \Omega$ be a universe. A pair of games $G, H \in \mathbb{U}$ are *distinguishable* modulo $\mathbb{U}$ if there is $X \in \mathbb{U}$ such that $o(G+X) \neq o(H+X)$. If G and H are indistinguishable modulo $\mathbb{U}$, we say they are *equal* and write $G = H \pmod{\mathbb{U}}$.

Definition 12 (Proper Extension). Let $\mathbb{U}, \mathbb{U}' \in \Omega$ be two universes such that $\mathbb{U} \subseteq \mathbb{U}'$. We say that $\mathbb{U}'$ is a *proper extension* of $\mathbb{U}$ if there is $G \in \mathbb{U}'$, such that, for all $H \in \mathbb{U}$, G and H are distinguishable modulo $\mathbb{U}'$. If $\mathbb{U}'$ is not a proper extension of $\mathbb{U}$, we write $\mathbb{U} \simeq \mathbb{U}'$, meaning that the universes $\mathbb{U}$ and $\mathbb{U}'$ are the same with respect to distinguishability.

Observation 13. Note that universes may differ with respect to game forms, but be the same with respect to distinguishability.

Returning to Problem 10, the question that arises concerns how to obtain proper extensions of the Dicot Kernel.

4.1. *Normal-play.*

Let us first analyze the game space Ω in normal-play.

Theorem 14. *Consider a normal-play parental universe* $\mathbb{U} \subseteq \Omega$. *If* $\mathbb{U}$ *contains a nondicot form that is distinguished from zero modulo* Ω, *then* $\mathbb{U} \simeq \Omega$.

Proof. In Definition 12, take $\mathbb{U}' = \Omega$; all comparisons will concern the full game space of normal-play games. We assume the existence of a nondicot form in $\mathbb{U}$, and hence it has a follower G that is Left-atomic or Right-atomic, but not both. Without loss of generality, suppose that G is Right-atomic and not Left-atomic. Then Left wins playing second, so $G \succcurlyeq \mathbf{0}$. But by assumption, $G \neq \mathbf{0}$, so $G \succ \mathbf{0}$.

We claim that $G = \boldsymbol{k}$ is a positive integer. Namely, $k = \min\{\ell : \boldsymbol{\ell} \succ G^L\}$, for all Left options of G, and where ℓ a positive integer. Suppose first that Right starts the game $G-\boldsymbol{k}$. Since G is Right-atomic, the only option is to $G-\boldsymbol{k}+\mathbf{1}$. By minimality of k, there is a left option G^L such that $G^L \not\succcurlyeq \boldsymbol{k}-\mathbf{1}$. Hence, Left wins when Right starts in $G^L-\boldsymbol{k}+\mathbf{1}$. Next, suppose that Left starts $G-\boldsymbol{k}$. By definition, for all Left options, $G^L-\boldsymbol{k} \prec 0$. Hence, Left loses.

We have the following possibilities:

(1) If $k=1$, then there is a form in $\mathbb{U}$ equal to $\mathbf{1}$.

(2) If $k>1$, then, since by the Simplicity Theorem, $\{\mathbf{0}\,|\,\boldsymbol{k}\}=\mathbf{1}$ and due to the fact that $\mathbb{U}$ is parental, there is a form in $\mathbb{U}$ equal to $\mathbf{1}$.

In both cases, there is a form in $\mathbb{U}$ equal to $\mathbf{1}$. But then, since $\mathbb{U}$ is closed for conjugation and for disjunctive sum, every integer has a form in $\mathbb{U}$.

Now, let $G=\{G^{\mathcal{L}}\,|\,G^{\mathcal{R}}\}$ be an arbitrary nonatomic canonical form under normal-play. Consider the form G' obtained by replacing all the integers of $G^{\mathcal{L}}\cup G^{\mathcal{R}}$ by equal forms of $\mathbb{U}$, whose existence we have already guaranteed, and by replacing all the other forms of $G^{\mathcal{L}}\cup G^{\mathcal{R}}$ by equal forms of $\mathbb{U}$, whose existence is assumed by induction. On the one hand, by construction, $G' =_{\mathbb{N}*} G$ since the two forms have equivalent sets of options modulo Ω. On the other hand, by parentality, $G'\in\mathbb{U}$. We conclude that each short game value has a form in $\mathbb{U}$, and hence $\mathbb{U}\simeq\Omega$. □

Thus, we have the following result.

Corollary 15. *The dicot* (*all-small*) *universe is the single proper absolute universe under normal-play.*

Proof. By Theorem 14, we know that an absolute universe is no smaller than the universe of all dicots. By Theorem 5 we know that adjoining any nondicot form to the dicots produces the full game space Ω. Hence the universe of dicots satisfies $\mathbb{U}\subseteq\Omega$; note that if $X\in\mathbb{U}$, then $X+\mathbf{1}\in\mathscr{L}$, but $-\mathbf{1}+\mathbf{1}\in\mathscr{P}$, so indeed $\mathbb{U}\subset\Omega$ is a proper extension. □

4.2. ***Misère-play.*** What about misère-play? Theorem 19 is the first step to the answer. We will give two different proofs, first a classical variation, by finding explicit distinguishing games, and then we use the modern approach, by absolute theory developed in [7].

We separate the outcome function in its two parts. The possible misère-play results are L (Left wins) and R (Right wins); by convention, they are totally ordered with L > R. The *Left-outcome* and *Right-outcome*, in optimal play from both players, of a misère-play game G are

$$o_L(G)=\begin{cases}\mathrm{L} & \text{if } G^{\mathcal{L}}=\varnothing,\\ \max o_R(G^L) & \text{otherwise,}\end{cases}\qquad o_R(G)=\begin{cases}\mathrm{R} & \text{if } G^{\mathcal{R}}=\varnothing,\\ \min o_L(G^R) & \text{otherwise,}\end{cases}$$

respectively.

In [7], we proved the following result.

Theorem 16 (Basic Order [7]). *Suppose $\mathbb{U} \subseteq (\Omega, \mathcal{A})$ is an absolute universe of combinatorial games and let $G, H \in \mathbb{U}$. Then $G \succcurlyeq H$ if and only if the following two conditions hold*:

Proviso: *$o_L(G+X) \geqslant o_L(H+X)$ for all Left-atomic $X \in \mathbb{U}$ and $o_R(G+X) \geqslant o_R(H+X)$ for all Right-atomic $X \in \mathbb{U}$.*

Maintenance: *For all G^R, there is an H^R such that $G^R \succcurlyeq H^R$, or there is a G^{RL} such that $G^{RL} \succcurlyeq H$, and for all H^L, there is a G^L such that $G^L \succcurlyeq H^L$, or there is an H^{LR} such that $G \succcurlyeq H^{LR}$.*

A consequence of this is the following useful result. Its usefulness is due to that normal-play theory is well known, and so it can be easy to see when the normal-play inequality fails to hold. If it fails to hold, then the inequality fails in any (!) absolute universe. And indeed, in this paper we prove that there are an infinite number of such universes.

Corollary 17 (Normal-play Order-Preserving [7]). *Let $G, H \in \mathbb{U}$, an absolute universe. If $G \succcurlyeq_{\mathbb{U}} H$ then $G \succcurlyeq_{\mathbb{N}^*} H$.*

The next results demonstrate that the misère-play convention has a richer structure than normal-play.

4.2.1. *Absolute universes strictly between $\mathbb{D}$ and $\mathbb{E}$.* First, we present two proofs for the existence of an absolute universe strictly between $\mathbb{D}$ and $\mathbb{E}$. We will have use for a concept closely related to the conjugate.

Definition 18 (Adjoint [10]). Consider a misère-play universe $\mathbb{U}$ and a game $G \in \mathbb{U}$. Then G° is the adjoint of G, where

$$G^\circ = \begin{cases} \{\mathbf{0} \mid \mathbf{0}\} & \text{if } G = \mathbf{0}, \\ \{G^{\mathcal{R}\circ} \mid \mathbf{0}\} & \text{if } |G^{\mathcal{R}}| > 0 \text{ and } G^{\mathcal{L}} = \varnothing, \\ \{\mathbf{0} \mid G^{\mathcal{L}\circ}\} & \text{if } |G^{\mathcal{L}}| > 0 \text{ and } G^{\mathcal{R}} = \varnothing, \\ \{G^{\mathcal{R}\circ} \mid G^{\mathcal{L}\circ}\} & \text{otherwise,} \end{cases}$$

and where $\circ$ applied to a set operates on its elements.

It is well known that, under the misère-play convention, $G + G^\circ \in \mathcal{P}$.

Theorem 19. *Let $S = \mathbb{D} \cup \{\mathbf{1}\}$, where $\mathbf{1} = \{\mathbf{0} \mid \varnothing\}$. Then $\overline{S}$ is a proper extension of $\mathbb{D}$ and $\mathbb{E}$ is a proper extension of $\overline{S}$.*

Proof 1. We observe that, by definition of closure of S, the forms $\mathbf{0}$, $\mathbf{1}$, $\mathbf{1}+\mathbf{1}, \ldots$ are the only Right-atomic games in $\overline{S}$.

First, we prove that each Right-atomic game is distinguished from any dicot game. This follows by the Maintenance, specifically by Corollary 17, because a strictly Right-atomic game is a positive number, and hence greater than any dicot in normal-play. Therefore, modulo $\bar{S}$, a Right-atomic game different than zero cannot be equal to a dicot. Hence $\bar{S}$ is a proper extension of $\mathbb{D}$.

Let us see that $\{\mathbf{0}, \mathbf{1} \mid \varnothing\}$ differs from all Right-atomic games of $\bar{S}$ modulo $\mathbb{E}$.

- The game $\{\mathbf{0}, \mathbf{1} \mid \varnothing\}$ is not equivalent to $\mathbf{0}$, because
$$o(\{\mathbf{0}, \mathbf{1} \mid \varnothing\}) = \mathscr{R} \quad \text{and} \quad o(\mathbf{0}) = \mathscr{N}.$$
- The game $\{\mathbf{0}, \mathbf{1} \mid \varnothing\}$ is not equivalent to $\mathbf{1}$, because
$$o(\{\mathbf{0}, \mathbf{1} \mid \varnothing\} + \{\mathbf{0} \mid *\}) = \mathscr{L} \quad \text{and} \quad o(\mathbf{1} + \{\mathbf{0} \mid *\}) = \mathscr{P}.$$
- The game $\{\mathbf{0}, \mathbf{1} \mid \varnothing\}$ is not equivalent to $\mathbf{1}+\mathbf{1}$, because
$$o(\{\mathbf{0}, \mathbf{1} \mid \,\} + \{\mathbf{0} \mid *\}) = \mathscr{L} \quad \text{and} \quad o(\mathbf{1} + \mathbf{1} + \{\mathbf{0} \mid *\}) = \mathscr{N}.$$
- The game $\{\mathbf{0}, \mathbf{1} \mid \varnothing\}$ is not equivalent to $\mathbf{1}+\mathbf{1}+\cdots+\mathbf{1}$ (more than two summands) because
$$o(\{\mathbf{0}, \mathbf{1} \mid \varnothing\} + \{\mathbf{0} \mid *\}) = \mathscr{L} \quad \text{and} \quad o(\mathbf{1} + \mathbf{1} + \cdots + \mathbf{1} + \{\mathbf{0} \mid *\}) = \mathscr{R}.$$

Now, let us consider a game G not Right-atomic. Because $o(\{\mathbf{0}, \mathbf{1} \mid \varnothing\}) = \mathscr{R}$, in case of equality, this forces $o(G) = \mathscr{R}$. Let
$$X = \big\{\{\mathbf{0} \mid \text{all adjoints of followers of } G\} \mid *, \{\mathbf{0} \mid *\}\big\}.$$
Playing first, Right wins $G + X$ by choosing the local misère-play winning line of G. This follows, since Left has only losing moves in X. Namely, whenever Left plays in X, Right responds to the appropriate adjoint-follower of G. In case of a final Left move to $\mathbf{0}+X$, Right answers $*$; in case of a final Left move to $\mathbf{1}+X$, Right answers $\mathbf{1}+\{\mathbf{0} \mid *\}$; in case of a final Left move to $\mathbf{1}+\cdots+\mathbf{1}+X$ (at least 2 summands), Right answers $\mathbf{1}+\cdots+\mathbf{1}+*$. On the other hand, playing first, Right loses $\{\mathbf{0}, \mathbf{1} \mid \varnothing\}+X$. So $\{\mathbf{0}, \mathbf{1} \mid \varnothing\} \neq G$. This proves $\mathbb{E}$ is a proper extension of $\bar{S}$. □

Proof 2. The first two paragraphs are the same as in the first proof.

Let us use Theorem 16 to show that $\{\mathbf{0}, \mathbf{1} \mid \varnothing\}$ differs from all Right-atomic games of $\bar{S}$ modulo $\mathbb{E}$. We get extensive use of Corollary 17. In the third item, we cannot use it, but instead Maintenance gives a contradiction in $\mathbb{E}$, and moreover, the Proviso gives another contradiction.

- The game $\{\mathbf{0}, \mathbf{1} \mid \varnothing\}$ is not equivalent to $\mathbf{0}$, because $\{\mathbf{0}, \mathbf{1} \mid \varnothing\} =_{\mathbb{N}*} \mathbf{2} \succ_{\mathbb{N}*} \mathbf{0}$.
- The game $\{\mathbf{0}, \mathbf{1} \mid \varnothing\}$ is not equivalent to $\mathbf{1}$, because $\{\mathbf{0}, \mathbf{1} \mid \varnothing\} =_{\mathbb{N}*} \mathbf{2} \succ_{\mathbb{N}*} \mathbf{1}$.
- The game $\{\mathbf{0}, \mathbf{1} \mid \varnothing\}$ is not equivalent $\mathbf{1}+\mathbf{1}$, because the Proviso is not satisfied with $\mathrm{L} = o_L(\{\mathbf{0}, \mathbf{1} \mid \varnothing\} - \mathbf{1}) > o_L(\mathbf{1}+\mathbf{1}-\mathbf{1}) = \mathrm{R}$. In fact, Maintenance is not satisfied, in spite of the normal-play equality. Namely, take $H = \{\mathbf{0}, \mathbf{1} \mid \varnothing\}$ and $G = \mathbf{1}+\mathbf{1}$. Choose $H^L = \mathbf{0}$. Then there is no G^L such that $G^L \succcurlyeq H^L$, because $\mathbf{1} \not\succcurlyeq \mathbf{0}$ in misère play, and there is no H^{LR} such that $G \succcurlyeq H^{LR}$.

- The game $\{\mathbf{0},\mathbf{1}\mid\varnothing\}$ is not equivalent to $\mathbf{1}+\mathbf{1}+\cdots+\mathbf{1}$ (more than two summands) because $\{\mathbf{0},\mathbf{1}\mid\varnothing\} =_{\mathbb{N}^*} 2 \prec_{\mathbb{N}^*} 3 \preccurlyeq \mathbf{1}+\mathbf{1}+\cdots+\mathbf{1}$.

Moreover, by normal-play theory, and Corollary 17, one may deduce that $\{\mathbf{0},\mathbf{1}\mid\varnothing\}\neq G$ whenever $G^{\mathcal{R}}\neq\varnothing$ and $G\neq_{\mathbb{N}^*}\mathbf{2}$. Otherwise, an argument similar to that used in the first proof is needed. □

Theorem 19 shows that the closure of $S=\mathbb{D}\cup\{\mathbf{1}\}$ is larger than the dicot universe but smaller than dead-ending universe. By extending this approach, we show that there are an infinite number of extensions strictly between the dicot and dead-ending universes.

Definition 20. Let n be a nonnegative integer. Then n *moves* for Left are defined recursively in the following way:

$$\mathbf{0}=\{\varnothing\mid\varnothing\} \qquad \text{and} \qquad \boldsymbol{n}=\{\boldsymbol{n}-\mathbf{1}\mid\varnothing\} \quad \text{if } n\geqslant 1.$$

The form $-\boldsymbol{n}$, n *moves* for Right, is defined recursively in a similar way.

Definition 21. Let n be a nonnegative integer. Then n *controlled moves* for Left are defined recursively in the following way:

$$\hat{0}=\mathbf{0} \qquad \text{and} \qquad \widehat{n}=\{\mathbf{0},\mathbf{1},\mathbf{2},\ldots,\boldsymbol{n}-1\mid\varnothing\} \quad \text{if } n\geqslant 1.$$

The form $\widehat{-n}$, n *controlled moves* for Right, is defined recursively in a similar way.

Notice that in normal-play, we don't need to distinguish moves from controlled moves. However, in misère-play, these forms are not equal. We will study the closure of the following universes.

Definition 22. For $n\geqslant 1$, let $S_n=\mathbb{D}\cup\{\hat{1},\ldots,\widehat{n+1}\}$.

Thus $\overline{S}_0$ is the universe studied in Theorem 19. The universes $\overline{S}_n$ are parental, since they are extensions of the dicot universe.

Definition 23. Recursively, let $\circledast_n$ be the game form

$$\circledast_0=* \qquad \text{and} \qquad \circledast_n=\{0\mid\circledast_{n-1}\} \quad \text{if } n>0.$$

Observation 24. Under the normal-play convention, these game forms are the canonical forms of the games $*$, $\uparrow$, $\Uparrow*$, $3.\!\uparrow$, $4.\!\uparrow*$, ...

Lemma 25. *Consider misère-play, with $n\geqslant 1$. Left wins $\boldsymbol{n}+\circledast_k$ playing second if and only if $n=k$.*

Proof. If k is zero, the result is trivial.

Suppose $k>0$ and $n=k$. After a Right's first move from $\boldsymbol{k}+\circledast_k$ to $\boldsymbol{k}+\circledast_{k-1}$, Left answers $\boldsymbol{k}-\mathbf{1}+\circledast_{k-1}$ and wins by induction.

Suppose $k>0$ and $n\neq k$. After a Right's first move from $\boldsymbol{n}+\circledast_k$ to $\boldsymbol{n}+\circledast_{k-1}$, there are two possibilities. If Left replies from $\boldsymbol{n}+\circledast_{k-1}$ to $\boldsymbol{n}$, Right wins since he has no more moves. If Left replies from $\boldsymbol{n}+\circledast_{k-1}$ to $\boldsymbol{n}-\mathbf{1}+\circledast_{k-1}$, Right answers $\boldsymbol{n}-\mathbf{1}+\circledast_{k-2}$ and, by induction, wins, or, if $k-1=0$, Right moves to $\boldsymbol{n}-\mathbf{1}$ and also wins, since it leaves the last move to be made by Left. In this second case, observe that, since $k=1$, $n\geqslant 1$, and $n\neq k$, we must have $n\geqslant 2$. □

The following result relies on equivalence classes in the misère-play convention. We give two proofs; in the first, we find an explicit distinguishing game, and in the second proof, we use absolute theory.

Theorem 26. *There are an infinite number of absolute universes strictly between* $\mathbb{D}$ *and* $\mathbb{E}$*.*

Proof 1. Let n be a nonnegative integer. It suffices to prove that if $G\in\overline{S}_n$ then $G\neq\widehat{n+2}$ mod $\overline{S}_{n+1}$, that is, $\overline{S}_{n+1}$ is a proper extension of $\overline{S}_n$.

We observe that, by definition of closure of S_n, the forms $\widehat{k}$ with $0\leqslant k\leqslant n+1$, and all disjunctive sums of these forms, are the only Right-atomic games in $\overline{S}_n$. First, let us see that $\widehat{n+2}$ does not equal any Right-atomic game of $\overline{S}_n$.

Trivially, $\widehat{n+2}\neq\mathbf{0}$, since $o(\widehat{n+2})=\mathscr{R}$ and $o(\mathbf{0})=\mathscr{N}$.

Let $X=\{*\mid\circledast_0,\dots,\circledast_{n+1}\}$. By Lemma 25, Left wins playing second in $\widehat{n+2}+X$. Namely, Right must play to $\circledast m$, for some $0\leqslant m\leqslant n+1$, and then Left responds with the game $\widehat{m}+\circledast m$.

Consider now an arbitrary Right-atomic disjunctive sum $G_1+\cdots+G_j\in\overline{S}_n$.

If this Right-atomic game has more than one summand, then Left loses $G_1+\cdots+G_j+X$ playing second, because Right goes to $G_1+\cdots+G_j+\circledast_0=G_1+\cdots+G_j+*$. Left loses because she has two or more moves in the Right-atomic summand.

If the disjunctive sum has only one summand, the best possible case for Left is $\widehat{n+1}+X$. (The “hand-tying principle” holds in misère-play for nonempty sets of options.) But, even so, Left loses playing second, because Right chooses $\widehat{n+1}+\circledast_{n+1}$ (Lemma 25). Therefore, in all cases, Left loses playing second in $G_1+\cdots+G_j+X$.

We conclude that $\widehat{n+2}$ is not equal to a Right-atomic game of $\overline{S}_n$ modulo $\overline{S}_{n+1}$.

Suppose now that $o(G)=\mathscr{R}$, and let

$$X=\big\{\{0\mid\text{all adjoints of followers of }G\}\mid\mid\circledast_0,\circledast_1,\dots,\circledast_{n+1}\big\}.$$

Playing first, Right wins $G+X$ by choosing the local misère-play winning line of G. By the adjoint construction, Left loses by playing in X, at any stage. In case of a Left move to $\mathbf{0}+X$, Right answers $\circledast_0$; in case of a Left move to $G_1+\cdots+G_j+X$, with more than one Right-atomic summand, Right answers $G_1+\cdots+G_j+\circledast_0$; in case of a Left move to G_1+X, with one Right-atomic summand, Right answers $G_1+\circledast_{n+1}$. On the other hand, playing first, Right loses $\widehat{n+2}+X$ (Lemma 25). Therefore, $\widehat{n+2}$ is different than G modulo $\bar{S}_{n+1}$, and the proof is finished. □

Proof 2. It suffices to prove that if $G \in \bar{S}_n$ then $G \neq \widehat{n+2}$ mod $\bar{S}_{n+1}$.

We observe that, by definition of closure of S_n, the forms $\widehat{k}$ with $0 \leqslant k \leqslant n+1$ and all disjunctive sums of these forms are the only Right-atomic games in $\bar{S}_n$. First, let us see that $\widehat{n+2}$ does not equal any Right-atomic game G of $\bar{S}_n$.

By Corollary 17, $\widehat{n+2} \neq G$, unless possibly if $G = \sum \widehat{k_i}$, with $\sum k_i = n+2$, because $\widehat{n+2} =_{\mathbb{N}*} n+2$. Thus, it suffices to prove that Maintenance mod $\bar{S}_{n+1}$ gives $G = \sum \widehat{k_i} \neq \widehat{n+2} = H$, whenever $\sum k_i = n+2$. Since $\widehat{n+2} \notin \bar{S}_n$, the disjunctive sum has at least two summands. There is a Left option $H^L = \mathbf{0}$, and for this option there is no G^L such that $H^L \succcurlyeq G^L$. Namely any G^L has at least one summand, $1 \preccurlyeq_{\mathbb{N}*} G^L$, and, so, that violates the normal-play inequality; by Corollary 17, we know that $H^L =_{\mathbb{N}*} 0 \prec_{\mathbb{N}*} 1 \preccurlyeq_{\mathbb{N}*} G^L$ cannot happen.

Moreover, the option $H^L = \mathbf{0}$, does not have a Right option, so we cannot have $H^{LR} \succcurlyeq G$.

We conclude that $\widehat{n+2}$ is not equal to a Right-atomic game of $\bar{S}_n$ modulo $\bar{S}_{n+1}$.

Also, $\widehat{n+2}$ does not equal any G with $G^{\mathcal{R}} \neq \varnothing$ modulo $\bar{S}_{n+1}$; the argument is the same as the proof of Theorem 19. □

4.2.2. *Absolute universes strictly between* $\mathbb{E}$ *and* Ω. First, we present a proof for the existence of an absolute universe strictly between $\mathbb{E}$ and Ω. From now on, for ease, we just present one proof.

Theorem 27. *Let* $S = \mathbb{E} \cup \{\{\varnothing \mid \mathbf{2}\}\}$. *Then* $\bar{S}$ *is a proper extension of* $\mathbb{E}$ *and* Ω *is a proper extension of* $\bar{S}$.

Proof. By definition of closure of S, the forms $\{\varnothing \mid \mathbf{2}\} + \cdots + \{\varnothing \mid \mathbf{2}\} + L$, where L is a Left-end form, are the only Left-atomic games of $\bar{S}$.

First, we prove that, modulo $\bar{S}$, $\{\varnothing \mid \mathbf{2}\}$ is distinguished from any Left-end form G. Let $X = \{-\mathbf{1} \mid \mathbf{0}\}$. This follows because, playing first, Left wins $G+X$ by moving to $G-\mathbf{1}$; her victory is explained by the fact that G is a Left-end. On the other hand, playing first, Left loses $\{\varnothing \mid \mathbf{2}\}+X$; she has to move to $\{\varnothing \mid \mathbf{2}\}-\mathbf{1}$ and Right replies $\mathbf{2}-\mathbf{1}$.

Second, let us consider a game $G \in \mathcal{N}$ that is not a Left-atomic form. Let $X = \{\varnothing \mid \mathbf{2}\} + \cdots + \{\varnothing \mid \mathbf{2}\}$ with a sufficiently large number of copies of $\{\varnothing \mid \mathbf{2}\}$, depending on the rank of G. Left loses playing first in $G+X$; that happens because Right can give a large number of moves to Left, by opening the components after a first move by Left in G. On the other hand, Left wins playing first in $\{\varnothing \mid \mathbf{2}\}+X$; she has no moves. Hence, $\bar{S}$ is a proper extension of $\mathbb{E}$.

Third, consider the $\mathcal{N}$-position $\{\varnothing \mid \mathbf{3}\} \in \Omega$. We prove that, for any $G \in \bar{S}$, the form $\{\varnothing \mid \mathbf{3}\}$ is different than G modulo Ω. Of course, we only need to consider the case $G \in \mathcal{N}$. Let

$$X = \big\{-\mathbf{2}, \{\{-\boldsymbol{n} \mid \varnothing\} \mid \varnothing\} \mid \{\text{all adjoints of followers of } G \mid \mathbf{0}\}\big\},$$

with a sufficiently large n. Playing first, Left loses $\{\varnothing \mid \mathbf{3}\} + X$ since both $\{\varnothing \mid \mathbf{3}\} - \mathbf{2}$ and $\{\varnothing \mid \mathbf{3}\} + \{\{-\boldsymbol{n} \mid \varnothing\} \mid \varnothing\}$ are losing moves; Right replies to $\mathbf{3} - \mathbf{2}$ or to $\mathbf{3} + \{\{-\boldsymbol{n} \mid \varnothing\} \mid \varnothing\}$. On the other hand, Left wins $G + X$ playing first. Since G is an $\mathcal{N}$-position, she can force a Right last move in the G component. That must produce a position like $\{\varnothing \mid \mathbf{2}\} + \cdots + \{\varnothing \mid \mathbf{2}\} + L + X$ with Left to play. There are two possibilities:

(1) Right has two or more consecutive moves in $\{\varnothing|\mathbf{2}\} + \cdots + \{\varnothing \mid \mathbf{2}\} + L$; in that case, Left answers $\{\varnothing \mid \mathbf{2}\} + \cdots + \{\varnothing \mid \mathbf{2}\} + L + \{\{-\boldsymbol{n} \mid \varnothing\} \mid \varnothing\}$ and she is in time to reach $-\boldsymbol{n}$.

(2) $\{\varnothing \mid \mathbf{2}\} + \cdots + \{\varnothing \mid \mathbf{2}\} + L$ is $\{\varnothing \mid \mathbf{2}\}$ or L; in that case, Left answers $\{\varnothing \mid \mathbf{2}\} - \mathbf{2}$ or $L - \mathbf{2}$ (L is a Left-end).

Hence, Ω is a proper extension of $\overline{S}$. □

Observation 28. It is easy to understand why the extension is made with $\{\varnothing \mid \mathbf{2}\}$ and not with $\{\varnothing \mid \mathbf{1}\}$. If Right plays on a disjunctive sum $\{\varnothing \mid \mathbf{2}\} + \cdots + \{\varnothing \mid \mathbf{2}\}$ with a large number of summands, he is able to "give" a large number of moves to Left, and that can be a big advantage under the misère-play convention. On the other hand, if Right plays on a disjunctive sum $\{\varnothing \mid \mathbf{1}\} + \cdots + \{\varnothing \mid \mathbf{1}\}$ with a large number of summands, he cannot "give" a large number of moves to Left, since whenever he opens a component, Left immediately finishes that component. In fact, $\overline{\mathbb{E} \cup \{\{\varnothing \mid \mathbf{1}\}\}}$ is not a proper extension of $\mathbb{E}$; using Theorem 16, it is possible to check that $\{\varnothing \mid \mathbf{1}\}$ is equal to $\mathbf{0}$ modulo $\overline{\mathbb{E} \cup \{\{\varnothing \mid \mathbf{1}\}\}}$.

By extending the previous approach, we show that there are an infinite number of extensions strictly between $\mathbb{E}$ and Ω. We need a preliminary lemma and a definition.

Lemma 29. *Let G be the disjunctive sum $\{\varnothing \mid \boldsymbol{n}_1\} + \cdots + \{\varnothing \mid \boldsymbol{n}_k\}$ where $k \geqslant 2$ and where, for all i, $n_i \geqslant 2$. Let n be a positive integer. If Right, playing first, wins $G - \boldsymbol{n}$, then Right, playing first, wins both*

(1) $G - (\boldsymbol{n} + \mathbf{1}) + *$, *and*

(2) $G + \{-(\boldsymbol{n} + \mathbf{1}) + * \mid \mathbf{0}\}$.

Proof. Case 1: In the game $G - (\boldsymbol{n} + \mathbf{1}) + *$, playing first, Right moves to $G - \boldsymbol{n} + *$. The only available answer for Left is to $G - \boldsymbol{n}$. Against that answer, by hypothesis, Right wins. Observe that, in total, Right makes $k + n + 1$ moves before the $n_1 + \cdots + n_k + 1$ moves available for Left (the summand 1 is the move in the star).

Case 2: In the game $G + \{-(\boldsymbol{n}+\boldsymbol{1}) + * \mid \boldsymbol{0}\}$, Right can move to $\boldsymbol{n}_1 + \{\varnothing \mid \boldsymbol{n}_2\} + \cdots + \{\varnothing \mid \boldsymbol{n}_k\} + \{-(\boldsymbol{n}+\boldsymbol{1}) + * \mid \boldsymbol{0}\}$. A Left move to $(\boldsymbol{n}_1 - \boldsymbol{1}) + \{\varnothing \mid \boldsymbol{n}_2\} + \cdots + \{\varnothing \mid \boldsymbol{n}_k\} + \{-(\boldsymbol{n}+\boldsymbol{1}) + * \mid \boldsymbol{0}\}$ would lose the game since Right can answer with $(\boldsymbol{n}_1 - \boldsymbol{1}) + \{\varnothing \mid \boldsymbol{n}_2\} + \cdots + \{\varnothing \mid \boldsymbol{n}_k\}$, opening all the remaining components after a move that Left still has in the first one. Hence, let us analyze a Left move to $\boldsymbol{n}_1 + \{\varnothing \mid \boldsymbol{n}_2\} + \cdots + \{\varnothing \mid \boldsymbol{n}_k\} - (\boldsymbol{n}+\boldsymbol{1}) + *$. Against that move, Right adopts the strategy of playing consecutively in the component $-(\boldsymbol{n}+\boldsymbol{1})$.

Two things may happen:

(1) Right runs out of moves in that component before Left has to play in the star, he opens the second summand of G and wins.

(2) Left spends all her moves in the components $\boldsymbol{n}_1$ and $*$ before Right runs out of moves in $-(\boldsymbol{n}+\boldsymbol{1})$, Right makes all the moves as in the analysis of $G-(\boldsymbol{n}+\boldsymbol{1})+*$. In total, Right makes $k+n+1$ moves before the $n_1 + \cdots + n_k + 2$ moves for Left (the summand 2 respects to the two Left moves in $\{-(\boldsymbol{n}+\boldsymbol{1}) + * \mid \boldsymbol{0}\}$ and in $*$). This is even one more move for Left than in the case of the analysis of the game $G - (\boldsymbol{n}+\boldsymbol{1}) + *$. Therefore, Right also wins in this second case. □

Definition 30. Let $n \geqslant 2$ be an integer. Then the *Left-hook* of order n is the game form $\widetilde{n} = \{\varnothing \mid \boldsymbol{n}\}$. The *Right-hook* of order n is the game form $\widetilde{-n} = \{-\boldsymbol{n} \mid \varnothing\}$.

Definition 31. For $n \geqslant 2$, let $Z_n = \mathbb{E} \cup \{\widetilde{2}, \ldots, \widetilde{n}\}$.

Theorem 32. *There are an infinite number of absolute universes strictly between* $\mathbb{E}$ *and* Ω.

Proof. It suffices to prove that, for all $n \geqslant 2$, if $G \in \overline{Z_n}$ then $G \not\equiv \widetilde{n+1} \bmod \overline{Z_{n+1}}$. Indeed, this proves that $\overline{Z_{n+1}}$ is a proper extension of $\overline{Z_n}$.

We observe that, by the definition of closure of Z_n (Definition 8), the forms $\widetilde{n}_1 + \cdots + \widetilde{n}_k + L$, with $2 \leqslant n_i \leqslant n$ and a Left-end L, are the only Left-atomic games in $\overline{Z_n}$. First, let us see that $\widetilde{n+1}$ does not equal any Left-atomic game of $\overline{Z_n}$.

If $L \not\cong \{\varnothing|\varnothing\}$ then $\widetilde{n}_1 + \cdots + \widetilde{n}_k + L$ is a negative integer under the normal-play convention and $\widetilde{n+1}$ is equal to zero under the normal-play convention. Due to that, by Corollary 17, the games $\widetilde{n}_1 + \cdots + \widetilde{n}_k + L$ and $\widetilde{n+1}$ are not equal modulo $\overline{Z_{n+1}}$. Hence, suppose that $L \cong \{\varnothing \mid \varnothing\}$, that is, consider only disjunctive sums of the type $\widetilde{n}_1 + \cdots + \widetilde{n}_k$.

If there is only one summand, let $X = -\boldsymbol{n}$. We have that, playing first, Right wins $\widetilde{n+1} + X$ by moving to $\boldsymbol{n}+\boldsymbol{1}-\boldsymbol{n}$, and loses $\widetilde{n}_1 + X$ since $n_1 \leqslant n$. Hence, suppose that there are two or more summands.

Let $X = -\boldsymbol{n}$. If, playing first, Right loses $\widetilde{n}_1 + \cdots + \widetilde{n}_k + X$, then, since Right wins $\widetilde{n+1} + X$, $\widetilde{n}_1 + \cdots + \widetilde{n}_k$ and $\widetilde{n+1}$ are not equal modulo $\overline{Z_{n+1}}$. Therefore, suppose that, playing first, Right wins $\widetilde{n}_1 + \cdots + \widetilde{n}_k - \boldsymbol{n}$. By Lemma 29, Right also wins $\widetilde{n}_1 + \cdots + \widetilde{n}_k + \{-(\boldsymbol{n}+\mathbf{1}) + * \,|\, \mathbf{0}\}$. But, playing first, Right loses $\widetilde{n+1} + \{-(\boldsymbol{n}+\mathbf{1}) + * \,|\, \mathbf{0}\}$; if he moves to $\widetilde{n+1}$, Left runs out of moves and wins; if he moves to $(\boldsymbol{n}+\mathbf{1}) + \{-(\boldsymbol{n}+\mathbf{1}) + * \,|\, \mathbf{0}\}$, Left wins with the move $(\boldsymbol{n}+\mathbf{1}) - (\boldsymbol{n}+\mathbf{1}) + *$ since she can force an extra Right's move in the star. Thus, in this last case, $X = \{-(\boldsymbol{n}+\mathbf{1}) + * \,|\, \mathbf{0}\}$ distinguishes $\widetilde{n}_1 + \cdots + \widetilde{n}_k$ from $\widetilde{n+1}$.

Suppose now that $G \in \mathcal{N}$ has a Left option. Let $X = \widetilde{2} + \cdots + \widetilde{2}$ be a disjunctive sum with a sufficiently large number of copies of $\widetilde{2}$, depending on the rank of G. Left loses playing first in $G + X$; that happens because Right is in time to give a very large number of moves to Left, by opening the components after a first move by Left in G. On the other hand, Left wins playing first in $\widetilde{n+1} + X$; she has no moves. $\overline{Z_{n+1}}$ is a proper extension of $\overline{Z_n}$. □

5. Discussion

Much like in Field Theory, where $\mathbb{Q}$ can be extended by the inclusion of irrationals, giving rise to fields strictly between $\mathbb{Q}$ and $\mathbb{R}$, in Combinatorial Game Theory, the misère dicots, $\mathcal{D}$, can be extended with dead-ends, giving rise to misère universes strictly between $\mathcal{D}$ and $\mathcal{E}$, and $\mathcal{E}$ can be extended with ends (not dead-ends), giving rise to universes strictly between $\mathcal{E}$ and $\mathcal{M}$. This fact alone has theoretical importance. However, there is also practical relevance. For example, all legal positions of DOMINEERING belong to $\mathcal{D}(G)$, where G is the dead-end $\{\,|\,0, -\mathbf{1}\}$. Consequently, an appropriate way to analyze rulesets under the misère-play convention may begin by first choosing a suitable universe among the multitude of universes we have proven to exist here. As a sequel to this paper, it will be important to analyze the algebra of these extensions, intimately related to the ends they contain. It will also be important to study local comparison procedures, as well as the possibility of using canonical forms or alternative approaches, such as selected representatives of reduced forms.

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larsson@iitb.ac.in — *Department of Industrial Engineering and Operations Research, Indian Institute of Technology Bombay, Mumbai, India*

r.nowakowski@dal.ca — *Mathematics and Statistics, Dalhousie University, Halifax, NS, Canada*

cmf.santos@fct.unl.pt — *Center for Mathematics and Applications (NovaMath), Universidade Nova de Lisboa Campus de Caparica, Lisbon, Portugal*