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Affine normal play

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There are many combinatorial games in which a move can terminate the game, such as a checkmate in CHESS. These moves give rise to diverse situations that fall outside the scope of the classical normal play theory (Conway 1976). We introduce an algebraic extension, by including infinities as *terminating games*, to analyze these type of structures. In this work, called *affine normal play*, we analyze the partial order, with respect to disjunctive sum, that results from this extension. We prove that it is possible to compare two affine games using only their forms. Furthermore, affine games can still be reduced, although the reduced forms are not unique. We establish that the classical normal play is order-embedded in the extended structure, constituting its substructure of invertible elements, which also implies the *conjugate property*: "every inverse is the conjugate". Additionally, as in the classical theory, affine games born by day n form a lattice with respect to the partial order of games.

1. Introduction

Combinatorial Game Theory (CGT) is the branch of mathematics that studies games with perfect information and no chance, where two players, Left and Right, take turns making moves. Standard references are [Albert et al. 2007; Berlekamp et al. 1982a; 1982b; Conway 1976; Siegel 2013]. In the early days, Conway [1976] developed a disjunctive sum theory for partizan normal play games, the convention where a player without moves loses, and proved that those games have a group structure, with respect to game addition. This group is here denoted by $\mathbb{Np}$. His theory does not cover well-known situations in game practice,[1] such as the CHESS rules of *checks* and *checkmates*. A checkmate is an

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[1]Here, the term "game practice" refers to recreational play, professional play, and many more situations.

absorbing or *terminating* move that immediately concludes a game, and awards victory to the player who checkmated the other player. At a first sight, one might assume that such an ending could be included into the normal play convention; naïvely the checkmated player cannot move, since their King cannot be rescued. However, when any similar situation appears in a disjunctive sum of games, one has to decide what happens to the other components in the disjunctive sum, if a checkmate appears in one of the components. We argue that the checkmate absorbs all the other components, and the full disjunctive sum is terminated. This cannot happen in the classical normal play convention.

Here we will develop an extension of the normal play disjunctive sum theory that includes checks, checkmates and many more features. This extension will be called *affine normal play*, and denoted by $\mathbb{Np}^\infty$.

The most fundamental issue, when introducing a new class of games, is to understand the partial order relations between its elements, given the standard notions of outcome $o(\cdot)$ and disjunctive sum "+"; for example, two normal play games G and H are equal if $o(G+X)=o(H+X)$ for all games X. Since we study an extension of the classical structure, we must always make sure that the surrounding context explains what the "for all X" means. The domain where X lives must at least include G and H, and the induced partial order may be sensitive to the choice of extent of this domain. However, it is known that in classical normal play, if we, for example, study two impartial games G and H, their order relation does not depend on the extent of "for all X", for example, whether it is "impartial ", "all-small" or "partizan" games. As we will see, the extended affine structure will be sensitive to the domain of X (there is a clarifying example in Section 2 on page 151).

Terminating games will be represented by the infinities ∞ and $\overline{\infty}$, where the first is positive (Left wins) and the second is negative (Right wins).[2] The sole goal of this document is to extend $\mathbb{Np}$ with these elements, and to explore the various properties of the resulting structure. The idea is similar to the compactification of the real number line with infinities, leading to the affinely extended real number line. Nevertheless, in the context of games, the analysis of the resulting algebra is considerably more sophisticated. It is by this analogy that we refer to $\mathbb{Np}^\infty$ as affine normal play, and its elements as affine normal play games, or just affine games. For a beginner's intuition, on page 150 we explain that the infinities have trivial equivalence classes. The impartial affine theory appears in [Larsson et al. 2021], where some basic partizan play results are included.

This study contains four main results, Theorems 33, 45, 46 and 63, to be reviewed below as we outline the content.

[2]In CGT, Left is a female and positive player, while Right is a male and negative player.

Section 1.1 elaborates on motivation. We make use of examples from classical recreational rulesets. In particular, we will use a variation of the classical ruleset GO, called ATARI GO, also known as FIRST CAPTURE GO.[3] In this study, we will also encounter variations of classical combinatorial games' rulesets such as BLUE-RED-HACKENBUSH, NIM, NIMSTRING, and AMAZONS [Berlekamp et al. 1982a; 1982b]. Whenever there are colored pieces, Left is b**L**ue and Right is **R**ed. If the colors are black and white, Left is b**L**ack and Right is White (clea**R**). Before we enter this motivational prelude, let us summarize the rest of the paper.

Section 2 gives basic definitions and introductory results; Theorem 5 addresses standard properties of the disjunctive sum, while Theorem 10 establishes that game equivalence and game order are indeed an equivalence relation and a partial order relation, respectively. At the end of this section, we exemplify that in $\mathbb{N}\mathbb{p}^\infty$ a restriction can alter the game equivalence (while this cannot occur in $\mathbb{N}\mathbb{p}$).

In Section 3 we recall Theorem 16 [Larsson et al. 2021], which establishes that, just like in the classical theory, a game $G \geqslant 0$ if and only if Left wins G playing second. As a consequence of this result, we have Corollary 17 (Order–Outcome Bijection), which confirms that the classical perfect matching between the partial order and the outcome classes continues to hold.

Recall that the classical structure $\mathbb{N}\mathbb{p}$ is an abelian group in which the inverse $-G$ of a game G is its conjugate $\overline{G}$, i.e., the game where the players switch roles (see Definition 13), and we can compare two games G and H without resorting to the universal statement in the definition of order. This is achieved through a *local* procedure, which consists of playing $G - H$, where $G \geqslant H$ if and only if Left wins $G - H$ playing second.

In Section 4 we find the first main theorem, Theorem 33. In $\mathbb{N}\mathbb{p}^\infty$, the classical normal play methods do not apply directly, because there are many noninvertible games. We prove that it is still possible to have a local procedure to compare G with H. Let us explain some background to this result here.

The following "Maintenance" property is, in normal play, an equivalent formulation of "Left wins $G - H$ playing second", and it becomes useful when analyzing structures that are not groups [Larsson et al. 2025]. Let us state it here, as its underlying idea remains fundamental to our approach.

Definition 1 [Larsson et al. 2025]. Let $G, H \in \mathbb{N}\mathbb{p}$. The pair (G, H) satisfies the maintenance property, $\boldsymbol{M}(G, H)$, if

(1) $\forall G^R$ ($\exists G^{RL}$ such that $G^{RL} \geqslant H$ or $\exists H^R$ such that $G^R \geqslant H^R$), and

(2) $\forall H^L$ ($\exists H^{LR}$ such that $G \geqslant H^{LR}$ or $\exists G^L$ such that $G^L \geqslant H^L$).

[3] See http://wikipedia.org/wiki/Capture_Go and http://wikipedia.org/wiki/Chess (consulted on December 15, 2023).

Let us illustrate this with a small example, namely that $\boldsymbol{M}\big(\frac{1}{2}, \uparrow*\big)$ holds. Here $G = \{0 \mid 1\}$ and $H = \{0, * \mid 0\}$. Item (1) holds because $G^L = 1 \geqslant 0 = H^R$, and item (2) holds, because $G^L = 0 \geqslant 0 = H^{L_1}$ and $G = \frac{1}{2} \geqslant 0 = H^{L_2R}$.

By inspecting these inequalities, and using induction, the reader may justify that, in $\mathbb{Np}$, we have $G \geqslant H$ if and only if $\boldsymbol{M}(G, H)$. In $\mathbb{Np}^\infty$, however, there exist forcing sequences of moves, and one curious consequence is that it is possible to have $G \geqslant H$ but not $\boldsymbol{M}(G, H)$; see the example in Section 4.1. In our first main result for affine normal play, $\mathbb{Np}^\infty$, we prove that $G \geqslant H$ if and only if $\boldsymbol{M}^\infty(G, H)$, a weaker version of the maintenance property; this is covered in Definitions 26 and 27 and Theorem 33.[4]

In Section 5, we prove that there are three ways to reduce a game. Theorem 37 introduces the notion of *absorbing reversibility*, the only one that does not exist in $\mathbb{Np}$. We will see, however, that the reduced forms cease to be unique.

Section 6 contains the second and third main results, Theorems 45 and 46. Theorem 45 states that the classical structure $\mathbb{Np}$ is effectively the substructure of the invertible elements of affine normal play, and Theorem 46 proves that $\mathbb{Np}$ is order-embedded in $\mathbb{Np}^\infty$. These results have far-reaching consequences. Alongside Theorem 16, Theorem 45 allows us to conclude that when H is a traditional Conway value, comparing G with H can be done by playing $G - H$, just as in the classical theory.

Section 7 contains a classification of affine normal play games. There is a particularly interesting game with a *check* option: as long as the current player is not mortally threatened in another summand, the first thing they should do is to deliver that check. This discovery led to the formalization of a *hammerzug* and the *pathetic infinitesimals* $+_\infty$ and $-_\infty$. In a way, ∞ and $\overline{\infty}$ close the game line, while $+_\infty$ and $-_\infty$ close the descending scales of infinitesimals.

Section 8 contains our fourth main result, Theorem 63, which establishes that the affine games born by day n form a lattice structure.

Finally, given that the analysis of combinatorial rulesets is a primary application of CGT, the document concludes with Section 9, exploring an ATARI GO endgame by determining the affine game values of its components.

1.1. ***Motivational prelude.*** The need to use infinities is directly related to the notion of "urgency", as measured through the CGT concept of temperature. As an example, consider the AMAZONS position G, illustrated in Figure 1.

While the players have more than one option, the ones depicted in Figure 1 dominate all the others. If Left plays first, she is in time to "conquer" territory

[4]We note that this is a theoretical result, perhaps with little practical relevance, as we estimate that in the vast majority of cases, the comparison can be made using only the maintenance property as stated in Definition 1.

Figure 1. A hot AMAZONS position.

(by moving to 3); if Right plays first, then it is Right who can play to -3. We have $G = \{3 \mid -3\} = \pm 3$, with a temperature equal to 3. Often, in a disjunctive sum $G + X$, both players aim to play in G to gain territory. However, it is important to note the use of "often" and not "always". Consider the disjunctive sum $G + X = \pm 3 + \pm 5$, shown in Figure 2. The component X (at the top of the board) is a component where Left has a move to 5 and Right has a move to -5; therefore, X is hotter than G. Of course, in this position, it is now more urgent to play in X than in G.

In the classical normal play convention, there are no supremely urgent games. More precisely, given a game G, it is always possible to find a disjunctive sum $G + X$ with a more urgent component X. However, certain rulesets include terminating moves; such moves are infinitely superior to all other moves, leading to an immediate victory.

The existence of terminating moves enables the existence of *terminating threats*; these are threats against which a player must defend. Any other move would be inferior. Terminating threats are commonly referred to as *checks*. Such situations have infinite temperature (with a supreme urgency).

It is worthwhile to reiterate this point, since it is central to all the analysis that follows: due to the fact that a terminating move has an absorbing nature, a player in check must always try to defend himself. In particular, this allows a player to interpose checks to "rearrange the position" before responding to the opponent's last move with a quiet move. This idea with intermediate checks is central to this study, as is discussed in Section 4. Even world-class chess players can forget

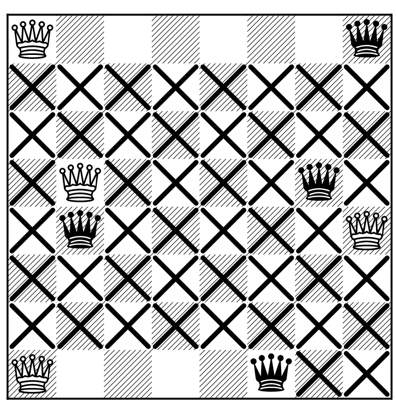


Figure 2. First things first.

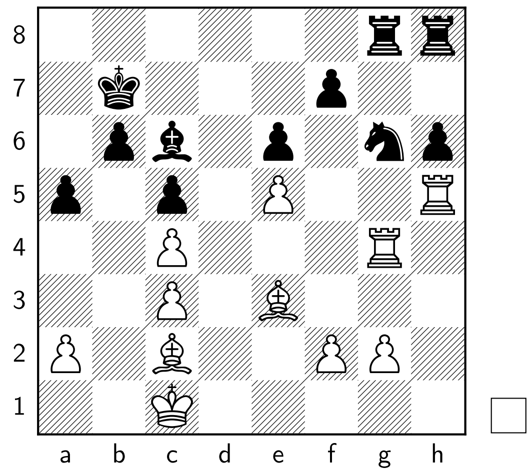


Figure 3. Carlsen–Anand, World Championship, RUS, 15 Nov 2014, Round 6. An eventual capture of the white pawn at e5 by the black knight is simply bad since that knight is captured by the white rook at h5 after two "automatic moves" resulting from the exchange of a pair of rooks at g8. Due to that, Carlsen thought he had the position under control and made the very bad move 26.Kd2?. Incredibly, Anand missed the opportunity and replied with another bad move 26...a4?. Anand could have explored the opportunity with a pair of zwischenzugs: 26.Kd2 Ne5! 27.Rg8 Nc4 (zwischenzug) 28.Kd3 Nb2 (one more zwischenzug) 29.Kd2 Rg8 (only now Black recaptures the white rook at g8, with a winning position).

the forcing nature of a *zwischenzug*, as can be seen in the example illustrated in Figure 3. "Zwischenzug" (German for "intermediate move") is a chess tactic where a player, instead of playing the expected move (commonly a recapture), first threatens the opponent with a check, and only after their response plays the expected move. Such a move is also called "in-between check" or *intermezzo*.

Checks and forcing sequences cannot be represented by elements of $\mathbb{Np}$. Consequently, the $\mathbb{Np}^\infty$ framework is needed to analyze some rulesets. In that framework, $\infty + \overline{\infty}$ is not defined. However, if $X \neq \overline{\infty}$, then $\infty + X = \infty$, and if $X \neq \infty$, then $\overline{\infty} + X = \overline{\infty}$. These follow from the absorbing nature of the terminating moves. There are three important situations of play that require $\mathbb{Np}^\infty$ for their analysis. Examples are given in the next subsections. In the first, the checks are overt, but they are hidden in the second and third.

1.1.1. *Terminating moves and infinitely hot components.* The ruleset CHESS is mentioned here because almost all readers are familiar with checks and checkmates within the context of this ruleset. However, it is a cyclic ruleset that allows ties. Additionally, CHESS positions do not tend to break down into disjoint components. In this work, from now on, we will use ATARI GO to exemplify some concepts, as it is a ruleset for which the use of $\mathbb{N}\mathbb{p}^{\infty}$ is particularly suitable.

In Figure 4, the black stones marked with triangles are "alive". This is a term in GO indicating that they cannot be captured; possibly, they are connected to some group with "eyes", as indicated by the arrow in the right corner. Thus, the corner region bounded by these stones is a component disjoint from the rest of the board.

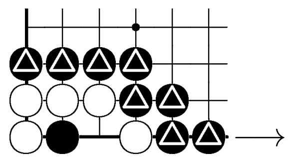


Figure 4. Infinite heat ATARI GO component.

Left, playing first, can capture the white stones, winning the game; Right, playing first, can capture the vulnerable black stone, winning the game. Thus, $H = \{\infty \mid \overline{\infty}\} = \pm\infty$ has infinite temperature; the player who has the move wins the game. For X different from ∞ and $\overline{\infty}$, in $H + X$, X is irrelevant, regardless of what X may be. This is in sharp contrast to the AMAZONS position in Figure 1.

Consider now the ATARI GO position W shown in Figure 5. The Left option W^L corresponding to placing a black stone on point "a" is a check since it threatens an immediate capture.[5]

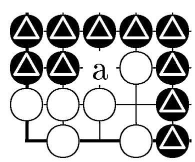


Figure 5. The Left option "a" is a check.

Right can defend with the option W^{LR}, as shown in Figure 6.

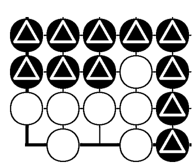


Figure 6. $W^{LR} = 0$.

[5]There is a much better move for Left than this check, which is left as an exercise for the reader.

Note that in W^{LR}, a Left option $W^{LRL} = \overline{\infty}$ is a *suicidal* move, that is, an option that Left never chooses unless it is the only thing she can do. Having this option or not makes no difference. Sometimes, in certain rulesets, such as GO or ATARI GO, these moves are considered illegal and cannot be made. For the sake of formalization, even in such cases, $\overline{\infty}$ is still included in the game form as if the move could be made, since this procedure does not change the nature of the ruleset. A single Left option $\overline{\infty}$ is the equivalent in $\mathbb{Np}$ to having an empty set of options; if there are other Left options, $\overline{\infty}$ is always dominated. On the other hand, the Right options in W^{LR} are moves that allow Left to immediately play to ∞. These moves take the form $\{\infty \mid \dots\}$, and Right never chooses them unless it is the only thing he can do. In practice, these Right options can be replaced by ∞ since they are also a form of suicide, just taking one more move. Thus, $W^{LR} = \{\overline{\infty} \mid \infty\}$ is a game in which neither player has moves except those that self-inflict defeat. As we will see later, this game is zero, the identity of $\mathbb{Np}^{\infty}$. It is now easy to understand that the initial check is $W^L = \{\infty \mid 0\}$.

In summary, if a Left option of a game G is $G^L = \infty$, then G^L is a checkmate; if a Left option is $G^L = \{\infty \mid G^{L\mathcal{R}}\}$, then G^L is a check; if a Left option is $G^L = \overline{\infty}$, then G^L is a suicidal move; if a Left option is $G^L = \{G^{L\mathcal{L}} \mid \overline{\infty}\}$, then the move allows being checkmated and can be replaced by $\overline{\infty}$. The same logic applies to Right options.

1.1.2. *Entailing moves.* A move is called *entailing* if it forces the opponent to respond locally in some specified sense.[6] It may be possible for the opponent to respond locally in more than one way; nevertheless, the opponent is restricted to those local options. Of course, this disrupts the usual logic of disjunctive sums. Consider a disjunctive sum $G_1 + G_2 + \cdots + G_k$. If Left is not lethally threatened in any component and makes an entailing move in G_1 to G_1^L, then the imposed restriction on Right prevents him from freely choosing any component from $G_1^L + G_2 + \cdots + G_k$ for his reply.

As an example, consider NIM played on sums of piles of stones $G_1 + \cdots + G_k$, but with the following change in the rules: whenever a player leaves three stones in one of the piles, the opponent must play in that very same pile on the next move.[7] Of course, a pile of four stones in traditional NIM is the game

$$*4 = \{0, *, *2, *3 \mid 0, *, *2, *3\}.$$

[6] The designation “entailing move” was proposed in [Berlekamp et al. 1982a; 1982b] precisely to convey the idea that a player is compelled (“entailed”) to play in a certain way.

[7] This ruleset does not have recreational interest, since a straightforward strategy-stealing argument shows that all initial positions with at least a pile of more than three stones are $\mathcal{N}$-positions. The representation of positions through game forms is what is of interest.

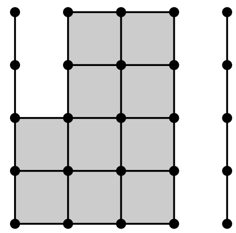


Figure 7. A disjunctive sum in NIMSTRING.

In the modified version, a pile of four stones can instead be described as

$$\{0, *, *2, *3_{\text{entail}} \mid 0, *, *2, *3_{\text{entail}}\}, \tag{1}$$

where the option $*3_{\text{entail}}$ compels the other player to continue playing in this component, even if the game is not played in isolation. Arguably, this notation is a bit heavy, and one of the benefits of $\mathbb{Np}^\infty$ is that it naturally provides an elegant abstract resolution to any such entailing situation (see also [Larsson et al. 2021]). Any CHESS player knows that a check compels the opponent to defend. In other words, a check is, by its nature, an entailing move. Since some of the elements of $\mathbb{Np}^\infty$ are checks, they can be employed as "gadgets" to describe situations similar to the modified NIM pile. The abstract form $\{0, *, *2, \{\infty \mid 0, *, *2\} \mid 0, *, *2, \{0, *, *2 \mid \overline{\infty}\}\}$ perfectly represents the pile of four stones in (1). As in the movie *The Godfather*, checks are "offers" (threats) that the opponent cannot refuse (cannot avoid defending). This explains why these gadgets work. When Left moves to $\{\infty \mid 0, *, *2\}$, Right has to defend by playing to 0, $*$, or $*2$. Since 0, $*$, or $*2$ are the options Right would have if forced to play in $*3$, the Left option $\{\infty \mid 0, *, *2\}$ is, in practice, a move that works as one that compels Right to play in $*3$.

In summary, if a Left entailing option compels Right to respond with an element from a set A, the way to represent this entailing move through a game form of $\mathbb{Np}^\infty$ is to use a check G^L where $A = G^{L\mathcal{R}}$, i.e., $G^L = \{\infty \mid A\}$. Obviously, Right entailing options are represented in a similar manner.

1.1.3. *Carry-on moves.* The ruleset NIMSTRING is a normal play variation of the popular children game DOTS & BOXES. It includes *carry-on* moves, moves that compel the player who moved to play again. The mandatory extra move can be carried out in the same component or any other component. In this particular ruleset, when a player "closes a box", the player must move again. Figure 7 shows a NIMSTRING position that consists of a disjunctive sum $G + H$, where G is the component on the left and H is the component on the right. For example, if Left closes a box in the left component, Left has to play again, being able to choose either of the two components to continue the play.

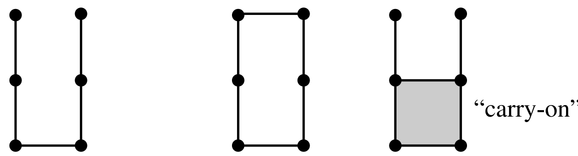


Figure 8. A NIMSTRING component G with its two options, a "double-box" and a carry-on.

In [Berlekamp et al. 1982a; 1982b], carry-on moves were called "complimenting moves". In this work, we prefer the designation carry-on move as we believe it to be much more descriptive. In Figure 8, we show the component G of the previous example, along with its two options, one of which is a carry-on move. If the top bar is drawn, the next player continues, but if the middle bar is drawn, then the current player has to carry on.

A carry-on move can be seen as a particular case of an entailing move. As previously mentioned, even when entailed, a player may have more than one option to respond. This happens when in a game form like $G^L = \{\infty \mid A\}$, the set A has more than one element.

When A has a single option, there is only one way for the player to escape the entailment. Suppose that in $G^L = \{\infty \mid A\}$, Right's only option is G^{LR}, i.e., $G^L = \{\infty \mid G^{LR}\}$. In this case, when Left opts for the entailing move, it is as if Left is "jumping" directly from G to G^{LR}, since there will be an opportunity to *play again* after replacing G by G^{LR} in the disjunctive sum. After the "jump", Left can play again on any component of the sum.

In the leftmost picture of Figure 8, suppose that Left closes a box by drawing the middle bar in G. The option is a 3-sided box indexed with a "carry-on", the rightmost picture. Note that a 3-sided box alone is a $\mathcal{P}$-position (zero). Namely, by drawing the final bar you must play again, but cannot, so the second player wins. Now, in view of the rightmost picture in Figure 8, playing to a zero, from which you have to play again, has a nice abstract representation; the middle-bar-drawing option in G is conveniently represented by Left's carry-on move $\{\infty \mid 0\}$ or Right's carry-on move $\{0 \mid \overline{\infty}\}$, respectively. If, on the other hand, a player draws the top bar in G, as in the middle picture, then they analogously place a $\mathcal{P}$-position (zero) in the disjunctive sum, but this time, by avoiding to close a box, passing the turn to the opponent. Thus, the game form that describes G is $\{0, \{\infty \mid 0\} \mid 0, \{0 \mid \overline{\infty}\}\}$.

For another example, let us consider the ruleset WHACKENBUSH, which prompts an interesting mathematical analysis. It is played like the classic BLUE-RED-HACKENBUSH but with an extra rule: whenever a player makes a move that drops one or more opponent's edges, they have to play again. Figure 9 illustrates an intriguing WHACKENBUSH position that cannot be adequately

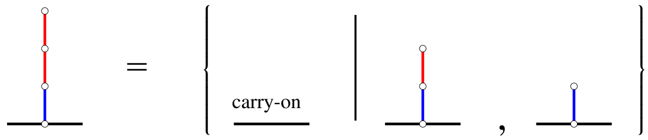


Figure 9. An intriguing WHACKENBUSH position.

described through an element of $\mathbb{N}\mathbb{p}$. In $\mathbb{N}\mathbb{p}^\infty$ the game value is $\{\{\infty \mid 0\} \mid 0, 1\} = \{\{\infty \mid 0\} \mid 0\}$.[8] This negative value does not exist in $\mathbb{N}\mathbb{p}$.

In summary, if, with a carry-on move, Left "jumps" directly from G to H, maintaining the right to play, then this option can be described as $G^L = \{\infty \mid H\}$. This is because it can be seen as a Left option (a "gadget") that compels Right to respond with H, returning the right to play again to Left. In game practice, what happens frequently is that Left moves from G to H, playing again without any intervention from Right. With the gadget, Left moves from G to G^L, and Right is forced to respond with H, allowing Left to play again. Of course, although Right intervenes, it is as if he had not, given the forced manner in which he had to do it. The carry-on option $G^L = \{\infty \mid H\}$ is symbolized by $\circlearrowright^H$. Analogously, Right carry-on options $\{H \mid \overline{\infty}\}$ are denoted as $\circlearrowleft^H$. For example, the NIMSTRING position shown in Figure 8 can be represented as $\{0, \circlearrowright^0 \mid 0, \circlearrowleft^0\}$. The WHACKENBUSH position shown in Figure 9 can be represented as $\{\circlearrowright^0 \mid 0\}$.

2. The structure of $\mathbb{N}\mathbb{p}^\infty$

In this section, we present the fundamental definitions and some initial results related to affine normal play. Before delving deeper into the structure, we define the objects under study, the set of abstract forms of the games. Like many others in CGT, it is a recursive definition. This is a first step towards the partially ordered structure, $\mathbb{N}\mathbb{p}^\infty$.

Definition 2 (Affine Game Forms). Let $\mathbb{A}_{-1} = \{\infty, \overline{\infty}\}$ be the base set, the set of affine game forms born on day -1; these are the only forms in which players have no options. For $i > -1$, $\mathbb{A}_i$ is the set of games of the form $\{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$, where $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ are nonempty finite sets of game forms contained in $\bigcup_{j<i} \mathbb{A}_j$; these are the affine forms born by day $i > -1$. Moreover, if $i > -1$, $G \in \mathbb{A}_i$, and $G \notin \mathbb{A}_{i-1}$, then G is born on day i; it is said to have a *formal birthday* of i, denoted by $\tilde{b}(G) = i$. The set of affine game forms is $\mathbb{A} = \bigcup_{i \geqslant -1} \mathbb{A}_i$.

The definition of affine game forms deserves some consideration. First of all, note that the affine forms are independent of any notion of outcome or order of games, and the infinities are symbolic games yet without any meaning or

[8]We note that $\{\{\infty \mid 0\} \mid 1\} = 0$.

interpretation. Secondly, observe that, except for games of day -1, both players have options. In other words, affine forms are dicotic, meaning either both players have options, or neither does; this latter scenario pertains to cases where the game is a terminating game, i.e., it is either ∞ or $\overline{\infty}$.

One might ask if the structure we are about to propose is limited by not being able to encompass rulesets with positions where one player has moves and the other does not. The answer to the question is easy: *the structure also covers those rulesets*. The only thing that needs to be done is a modification by choosing $G^{\mathcal{L}} = \{\overline{\infty}\}$ or $G^{\mathcal{R}} = \{\infty\}$ for the player without moves. In practice, if the empty set is replaced with a single suicidal move, the same effect is achieved, as the existence of that move is equivalent to having no moves at all. It should also be mentioned that the dicotic nature of the structure streamlines the thinking process in many proofs, as one can assume for a nonterminating game that both players have moves. Note also that all games end with a move to ∞ or to $\overline{\infty}$, i.e., all games end with a "checkmate", which can be self-inflicted or not. In the case where the move is a suicidal move, it may seem that the name "normal play" is poorly chosen, as in that case the last player loses. However, this is not the case, as it is again a matter of interpretation. Before making the suicidal move, in practice, the player was deprived of satisfactory moves, and that is the fundamental reason for their loss. In other words, the structure preserves the logic of the normal play convention.

The recursion begins on day -1 instead of day 0 in consistency with the $\mathbb{Np}$-embedding. In Section 6, we will prove that the classical structure $\mathbb{Np}$ is order-embedded in $\mathbb{Np}^{\infty}$. Starting the recursion on day -1 allows a game's formal birthday to match in both structures if the game belongs to both. In a way, the infinities play a role analogous to the empty set in the classical structure, with a difference: in the classical structure, the empty set is always "horrible" for a player, whereas in the structure we are proposing, an infinity can be "horrible" or "wonderful", depending on its sign. Somehow, in the classical structure, the empty set "exists" on day -1, before day zero, as it is used to make the only game of day zero. Similarly, infinities exist on day -1. Since there are two infinities and not just one empty set, there is more than one game born by day zero. There is another difference, but it is not very crucial. In the classical structure, the empty set is a useful *atom* for constructing game forms, but it is not a game itself. The infinities are games. However, in a certain sense, the infinities also serve as atoms for building game forms. This subject will be revisited in Section 8.

Moving on to the definition of disjunctive sum, it makes sense to establish an initial parallel with what happens in the extended real number line. The disjunctive sum is well-defined except in the case where there are infinities of different signs in the sum. Regarding the game practice, this poses no issue because there is never more than one terminating move in a game, given its absorbing nature; the game ends as soon as one occurs.

Definition 3 (Disjunctive Sum). The disjunctive sum of two affine game forms G and H is given by

$$G+H=\begin{cases}\text{not defined} & \text{if } (G=\infty \text{ and } H=\overline{\infty}) \text{ or } (G=\overline{\infty} \text{ and } H=\infty),\\ \infty & \text{if } (G=\infty \text{ and } H\neq\overline{\infty}) \text{ or } (G\neq\overline{\infty} \text{ and } H=\infty),\\ \overline{\infty} & \text{if } (G=\overline{\infty} \text{ and } H\neq\infty) \text{ or } (G\neq\infty \text{ and } H=\overline{\infty}),\\ \{G^{\mathcal{L}}+H, G+H^{\mathcal{L}} \mid G^{\mathcal{R}}+H, G+H^{\mathcal{R}}\} & \text{otherwise.}\end{cases}$$

Observation 4. The definition of disjunctive sum is recursive and is as usual presented with some abuse of notation: $G^{\mathcal{L}}+H,\ G+H^{\mathcal{L}}$ is shorthand for $\{G^L+H \mid G^L\in G^{\mathcal{L}}\}\cup\{G+H^L \mid H^L\in H^{\mathcal{L}}\}$ (and the same for Right's options).

A second parallel concerns the following theorem. The extended real number line is "almost" an abelian group, with only one issue when infinities are involved. Similarly, the affine normal play structure is "almost" an abelian monoid.

Theorem 5. *Let G, H, and J be affine game forms. Then*

(1) *$G+(H+J)$ and $(G+H)+J$ are either equal or both undefined*;

(2) *$G+H$ and $H+G$ are either equal or both undefined.*

Proof. The items are established as follows.

Item (1): If one of the summands is ∞ and one of the others is $\overline{\infty}$, it is easy to verify that both $G+(H+J)$ and $(G+H)+J$ are undefined. If one of the summands is ∞ without either of the others being $\overline{\infty}$, then both $G+(H+J)$ and $(G+H)+J$ are equal to ∞. If one of the summands is $\overline{\infty}$ without either of the others being ∞, then both $G+(H+J)$ and $(G+H)+J$ are equal to $\overline{\infty}$. Thus, assume that none of G, H, and J are infinities. In that case,

$$\begin{aligned}(G+(H+J))^{\mathcal{L}}&\\ &=\big(G^{\mathcal{L}}+(H+J)\big)\cup\big(G+(H+J)^{\mathcal{L}}\big)\\ &=\big(G^{\mathcal{L}}+(H+J)\big)\cup\big(G+(H^{\mathcal{L}}+J)\big)\cup\big(G+(H+J^{\mathcal{L}})\big)\\ &=\big((G^{\mathcal{L}}+H)+J\big)\cup\big((G+H^{\mathcal{L}})+J\big)\cup\big((G+H)+J^{\mathcal{L}}\big)\quad\text{(induction)}\\ &=\big((G+H)^{\mathcal{L}}+J\big)\cup\big((G+H)+J^{\mathcal{L}}\big)\\ &=((G+H)+J)^{\mathcal{L}}.\end{aligned}$$

Note that the induction works well, given that the sums are defined because none of G, H, and J are infinities. Similarly one can prove

$$(G+(H+J))^{\mathcal{R}}=((G+H)+J)^{\mathcal{R}}.$$

Hence,

$$\begin{aligned} G+(H+J) &= \{(G+(H+J))^{\mathcal{L}} \mid (G+(H+J))^{\mathcal{R}}\} \\ &= \{((G+H)+J)^{\mathcal{L}} \mid ((G+H)+J)^{\mathcal{R}}\} \\ &= (G+H)+J. \end{aligned}$$

Item (2): If one of the summands is ∞ and the other is $\overline{\infty}$, then both $G+H$ and $H+G$ are undefined. If one of the summands is ∞ without the other being $\overline{\infty}$, then both $G+H$ and $H+G$ are equal to ∞. If one of the summands is $\overline{\infty}$ without the other being ∞, then both $G+H$ and $H+G$ are equal to $\overline{\infty}$. Thus, assume that neither G nor H are infinities. In that case,

$$\begin{aligned} G+H &= \{G^{\mathcal{L}}+H, G+H^{\mathcal{L}} \mid G^{\mathcal{R}}+H, G+H^{\mathcal{R}}\} \\ &= \{H^{\mathcal{L}}+G, H+G^{\mathcal{L}} \mid H^{\mathcal{R}}+G, H+G^{\mathcal{R}}\} \quad \text{(induction)} \\ &= H+G. \end{aligned}$$

The induction applies because neither G nor H are infinities, so the sums are defined. □

In order to define an equivalence relation and a partial order of games, we first define the concept of an outcome; in some proofs to come it is beneficial to individualize the notion in the following manner.

Definition 6 (Individualized Outcomes). The *individualized outcomes* of a game G are

$$o_L(G) = \begin{cases} \mathrm{L} & \text{if } G=\infty \text{ or there exists } G^L \in G^{\mathcal{L}} \text{ such that } o_R(G^L)=\mathrm{L}, \\ \mathrm{R} & \text{otherwise,} \end{cases}$$

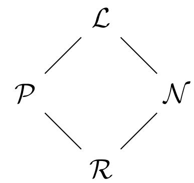
$$o_R(G) = \begin{cases} \mathrm{R} & \text{if } G=\overline{\infty} \text{ or there exists } G^R \in G^{\mathcal{R}} \text{ such that } o_L(G^R)=\mathrm{R}, \\ \mathrm{L} & \text{otherwise.} \end{cases}$$

The symbols L and R have a total order as L > R, and they mean "Left wins" and "Right wins", respectively. Note that Definition 6 formalizes the idea of *optimal play* in that, having that move at their disposal, a player chooses a winning move. In CGT, the number of moves in which a player wins a game is not relevant; what matters is winning. In other words, speed is irrelevant.

Definition 7 (Outcomes). The perfect play *outcome* of a game G is the pair $o(G)=(o_L(G), o_R(G))$, which can be one of the four possibilities (L, L), (L, R), (R, L), and (R, R), each one indicating the winner depending on who starts.

We identify the range of $o(G)$ with the standard *outcome classes* $\mathcal{L}=$ (L, L), $\mathcal{N}=$ (L, R), $\mathcal{R}=$ (R, R) and $\mathcal{P}=$ (R, L), the sets of all games with the indicated outcome, so that we can write $G \in \mathcal{L}$, and so on.

From Left's perspective, $\mathcal{L}$ is the best outcome (she wins, regardless of whether playing first or second) and $\mathcal{R}$ is the worst (she loses, regardless of whether playing first or second).[9] On the other hand, regarding $\mathcal{N}$ and $\mathcal{P}$, the victory depends on playing first or second, so these outcomes are not comparable. These considerations explain the partial order of outcomes:



Note that we have a partial order of games in $\mathbb{A}_0$ corresponding to that of the outcome classes (see the middle part of Figure 13).

As mentioned, Left wins a game if the last move is to ∞, by either of the players. Of course, Right only makes this move when he has no other option, and Left makes this move independently of any other option. Analogously, Right wins if the last move is to $\overline{\infty}$, by either player.

With a partial order of outcomes defined, we write $o(G)=o(H)$, $o(G)\geqslant o(H)$ or $o(G) \parallel o(H)$, and so on. With the concept of outcome established, we can define the partial order and equivalence of games.

Definition 8 (Partial Order and Game Equivalence). Let G and H be affine games. Then $G \geqslant H$ if $o(G+X) \geqslant o(H+X)$ for all games $X \in \mathbb{A} \setminus \{\infty, \overline{\infty}\}$, and $G = H$ if $G \geqslant H$ and $H \geqslant G$.

Sometimes we want to study other ranges of the "for all X" part of this definition. For example, in case of classical normal play, if $G, H \in \mathbb{Np}$ we would write $G \geqslant H$ modulo $\mathbb{Np}$ (see also Section 6 for the Conway-embedding and more). If we want to emphasize the inequality in Definition 8, then we write instead $G \geqslant H$ modulo $\mathbb{Np}^{\infty}$, and so on.

Observation 9. The relation $G \geqslant H$ means that replacing H by G can never hurt Left, no matter what the disjunctive sum is; the relation $G = H$ means that replacing H by G can never hurt Left or Right in any disjunctive sum.

Theorem 10. *The game relations in Definition 8 are a partial order relation and an equivalence relation, respectively.*

Proof. We begin with the proof of the equivalence relation. Game equality is reflexive and symmetric as direct consequences of Definition 8. Suppose now that $G = H$ and $H = J$. For every $X \in \mathbb{A} \setminus \{\infty, \overline{\infty}\}$, by $G = H$, then

[9] In CGT, Left is the "positive player", and Right is the "negative player": Left aims for the highest possible, while Right seeks the lowest possible. Of course, this reflects those recreational two-player games, where each player pulls in their own direction,.

$o(G+X)=o(H+X)$, and by $H=J$, then $o(H+X)=o(J+X)$. Consequently, as $o(G+X)=o(H+X)=o(J+X)$, then $G=J$, which establishes transitivity.

Game inequality is reflexive as a direct consequence of Definition 8. Suppose now that $G \geqslant H$ and $H \geqslant G$. For every $X \in \mathbb{A} \setminus \{\infty, \overline{\infty}\}$, by $G \geqslant H$ then $o(G+X) \geqslant o(H+X)$. On the other hand, by $H \geqslant G$ it follows $o(H+X) \geqslant o(G+X)$. Consequently, since $o(G+X) = o(H+X)$, then $G = H$, and antisymmetry is established. Finally, suppose that $G \geqslant H$ and $H \geqslant J$. For every $X \in \mathbb{A} \setminus \{\infty, \overline{\infty}\}$, by $G \geqslant H$ then $o(G+X) \geqslant o(H+X)$. On the other hand, by $H \geqslant J$ then $o(H+X) \geqslant o(J+X)$. Thus, as, for every $X \in \mathbb{A} \setminus \{\infty, \overline{\infty}\}$, $o(G+X) \geqslant o(H+X) \geqslant o(J+X)$, then $G \geqslant J$ and transitivity is established. □

As already indicated, $\mathbb{Np}^{\infty}$ denotes $(\mathbb{A}, +, \geqslant)$, where the inequality is game inequivalence. Furthermore, $G > H$ means that $G \geqslant H$ and $G \neq H$. When two games have exactly the same game form, they are *isomorphic*, denoted as $G \cong H$. Naturally, if $G \cong H$, then $G = H$, but the reverse may not hold.

For a beginner's intuition, let us justify that the equivalence classes of the infinities are the trivial ones. Suppose that $X \ncong \infty$. Then X does not terminate the game in Left's favor. So, in $X + \{\overline{\infty} \mid \overline{\infty}\}$ Right wins playing first, but in $\infty + \{\overline{\infty} \mid \overline{\infty}\} = \infty$ the game is terminated and Left won. The symmetric argument holds for $\overline{\infty}$.

Let us present some more basic concepts.

Definition 11 (Check Games). If ∞ is a Left option of a game, then that game is a *Left check*. A Left check that is a Left option of a game G is denoted by $G^{\overrightarrow{L}}$. A *Right check* and the notation $G^{\overleftarrow{R}}$ are defined symmetrically. If a game is either a Left check or a Right check, then it is a *check*.

Definition 12 (Quiet Games). Let $G \in \mathbb{Np}^{\infty}$. If G is not a Left check, then G is *Left quiet*. If G is not a Right check, then G is *Right quiet*. If G is Left quiet and Right quiet then G is *quiet*, and G is *strictly quiet* if it is quiet but not an infinity.

Observe that a quiet game may be ∞ or $\overline{\infty}$. By convention, a Left option G^L can be a Left check (or quiet), but a Left option $G^{\overrightarrow{L}}$ must be a Left check.

Suppose that $G \geqslant H$, and Left wins playing first in $H+X$. Hence there must also be a winning move for Left in $G+X$. It can be directly a quiet move to G^L+X, but it can also be a quiet move $G^{\overrightarrow{L}R\ldots\overrightarrow{L}RL}+X$ after a sequence of Left checks and the responses that Right gives.

We are now ready to mention the fundamental difference between classical normal play and affine normal play. In the first case, all moves are quiet. Thus, for each H^L, either there is a quiet move H^{LR} such that $G \geqslant H^{LR}$ or there is a quiet move G^L such that $G^L \geqslant H^L$. In affine normal play, it is Left who interrupts the forcing sequences in G, and it is Right who interrupts the forcing sequences in H^L. In this spirit, we will present a weaker version of the maintenance property, for which it is not necessary to have immediately $G \geqslant H^{LR}$ or $G^L \geqslant H^L$. It suffices to achieve the inequalities after sequences of checks. We will return to this topic in Section 4, concerning our first main result.

The *conjugate* of a given game switches the roles of the players. It has a recursive definition.

Definition 13 (Conjugate). The conjugate of $G \in \mathbb{Np}^\infty$ is

$$\overline{G} = \begin{cases} \overline{\infty} & \text{if } G = \infty, \\ \infty & \text{if } G = \overline{\infty}, \\ \{\overline{G^{\mathcal{R}}} \mid \overline{G^{\mathcal{L}}}\} & \text{otherwise,} \end{cases}$$

where $\overline{G^{\mathcal{R}}}$ denotes the set of games $\overline{G^R}$, for $G^R \in G^{\mathcal{R}}$, and similarly for $\overline{G^{\mathcal{L}}}$.

We will see in Section 6, Theorem 45, that $\mathbb{Np}^\infty$ does not deviate from the standard *conjugate property*: whenever a game is invertible, its inverse is its conjugate.

Recall that to have $G \geqslant H$, it is necessary to have $o_L(G+X) \geqslant o_L(H+X)$ and $o_R(G+X) \geqslant o_R(H+X)$ for every game X different from ∞ and $\overline{\infty}$. In other words, if Left, playing first, wins $H+X$, she must also win $G+X$; if Right, playing first, wins $G+X$, he must also win $H+X$. Often, moves in X are covered by simple induction. Therefore, the sensible moves are in G or in H.

Let us recall some notation from Section 1.1.3. Let $G \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$. Then the Left check is denoted by $\circlearrowleft^G = \{\infty \mid G\}$, and analogously for Right checks.

It is pertinent to observe that, although this does not happen in $\mathbb{Np}$, in $\mathbb{Np}^\infty$ a restriction can alter the game equivalence. To clarify what is meant by this, consider the games ☾ $:= \{0, \circlearrowleft^0 \mid 0, \circlearrowright^0\}$ and $\{*, \circlearrowleft^* \mid *, \circlearrowright^*\}$. These two games are not equivalent modulo $\mathbb{Np}^\infty$, since, if $X = \{0 \mid -1\}$, then Left, playing first, wins ☾ $+\, X$ but loses $\{*, \circlearrowleft^* \mid *, \circlearrowright^*\} + X$.

To find a natural restriction in which the two games are equivalent, let us recall some notation. We define an affine game form $G \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$ as *symmetric* if $G^{\mathcal{R}} = \overline{G^{\mathcal{L}}}$ (the set of Right options is the same as the set of the conjugates of Left options), and $G \in \mathbb{Np}^\infty$ is *impartial* if it is symmetric and all strictly quiet followers are symmetric. The class of impartial games is denoted by $\mathbb{I}$; see [Larsson et al. 2021; 2023]. For example, the game $\{\{\infty \mid 0\} \mid \{0 \mid \overline{\infty}\}\}$ is symmetric, and impartial.

Following the establishment of these concepts, the game equivalence *modulo* $\mathbb{I}$, denoted by $=_{\mathbb{I}}$, is defined exactly the same way as in Definition 8, with the difference that the distinguishing games X have to be elements of $\mathbb{I}$. In [Larsson et al. 2021], it was shown that the game values of this restricted structure are the well-known nimbers plus an extra value, which is ☾ . In this restriction, we already have ☾ $=_{\mathbb{I}} \{*, \circlearrowleft^* \mid *, \circlearrowright^*\}$.

In fact, we even have ☾ $=_{\mathbb{I}} \pm\infty := \{\infty \mid \overline{\infty}\}$. In other words, there is no distinction between ☾ and $\{*, \circlearrowright^* \mid *, \circlearrowleft^*\}$ by comparing only with other impartial games; these games are equivalent modulo $\mathbb{I}$, but they are not equivalent modulo $\mathbb{Np}^\infty$.

3. More structure

This section is dedicated to some initial results on the partial order of affine games. We begin by identifying the neutral element of $\mathbb{Np}^\infty$, with respect to disjunctive sum.

Theorem 14. *The game* $\{\overline{\infty} \mid \infty\}$, *denoted by* 0, *is the identity of* $\mathbb{Np}^\infty$.

Proof. According to Definition 3, $G + \{\overline{\infty} \mid \infty\}$ commutes, and thus it suffices to prove that $G + \{\overline{\infty} \mid \infty\} = G$, for all G.

If $G = \infty$, according to Definition 3, $G + \{\overline{\infty} \mid \infty\} = \infty + \{\overline{\infty} \mid \infty\} = \infty = G$, and analogously for $G = \overline{\infty}$.

Hence, assume that $G \neq \infty$ and $G \neq \overline{\infty}$. Let $X \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$, and suppose that, playing first, a player wins $G + X$. Without loss of generality, assume that the player is Left, and she wins with some option $(G + X)^L$. Then, she also wins $G + X + \{\overline{\infty} \mid \infty\}$ with the option $(G + X)^L + \{\overline{\infty} \mid \infty\}$. Essentially, she mimics the strategy used when $G + X$ is played alone, since neither player can make a move in $\{\overline{\infty} \mid \infty\}$ without immediately leading to defeat. On the other hand, if, playing first, Left wins $G + X + \{\overline{\infty} \mid \infty\}$, this must be done with an option $(G + X)^L + \{\overline{\infty} \mid \infty\}$. She also wins $G + X$ with the option $(G + X)^L$, since, once more, one can never touch the component $\{\overline{\infty} \mid \infty\}$, and the mimicry can be employed again. Thus, for all games G, for all games X, $o(G + X + \{\overline{\infty} \mid \infty\}) = o(G + X)$. Thus, $G + \{\overline{\infty} \mid \infty\} = G$, for all games G. □

The next theorem follows directly from the absorbing nature of the infinities.

Theorem 15. *If* $G \in \mathbb{Np}^\infty$, *then* $\infty \geqslant G$ *and* $G \geqslant \overline{\infty}$.

Proof. Let $G \in \mathbb{Np}^\infty$. If $X \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$ then, according to Definition 3, $\infty + X = \infty$. Hence, by Definition 6, $o(\infty + X) = o(\infty) = (\mathrm{L}, \mathrm{L})$. Thus, for every $X \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$, we have $o(\infty + X) \geqslant o(G + X)$. That is, $\infty \geqslant G$. Proving that $G \geqslant \overline{\infty}$ is analogous. □

Regarding the classical structure $\mathbb{Np}$, a well-known theorem states that $G \geqslant 0$ if and only if Left wins G playing second. The ultimate reason for this result is that if Left has a winning strategy in a game X, she also has it in $G + X$. She can use a "local response strategy", meaning that in $G + X$, Left responds to Right's moves in a component with a move in that same component, as if she were playing it in isolation. Under the normal play convention, Left makes the last move in both components and, consequently, in the disjunctive sum as a whole. The following theorem, which is proved in [Larsson et al. 2021], and called the Fundamental Theorem of Affine Normal Play, states that this result remains valid in $\mathbb{Np}^\infty$. We include a proof for completeness.

Theorem 16 [Larsson et al. 2021]. *Let* $G \in \mathbb{Np}^\infty$. *Then* $G \geqslant 0$ *if and only if* $G \in \mathcal{L} \cup \mathcal{P}$.

Proof. Assume $G \geqslant 0$. We have $0 \in \mathcal{P}$, and so, by order of outcomes, $G \in \mathcal{L} \cup \mathcal{P}$.

Suppose now that $G \in \mathcal{L} \cup \mathcal{P}$. If $G = \infty$, according to Theorem 15, $G \geqslant 0$; hence, assume $G \neq \infty$ and let $X \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$. If, playing first, Left wins X with the option X^L, then she also wins $G+X$ with the option $G+X^L$. Essentially, she mimics the strategy used when X is played alone, answering locally when Right plays in G. Due to the assumption $G \in \mathcal{L} \cup \mathcal{P}$, this is a winning strategy for Left in $G+X$. If Left, playing second, wins X. Then, on $G+X$, she can respond to each of Right's moves locally, with a winning move on the same component, because $G \in \mathcal{L} \cup \mathcal{P}$. Thus Left can win $G+X$ playing second. Consequently, $o(G+X) \geqslant o(X)$, for all $X \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$, and hence $G \geqslant 0$. □

In analogy to $\mathbb{Np}$, Theorem 16 implies the usual correspondence between the partial order and the outcome classes.

Corollary 17 (Order–Outcome Bijection). *If* $G \in \mathbb{Np}^\infty$, *then*

- $G > 0$ *if and only if* $G \in \mathcal{L}$;
- $G = 0$ *if and only if* $G \in \mathcal{P}$;
- $G \parallel 0$ *if and only if* $G \in \mathcal{N}$;
- $G < 0$ *if and only if* $G \in \mathcal{R}$.

Proof. In this proof, we also use the dual version of Theorem 16: "$G \leqslant 0$ if and only if $G \in \mathcal{R} \cup \mathcal{P}$".

Suppose that $G > 0$. By Theorem 16, $G \in \mathcal{L} \cup \mathcal{P}$. But, we cannot have $G \in \mathcal{P}$, for otherwise $G \in \mathcal{R} \cup \mathcal{P}$ and $G \leqslant 0$. Therefore, $G \in \mathcal{L}$. Conversely, suppose that $G \in \mathcal{L}$. By Theorem 16, we have $G \geqslant 0$. But, we cannot have $G = 0$, for otherwise $G \leqslant 0$, and $G \in \mathcal{R} \cup \mathcal{P}$. Hence, $G > 0$. Thus, the first equivalence holds.

The proof of the fourth equivalence is analogous.

For the second equivalence, we have that if $G = 0$, then $G \geqslant 0$ and $G \leqslant 0$. Therefore, $G \in (\mathcal{L} \cup \mathcal{P}) \cap (\mathcal{R} \cup \mathcal{P}) = \mathcal{P}$.

The third equivalence is a consequence of eliminating all other possibilities. □

The subsequent theorems offer a set of valuable results that relate the partial order of games to the disjunctive sum.

Theorem 18. *If* $G, H \in \mathbb{Np}^\infty$ *and* $J \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$, *then*

$$G \geqslant H \implies G + J \geqslant H + J.$$

Proof. Consider any $X \in \mathbb{Np}^{\infty} \setminus \{\infty, \overline{\infty}\}$ and let $X' = J + X$. Since X and J are neither ∞ nor $\overline{\infty}$, we have that X' is neither ∞ nor $\overline{\infty}$. Thus Definition 8 implies $o(G + X') \geqslant o(H + X')$, that is, $o(G + J + X) \geqslant o(H + J + X)$. Therefore the arbitrariness of X implies $G + J \geqslant H + J$. □

As before, if $G \in \mathbb{Np}^{\infty}$ has an inverse, we call the inverse $-G$.

Theorem 19. *If $G, H, J \in \mathbb{Np}^{\infty}$ and J is an invertible element, then*

$$G \geqslant H \iff G + J \geqslant H + J.$$

Proof. Firstly, note that the fact that J is invertible implies that J is neither ∞ nor $\overline{\infty}$, since, given the absorbing nature of infinities, there cannot be any element that, when added to these elements, equals 0. Thus, the implication $G \geqslant H \implies G + J \geqslant H + J$ directly follows from Theorem 18.

Regarding the reciprocal implication, consider any $X \in \mathbb{Np}^{\infty} \setminus \{\infty, \overline{\infty}\}$ and let $X' = -J + X$ (J is invertible, i.e., $-J$ exists and $J + (-J) = 0$). Since $-J$ is invertible, $-J$ is neither ∞ nor $\overline{\infty}$, so X' is neither ∞ nor $\overline{\infty}$. By Definition 8, $o(G + J + X') \geqslant o(H + J + X')$, that is, $o(G + J - J + X) \geqslant o(H + J - J + X)$. Hence, $o(G + X) \geqslant o(H + X)$, so, by the arbitrariness of X, we have $G \geqslant H$. □

The following theorem is highly relevant as it establishes that the comparison between two games, one of which is invertible, can be conducted by playing, exactly in the same manner as in the classical structure. In Section 6, an elegant characterization of invertible elements will be presented, reinforcing the significance of the theorem. Recall that if H is invertible, we call the inverse $-H$, and as usual, we use the abbreviation $G + (-H) = G - H$.

Theorem 20. *If $G, H \in \mathbb{Np}^{\infty}$ and H is invertible, then $G \geqslant H$ if and only if Left wins $G - H$ playing second, and $G = H$ if and only if the second player wins $G - H$.*

Proof. By Theorem 19, $G \geqslant H \iff G - H \geqslant H - H$. Therefore, $G \geqslant H \iff G - H \geqslant 0$. By Theorem 16, this leads to $G \geqslant H \iff G - H \in \mathcal{L} \cup \mathcal{P}$. Finally, $G = H \iff G - H \in \mathcal{P}$, since $G \geqslant H$ and $H \leqslant G$. □

The two following theorems are straightforward consequences of the order and equivalence relations of affine games.

Theorem 21. *If $G, H, J, W \in \mathbb{Np}^{\infty} \setminus \{\infty, \overline{\infty}\}$, then*

- *$G \geqslant H$ and $J \geqslant W$ implies $G + J \geqslant H + W$;*
- *$G = H$ and $J = W$ implies $G + J = H + W$.*

Proof. Starting with the first implication, it follows from Theorem 18 that $G + J \geqslant H + J$ and $H + J \geqslant H + W$. Next, it follows from the fact that we have an order relation that $G + J \geqslant H + W$. The second implication is a trivial consequence of the first one. □

Theorem 22. *Let* $G, H \in \mathbb{N}\mathbb{p}^\infty$. *If* $G > 0$ *and* $H \geqslant 0$ *then* $G + H > 0$.

Proof. If either G or H is ∞, the theorem holds trivially. Assuming that neither is the case, by Theorem 21, we already know that $G + H \geqslant 0$. So, it is enough to show that $G + H \neq 0$. Since $G > 0$ then, without loss of generality, we may assume that $o_L(G+X) = \mathrm{L}$ and $o_L(X) = \mathrm{R}$ for some X different from ∞ and $\overline{\infty}$. Because $H \geqslant 0$, we have $o_L(G + X + H) \geqslant o_L(G + X + 0) = o_L(G + X) = \mathrm{L}$. Since $o_L(G + H + X) = \mathrm{L}$ and $o_L(X) = \mathrm{R}$, we have $G + H \neq 0$. □

We conclude this section with a very simple theorem that encapsulates the idea that having extra options can never be a disadvantage.

Theorem 23 (Monotonicity Principle). *If* $G \in \mathbb{N}\mathbb{p}^\infty \setminus \{\infty, \overline{\infty}\}$, *then, for any* $H \in \mathbb{N}\mathbb{p}^\infty$, *we have* $\{G^{\mathcal{L}} \cup \{H\} \mid G^{\mathcal{R}}\} \geqslant G$.

Proof. Let $G' = \{G^{\mathcal{L}} \cup \{H\} \mid G^{\mathcal{R}}\}$ and consider any $X \in \mathbb{N}\mathbb{p}^\infty \setminus \{\infty, \overline{\infty}\}$. If Left has a winning strategy in $G + X$ (playing first or second), then Left also has a winning strategy in $G' + X$ since she never has to use the extra option, simply sticking to the winning strategy that exists in $G + X$. □

4. Local comparison

As mentioned earlier, it is possible to compare two games $G, H \in \mathbb{N}\mathbb{p}$ by using only their followers. In this section, we will see that the local comparison in $\mathbb{N}\mathbb{p}^\infty$ is not as straightforward. Namely, it is possible to have $G, H \in \mathbb{N}\mathbb{p}^\infty$ with $G \geqslant H$, but without satisfying the maintenance property $\boldsymbol{M}(G, H)$, from Definition 1. This is due to the forcing nature of checks and sequences of checks. For most established CGT monoids this cannot happen (see [Larsson et al. 2025]).

Nevertheless, we have found a local procedure, which uses a weaker version of the maintenance property, that takes into account any issue regarding the forcing sequences. Thus, we begin this section by presenting an example of two games G and H such that $G \geqslant H$, but where the usual maintenance property fails.

4.1. *A motivating example.*

Let

$$G = \Big\{*3, \{\infty \mid \{0 \mid \overline{\infty}\}, \{*2 \mid \overline{\infty}\}\} \;\|\; 0\Big\},$$
$$H = \Big\{\{1 \mid \{0 \mid \overline{\infty}\}, \{*2, *3 \mid \overline{\infty}\}\} \;\|\; 0\Big\}.$$

In these game forms, 0 is the game $\{\overline{\infty} \mid \infty\}$, 1 is the game $\{0 \mid \infty\}$, and the nimbers are the games defined recursively from 0 in the traditional way. Both games G and H have 0 as the only Right option, while G has two Left options, $*3$ and the Left check $\{\infty \mid \{0 \mid \overline{\infty}\}, \{*2 \mid \overline{\infty}\}\}$. Note that Right will respond to this check by one of his two check options. On the other hand, H has a single Left option, namely $\{1 \mid \{0 \mid \overline{\infty}\}, \{*2, *3 \mid \overline{\infty}\}\}$, which in its turn has two Right checks as Right options.

Claim 1: $G \geqslant H$.

Proof of Claim 1. Let $X \in \mathbb{N}p^{\infty} \setminus \{\infty, \overline{\infty}\}$. If Right wins $G+X$ by moving to $G+X^R$, then, by induction, he wins $H+X$ by moving to $H+X^R$. On the other hand, if Right wins $G+X$ by moving to $0+X$, then the exact same move is available in $H+X$. Therefore, for every $X \in \mathbb{N}p^{\infty} \setminus \{\infty, \overline{\infty}\}$, if Right wins $G+X$ when playing first, then Right also wins $H+X$.

If Left wins $H+X$ by moving to $H+X^L$, then, by induction, she wins $G+X$ by moving to $G+X^L$. The sensible option happens when Left wins $H+X$ by moving to H^L+X. In that case, she must have a winning move against both Right checks $\{0 \mid \overline{\infty}\}+X$ and $\{*2, *3 \mid \overline{\infty}\}+X$. The first implies that $0+X$ is a winning move for Left, while the second implies that either $*2+X$ or $*3+X$ is a winning move for Left. Hence, we conclude that either both $0+X$ and $*2+X$ are winning moves for Left or both $0+X$ and $*3+X$ are winning moves for Left. In the latter case, Left, playing first, secures a win in $G+X$ by moving to $*3+X$. In the former case, Left, playing first, wins $G+X$ by giving the check $\{\infty \mid \{0 \mid \overline{\infty}\}, \{*2 \mid \overline{\infty}\}\}+X$. Thus, for every $X \in \mathbb{N}p^{\infty} \setminus \{\infty, \overline{\infty}\}$, if Left wins $H+X$ when playing first, then she also wins $G+X$.

Claim 2: The maintenance property $\boldsymbol{M}(G, H)$ is not satisfied.

Proof of Claim 2. Given the games G and H, we have

$$G^{L_1} = *3, \qquad G^{L_2} = \{\infty \mid \{0 \mid \overline{\infty}\}, \{*2 \mid \overline{\infty}\}\},$$
$$H^{LR_1} = \{0 \mid \overline{\infty}\}, \quad H^{LR_2} = \{*2, *3 \mid \overline{\infty}\}.$$

We will show that $G^{L_1} \not\geqslant H^L$, $G^{L_2} \not\geqslant H^L$, $G \not\geqslant H^{LR_1}$, and $G \not\geqslant H^{LR_2}$.

- $G^{L_1} \not\geqslant H^L$, because, playing first, Left loses $G^{L_1}-1 = *3-1$, but wins $H^L - 1 = \{1 \mid \{0 \mid \overline{\infty}\}, \{*2, *3 \mid \overline{\infty}\}\} - 1$.
- $G^{L_2} \not\geqslant H^L$, because, playing second, Left loses

$$G^{L_2} + \{0 \mid *2\} = \big\{\infty \mid \{0 \mid \overline{\infty}\}, \{*2 \mid \overline{\infty}\}\big\} + \{0 \mid *2\},$$

but wins $H^L + \{0 \mid *2\} = \big\{1 \mid \{0 \mid \overline{\infty}\}, \{*2, *3 \mid \overline{\infty}\}\big\} + \{0 \mid *2\}$.

- $G \not\geqslant H^{LR_1}$, because, playing first, Left loses

$$G = \big\{*3, \{\infty \mid \{0 \mid \overline{\infty}\}, \{*2 \mid \overline{\infty}\}\} \parallel 0\big\},$$

but wins $H^{LR_1} = \{0 \mid \overline{\infty}\}$.

- $G \not\geqslant H^{LR_2}$, because, playing first, Left loses

$$G + *2 = \big\{*3, \{\infty \mid \{0 \mid \overline{\infty}\}, \{*2 \mid \overline{\infty}\}\} \parallel 0\big\} + *2,$$

but wins $H^{LR_2} + *2 = \{*2, *3 \mid \overline{\infty}\} + *2$.

This finishes the proof of Claim 2.

4.2. ***Strategies of responses and game trees.*** Let us define the concept of a "strategy of responses" to forcing sequences.

Definition 24 (Strategy of Responses to Checks). Let $G \in \mathbb{N}\mathbb{P}^{\infty}$. A strategy of Right responses to Left checks in G is a network of decisions in response to forcing sequences carried out by Left, constructed as follows:

- For $i = 0$, Right's strategy set $\mathbb{R}^0(G) = \{G\}$ corresponds to no Left check at all.
- For $i > 0$, for all $K \in \mathbb{R}^{i-1}(G)$, for each Left check $K^{\overrightarrow{L}}$, Right's strategy set $\mathbb{R}^i(G)$ contains exactly one Right choice, $K^{\overrightarrow{L}R} \neq \infty$, whenever such a choice can be made.

The set $\mathbb{R}(G) = \bigcup_{i \geqslant 0} \mathbb{R}^i(G)$ is a strategy of Right responses to Left checks. A strategy of Left responses to Right checks $\mathbb{L}(G)$ is defined analogously. The set of all strategies of Right check-responses in G is denoted by $\mathfrak{R}(G)$. Analogously, the set of all strategies of Left check-responses in G is denoted by $\mathfrak{L}(G)$.

Note that, for some i, $\mathbb{R}^i(G)$ may be empty; when this happens, for every $k > i$, $\mathbb{R}^k(G)$ is also empty, which indicates the end of Right's decision-making process.

Given a game G, a strategy of Right responses to Left checks is a set of Right choices in response to all forcing sequences by Left (a strategy of Left responses to Right checks is the symmetrical concept). We will be interested in Left's first quiet move after such a sequence of Right responses. When comparing games, there is the possibility for Left to play in G itself, so G must trivially belong to the strategy of Right check-responses. That explains the base case $\mathbb{R}^0(G) = \{G\}$. While Left is forcing with checks, Right must answer, and these answers belong to the strategy. That explains the recursion. What is new in $\mathbb{N}\mathbb{P}^{\infty}$ is that the Left quiet options in G are not the only Left quiet options that matter. We can have relevant Left quiet options after a sequence of checks (zwischenzugs), and these options depend on the strategy chosen by Right.

Example 25. Consider once again G and H^L as given in Section 4.1. Starting with $G = \{*3, \{\infty \mid \{0 \mid \overline{\infty}\}, \{*2 \mid \overline{\infty}\}\} \parallel 0\}$, there are two Right strategies of responses to Left checks, $\mathfrak{R}(G) = \{\mathbb{R}_1(G), \mathbb{R}_2(G)\}$, where

$$\mathbb{R}_1(G) = \{G, \{0 \mid \overline{\infty}\}\} \quad \text{and} \quad \mathbb{R}_2(G) = \{G, \{*2 \mid \overline{\infty}\}\}.$$

In $H^L = \{1 \mid \{0 \mid \overline{\infty}\}, \{*2, *3 \mid \overline{\infty}\}\}$, there are two Left strategies of responses to Right checks, $\mathfrak{L}(H^L) = \{\mathbb{R}_1(H^L), \mathbb{R}_2(H^L)\}$, where

$$\mathbb{L}_1(H^L) = \{H^L, 0, *2\} \quad \text{and} \quad \mathbb{L}_2(H^L) = \{H^L, 0, *3\}.$$

Notation. Let $\overrightarrow{G}$ denote a typical element of $\mathbb{R}(G)$ and let $\overleftarrow{G}$ denote a typical element of $\mathbb{L}(G)$.

Hence, in $\mathbb{R}_1(G)$ in Example 25, $\overrightarrow{G}_0 = G$ and $\overrightarrow{G}_1 = \{0 \,|\, \overline{\infty}\}$, say. Observe that the direction of the arrow is consistent with Definition 11.

A game tree of an affine game is drawn the same way as in the standard theory for partizan games, with two nuances. The first is that whenever a player's only option is a suicidal move, no edge is placed in the tree (as having or not having that move is equivalent). The second nuance is that whenever a player has at their disposal a winning terminal, a small arrow is placed in the game tree. That said, as in classical theory, the followers of a game are identified with the nodes of its game tree, as these nodes are the roots of the subtrees that represent them.

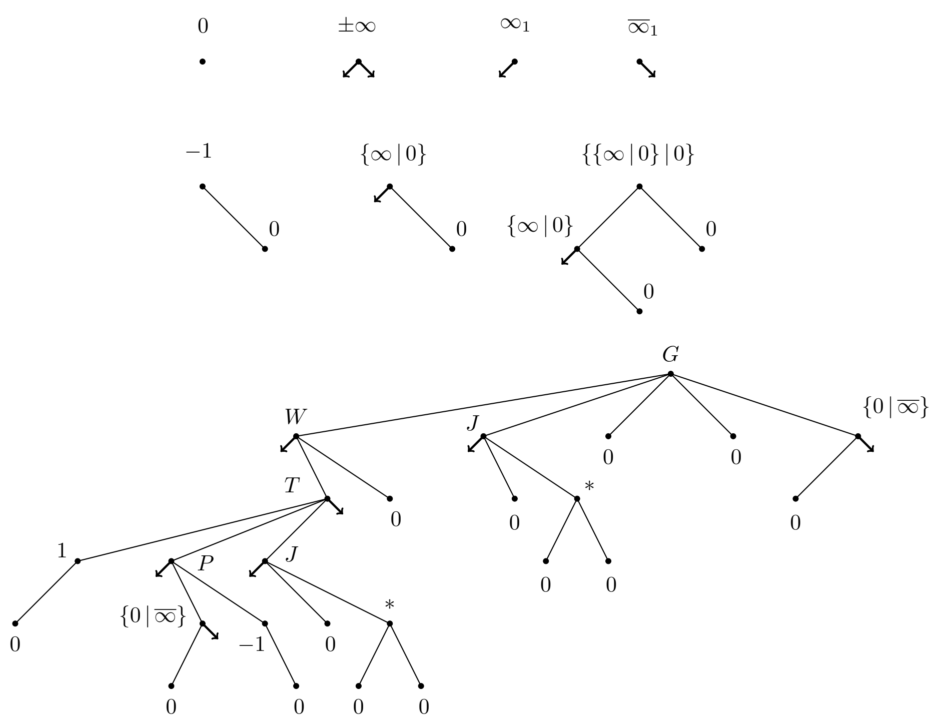


Figure 10. Top: The game trees of the four games $0 = \{\overline{\infty} \,|\, \infty\}$, $\pm\infty = \{\infty \,|\, \overline{\infty}\}$, $\infty_1 = \{\infty \,|\, \infty\}$, and $\overline{\infty}_1 = \{\overline{\infty} \,|\, \overline{\infty}\}$. Center: The game trees of the games -1, $\{\infty \,|\, 0\}$, and $\{\{\infty \,|\, 0\} \,|\, 0\}$, which are easy to understand. Bottom: In this game tree, $J = \{\infty \,|\, 0, *\}$, $P = \{\infty \,|\, \{0 \,|\, \overline{\infty}\}, -1\}$, $T = \{1, P, J \,|\, \overline{\infty}\}$, and $W = \{\infty \,|\, T, 0\}$. Since $G = \{W, J, 0 \,|\, 0, \{0 \,|\, \overline{\infty}\}\}$, its game tree pertains to the game form $\big\{\big\{\infty|||\big\{1, \{\infty|\{0|\overline{\infty}\}, -1\}, \{\infty|0, *\}||\overline{\infty}\big\}, 0\big\}, \{\infty|0, *\}, 0||||0, \{0|\overline{\infty}\}\big\}$.

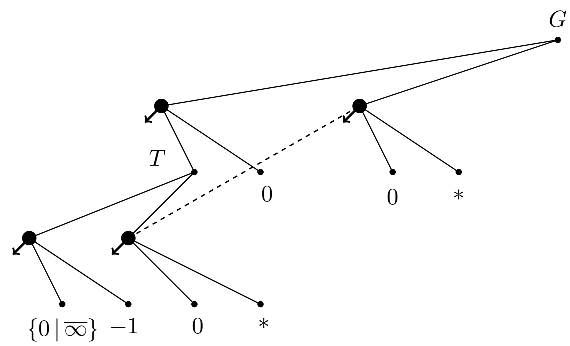


Figure 11. The Left-forcing-tree of the game G shown in Figure 10.

The games ∞ and $\overline{\infty}$ do not have a game tree, as in these cases, the play has already ended. Figure 10 shows some examples.

Given an affine game G, it is possible to consider the *Left-forcing-tree* of G with all the information about the forcing sequences that can be carried out by Left. To do this, the following four steps are taken:

(1) The root, identified with the game G itself, is not removed from the game tree.

(2) All nodes related to the elements of $G^{\mathcal{R}}$ and their followers are removed from the game tree.

(3) Regarding the Left moves, only the nodes identified with followers of the type $G^{\overrightarrow{L}R\overrightarrow{L}R\ldots\overrightarrow{L}}$ are retained.

(4) Regarding the Right moves, only the nodes identified with the responses to the Left checks from the previous item are retained.

Note that a Left-forcing-tree *is not* a game tree. It is an object constructed from a game tree, keeping games as labels for some chosen nodes. Figure 11 illustrates the Left-forcing-tree of the game G shown in Figure 10.

It is possible to think of $\mathfrak{R}(G)$ by considering the Left-forcing-tree of G. The larger ones (in black) are decision nodes, nodes where Right can be faced with a decision. The smaller nodes are possible choices, and a strategy of Right check-responses is a set containing exactly one choice per node.

The dashed line indicates that Right's decision can be considered the same in both nodes (the check is identical). Making different decisions against the same check, only because it occurs at different places of the game tree, merely increases the number of elements in the strategy, benefiting Left. Therefore, assuming that Right always responds in the same way against the same check reduces the work in determining strategies. Keeping that in mind, there are six strategies of Right responses in G:

- $\mathbb{R}_1(G) = \{G, 0\}$: Right moves to 0 against both checks.
- $\mathbb{R}_2(G) = \{G, 0, *\}$: Right moves to 0 against the first check and to $*$ against the second.
- $\mathbb{R}_3(G) = \{G, T, \{0 \mid \overline{\infty}\}, 0\}$: Against the first check, Right moves to T with the intention of continuing with $\{0 \mid \overline{\infty}\}$ or with 0. The response to the second check does not need to be specified since that check has already been analyzed.
- $\mathbb{R}_4(G) = \{G, T, \{0 \mid \overline{\infty}\}, *\}$: Against the first check, Right moves to T with the intention of continuing with $\{0 \mid \overline{\infty}\}$ or with $*$. The response to the second check does not need to be specified.
- $\mathbb{R}_5(G) = \{G, T, -1, 0\}$: Against the first check, Right moves to T with the intention of continuing with -1 or with 0. The response to the second check does not need to be specified.
- $\mathbb{R}_6(G) = \{G, T, -1, *\}$: Against the first check, Right moves to T with the intention of continuing with -1 or with $*$. The response to the second check does not need to be specified.

The set of strategies of Right responses in G is

$$\mathfrak{R}(G) = \{\mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_3, \mathbb{R}_4, \mathbb{R}_5, \mathbb{R}_6\}.$$

4.3. ***Local comparison in*** $\mathbb{Np}^{\infty}$**.** When we generalize the maintenance property to affine normal play, we combine two strategies of check-responses, one for each player, and we study what happens at a first quiet move, respectively.

Definition 26 (Check-maintained Pairs). Let $G, H \in \mathbb{Np}^{\infty}$. The pair of games (G, H) is *check-maintained* if, for all $(\mathbb{R}, \mathbb{L}) \in \mathfrak{R}(G) \times \mathfrak{L}(H)$, there are two games $G' \in \mathbb{R}$ and $H' \in \mathbb{L}$ for which

- there exists a Right quiet game H'^R such that $G' \geqslant H'^R$, or
- there exists a Left quiet G'^L such that $G'^L \geqslant H'$.

Definition 27 (∞-Maintenance). Let $G, H \in \mathbb{Np}^{\infty}$ be two affine games. Then (G, H) satisfies the ∞-maintenance property if (G^R, H) is check-maintained for all $G^R \in G^{\mathcal{R}}$, and (G, H^L) is check-maintained for all $H^L \in H^{\mathcal{L}}$. We write $\boldsymbol{M}^{\infty}(G, H)$ to denote that (G, H) satisfies the ∞-maintenance property.

Observation 28. If (G, H) satisfies the ∞-maintenance property, then, for all G^R and for each pair $(\mathbb{R}(G^R), \mathbb{L}(H))$, there are two games

$$\overrightarrow{G^R_a} \in \mathbb{R}(G^R) \quad \text{and} \quad \overleftarrow{H_b} \in \mathbb{L}(H)$$

compatible with the traditional maintenance property. That means that there exists a Right quiet game $(\overleftarrow{H}_b)^R$ for which

$$\overrightarrow{G}^R_a \geqslant (\overleftarrow{H}_b)^R,$$

or there exists a Left quiet game $(\overrightarrow{G}^R_a)^L$ for which

$$(\overrightarrow{G}^R_a)^L \geqslant \overleftarrow{H}_b.$$

The same happens with any H^L. In that sense, we have a weaker version of the traditional maintenance property, as will be stated in Theorem 34.

Example 29. Consider again the games G and H from Section 4.1. We will verify that H^L poses no issues regarding the ∞-maintenance property (G^R clearly does not cause any trouble). We have two Right-strategies of check-responses, $\mathbb{R}_1(G) = \{G, \{0 \mid \overline{\infty}\}\}$ and $\mathbb{R}_2(G) = \{G, \{*2 \mid \overline{\infty}\}\}$, and two Left-strategies of check-responses, $\mathbb{L}_1(H^L) = \{H^L, 0, *2\}$ and $\mathbb{L}_2(H^L) = \{H^L, 0, *3\}$. There are four pairs to consider. Recall Definitions 26 and 27.

- For $(\mathbb{R}_1(G), \mathbb{L}_1(H^L))$, we have the pair $(\{0 \mid \overline{\infty}\}, 0)$ and the Left quiet game $0 \in \{0 \mid \overline{\infty}\}^{\mathcal{L}}$ such that $0 \geqslant 0$.
- For $(\mathbb{R}_1(G), \mathbb{L}_2(H^L))$, we have the pair $(\{0 \mid \overline{\infty}\}, 0)$ and the Left quiet game $0 \in \{0 \mid \overline{\infty}\}^{\mathcal{L}}$ such that $0 \geqslant 0$.
- For $(\mathbb{R}_2(G), \mathbb{L}_1(H^L))$, we have the pair $(\{*2 \mid \overline{\infty}\}, *2)$ and the Left quiet game $*2 \in \{*2 \mid \overline{\infty}\}^{\mathcal{L}}$ such that $*2 \geqslant *2$.
- For $(\mathbb{R}_2(G), \mathbb{L}_2(H^L))$, we have the pair $(G, *3)$ and the Left quiet game $*3 \in G^{\mathcal{L}}$ such that $*3 \geqslant *3$.

We are now well prepared to prove the two ∞-maintenance implications, Theorems 30 and 32, that underpin the main result of this section, Theorem 33.

Theorem 30. *Let $G, H \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$. If $G \geqslant H$ then $\boldsymbol{M}^\infty(G, H)$.*

The following lemma assists the proof of Theorem 30.

Lemma 31. *Let $G, H \in \mathbb{Np}^\infty$.*

- *If $G \not\geqslant H$ and G is Left quiet, then there is some $X \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$ such that $o_L(G + X) = \mathrm{R} < o_L(H + X) = \mathrm{L}$.*
- *If $G \not\geqslant H$ and H is Right quiet, then there is some $X \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$ such that $o_R(G + X) = \mathrm{R} < o_R(H + X) = \mathrm{L}$.*

Proof. It is enough to prove the first item, as the proof for the second is entirely analogous. By Definition 8, $G \not\geqslant H$ implies that

- $\exists X \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$ such that $o_L(G + X) = \mathrm{R} < o_L(H + X) = \mathrm{L}$, or
- $\exists Y \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$ such that $o_R(G + Y) = \mathrm{R} < o_R(H + Y) = \mathrm{L}$.

If the first case is satisfied, then the proof is complete. If the second case is satisfied, then let X be the game $\{Y \mid \overline{\infty}\}$. Cases where $G = \overline{\infty}$ or $H = \infty$ are very simple to handle. Otherwise, we have $o_L(G + X) = \mathrm{R} < o_L(H + X) = \mathrm{L}$. These individualized outcomes arise from the fact that the moves to $G + Y$ and $H + Y$ are mandatory, and $o_R(G + Y) = \mathrm{R} < o_R(H + Y) = \mathrm{L}$. □

Proof of Theorem 30. For the sake of contradiction, suppose that we have $G \geqslant H$ and that (G, H) does not satisfy the ∞-maintenance property as stated in Definition 27.

Without loss of generality, suppose that the ∞-maintenance property fails due to the existence of some $H^L \in H^{\mathcal{L}}$ opposing it. If the failure were due to the existence of some $G^R \in G^{\mathcal{R}}$ opposing it, the proof would be similar.

Saying that is the same as saying that there is a particular *pernicious pair*

$$(\mathbb{R}(G), \mathbb{L}(H^L)) = (\{\overrightarrow{G}_1, \dots, \overrightarrow{G}_a, \dots\}, \{\overleftarrow{H^L_1}, \dots, \overleftarrow{H^L_b}, \dots\}),$$

where both of the following hold for each pair $(\overrightarrow{G}_a, \overleftarrow{H^L_b})$:

(1) There are no Left quiet games $(\overrightarrow{G}_a)^{L_i}$ or

$$\underbrace{(\overrightarrow{G}_a)^{L_i}}_{\text{Left quiet}} \not\geqslant \overleftarrow{H^L_b} \quad \text{for all } i.$$

(2) There are no Right quiet games $(\overleftarrow{H^L_b})^{R_j}$ or

$$\overrightarrow{G}_a \not\geqslant \underbrace{(\overleftarrow{H^L_b})^{R_j}}_{\text{Right quiet}} \quad \text{for all } j.$$

Note that there is a slight abuse of notation in the previous expressions. The games $(\overrightarrow{G}_a)^{L_i}$ should ideally be expressed as $(\overrightarrow{G}_a)^{L_{i,a}}$, where $i \in \{1, \dots, k_a\}$, given that we are enumerating the, say k_a, Left quiet options of $\overrightarrow{G}_a$. However, to prevent an overwhelming use of subscripts and superscripts, we have chosen a simplified writing style.

Fact 1: Regarding the Left quiet games, Lemma 31 ensures that, for all a, b, i, there exist distinguishing games $X^i_{a,b}$ such that

$$o_L(\underbrace{(\overrightarrow{G}_a)^{L_i}}_{\text{Left quiet}} + X^i_{a,b}) = \mathrm{R} < o_L(\overleftarrow{H^L_b} + X^i_{a,b}) = \mathrm{L}.$$

Fact 2: Regarding the Right quiet games, Lemma 31 ensures that, for all a, b, j, there exist distinguishing games $Y^j_{a,b}$ such that

$$o_R(\overrightarrow{G}_a + Y^j_{a,b}) = \mathrm{R} < o_R(\underbrace{(\overleftarrow{H^L_b})^{R_j}}_{\text{Right quiet}} + Y^j_{a,b}) = \mathrm{L}.$$

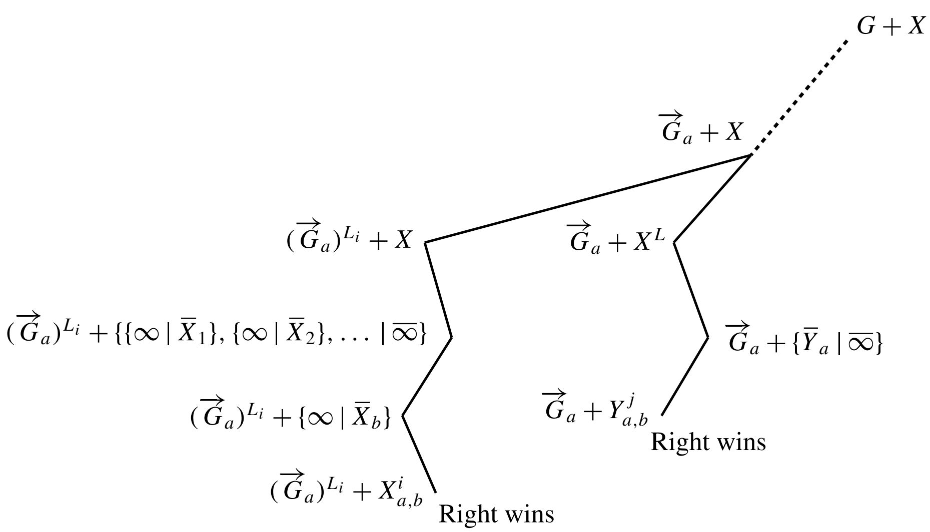
With the aim of constructing a distinguishing game leading to the desired contradiction, for each b, let $\overline{X}_b = \bigcup_{a,i}\{X^i_{a,b}\}$, and for each a, let $\overline{Y}_a = \bigcup_{b,j}\{Y^j_{a,b}\}$. The slight abuse of notation is repeated. The set $\bigcup_{a,i}\{X^i_{a,b}\}$ is $\bigcup_a \bigcup_{i\in\{1,\dots,k_a\}}\{X^i_{a,b}\}$, and the same for $\bigcup_{b,j}\{Y^j_{a,b}\}$. Supported by these sets, let

$$X^L = \begin{cases} \overline{\infty} & \text{if there are no Right quiet } (\overleftarrow{H^L_b})^{R_j}, \\ \{\infty \mid \{\overline{Y}_1 \mid \overline{\infty}\}, \{\overline{Y}_2 \mid \overline{\infty}\}, \dots\} & \text{if there are Right quiet } (\overleftarrow{H^L_b})^{R_j}, \end{cases}$$

$$X^R = \begin{cases} \infty & \text{if there are no Left quiet } (\overrightarrow{G}_a)^{L_i}, \\ \{\{\infty \mid \overline{X}_1\}, \{\infty \mid \overline{X}_2\}, \dots \mid \overline{\infty}\} & \text{if there are Left quiet } (\overrightarrow{G}_a)^{L_i}, \end{cases}$$

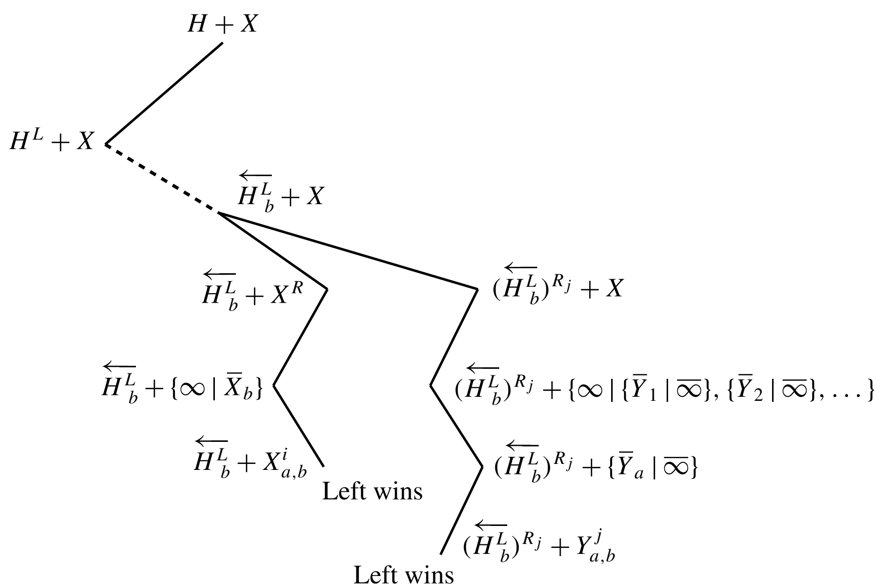
and consider the distinguishing game $X = \{X^L \mid X^R\}$.

Claim 1: Left, playing first, loses $G + X$. Right follows his strategy of the pernicious pair, described in the following picture. The dashed line means "forcing sequence". After that, Left either chooses a Left quiet game on the first component or makes a move on the second component. The "Right wins" are supported by Facts 1 and 2. If there are no Left quiet games $(\overrightarrow{G}_a)^{L_i}$, then only the branch on the right matters. According to Definition 24, Right never moves to ∞, and, if Right moves to $\overline{\infty}$, then Left loses anyway. If $X^L = \overline{\infty}$, then Left loses when playing to $(\overrightarrow{G}_a) + X^L$.



Claim 2: Left, playing first, wins $H + X$ by moving to $H^L + X$. Left follows her strategy of the pernicious pair, described in the following picture. The dashed line means "forcing sequence". After that, Right either makes a move on the second component or chooses a Right quiet game on the first component. The "Left wins" are supported by Facts 1 and 2. If there are no Right quiet games $(\overleftarrow{H^L_b})^{R_j}$

then only the branch on the left matters. According to Definition 24, Left never moves to $\overline{\infty}$, and, if Left moves to ∞, then Left wins anyway. If $X^R = \infty$ then Right loses when playing to $(\overleftarrow{H^L_b}) + X^R$.



It follows from Claims 1 and 2 that $G \not\geqslant H$, resulting in a contradiction. □

The opposite direction is also true, as the following theorem shows.

Theorem 32. *Let $G, H \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$. If $\boldsymbol{M}^\infty(G, H)$ then $G \geqslant H$.*

Proof. For the sake of contradiction, suppose that $\boldsymbol{M}^\infty(G, H)$ and $G \not\geqslant H$. According to Definition 8, there exists a distinguishing game $X \in \mathbb{Np}^\infty \setminus \{\infty, \overline{\infty}\}$ such that either $o_L(G + X) = \mathrm{R} < o_L(H + X) = \mathrm{L}$ or $o_R(G + X) = \mathrm{R} < o_R(H + X) = \mathrm{L}$.

Without loss of generality, assume that X has the smallest formal birthday, and such that $o_L(G + X) = \mathrm{R} < o_L(H + X) = \mathrm{L}$. We can establish the following three facts:

(a) We may assume that X is Left quiet since $o_L(G + X) = \mathrm{R}$.

(b) If $o_L(H + X) = o_R(H^L + X) = \mathrm{L}$ for some $H^L \in H^{\mathcal{L}}$, then there exists a strategy $\mathbb{L}(H^L)$ such that

$$o_R(\overleftarrow{H^L} + X) = \mathrm{L} \quad \text{for all } \overleftarrow{H^L} \in \mathbb{L}(H^L).$$

If this were not the case, Right could maneuver with checks, and we would not have $o_R(H^L + X) = \mathrm{L}$. We can also conclude that

$$o_L((\overleftarrow{H^L})^R + X) = \mathrm{L} \quad \text{for all } \overleftarrow{H^L} \in \mathbb{L}(H^L),$$

regardless of the Right option $(\overleftarrow{H^L})^R$. On the other hand, there exists a strategy $\mathbb{R}(G)$ such that

$$o_L(\overrightarrow{G}+X)=\mathrm{R}\quad\text{for all }\ \overrightarrow{G}\in\mathbb{R}(G).$$

If this were not the case, Left could maneuver with checks and we would not have $o_L(G+X)=\mathrm{R}$. We can also conclude that

$$o_R(\overrightarrow{G}^L+X)=\mathrm{R}\quad\text{for all }\ \overrightarrow{G}\in\mathbb{R}(G),$$

regardless of the Left option $\overrightarrow{G}^L$. Thus, if $\overrightarrow{G}\in\mathbb{R}(G)$ and $\overleftarrow{H^L}\in\mathbb{L}(H^L)$, there are neither

$$\overrightarrow{G}^L\geqslant\overleftarrow{H^L}\quad\text{nor}\quad\overrightarrow{G}\geqslant(\overleftarrow{H^L})^R.$$

Equivalently, the pair $(\mathbb{R}(G),\mathbb{L}(H^L))$ breaks the ∞-maintenance property, which constitutes a contradiction.

(c) Suppose next that $o_L(H+X)=o_R(H+X^L)=\mathrm{L}$ for some game $X^L\in X^{\mathcal{L}}$. By the initial smallest birthday assumption, together with (a), $o_R(G+X^L)\geqslant o_R(H+X^L)$. Thus we conclude that $o_L(G+X)\geqslant o_R(G+X^L)\geqslant o_R(H+X^L)=o_L(H+X)$. Consequently, in that case, $o_L(G+X)=\mathrm{L}$, which is once again a contradiction.

A contradiction is unavoidable given the assumptions. Hence, we have shown that $G\geqslant H$. □

We have arrived at the main result of this section.

Theorem 33 (Main Theorem). *Let $G, H\in\mathbb{Np}^\infty\setminus\{\infty,\overline{\infty}\}$. Then $G\geqslant H$ if and only if $\boldsymbol{M}^\infty(G,H)$.*

Proof. Combine Theorems 30 and 32. □

Situations with many checks are relatively rare in game practice. In most cases, the traditional maintenance property, as stated in Definition 1, suffices to verify whether $G\geqslant H$.

Theorem 34. *Let $G, H\in\mathbb{Np}^\infty\setminus\{\infty,\overline{\infty}\}$. If $\boldsymbol{M}(G,H)$ then $G\geqslant H$.*

Proof. If $\boldsymbol{M}(G,H)$, then $\boldsymbol{M}^\infty(G,H)$, since G belongs to every strategy $\mathbb{R}(G)$, H^L belongs to every strategy $\mathbb{L}(H^L)$, G^R belongs to every strategy $\mathbb{R}(G^R)$, and H belongs to every strategy $\mathbb{L}(H)$.

As $\boldsymbol{M}^\infty(G,H)$, Theorem 33 ensures that $G\geqslant H$. □

5. Reductions and reduced forms

In $\mathbb{Np}^\infty$, the game reductions are essentially the same as in $\mathbb{Np}$. The following three theorems are stated from Left's perspective, but, of course, there are dual statements from Right's perspective.

Domination captures the idea that if a player has an option at least as good as another, that other option becomes superfluous.

Theorem 35 (Domination). *If $G \in \mathbb{N}\mathbb{p}^{\infty}$ is a game with two Left options $G^{L_1}, G^{L_2} \in G^{\mathcal{L}}$, where $G^{L_2} \geqslant G^{L_1}$, then $G = \{G^{\mathcal{L}} \setminus \{G^{L_1}\} \mid G^{\mathcal{R}}\}$.*

Proof. Let $G' = \{G^{\mathcal{L}} \setminus \{G^{L_1}\} \mid G^{\mathcal{R}}\}$. We want to verify that $o_L(G + X) = o_L(G' + X)$ and $o_R(G + X) = o_R(G' + X)$ for every game X different from ∞ and $\overline{\infty}$.

If Left wins $G' + X$ with a move $G' + X^L$, then, by induction, she also wins $G + X$ with the move $G + X^L$. Analogously, if Left wins $G + X$ with a move $G + X^L$, then, by induction, she also wins $G' + X$ with the move $G' + X^L$. The argument for the case where Right plays first in the component X is similar.

Regarding moves other than in the component X, the only situation that escapes direct mimicry pertains to a possible winning move by Left in $G + X$ to $G^{L_1} + X$. In that case, playing first, she also wins $G' + X$ by moving to $G^{L_2} + X$, since $G^{L_2} \geqslant G^{L_1}$. □

Open reversibility captures the idea that if a player has an option in a component against which the opponent, in a worst-case scenario, can respond in a way to not harm their situation, the player chooses that option only if they intend to continue playing locally. If the intention is to play in another component, it is better to play in that component right from the start, thus avoiding the opportunity for the mentioned opponent's improvement.

Theorem 36 (Open Reversibility). *If $G \in \mathbb{N}\mathbb{p}^{\infty}$ is a game with $G^L \in G^{\mathcal{L}}$ and $G^{LR} \neq \overline{\infty}$ such that $G \geqslant G^{LR}$, then $G = \{G^{\mathcal{L}} \setminus \{G^L\}, G^{LR\mathcal{L}} \mid G^{\mathcal{R}}\}$.*

Proof. We note that $G^{LR\mathcal{L}}$ is not empty. This happens because G^{LR} cannot be ∞ due to the condition $G \geqslant G^{LR}$ and it cannot be $\overline{\infty}$ due to the condition $G^{LR} \neq \overline{\infty}$. Let $H = \{G^{\mathcal{L}} \setminus \{G^L\}, G^{LR\mathcal{L}} \mid G^{\mathcal{R}}\}$. We want to prove that $H = G$, i.e., $G \geqslant H$ and $H \geqslant G$.

First, we prove that $H \geqslant G^{LR}$. Let X be a game different from ∞ and $\overline{\infty}$. If Left wins $G^{LR} + X$ with $G^{LR} + X^L$, then, by induction, she also wins $H + X$ with the move $H + X^L$. If Left wins $G^{LR} + X$ with a move $G^{LRL} + X$, she also wins $H + X$ with the move $G^{LRL} + X$, since $G^{LRL} \in H^{\mathcal{L}}$. If Right, playing first, wins $H + X$ with $H + X^R$, then, by induction, she also wins $G^{LR} + X$ with the move $G^{LR} + X^R$. Suppose now that Right, playing first, wins $H + X$ with $H^R + X$. Since $H^R \in G^{\mathcal{R}}$, we have that $H^R + X \in (G + X)^{\mathcal{R}}$. Consequently, Right has a winning move in $G + X$, and due to the fact that $G \geqslant G^{LR}$, he also has a winning move in $G^{LR} + X$.

Second, we prove that $H \geqslant G$. The only case whose analysis is not straightforward is a possible winning move for Left from $G + X$ to $G^L + X$. In that case, it is mandatory for Left, playing first, to have a winning move in $G^{LR}+X$. As we already know that $H \geqslant G^{LR}$, Left must have a winning move in $H+X$ as well.

Third, we prove that $G \geqslant H$. The only cases whose analysis is not straightforward are possible winning moves for Left from $H+X$ to $G^{LRL}+X$. Since these winning moves are also available in $G^{LR}+X$, the fact that $G \geqslant G^{LR}$ implies that Left must also have a winning move in $G+X$. □

Absorbing reversibility captures the idea that if a player has an option that allows the opponent to make a winning terminating move, in practice, that option works as if the player were directly resigning.

Theorem 37 (Absorbing Reversibility). *If* $G \in \mathbb{N}\mathbb{p}^{\infty}$ *is a game with a Left option* $G^L = \{G^{L\mathcal{L}} \mid \overline{\infty}\}$, *then* $G = \{G^{\mathcal{L}} \setminus \{G^L\}, \overline{\infty} \mid G^{\mathcal{R}}\}$.

Proof. Let $H = \{G^{\mathcal{L}} \setminus \{G^L\}, \overline{\infty} \mid G^{\mathcal{R}}\}$. We want to prove that $H = G$, i.e., $G \geqslant H$ and $H \geqslant G$.

First, we prove that $G \geqslant H$. Let X be a game different from ∞ and $\overline{\infty}$. If Left wins $H+X$ with $H+X^L$, then, by induction, she also wins $G+X$ with the move $G+X^L$. Regarding moves other than in the component X, the only Left option in $H+X$ that escapes direct mimicry is $\overline{\infty}+X$. However, there is no need to consider that possibility, as it is a suicidal move for Left. To prove that the existence of a winning move for Right in $G+X$ implies the existence of a winning move for Right in $H+X$, it is sufficient to use induction and mimicry.

Second, we prove that $H \geqslant G$. Once again, winning moves in component X are addressed by induction. Regarding moves other than in the component X, the only Left option in $G+X$ that escapes direct mimicry is $\{G^{L\mathcal{L}} \mid \overline{\infty}\}+X$. However, there is no need to consider that possibility, as Right can win by responding to $\overline{\infty}+X$. To prove that the existence of a winning move for Right in $H+X$ implies the existence of a winning move for Right in $G+X$, it is sufficient to use induction and mimicry. □

Similar to what is formalized in classical theory, a game form G is reduced if none of its followers admits any of the three reductions stated in Theorems 35, 36, and 37 (also in terms of dual formulations according to Right's perspective). Interestingly, in $\mathbb{N}\mathbb{p}^{\infty}$, there can be two reduced forms, G and G', such that $G = G'$ and $G \not\cong G'$, meaning that the uniqueness of reduced forms is lost. Once again, it is a consequence of checks and forcing sequences. Still, this is not a serious issue, since reduced forms can still be used in proofs and for the purpose of simplifications. The following games constitute an example of this occurrence:

$$G = \Big\{\{\infty \mid \{0, *2 \mid \overline{\infty}\}, \{*, *3 \mid \overline{\infty}\}\}, \Big\{\infty \mid \{0, * \mid \overline{\infty}\}, \{*2, *3 \mid \overline{\infty}\}\Big\} \parallel \overline{\infty}\Big\},$$
$$H = \Big\{\{\infty \mid \{0, *2 \mid \overline{\infty}\}, \{*, *3 \mid \overline{\infty}\}\}, \Big\{\infty \mid \{0, *3 \mid \overline{\infty}\}, \{*, *2 \mid \overline{\infty}\}\Big\} \parallel \overline{\infty}\Big\}.$$

We have that $G \ncong H$ because, although the game forms share a Left check, there is a pair of distinct Left checks. On the other hand, a careful application of Theorem 33 allows us to verify that $G \geqslant H$ and $H \geqslant G$, i.e., $G = H$. The same theorem allows us to check that there are no reducible followers in either of the two game forms.

In practice, what happens is that, although there are different checks, the control provided by its forcing nature is essentially the same. The following definition formalizes the concept of reduced form.

Definition 38 (Reduced Forms). An affine game form G is a *reduced form* if the following two conditions are satisfied:

- None of the followers of G admits any of the three reductions stated in Theorems 35, 36, and 37.
- There is no G' such that $G' = G$ and $\tilde{b}(G') < \tilde{b}(G)$.

6. Conway-embedding and invertibility

Recall that the effect of the absence of moves (empty set) can be achieved in $\mathbb{N}\mathbb{p}^{\infty}$ by choosing $G^{\mathcal{L}} = \{\overline{\infty}\}$ or $G^{\mathcal{R}} = \{\infty\}$ for the player without moves; namely, a single resigning move is the same as having no moves at all. In this section, we formalize this idea, proving that the traditional Conway values are order-embedded in the affine structure. In fact, we will prove that the group structure of Conway values is exactly the substructure of the invertible elements of $\mathbb{N}\mathbb{p}^{\infty}$.

Definition 39 (Conway Forms and Games). A game $G \in \mathbb{N}\mathbb{p}^{\infty}$ is a *Conway Form* if G is distinct from ∞ and $\overline{\infty}$, and has no checks as followers. Let $\mathbb{N}\mathbb{p}^{C} \subseteq \mathbb{N}\mathbb{p}^{\infty}$ denote the substructure of Conway Forms. A game $G \in \mathbb{N}\mathbb{p}^{\infty}$ is a *Conway Game* if it is equal to a Conway Form in terms of equivalence of games modulo $\mathbb{N}\mathbb{p}^{\infty}$. A game $G \in \mathbb{N}\mathbb{p}^{\infty}$ is a *pressing game* if it is not a Conway Game.

Example 40. The game $G = \{\{\overline{\infty} \mid \infty\} \mid \{\overline{\infty} \mid \infty\}\} = \{0 \mid 0\} = *$ is a Conway Form (the game form has no checks as followers). The game $G' = \{\{\infty \mid *\} \mid \{* \mid \overline{\infty}\}\}$ is not a Conway Form because it has checks as followers. However, since $G' = G$, it is a Conway Game.

Definition 41 (Conway Map). The *Conway map* $c : \mathbb{N}\mathbb{p}^{C} \to \mathbb{N}\mathbb{p}$ is recursively defined in the following way:

$$c(G) = \begin{cases} 0 = \{\varnothing \mid \varnothing\} & \text{if } \; G = 0 = \{\overline{\infty} \mid \infty\}, \\ \{\varnothing \mid c(G^{\mathcal{R}})\} & \text{if } \; G = \{\overline{\infty} \mid G^{\mathcal{R}}\} \; \text{ and } \; G^{\mathcal{R}} \neq \{\infty\}, \\ \{c(G^{\mathcal{L}}) \mid \varnothing\} & \text{if } \; G = \{G^{\mathcal{L}} \mid \infty\} \; \text{ and } \; G^{\mathcal{L}} \neq \{\overline{\infty}\}, \\ \{c(G^{\mathcal{L}}) \mid c(G^{\mathcal{R}})\} & \text{otherwise.} \end{cases}$$

Here, $c(G^{\mathcal{L}})$ means the set $\{c(G^{L}) : G^{L} \in G^{\mathcal{L}}\}$, and analogously for $c(G^{\mathcal{R}})$.

Lemma 42. *If* $G \in \mathbb{Np}^{\boldsymbol{C}}$*, then* $c(\overline{G}) = \overline{c(G)}$.

Proof. This is an immediate consequence of Definitions 13 and 41. □

Theorem 43 (Conway Map). *The Conway map is a one-to-one map.*

Proof. We define first the following map from $\mathbb{Np}$ to $\mathbb{Np}^{\mathrm{C}}$:

$$c^{-1}(G) = \begin{cases} 0 = \{\overline{\infty} \mid \infty\} & \text{if } G = 0 = \{\varnothing \mid \varnothing\}, \\ \{\overline{\infty} \mid c^{-1}(G^{\mathcal{R}})\} & \text{if } G = \{\varnothing \mid G^{\mathcal{R}}\} \text{ and } G^{\mathcal{R}} \neq \varnothing, \\ \{c^{-1}(G^{\mathcal{L}}) \mid \infty\} & \text{if } G = \{G^{\mathcal{L}} \mid \varnothing\} \text{ and } G^{\mathcal{L}} \neq \varnothing, \\ \{c^{-1}(G^{\mathcal{L}}) \mid c^{-1}(G^{\mathcal{R}})\} & \text{otherwise.} \end{cases}$$

Now, we prove that $c^{-1}c = \mathbf{1}$, where $\mathbf{1}$ is the identity map. Proving that cc^{-1} is also the identity is entirely analogous, so we refrain from doing it. The proof is carried out for each of the four cases of the definitions of the maps, using induction:

Case 1: $$cc^{-1}(0) = cc^{-1}(\{\varnothing \mid \varnothing\}) = c(\{\overline{\infty} \mid \infty\}) = \{\varnothing \mid \varnothing\} = 0;$$

Case 2: $$\begin{aligned} cc^{-1}(\{\varnothing \mid G^{\mathcal{R}}\}) &= c(\{\overline{\infty} \mid c^{-1}(G^{\mathcal{R}})\}) \\ &= \{\varnothing \mid cc^{-1}(G^{\mathcal{R}})\} \\ &= \{\varnothing \mid G^{\mathcal{R}}\} \quad \text{(induction);} \end{aligned}$$

Case 3: $$\begin{aligned} cc^{-1}(\{G^{\mathcal{L}} \mid \varnothing\}) &= c(\{c^{-1}(G^{\mathcal{R}}) \mid \infty\}) \\ &= \{cc^{-1}(G^{\mathcal{L}}) \mid \varnothing\} \\ &= \{G^{\mathcal{L}} \mid \varnothing\} \quad \text{(induction);} \end{aligned}$$

Case 4: $$\begin{aligned} cc^{-1}(\{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}) &= c(\{c^{-1}(G^{\mathcal{L}}) \mid c^{-1}(G^{\mathcal{R}})\}) \\ &= \{cc^{-1}(G^{\mathcal{L}}) \mid cc^{-1}(G^{\mathcal{R}})\} \\ &= \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\} \quad \text{(induction).} \end{aligned}$$ □

Observation 44. The game tree of a Conway Form has no "small arrows" and is identical to the game tree of the corresponding form in $\mathbb{Np}$.

The following theorem constitutes a fundamental result, directly related to Theorem 20. Since Conway Games are invertible, with their inverses being their conjugates, the comparison involving these elements can be done by playing, in the traditional way.

One can also establish a third parallel with what happens on the extended real number line. In that structure, infinities are the only noninvertible elements. In $\mathbb{Np}^{\infty}$, noninvertible elements are those where there is some pressure exerted by an infinity in one of their followers (this also explains the choice of the term "pressing game").

Theorem 45. *Let $G \in \mathbb{N}\mathbb{p}^{\infty}$. We have that G is invertible if and only if G is a Conway Game. Moreover, if G is a Conway Game, then $\overline{G} = -G$.*

Proof. ($\Leftarrow$) Suppose first that G is a Conway Form. If $G = 0$, then $\overline{G} = 0$ and hence $G + \overline{G} = 0$. Otherwise, according to Corollary 17, it suffices to verify that $G + \overline{G}$ is a $\mathcal{P}$-position. If Left, playing first, chooses $G^L + \overline{G}$, because this game is not ∞ (G is not a check), Right can answer with $G^L + \overline{G^L}$, and, by induction, because G^L is a Conway Form with no checks as followers, that option is equal to zero. Because of that, by Corollary 17, that option is a $\mathcal{P}$-position, and Right wins. Analogous arguments work for the other options of the first player, and so, $G + \overline{G}$ is a $\mathcal{P}$-position. Again, by Corollary 17, $\overline{G} = -G$.

Suppose now that G is a Conway Game, but not a Conway Form. Because it is a Conway Game, by definition, it is equal to some $H \in \mathbb{N}\mathbb{p}^{\mathrm{C}}$. The first paragraph proved that $H + \overline{H} = 0$. Also, by symmetry, $\overline{G} = \overline{H}$. Therefore, $-H = \overline{H}$ implies $-G = \overline{G}$.

($\Rightarrow$) Let $G \in \mathbb{N}\mathbb{p}^{\infty}$ be a reduced form of a pressing game. Suppose that there is $H \in \mathbb{N}\mathbb{p}^{\infty}$ such that $G + H = 0$, i.e., $G + H \in \mathcal{P}$. Assume that H is also a reduced form and that $\tilde{b}(G) + \tilde{b}(H)$ is the smallest possible for G and H satisfying these conditions.

If G has no options, G is either ∞ or $\overline{\infty}$, and $G + H \notin \mathcal{P}$, which is a contradiction. If all options of G are Conway Games, then G itself must be a Conway Game, leading to a contradiction as well. If $G = \{\overline{\infty} \mid \infty\}$, then $G = 0$, leading to another contradiction. So, without loss of generality, let $G^{L_1} + H$ be a move such that $G^{L_1} \neq \overline{\infty}$ is a pressing game.

Against $G^{L_1} + H$, Right must have a winning response less than or equal to zero. Suppose that Right's winning answer is some $G^{L_1 R} + H \leqslant 0$. If so, by adding G to both sides, we have $G^{L_1 R} \leqslant G$. This contradicts the assumption (reduced forms), as if $G^{L_1 R}$ is $\overline{\infty}$, then G^{L_1} is absorbing reversible, and if $G^{L_1 R}$ is not $\overline{\infty}$, then G^{L_1} is open reversible.

Suppose now that Right's winning response is some $G^{L_1} + H^{R_1} \leqslant 0$. If $G^{L_1} + H^{R_1} = 0$, then it contradicts the assumption once more (smallest possible $\tilde{b}(G) + \tilde{b}(H)$), since G^{L_1} is a pressing game. Therefore, we have $G^{L_1} + H^{R_1} < 0$.

In order to initiate a "carousel argument", often used in CGT, consider the first move by Right in $G + H$ to $G + H^{R_1}$. By similar reasons, Left must have an option $G^{L_2} + H^{R_1} \geqslant 0$. If H^{R_1} is a Conway Game, by the first part, it is invertible, and we have $G^{L_1} < -H^{R_1} \leqslant G^{L_2}$. In this case, G^{L_1} is a dominated option, contradicting the assumption (reduced forms). On the other hand, if H^{R_1} is not a Conway Game, we cannot have $G^{L_2} + H^{R_1} = 0$ as it once again contradicts the assumption (smallest possible $\tilde{b}(G)+\tilde{b}(H)$). Thus, we have $G^{L_2}+H^{R_1}>0$. Turning the carousel, a first move by Left in $G+H$ to $G^{L_2}+H$ must be answered with some $G^{L_2}+H^{R_2}<0$. And turning again, a first move by Right in $G+H$ to $G+H^{R_2}$ must be answered with some $G^{L_3}+H^{R_2}>0$. Note that, in this case, it is not possible to have $G^{L_1}+H^{R_2}>0$, since we would have

- $G^{L_1}+H^{R_1}<0$;
- $\overline{G^{L_2}}+\overline{H^{R_1}}<0$ (since $G^{L_2}+H^{R_1}>0$);
- $G^{L_2}+H^{R_2}<0$;
- $\overline{G^{L_1}}+\overline{H^{R_2}}<0$ (since $G^{L_1}+H^{R_2}>0$).

Combining these inequalities by making use of Theorem 22 would lead to the inequality

$$G^{L_1}+H^{R_1}+G^{L_2}+H^{R_2}+\overline{G^{L_1}}+\overline{H^{R_1}}+\overline{G^{L_2}}+\overline{H^{R_2}}<0.$$

That cannot occur, given that the specified game, due to its symmetry, can belong to $\mathcal{P}$ or $\mathcal{N}$, but never to $\mathcal{R}$. Entirely analogous arguments are sufficient to show that the winning responses must always be obtained with options different from the previous ones. Since the carousel cannot rotate indefinitely, it is not possible to have G and H such that $G+H\in\mathcal{P}$, i.e., such that $G+H=0$. □

We conclude this section by proving the Conway-embedding.

Theorem 46 (Conway-embedding). *For any $G, H \in \mathbb{Np}^{C}$, if $G \geqslant H$ then $c(G) \geqslant c(H)$, where the first is the order relation of $\mathbb{Np}^{\infty}$ and the second is the order relation of $\mathbb{Np}$. Similarly, for any $G, H \in \mathbb{Np}$, if $G \geqslant H$ then $c^{-1}(G) \geqslant c^{-1}(H)$, where the first is the order relation of $\mathbb{Np}$ and the second is the order relation of $\mathbb{Np}^{\infty}$.*

Proof. Let G and H be two Conway Forms such that $G \geqslant H$. According to Theorem 45, H is invertible and, by Theorem 20, we have $G-H\in\mathcal{L}\cup\mathcal{P}$. Let us verify that, in $\mathbb{Np}$, we also have $c(G)+c(\overline{H})\in\mathcal{L}\cup\mathcal{P}$. With that verified, Lemma 42 ensures that $c(G)+c(\overline{H})\geqslant 0 \iff c(G)-c(H)\geqslant 0 \iff c(G)\geqslant c(H)$. Suppose that Right moves to some $c(G^R)+c(\overline{H})$. Considering the game $G^R-H\in\mathbb{Np}^{\infty}$, Left must have a winning move, which can be either some $G^{RL}-H\geqslant 0$ or some $G^R-H^R\geqslant 0$. Concerning $\mathbb{Np}$, by induction, in the first case, we have $c(G^{RL})+c(\overline{H})\geqslant 0$, and in the second case, $c(G^R)+c(-H^R)\geqslant 0$. In other words, Left has a winning move in $c(G^R)+c(\overline{H})$. If Right moves to some $c(G)+c(-H^L)$, the inductive argument works exactly the same way, and Left has a winning response. We can conclude that $c(G)\geqslant c(H)$.

The proof for the second implication in the theorem can be carried out by induction in a completely analogous manner. □

7. Classification of affine games

In the classical structure $\mathbb{Np}$ games are classified as *cold*, *tepid*, or *hot*. This classification is based on the concept of *numbers* and the related Number Avoidance Theorem. When a game is a number (cold), both players have a negative incentive; they are in a *zugzwang* in the sense that making a move leads to a "loss" in terms of guaranteed moves (if there are any moves available at all).[10] For example, $\frac{1}{2} = \{0 \mid 1\}$; if Left plays, she reduces $\frac{1}{2}$ to 0, and if Right plays, he increases $\frac{1}{2}$ to 1. In other words, both players prefer the opponent to play rather than playing themselves. When a game is hot, the opposite occurs. There is urgency in the sense that making a move leads to a "gain" of guaranteed moves. An example of a hot game is $\pm 1 = \{1 \mid -1\}$. When a game is tepid, a move brings neither gains nor losses in terms of guaranteed moves. Playing may be important for the sake of making the last move per se, but not in terms of gaining guaranteed moves. An example of a tepid game is $\{1 \mid 1\}$; Left already has one guaranteed move regardless of whether she moves first or not. Thus, intuitively, the Number Avoidance Theorem establishes that a player should not play on numbers in order to avoid losing what is already guaranteed; playing in hot games or tepid games avoids such a loss.

In this section, we extend these concepts, considering that in $\mathbb{Np}^\infty$, infinities come into play. Numbers in $\mathbb{Np}^\infty$ are the same as the numbers in $\mathbb{Np}$, since $\overline{\infty}$ (and ∞) for Left (Right) are equivalent to having no moves at all. For this purpose, it is important to start with the Number Avoidance Theorem, which still holds in affine normal play.

Theorem 47 (Number Avoidance Theorem). *Let $G, x \in \mathbb{Np}^\infty$, and suppose x is a number and G is not. If Left* (*resp. Right*) *has a winning move on $G + x$, then she has a winning move of the form $G^L + x$* (*resp. $G^R + x$*).

Proof. We may assume that x is in the classical reduced form via Conway-embedding, since the outcomes of the considered sums do not depend on its form. If $G = \infty$ or $G = \overline{\infty}$ there are no moves and there is nothing to prove. Now suppose that $G + x^L$ is a winning move for Left. According to Theorem 16, we have $G + x^L \geqslant 0$. But G is not equal to a number, and thus $G \neq -x^L$ (x^L is invertible), so in fact $G + x^L > 0$ is mandatory. Again by Theorem 16, Left has a winning move in $G + x^L$. By induction, we may assume that it is some $G^L + x^L \geqslant 0$. Since x is in the classical reduced form, we have $x^L < x$ and $G^L + x > G^L + x^L \geqslant 0$. Hence, $G^L + x$ is a winning move for Left in $G + x$. The proof from Right's perspective is analogous. □

[10]Zugzwang (from German "compulsion to move") refers to a situation where a player is placed at a disadvantage because they must make a move.

The concept of *stop* in classical theory is based on the assumption that two players move in a component, and, considering the Number Avoidance Theorem, stop playing on that component as soon as a number has been reached. Left attempts to have this stopping number be as large as possible, and Right wants it to be as small as possible. Our next step consists of defining this concept for $\mathbb{Np}^\infty$, in order to classify affine games similarly to the classical classification, with just a few additions due to infinities. An obvious difference when comparing with the classical structure is that the stopping point may not be a number, but rather ∞ or $\overline{\infty}$. This is because there is the possibility for one of the players to force a terminating move.

Definition 48 (Stops). *Left* and *Right stops* of G are recursively defined by

$$\mathrm{Ls}(G) = \begin{cases} \infty & \text{if } G = \infty, \\ \overline{\infty} & \text{if } G = \overline{\infty}, \\ x & \text{if } G = x \text{ is a number}, \\ \max\{\mathrm{Rs}(G^L) \mid G^L \in G^{\mathcal{L}}\} & \text{otherwise}, \end{cases}$$

$$\mathrm{Rs}(G) = \begin{cases} \infty & \text{if } G = \infty, \\ \overline{\infty} & \text{if } G = \overline{\infty}, \\ x & \text{if } G = x \text{ is a number}, \\ \min\{\mathrm{Ls}(G^R) \mid G^R \in G^{\mathcal{R}}\} & \text{otherwise}. \end{cases}$$

Definition 49. Let $G \in \mathbb{Np}^\infty$. Then

- G is *cold* if it is a number;
- G is *tepid* if the stops are finite and $\mathrm{Ls}(G) = \mathrm{Rs}(G)$, but G is not a number;
- G is *hot* if $\mathrm{Ls}(G) > \mathrm{Rs}(G)$;
- G is *bubbling* if both stops are ∞ or both stops are $\overline{\infty}$.

Example 50. The following list has an example of each of the four types of affine games.

- $G = \{\{\infty \mid -1*\} \mid 0\}$ is a number since $\{\infty \mid -1*\}$ is reversible through $-1*$, and G reduces to $-\frac{1}{2} = \{-1 \mid 0\}$.
- $G = \{\{\infty \mid 0\} \mid 0\}$ is tepid since both stops are finite and $\mathrm{Ls}(G) = \mathrm{Rs}(G) = 0$. It is a reduced form of a pressing game, as it is not equivalent to any Conway Form.
- $G = \{\{\infty \mid 1\} \mid 0\}$ is hot since $\mathrm{Ls}(G) = 1$, $\mathrm{Rs}(G) = 0$, and $\mathrm{Ls}(G) > \mathrm{Rs}(G)$. The game G is a pressing game, as it is not equivalent to any Conway Form.
- $G = \{\infty \mid \infty\}$ is bubbling since $\mathrm{Ls}(G) = \mathrm{Rs}(G) = \infty$. It is inevitable that G ends with Left delivering a checkmate, regardless of whether she plays first

or not. In this sense, G has already reached an irreversible boiling state, which explains the choice of the word "bubbling".

The class of *mate games* is a subclass of bubbling games. Its formalization is needed given the frequency with which these situations occur. In an n-Leftmate (n-Rightmate), Left (Right) is guaranteed the possibility of checkmating in n moves, without the opponent being able to defend. However, the checkmate cannot occur any faster than that.

Definition 51 (Mate Games). Let n be a nonnegative integer. An *n-Leftmate* ∞_n is recursively defined by

$$\infty_n = \begin{cases} \infty & \text{if } n = 0, \\ \{\infty_{n-1} \mid \infty\} & \text{if } n > 0. \end{cases}$$

The definition of an *n-Rightmate*, denoted by $\overline{\infty}_n$, is symmetric.

Note that $\infty_n \neq \infty$, because in a sum with ∞_n, Right can still win, if he can mate faster. In particular, $\cdots < \infty_n < \infty_{n-1} < \cdots < \infty_1 < \infty_0 = \infty$.

Observation 52. The game $\{\infty_3 \mid \infty_8\}$ is a bubbling game, but it is not an n-Leftmate. The reason for this is that the inevitable checkmate can occur at different speeds, depending on whether it is Left or Right who starts. If Right starts, he can delay his fateful destiny in the component a little, and this can be crucial in certain disjunctive sums.

Nomenclature. When a game is tepid and the stops are equal to zero, it is an *infinitesimal*. When a game is tepid and the stops are not zero, it is a *translation of an infinitesimal*. When a game is hot and at least one of the stops is infinite, it is *scalding*.

Observation 53. There are infinitesimals that are not Conway Games, such as $\{\{\infty \mid 0\} \mid 0\}$ already used in Example 50 and materialized in Figure 9. On the other hand, there are no scalding games that are Conway Games. For example, $\{\infty_3 \mid 0\}$ is a scalding game. In this game, Right is not yet condemned to being checkmated by the opponent as long as he retains the right to make the move. The temperature is infinite, but the boiling state of a bubbling game has not been reached yet, as in a scalding game neither player is definitively condemned to be checkmated.

Now, let us move on to an important concept regarding affine normal play. If we consider the classical structure, all reductions are based on the fundamental idea of "moves that will never be used". A dominated option never needs to be used. Regarding a reversible option, since it is only chosen with the intention of continuing to play locally; if a move is not related to this local continuation, it will never be used.

When we move to the affine structure $\mathbb{Np}^\infty$, something entirely new happens. There are checks that should "always" be used, with one single exception. Unless a player is facing a check in another component, the first thing to do is to deliver a check in the specified component. In affine normal play, in addition to "never", "always" also comes into play. Thus, there are games that, although they cannot be reduced, already contain a certain type of "intrinsic reduction": they will be the target of a move at the first opportunity. In a way, these components are ephemeral.[11] The following definition formalizes this idea, capturing the concept of a *hammerzug*. A hammerzug (from German "move with a hammer") is a move that a player should always make, unless they are under lethal threat elsewhere, as it can only help without having any negative consequences. That is, when a hammerzug is a component of a disjunctive sum, a player should, without any delay, deliver a check in that component, as long as they are not already facing a check elsewhere.

Definition 54 (Hammerzug). A game $G \in \mathbb{Np}^\infty$ is a *Left hammerzug* if there exists a Left check $G^L = \{\infty \mid G^{L\mathcal{R}}\}$ such that $G^{LR} \geqslant G$ for all $G^{LR} \in G^{L\mathcal{R}}$. The definition of a Right hammerzug is similar. A game $G \in \mathbb{Np}^\infty$ is a *hammerzug* if it is both a Left hammerzug and a Right hammerzug.

The game $\{\{\infty \mid 0\} \mid 0\}$ is a Left hammerzug (see also Definition 57). Of course, there is no dominated option, and standard reversibility does not apply. Moreover, $\{\infty \mid 0\}$ is not absorbing reversible, since $\{\{\infty \mid 0\} \mid 0\} < 0$. So, $\{\{\infty \mid 0\} \mid 0\}$ is in reduced form. It is an example of a Left hammerzug that does not allow further reductions.

Example 55. The game $G = \{\{\infty \mid 1\} \mid 0\} = \{\circlearrowright^1 \mid 0\}$ is a Left hammerzug since $1 > G$. The game $G = \{\{\infty \mid 1\} \mid \{-1 \mid \overline{\infty}\}\} = \{\circlearrowright^1 \mid \circlearrowleft^{-1}\}$ is a hammerzug since $1 > G > -1$.

Theorem 56 (Extreme Urgency). *If $G \in \mathbb{Np}^\infty$ is a Left hammerzug, then, for every Right quiet game X, $o_L(G+X) = o_R(G^L+X)$, where G^L is the Left check specified in Definition 54. There is a dual result in terms of Right hammerzugs.*

Proof. We prove that $o_L(G+X) \geqslant o_R(G^L+X)$ and $o_R(G^L+X) \geqslant o_L(G+X)$. The first inequality is evident, since, in a worst-case scenario, Left can move to G^L+X in $G+X$. Regarding the second inequality, we have $o_R(G^L+X) = o_L(G^{LR}+X)$, where G^{LR} is the best defense for Right against the Left check. By Definition 54, $o_L(G^{LR}+X) \geqslant o_L(G+X)$, so $o_R(G^L+X) \geqslant o_L(G+X)$. □

[11] One might draw a parallel with the extremely short-lived Top Quark and Higgs Boson etc. in Physics.

In the classical structure $\mathbb{Np}$, there are classes of infinitesimals of great relevance. The powers of up,

$$\uparrow = \{0 \mid *\}, \quad \uparrow^2 = \{0 \mid \downarrow + *\}, \quad \uparrow^3 = \{0 \mid \downarrow + \downarrow^2 + *\}, \ldots,$$

are infinitesimals with respect to the previous ones. This means that $n{\cdot}\uparrow^2 < \uparrow$, regardless of the number of copies n. The same applies to $n{\cdot}\uparrow^3 < \uparrow^2$, $n{\cdot}\uparrow^4 < \uparrow^3$, and so on. The tinies and minies exhibit a similar behavior. The game tiny, denoted by $⧾_1$, is the positive game $\{0 \mid \{0 \mid -1\}\}$. Similarly, the game miny, denoted by $⧿_1$, is the negative game $\{\{1 \mid 0\} \mid 0\}$. In general, given a positive integer n, we have $⧾_n = \{0 \mid \{0 \mid -n\}\}$ and $⧿_n = \{\{n \mid 0\} \mid 0\}$. These are infinitesimals of a "magnitude" different from the powers of up. Saying the same rigorously, in addition to the facts, for all positive numbers n, $n \cdot ⧾_2 < ⧾_1$, $n \cdot ⧾_3 < ⧾_2$, and so on, $⧾_1$ is infinitesimal with respect to $\uparrow^k$, for any positive integer k. Now, for a better understanding of what follows, it is helpful to make a brief observation about tinies and minies.

In a game like $\{0 \mid -80\} + ⧿_{100}$, Left, playing first, wins by playing in the infinitesimal component and not in the hot component. This happens because Left can make a significant threat in $⧿_{100}$. This threat is powerful enough to counteract the gain that Right can achieve in the other component. However, in game practice, it is not common to have a component with the value $⧿_{100}$. This is because it would require a threat of 100 in "territorial" terms to be defended with a move to 0, a situation that would require a ruleset with very special characteristics. However, in $\mathbb{Np}^\infty$, "powerful threat" has a worst case scenario in a "terminal threat", and given that mate threats are very frequent, there is a certain type of games that are very common. The ultimate tiny (miny) can be defined as follows.

Definition 57 (Pathetic Infinitesimals). The game $\{0 \mid \{0 \mid \overline{\infty}\}\} = \{0 \mid \circlearrowleft^0\}$ is called *pathetic tiny* and denoted by $⧾_\infty$. The game $\{\{\infty \mid 0\} \mid 0\} = \{\circlearrowleft^0 \mid 0\}$ is called *pathetic miny* and denoted by $⧿_\infty$.

It is trivial to verify that $⧾_\infty > 0 > ⧿_\infty$. Furthermore, the name "pathetic" is easily explainable. As observed in the previous paragraph, the presence of $⧿_{100}$ in a disjunctive sum tends to be ephemeral, as in most cases, Left uses her threat to replace, to her advantage, a negative game, by zero. Once again, in $\mathbb{Np}^\infty$, "in most cases" can be transformed into "always", as the following theorem establishes; this is the first of three notable facts about the pathetic infinitesimals. The "pathetic nature" of these positive and negative values is due to their ephemerality. Both a pathetic tiny and a pathetic miny have a very limited lifespan in a disjunctive sum, as, at the first opportunity, one of the players makes them disappear.

Theorem 58. *The pathetic tiny* $⧾_{\infty}$ *is a Right hammerzug. Similarly, the pathetic miny* $⧿_{\infty}$ *is a Left hammerzug.*

Proof. This is a direct consequence of Definition 54 and $⧾_{\infty} > 0 > ⧿_{\infty}$. □

The second notable fact concerns a certain "lack of continuity" exhibited by $\mathbb{Np}^{\infty}$. The smallest positive element and the largest negative element are effectively reached.

Theorem 59. *The game* $⧾_{\infty}$ *is the smallest positive game of all, i.e., if* $G \in \mathbb{Np}^{\infty}$ *is such that* $G > 0$, *then* $G \geqslant ⧾_{\infty}$. *Similarly, the game* $⧿_{\infty}$ *is the largest negative game of all.*

Proof. If $G = \infty$, there is nothing to prove, so assume otherwise. According to Theorem 16, $G > 0$ implies the existence of some G^L such that $G^L \geqslant 0$ (Fact 1). On the other hand, the same theorem guarantees that for each Right option $G^R \neq \infty$, there exists G^{RL} such that $G^{RL} \geqslant 0$. For these cases, it is straightforward to verify $\boldsymbol{M}(G^R, \{0 \mid \overline{\infty}\})$, since for the Left option to 0 in $\{0 \mid \overline{\infty}\}$, there exists some $G^{RL} \geqslant 0$. Consequently, according to Theorem 34, we have $G^R \geqslant \{0 \mid \overline{\infty}\}$ (Fact 2).

With the help of these two facts, let us check that the pair $(G, ⧾_{\infty})$ satisfies also the maintenance property as established in Definition 1, and, therefore, by Theorem 34, $G \geqslant ⧾_{\infty}$. For the Left move to 0 in $⧾_{\infty}$, there exists some $G^L \geqslant 0$ (Fact 1). For each Right option G^R, we have $G^R \geqslant \{0 \mid \overline{\infty}\}$ (Fact 2).

Proving that $⧿_{\infty}$ is the largest negative game of all is analogous. □

To better understand the third notable fact, recall that $\uparrow^2$ is infinitesimal with respect to $\uparrow$, i.e., any number of copies of $\uparrow^2$ is less than a single copy of $\uparrow$. But, it is important to note that this is entirely different from saying that having two copies or just one copy of $\uparrow^2$ is irrelevant. If only $\downarrow$ is present, Left loses all disjunctive sums of the form $n.\uparrow^2 + \downarrow$. However, in other sums, having two copies instead of one can be the difference between winning or losing. In fact, we have $2.\uparrow^2 > \uparrow^2$. Furthermore, in the classical structure $\mathbb{Np}$, if $G > 0$, then $2.G > G$.

The terminating games ∞ and $\overline{\infty}$ bring the existence of nontrivial finite idempotents. Having many or few copies of a pathetic tiny is irrelevant since, at the slightest opportunity, Right disposes of them all at once through a sequence of checks.

Theorem 60 (Pathetic Idempotents). *The games* $⧾_{\infty}$ *and* $⧿_{\infty}$ *are idempotents.*

Proof. We prove only the equality $⧾_{\infty} + ⧾_{\infty} = ⧾_{\infty}$, because the proof for $⧿_{\infty} + ⧿_{\infty} = ⧿_{\infty}$ is symmetric.

The inequality $⧾_{\infty} + ⧾_{\infty} \geqslant ⧾_{\infty}$ is a trivial consequence of Theorem 18, since $⧾_{\infty} > 0$.

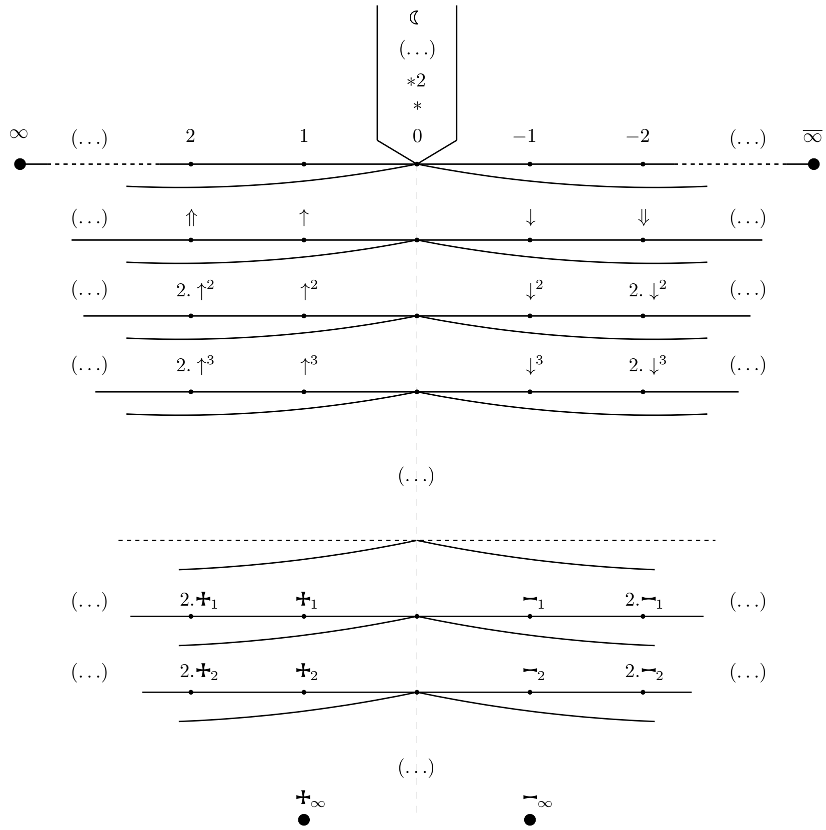


Figure 12. Affine game hierarchy.

Regarding the inequality $\boldsymbol{+}_\infty + \boldsymbol{+}_\infty \leqslant \boldsymbol{+}_\infty$, we begin by observing that the game $\boldsymbol{+}_\infty + \boldsymbol{+}_\infty$ is $\{\boldsymbol{+}_\infty \mid \{\boldsymbol{+}_\infty \mid \overline{\infty}\}\}$. Additionally, $\boldsymbol{M}(\{0 \mid \overline{\infty}\}, \{\boldsymbol{+}_\infty \mid \overline{\infty}\})$ because, for the Left move to $\boldsymbol{+}_\infty$ in $\{\boldsymbol{+}_\infty \mid \overline{\infty}\}$, there is $\{0 \mid \overline{\infty}\} \geqslant \{0 \mid \overline{\infty}\}$, and for the Right move to $\overline{\infty}$ in $\{0 \mid \overline{\infty}\}$, there is $\overline{\infty} \geqslant \overline{\infty}$. Hence, according to Theorem 34, we have $\{0 \mid \overline{\infty}\} \geqslant \{\boldsymbol{+}_\infty \mid \overline{\infty}\}$ (Fact 1). Moreover, $\boldsymbol{M}(\boldsymbol{+}_\infty, \{0 \mid \overline{\infty}\})$ because, for the Left move to 0 in $\{0 \mid \overline{\infty}\}$, there is $0 \geqslant 0$, and for the Right move to $\{0 \mid \overline{\infty}\}$ in $\boldsymbol{+}_\infty$, there is $\{0 \mid \overline{\infty}\} \geqslant \overline{\infty}$. Hence, according to Theorem 34, we have $\boldsymbol{+}_\infty \geqslant \{0 \mid \overline{\infty}\}$ (Fact 2).

With the help of these facts, let us check that $\boldsymbol{M}(\boldsymbol{+}_\infty, \{\boldsymbol{+}_\infty \mid \{\boldsymbol{+}_\infty \mid \overline{\infty}\}\})$. For the Left move to $\boldsymbol{+}_\infty$ in $\{\boldsymbol{+}_\infty \mid \{\boldsymbol{+}_\infty \mid \overline{\infty}\}\}$, there is $\boldsymbol{+}_\infty \geqslant \{0 \mid \overline{\infty}\}$ (Fact 2). For the Right move to $\{0 \mid \overline{\infty}\}$ in $\boldsymbol{+}_\infty$, there is $\{0 \mid \overline{\infty}\} \geqslant \{\boldsymbol{+}_\infty \mid \overline{\infty}\}$ (Fact 1). Hence, according to Theorem 34, we have $\boldsymbol{+}_\infty \geqslant \{\boldsymbol{+}_\infty \mid \{\boldsymbol{+}_\infty \mid \overline{\infty}\}\}$. □

Observation 61. Although, by Theorem 22, it is true in the case where $J = 0$, in general, it is not true that $G > 0$ and $H \geqslant J$ implies $G + H > J$. Just consider $G = H = J = \boldsymbol{+}_\infty$ and Theorem 60 to observe this fact.

Figure 12 contains a visualization of the affine game line. On one hand, the Conway Game line is closed with ∞ and $\overline{\infty}$ (larger vertices). On the other hand, the descending set of lines respecting some scales of infinitesimals is also closed with $+_\infty$ and $-_\infty$ (larger vertices). Note that the way to close this set is not with a line but only with two elements. This happens because these are idempotent. The game $ℂ = \{0, ⥁^0 \mid 0, ⥀^0\}$ will be discussed in Section 8.

8. Lattice structure of affine games born by day *n*

In this section, we discuss the lattice structure of the affine games born by day n. To this end, it is important to make a preliminary observation about the notion of birthday. The formalization, $\mathbb{A}$, presented in Definition 2 gives all literal form games, and sometimes we refer to the "formal birthday" of games in this context. Since the equality of games is an equivalence relation, one can consider the quotient space $\mathbb{Np}^\infty/{=}$ and it makes sense to define the *birthday* of G, $b(G)$, as the "birthday of the equivalence class of G". More precisely, the birthday of G is the smallest formal birthday among the games G' that are equivalent to G. We observe that, in $\mathbb{A}_n$, all games have a formal birthday less than or equal to n, but the formal birthday can be greater than the birthday. For example, $\{-2 \mid 2\} \in \mathbb{A}_3$, with a formal birthday of 3 and a birthday of 0, since $\{-2 \mid 2\} = 0$.

We will prove that the affine games born by day n form a lattice structure; the poset $(\mathbb{G}_n, \geqslant)$, consisting of equivalence classes with a birthday less than or equal to n, is a lattice structure. Whenever we write $G \in \mathbb{G}_n$ or $G \geqslant H$, we are using representatives from the equivalence classes. That said, we begin with a definition necessary for the main proof.

Definition 62. Let $G \in \mathbb{G}_n$. The sets $\lfloor G \rfloor_n$ and $\lceil G \rceil_n$ are

$$\lfloor G \rfloor_n = \{W : \text{there is some } \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\} \text{ equal to } G, \text{ where the birthdays of the games in } G^{\mathcal{L}} \text{ and } G^{\mathcal{R}} \text{ are less than } n, \text{ and } W \in G^{\mathcal{L}}\};$$

$$\lceil G \rceil_n = \{W : \text{there is some } \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\} \text{ equal to } G, \text{ where the birthdays of the games in } G^{\mathcal{L}} \text{ and } G^{\mathcal{R}} \text{ are less than } n, \text{ and } W \in G^{\mathcal{R}}\}.$$

We are now ready for the main result of this section.

Theorem 63 (Lattice Structure). *Let $n \geqslant -1$, and $G, H \in \mathbb{G}_n \setminus \{\infty, \overline{\infty}\}$. The poset $(\mathbb{G}_n, \geqslant)$ is a lattice where*

$$G \vee H = \{\lfloor G \rfloor_n \cup \lfloor H \rfloor_n \mid (\lceil G \rceil_n \cap \lceil H \rceil_n) \cup \{\infty\}\},$$
$$G \wedge H = \{(\lfloor G \rfloor_n \cap \lfloor H \rfloor_n) \cup \{\overline{\infty}\} \mid \lceil G \rceil_n \cup \lceil H \rceil_n\}.$$

Proof. If either of the games G or H is ∞ or $\overline{\infty}$, the elements $G \vee H$ and $G \wedge H$ arise from the absorbing nature of infinities. It is an easy application of Theorem 15 that, if $G = \infty$, then $G \vee H = \infty$ and $G \wedge H = H$ regardless of H, and if $G = \overline{\infty}$, then $G \vee H = G$ and $G \wedge H = \overline{\infty}$ regardless of H. In other words, $\top = \infty$ and $\bot = \overline{\infty}$. Hence, it is enough to consider $G, H \in \mathbb{A}_n \setminus \{\infty, \overline{\infty}\}$ and prove the facts stated in the theorem.

Firstly, let us prove that $G \vee H \geqslant G$ and $G \vee H \geqslant H$, where the join is

$$G \vee H = \{\lfloor G \rfloor_n \cup \lfloor H \rfloor_n \mid (\lceil G \rceil_n \cap \lceil H \rceil_n) \cup \{\infty\}\}.$$

We only need to prove that $G \vee H \geqslant G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$, since the proof for $G \vee H \geqslant H$ is analogous. The sets $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ are chosen in such a way that their elements have a birthday less than or equal to n. If Left, playing first, wins $G + X$ with some $G^L + X$, since $G^L \in \lfloor G \rfloor \subseteq \lfloor G \rfloor \cup \lfloor H \rfloor$, Left also wins $G \vee H + X$ by moving to $G^L + X$. If Left, playing first, wins $G + X$ with some $G + X^L$, then, by induction, Left also wins $G \vee H + X$ with $G \vee H + X^L$. On the other hand, if Right, playing first, wins $G \vee H + X$ with some $(G \vee H)^R + X$, due to $(G \vee H)^R \in \lceil G \rceil_n$, $(G \vee H)^R$ is a Right option of some G', equivalent to G. Thus, Right also wins $G' + X$ with the move $(G \vee H)^R + X$. Given the equivalence of G' and G, Right, playing first, must also win $G + X$. If Right, playing first, wins $G \vee H + X$ with some $G \vee H + X^R$, then, by induction, Right also wins $G \vee H + X$ with $G \vee H + X^R$.

Secondly, for $W \in \mathbb{G}_n$, let us prove that if $W \geqslant G$ and $W \geqslant H$, then $W \geqslant G \vee H$. Since it is a very straightforward fact to justify if W is ∞ or $\overline{\infty}$, let us assume that neither is the case. Let us start by proving that any $W^R \in W^{\mathcal{R}}$ is an element of $\lceil G \rceil_n \cap \lceil H \rceil_n$. To do so, it is sufficient to prove that any element W^R is an element of $\lceil G \rceil_n$, as doing the same in relation to $\lceil H \rceil_n$ is similar.

If $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$, then $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}} \cup \{W^R\}\}$. This happens because, on one hand, according to Theorem 23, $\{G^{\mathcal{L}} \mid G^{\mathcal{R}}\} \geqslant \{G^{\mathcal{L}} \mid G^{\mathcal{R}} \cup \{W^R\}\}$. On the other hand, $\{G^{\mathcal{L}} \mid G^{\mathcal{R}} \cup \{W^R\}\} \geqslant \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$ is a consequence that if $W^R + X$ is a winning move for Right, given $W \geqslant G$, there must also be a winning move for Right in $G + X$. Once $\{G^{\mathcal{L}} \mid G^{\mathcal{R}} \cup \{W^R\}\}$ is a possible form of G, by Definition 62, $W^R \in \lceil G \rceil_n$.

Now that we have established that $W^R \in \lceil G \rceil_n \cap \lceil H \rceil_n$, we prove $W \geqslant G \vee H$. If Left, playing first, wins $G \vee H + X$ with some $(G \vee H)^L + X$, then that move is a winning move in $G + X$ or in $H + X$, given that $(G \vee H)^{\mathcal{L}} = \lfloor G \rfloor_n \cup \lfloor H \rfloor_n$. Since $W \geqslant G$ and $W \geqslant H$, Left must also have a winning move in $W + X$. If Left, playing first, wins $G \vee H + X$ with some $G \vee H + X^L$, then, by induction, Left also wins $W + X$ with $W + X^L$. If Right, playing first, wins $W + X$ with some $W^R + X$, due to the fact $W^R \in \lceil G \rceil_n \cap \lceil H \rceil_n$, Right also wins $G \vee H + X$ by moving to $W^R + X$. If Right, playing first, wins $W + X$ with some $W + X^R$, then, by induction, Right also wins $G \vee H + X$ with $G \vee H + X^R$.

The same type of reasoning applies to $G \wedge H$. □

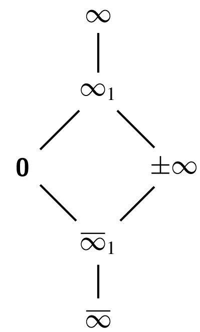


Figure 13. The lattice structure of the poset $(\mathbb{G}_0, \geqslant)$.

We conclude this section with some examples. With two atoms instead of one, the lattice structure of the affine games born by day zero has more elements than that of classical theory (Figure 13).

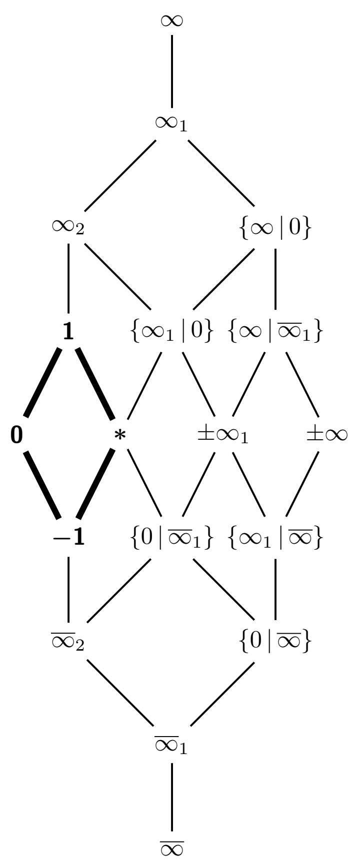


Figure 14. The lattice structure of the poset $(\mathbb{G}_1, \geqslant)$.

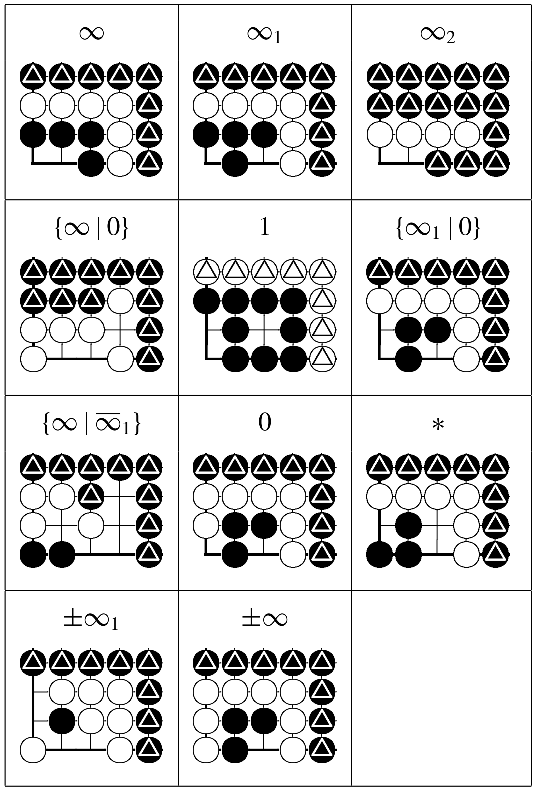


Figure 15. Materialization of $\mathbb{G}_1$ through ATARI GO positions. (The infinity shows the board before the removal of the white stones occurs.)

Figure 14 shows the lattice structure of the affine games born by day one. The bold represents the substructure of Conway Games born by day one, which is order-embedded. Figure 15 shows an ATARI GO position for each game value in $\mathbb{G}_1$. Note that some games fall out because they equal simpler games, such as $\{0 \mid \infty_1\} = 1$.

9. A recreational game example

The previous sections established the structure $\mathbb{Np}^\infty$, along with its properties. In this section, we discuss its application in game practice. The analysis of combinatorial games is important not only for the games themselves, but also for the paths it opens up. For example, recent projects carried out by DeepMind, such as ALPHAGO and ALPHAZERO, aimed to improve AI capabilities in playing combinatorial games. However, the goal was not only to analyze specific games, but also to push AI and mathematics, promoting algorithms capable of dealing with important real-world problems.

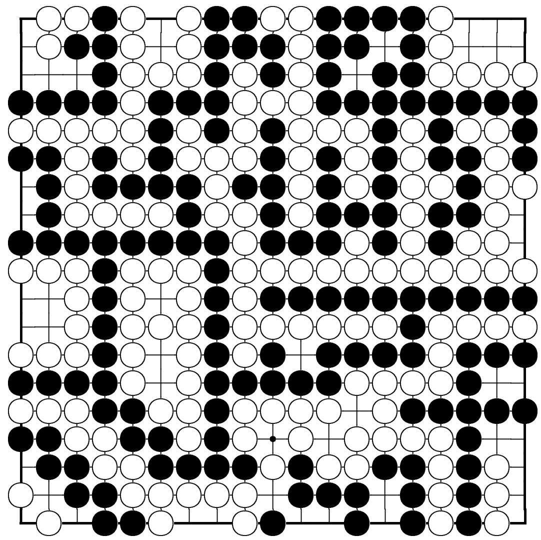


Figure 16. What is the outcome of this game?

Nevertheless, one should never forget that games are in themselves a primary application of CGT. At this very moment as the reader is reading this sentence, hundreds of millions of people are playing games, and many of them make gaming their professional activity. Games constitute a primordial activity like ceremonial burial, arts, education, and so on, playing a role in human development [Huizinga 1949]. We have chosen ATARI GO for this final section because, due to its characteristics, it is very suitable for the purpose of showing the pertinence of affine normal play theory. However that is not the only reason. This simplified version of GO has been used in various countries due to its educational and therapeutic value. Yasuda Yasutoshi (a 9 dan professional GO master player) was concerned about various types of problems in Japanese schools; he argues that the use of ATARI GO has measurable positive effects [Yasuda 2021].

Without further delay, let us delve into the analysis of a carefully chosen example. We propose an endgame reached in a game with many mistakes made by both players. In the position shown in Figure 16, it is easy to verify that Right wins when playing first. The interesting question is whether Left wins playing first, i.e., determining if the outcome class of the game is $\mathcal{N}$ or $\mathcal{R}$. As usual in CGT, the first step is to divide and conquer.

In Figure 17, there is a large set of alive pieces. Moves on intersections marked with “s” are suicidal moves and should not be played by either player. Moves on intersections marked with “t” are also terrible, as a response in “m” leads to a catastrophe. Finally, moves on intersections marked with “h” are horrible, weakening the black group.

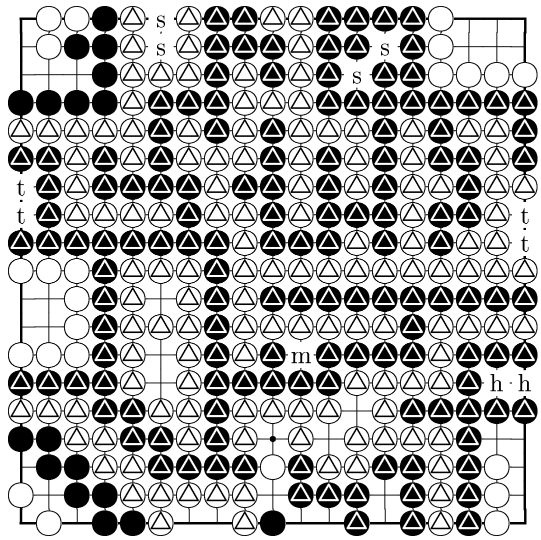


Figure 17. Alive pieces.

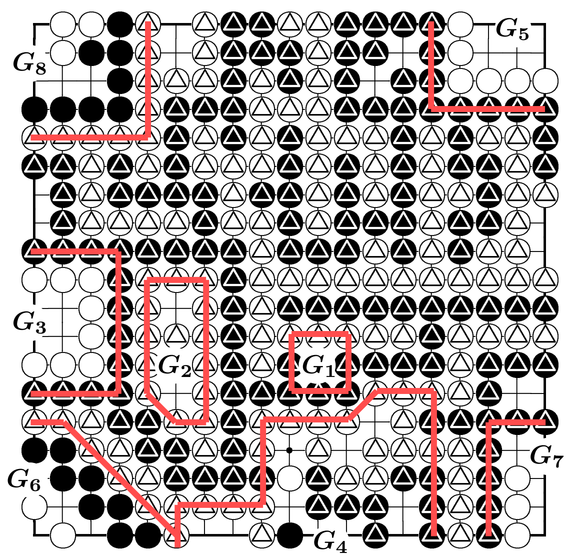


Figure 18. Disjoint components.

Given the set of alive pieces, it is possible to identify eight disjoint components (Figure 18). The next step is to determine the game value of each of them.

Starting with the easiest components, G_1 is equal to $*$, since either player can place a stone at that intersection. The component G_2 is equal to -1, as Left cannot play in that region, and Right can make a last move there. Finally, G_3 is equal to 0 because it is a $\mathcal{P}$-position. Left cannot play inside that region, and

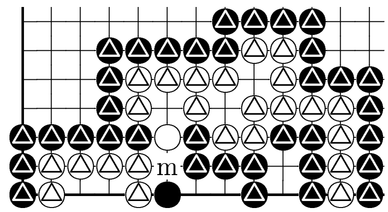


Figure 19. The component G_4 is a hammerzug.

any move by Right can be answered with a symmetric move diagonally. In this way, these three components are Conway Games, with the first one being tepid (infinitesimal) and the second and third being cold (numbers).

Advancing to the analysis of G_4, a move at the intersection marked with an "m" in Figure 19 is mandatory for either player, dominating all others (for ease of drawing, we slightly adjusted the positions of the surrounding pieces without changing the nature of the component). Regardless of whether Left or Right makes the move, it is a check. It is reasonably straightforward to see that $G_4 = \{\{\infty \mid 2\} \mid \{-2 \mid \overline{\infty}\}\} = \{\circlearrowright^2 \mid \circlearrowright^{-2}\}$. Since $2 > G_4 > -2$, this component is a hammerzug. As Left is not in check in any other component, Theorem 56 ensures that the first move can be made in this component without having to think about anything else (collecting two guaranteed moves).

The component G_5 does not have as clear of an analysis as the previous ones. Consider Figure 20. If Right starts and places a stone at intersection "a", he wins 3 guaranteed moves. We leave it to the reader to conclude that if Left starts and places a stone at "a", then the resulting game is $\{\{\infty_1 \mid *\} \mid \{* \mid \overline{\infty}_1\}\}$, and if she places a stone at "b", then the resulting game is $\{\{\infty_2 \mid *\} \mid \{* \mid \overline{\infty}_1\}\}$. Both games can be reduced and are equal to $*$. Thus, $G_5 = \{* \mid -3\}$, and we have again a Conway Game, but this time it is hot.

The component G_6 is very interesting. Both players are in a rush to place a stone at the intersection marked with an "m" in Figure 21. If Right does it, he ensures a winning terminating move in that component. If Left does it, she manages to defend, albeit at the cost of giving one guaranteed move to the opponent. Therefore, we have $G_6 = \{-1 \mid \overline{\infty}_2\}$. This negative component is scalding.

The component G_7 is also scalding, but it is Left who is threatening to cause a bubbling situation. If Left places a stone at the intersection marked with an "a"

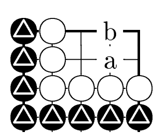


Figure 20. The component G_5 is equal to $\{* \mid -3\}$.

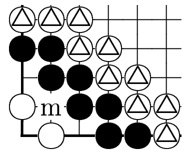


Figure 21. The negative component G_6 is scalding.

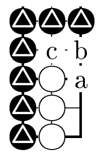


Figure 22. The fuzzy component G_7 is scalding.

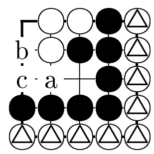


Figure 23. The component G_8 is scalding and Left can defend in sente.

in Figure 22, she guarantees a winning terminating option in 3 moves. If Right plays first, placing a stone at “a”, “b” or “c” results in a $\mathcal{P}$-position, so Right has a move to 0. Hence, we have $G_7 = \{\infty_3 \,|\, 0\}$.

The component G_8 has a very interesting peculiarity. If Right places a stone at the intersection marked with an “a” in Figure 23, he guarantees a winning terminating option in 2 moves. What is different from the previous cases is that Left has *two distinct ways to defend*. If Left places a stone at the intersection ”b”, then she threatens a move that ensures the victory after one move. Against that move, Right must place a stone at “a”, also threatening to win. To defend, Left places a stone at “c”, achieving a $\mathcal{P}$-position, i.e., 0. It follows that placing a black stone at ”b” is the option $\{\infty_1 \,|\, \{0 \,|\, \overline{\infty}_1\}\}$. On the other hand, if the first move is the placement of a black stone at “a”, Right must defend by placing a stone at “b”, achieving a $\mathcal{P}$-position, i.e., 0. Thus, this second option for Left is $\{\infty_1 \,|\, 0\}$. We have $G_8 = \big\{\{\infty_1 \,|\, \{0 \,|\, \overline{\infty}_1\}\}, \{\infty_1 \,|\, 0\} \,|\, \overline{\infty}_2\big\}$, which can be reduced to $G_8 = \{0, \{\infty_1 \,|\, 0\} \,|\, \overline{\infty}_2\}$. If G_8 were the only component in the disjunctive sum, the first option (placement at “b”) would be the winning move for Left. With more components, the second option defends *in sente*,[12] making it a potentially good choice.

[12] From Japanese “before hand”, playing *sente* (先手) means to keep the initiative.

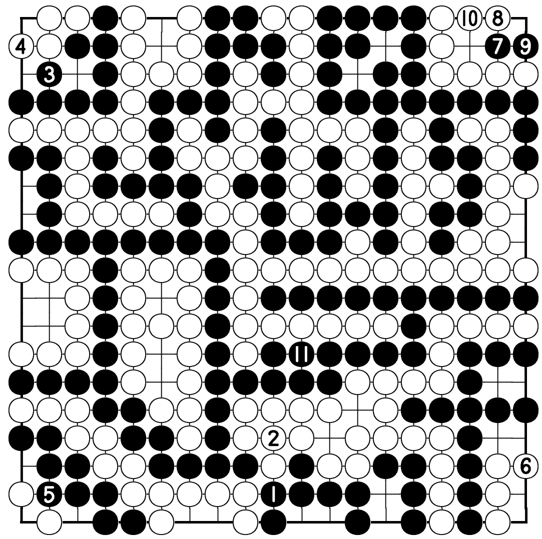


Figure 24. Left, playing first, wins.

Given the analysis of the components, the position shown in Figure 16 is the following disjunctive sum:

$$G = * + (-1) + 0 + \{\circlearrowleft^{2} \mid \circlearrowleft^{-2}\} + \{* \mid -3\} + \{-1 \mid \overline{\infty}_2\} + \{\infty_3 \mid 0\} + \{0, \{\infty_1 \mid 0\} \mid \overline{\infty}_2\}.$$

The presence of multiple scalding components, as well as a hammerzug, leads to the conjecture that there must have been several previous errors. Typically, when a scalding component comes into play, it cools down in the next move as the threatened player attempts to defend. However, this is not one hundred percent certain, as the best defense can be the attack, which might be executed with a "faster" threat in another component. Returning to the game G, after the moves in the fourth and eighth components (in this order), Right has to respond, and the following sum is obtained (it is worth noting that Left employed the defense in sente in the eighth component):

$$G = * + (-1) + 0 + 2 + \{* \mid -3\} + \{-1 \mid \overline{\infty}_2\} + \{\infty_3 \mid 0\} + 0.$$

At that point, Left defends in the component $\{-1 \mid \overline{\infty}_2\}$, obtaining the sum

$$G = * + (-1) + 0 + 2 + \{* \mid -3\} + (-1) + \{\infty_3 \mid 0\} + 0.$$

Now that Right is under threat in the scalding component $\{\infty_3 \mid 0\}$, he has no time to play in the hot Conway Game and must respond with

$$G = * + (-1) + 0 + 2 + \{* \mid -3\} + (-1) + 0 + 0.$$

Left finishes gloriously by playing to

$$G = * + (-1) + 0 + 2 + * + (-1) + 0 + 0.$$

Note that if the empty intersection concerning G_1 did not exist, Left would not be able to win. Figure 24 shows the exact winning sequence.

10. Open problems

With the algebraic structure $\mathbb{Np}^\infty$ established, it is important to illustrate its applicability through the analysis of concrete rulesets. The ruleset WHACKENBUSH is introduced in Section 1.1.3 to exemplify pathetic infinitesimals and more.

Problem 1. Provide a complete solution for WHACKENBUSH.

In the first arXiv manuscript of this paper, we sketch the idea of a ruleset formalization of ∞-maintenance, called the INQUISITOR. This is a generalization of the classical Left-wins-playing-second implementation of the maintenance in $\mathbb{Np}$.

Problem 2. Formalize the INQUISITOR.

Reduced canonical form, atomic weight theory and temperature theory are great established tools in $\mathbb{Np}$.

Problem 3. Extend reduced canonical form, atomic weight theory and temperature theory to encompass affine normal play.

In this paper we prove the weak number avoidance theorem for affine normal play.

Problem 4. Prove the translation property for numbers, the strong number avoidance theorem, in the setting of $\mathbb{Np}^\infty$.

When you have a group, of course $G + J \geqslant H + J$ is equivalent to $G \geqslant H$. When you have a monoid $G = H$ implies $G + J \geqslant H + J$, but you are only sure that $G + J \geqslant H + J$ implies $G \geqslant H$ if J is invertible. This is Theorem 19 in our setting. (Similar results appear elsewhere, for misère monoids and so on.) However, whether $G + J \geqslant H + J$ implies $G \geqslant H$ for all J often remains an open problem; such results are usually referred to as Pocancellation Theorems.

Problem 5. Does pocancellation hold in $\mathbb{Np}^\infty$?

When reduced forms are unique, and the reductions do not increase the formal birthday, then the reduced forms are simplest forms. But, if they are not unique, the argument cannot be applied. So, we have defined that G is a reduced form if all reductions were made and G has the smallest formal birthday in those conditions. Now, perhaps it is true that “all reductions were made” implies “smallest formal birthday”. But we do not yet know.

Problem 6. Does the first item in the definition of reduced form, Definition 38, suffice?

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