Absolute combinatorial game theory

URBAN LARSSON, RICHARD J. NOWAKOWSKI
AND CARLOS P. SANTOS

We propose a unifying additive theory for standard conventions in combinatorial game theory, including normal-, misère- and scoring-play, studied by Berlekamp, Conway, Dorbec, Ettinger, Guy, Larsson, Milley, Neto, Nowakowski, Renault, Santos, Siegel, Sopena, and Stewart — in works spanning the years 1976 through 2019 — and others. A game *universe* is a set of games that satisfies some standard closure properties. Here, we reveal when the fundamental game comparison problem, "Is $G \succcurlyeq H$?", simplifies to a constructive "local" solution, which generalizes Conway's foundational result in *On Numbers and Games* (1976) for normal-play games. This happens in a broad and general fashion whenever a given game universe is *absolute*. Games in an absolute universe satisfy two properties, dubbed *parentality* and *saturation*, and we prove that the latter is implied by the former. Parentality means that any pair of nonempty finite sets of games is admissible as options, and saturation means that, given any game, the first player can be favored in a disjunctive sum. Game comparison is at the core of combinatorial game theory, and for example, efficiency of potential reduction theorems rely on a local comparison. We distinguish between three levels of game comparison; superordinate (global), basic (semiconstructive) and subordinate (local) comparison. In proofs, a sometimes tedious challenge faces a researcher in combinatorial game theory: in order to disprove an inequality, an explicit distinguishing game might be required. Here, we explain how this job becomes obsolete whenever a universe is absolute. Namely, it suffices to see if a pair of games satisfies a certain Proviso together with a Maintenance of an inequality.

1. Introduction

A combinatorial game is a (terminating) two-player zero-sum game of perfect information and no randomness. Given a game and a starting player, the two players alternate moving, until the current player has no options, and a winner is

declared. Amongst the most popular conventions, we find *normal-play*, in which a player who cannot move loses, and *misère-play*, in which such a player wins; these two conventions will henceforth be called the *classical conventions*. The third main track is *scoring-play*, a convention with real-valued "scores" attached to the terminal positions, and the two players, Left and Right, seek to maximize and minimize this score, respectively. Combinatorial game theory concerns optimal play, captured in so-called *outcome functions*, and traditionally they have distinct definitions, depending on the convention at hand. Likewise, the notion of a *disjunctive sum* of games and the definition of *game comparison* used to require slightly different definitions in classical vs. scoring settings. This study concerns a unification of terminating games in an *absolute theory* that combines the best of all, and more. The absolute theory provides recursive solutions for game comparison, a notion that is nonconstructive by definition. Recursive solutions were known before in the literature, but they were always treated individually; here we unify the approach.

The two players are usually called *Left* (female) and *Right* (male).[1] We distinguish between the set of Left options, those positions that Left can reach in one move, and similarly the set of Right options, denoted by $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ respectively. Any game G can be represented by two such sets, and we use the standard recursive definition of a game, $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$. The games are commonly represented by a rooted (finite or infinite) tree, with two types of edges (Left- and Right-slanting respectively), called the "game tree". The nodes are positions that can be reached during play, the edges represent the possible moves, and the root is the current position. The children of a node are all the positions that can be reached in one move and these are called the *options*.

We restrict attention to those game trees for which each node, a.k.a. *follower*, has a finite number of edges (options) and where any, not necessarily alternating, play sequence is finite; this is sometimes called the class of *short* combinatorial games. Thus, any such game can be expanded in terms of elements of its *terminal* positions. This gives the common proof technique "induction on the options" since the depth of the game tree of an option is at least one less than that of the original position.

The reader is invited to consult the standard textbooks in combinatorial game theory: [1; 3; 7; 30]. Although they do not cover scoring-play, they lay the foundation for the field, in particular through their thorough development of the normal-play theory. Here, we will demonstrate that normal-play ideas are at the core of many other conventions, in particular misère- and scoring-play, and where suitable restrictions may be imposed on the games within a convention.

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[1]After Louise and Richard Guy, who contributed so much to this field.

Let us recall the foundations of the field. Given a classical convention, there are four *outcome* classes. The starting player, Left or Right, wins a game G if the outcome satisfies $o(G) = \mathscr{N}$, the second player wins if $o(G) = \mathscr{P}$, player Left wins independently of who starts if $o(G) = \mathscr{L}$, and player Right wins independently of who starts if $o(G) = \mathscr{R}$. The First Fundamental Theorem of combinatorial games states that these definitions induce a tetra partition of games.

Theorem 1 (First Fundamental Theorem, folklore). *If G is a classical game, then $o(G) \in \{\mathscr{N}, \mathscr{P}, \mathscr{L}, \mathscr{R}\}$.*

These outcomes inherit a partial order from a given total order of the possible results: "Left wins" $= \mathrm{L} > \mathrm{R} =$ "Right wins". We get the order between outcomes as $\mathscr{L} > \mathscr{N} > \mathscr{R}$, and $\mathscr{L} > \mathscr{P} > \mathscr{R}$, but where $\mathscr{N}$ and $\mathscr{P}$ are incomparable because the winner depends on who starts.

In scoring-play the situation is more varied and seemingly more challenging; instead of a tetra partition of outcomes, outcomes are represented by ordered pairs of real numbers, thus resulting in an uncountable number of outcome classes. The purpose of this work is to demonstrate how to overcome this apparent complication, and show that, in a fundamental sense, the similarities are profounder than the differences. Formal definitions suitable for absolute study will be the content of Section 3, and onwards, while this section and the next one aim to provide a sufficient intuition to motivate a thorough study of the subject.

The game addition operator "+", called *disjunctive sum*, conveys that the current player plays in exactly one game component out of a finite sum of games (Section 3.1). For any specified universe, this induces a partially ordered monoid of games, with the neutral element the empty game, henceforth denoted by **0**.

A *universe* of combinatorial games is a set of games that satisfies standard closure properties (closure under disjunctive sum, closure under taking options, closure under conjugate) and has a neutral element in "the empty game" (Section 3.2).

Definition 2 (Partial Order). Consider two games G and H in a specified universe $\mathbb{U}$. Then $G \succcurlyeq_{\mathbb{U}} H$ if, for all games X in this universe, $o(G+X) \succcurlyeq o(H+X)$. If $G \succcurlyeq_{\mathbb{U}} H$ and $H \succcurlyeq_{\mathbb{U}} G$, then $G =_{\mathbb{U}} H$.[2]

The subscript $\mathbb{U}$ is omitted when the universe is understood. This is the standard definition for order of combinatorial games. Its motivation stems from games like the oriental game of GO, where the game decomposes into independent components during play. Are two given game components comparable in *any* possible play situation? In this sense, Definition 2 leaves no ambiguity of preference.

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[2]We use "$\succcurlyeq$" for partial order of games.

A note on terminology. We often refer to the games X in Definition 2 as *distinguishing* games, in this sense: if there is a distinguishing game, then the inequality (equality) does not hold, and otherwise, if there is no game X to distinguish the games G and H, then the inequality (equality) holds.

But, given this seemingly overwhelming requirement for games to be comparable, one might guess that for most universes, all games would be incomparable. When, if ever, could one hope for an interesting structure in the induced partial order of games?

1.1. *The appeal of normal-play, and the Second Fundamental Theorem.* One of the most beautiful discoveries of combinatorial game theory is that, in normal-play, the maximizer, Left, wins playing second in the game G if and only if $G \succcurlyeq \mathbf{0}$. Moreover, it is well known that normal-play games constitute a group structure. This leads to a *local*, and therefore *constructive*, game comparison:[3] namely, it suffices to understand G and H in terms of their followers.[4] Specifically, $G \succcurlyeq H$ if and only if Left wins the game $G - H = G + (-H)$ playing second. Here, the negative of a game is the game where the players have swapped roles.

There are three equivalent ways of expressing this result. A typical Left (Right) option of a game G, is denoted by G^L (G^R).

Theorem 3 (Second Fundamental Theorem [2; 5]). *Consider a pair of normal-play games* G, H. *The following items are equivalent*:

(1) $G \succcurlyeq H$.

(2) *Left wins* $G - H$ *playing second.*

(3) *Neither* $G^R \preccurlyeq H$ *nor* $H^L \succcurlyeq G$ *ever holds.*

(4) (a) *For all* G^R, *there is an* H^R *such that* $G^R \succcurlyeq H^R$ *or there is a* G^{RL} *such that* $G^{RL} \succcurlyeq H$.

   (b) *For all* H^L, *there is a* G^L *such that* $G^L \succcurlyeq H^L$ *or there is an* H^{LR} *such that* $G \succcurlyeq H^{LR}$.

The fourth item is of special interest to this work, and although it is intuitively clear, we will include a proof on page 120. In combination with the second item, the interpretation is a "maintenance" of game preference via the possible Left responses to Right's opening move, either G^R or $-H^L$. In the pursuit of a generalization of the Second Fundamental Theorem, the main task is to show

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[3]We will use the terms "local" and "constructive" interchangeably depending on which aspect we wish to emphasize. Other terms in the literature with the same meaning are "option-only", "play-comparison", "algorithmic" and "recursive".

[4]A game G' is a *follower* of the game G if there is a not necessarily alternating (and perhaps empty) play sequence from G to G'.

that an analogous reasoning holds in a much more general context, where group structure cannot be assumed.

1.2. *The fundamental theorem of absolute universes.* In this study, we propose an extension to the Second Fundamental Theorem. To this purpose, it will be necessary to decompose the outcome function into its two parts, depending on who starts. We write $o(G) = (o_L(G), o_R(G))$, where $o_L(G)$ ($o_R(G)$) denotes the Left-outcome (Right-outcome): the result in optimal play when Left (Right) starts. The formal definition of outcome will require some preliminary machinery (see Section 3), but the main idea is transparent by this notion, and it suffices for this section and Section 2.

A *Left-atomic* (*Right-atomic*) game is a game where Left (Right) cannot move, and a game is *atomic* if it is Left-atomic or Right-atomic. A game is *purely atomic* if it is both Left- and Right-atomic.

By combining the fourth item in Theorem 3, henceforth called the Maintenance (Common Normal Part in [16]), with a Proviso that concerns addition with atomic games, we obtain a unifying theory for many short combinatorial games.

A universe of combinatorial games is *absolute* if it is *parental* (Definition 13) and *outcome saturated* (Definition 22). In essence, parental means that any pair of nonempty sets of game forms of the universe makes a game form of the universe, and saturated means that given a game G, the there is a game H such that the first player is favored in the disjunctive sum $G + H$.

Theorem 4 (Absolute Fundamental Theorem). *Consider a pair of games G, H in an absolute universe, $\mathbb{U}$. The relationship $G \succcurlyeq H$ holds if and only if the following two conditions are satisfied*:

Proviso: $o_L(G + X) \geqslant o_L(H + X)$ *for all Left-atomic $X \in \mathbb{U}$ and $o_R(G + X) \geqslant o_R(H + X)$ for all Right-atomic $X \in \mathbb{U}$.*

Maintenance: *For all G^R, there is an H^R such that $G^R \succcurlyeq H^R$, or there is a G^{RL} such that $G^{RL} \succcurlyeq H$, and for all H^L, there is a G^L such that $G^L \succcurlyeq H^L$, or there is an H^{LR} such that $G \succcurlyeq H^{LR}$.*

The Proviso adjusts for a terminal situation that is more complicated than the normal-play ending. It is not yet a purely local consideration. But in all the universes we have encountered, it can be simplified, which is the content of Section 6. The Maintenance of Theorem 4 clearly replaces the infinite number of comparisons by a local condition.

We propose a taxonomy of "game comparison":[5]

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[5]Taxonomy references: W. Croft and D. A. Cruse, *Cognitive linguistics*, Cambridge Univ. Press, 2004. F. Ungerer and H.-J. Schmid, *An introduction to cognitive linguistics*, Longman, London, 1996.

- Superordinate: This is the standard nonconstructive game comparison by Definition 2.

- Basic: Absolute universes have a semiconstructive game comparison, as detailed in Theorem 4.

- Subordinate: Many absolute universes have local/constructive comparison. Here we find, for example, the Second Fundamental Theorem, which concerns normal-play, a special lucky case, where the Proviso returns an empty condition.

Some relevant literature is displayed in Table 1.

1.3. *Outline.* In Section 2, we recall some (absolute) universes from the literature that motivated this work; this section can be omitted for a reader who already feels sufficiently motivated, and prefers to dive directly into the more technical definitions of absolute play, required in the proofs to come. The subsequent sections do not depend on Section 2.

Section 3 conceptualizes absolute combinatorial game theory, which leads to a partial order for universes at a superordinate level of taxonomies; that is, we convert Definition 2 to our setting. Section 4 concerns a proof technique via so-called *downlinked* games [30]. In Section 5, we prove the main result, Theorem 4, which promises a semiconstructive game comparison in absolute universes, via the Proviso; thus this section deals with a basic taxonomic level. In Section 6, we discuss the translation of the Proviso in some familiar universes of games, corresponding to a subordinate and constructive level of game comparison.

2. A motivation via standard universes of games

This section motivates the absolute framework as presented in Section 3 and onwards. The terminology and notation introduced here is local, i.e., not yet unifying, and the rest of the paper does not depend on it.

Independently of the winning convention, a reader may be acquainted with other restrictions on the set of permitted games. For example, a game is *impartial* if both players have the same options, and each option is impartial. Note that, because of the parentality condition, as outlined in Section 1.2 before Theorem 4, this restriction gives a too small universe of games. On the other hand impartial games have a more rudimental partial order in the sense that all games are either equal or incomparable; thus absolute theory is not needed, but other methods are required, such as the famous misére quotients [25; 26; 27].

In order to illuminate Theorem 4 further, in Section 2.1 (Example 5) we begin by comparing some simple games in a well-known absolute universe, namely the

class	play convention		
	normal	misère	scoring
partizan	[2; 5]	[21; 31]	[15; 17; 33]
dead-ending	[2; 5]	[18; 22; 23]	
dicot/all-small	[5]　　　[20]	[8]	[9; 10; 24]
impartial	[12; 32]	[13]　[25; 26; 27]	
loopy	[6; 11]　[28; 29]		

Table 1. Summary of important developments in the theory of additive combinatorial games. Works published since 2007 are listed on the right side of each column. Cyclic (or loopy) games are not (yet) included in this framework. Section 2 gives a detailed but still informal account of how some of these game universes interact with the absolute theory.

misère *dicots*, here denoted by $\mathbb{D}$; see, e.g., [8].[6] Dicot means that if one of the players has an option, then so does the other player, and each option is a dicot.

Dicot universes have nice properties, but they do not capture the richness of absolute theory. The *dead-ending* universe (Sections 2.2 and 2.3) and the *guaranteed universe* (Section 2.4) give appealing motivations for the theory, with formal treatments in Section 6. Here they are used to explain the birth of the absolute ideas, and to highlight the similarities and differences in the classical, respectively scoring, settings. A game is *dead-ending* if whenever one of the players runs out of options, then this player will never again be able to move in this game (regarded as a component in a disjunctive sum of games). The universe of dead-ending games will be denoted by $\mathbb{E}$. Scoring games have scores attached to the terminal situations, and it is possible to define a *guaranteed* property where a player who runs out of options in one component can never score worse in this component, if the disjunctive sum play continues. This property will define the universe of guaranteed scoring games, $\mathbb{Gs}$. A preliminary observation here is that both these restrictions limit what can happen after a terminating situation for one of the players, and similar to normal-play this player would typically not be favored in the full disjunctive sum, by running out of moves in one of the components.

2.1. *A dicot universe.* The dicot restriction leads to a tremendous simplification of the Proviso, namely it reduces to $o(G) \succcurlyeq_{\mathbb{D}} o(H)$, because, in $\mathbb{D}$, the only possibility for an atomic game X is $X = \mathbf{0}$, the empty game.

So, why is the Proviso necessary? We let $* = \{\mathbf{0}|\mathbf{0}\}$, and $*2 = \{\mathbf{0}, *|\mathbf{0}, *\}$ be literal forms. The games $G = \{*2|*2\}$ and $H = 0$ maintain inequality modulo

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[6]The authors prove uniqueness of canonical forms, but they omit to deduce the affiliated local game comparison.

dicot misère, but $G \not\succcurlyeq_{\mathbb{D}} H$, because $o_L(G) =_{\mathbb{D}}=$ R, but $o_L(H) =_{\mathbb{D}}$ L, which indeed is captured by the Proviso.

Observe that Maintenance in normal-play does not imply Maintenance in an absolute universe. For example, take the literal forms $G = \{\mathbf{0} \,|\, \{\ast 2 \,|\, \ast 2\}\}$ and $H = \ast$. Then $G \succcurlyeq_{\mathbb{N}_{\mathbb{P}}} H$, with $\mathbb{N}_{\mathbb{P}}$ the normal-play universe. But the pair G, H does not satisfy Maintenance modulo dicot misère.

Let us view some more examples of dicot misère-play.

Example 5 (Theorem 4, Dicot Misère). In both normal- and misère-play, define the literal form games $\ast = \{\mathbf{0}|\mathbf{0}\}$, $\uparrow = \{\mathbf{0} \,|\, \ast\}$ and $\curlywedge = \{\mathbf{0}, \ast \,|\, \ast\}$. Observe that, by standard reversibility, in normal-play, $\curlywedge =_{\mathbb{N}_{\mathbb{P}}} \uparrow$. However, in dicot misère we have that $\curlywedge \succ_{\mathbb{D}} \uparrow$. To prove this, observe that the Maintenance holds (with say $G = \curlywedge$ and $H = \uparrow$), and L $= o_L(\curlywedge) >_{\mathbb{D}} o_L(\uparrow) =$ R. Thus, we get $\curlywedge \succcurlyeq_{\mathbb{D}} \uparrow$ but $\uparrow \not\succcurlyeq_{\mathbb{D}} \curlywedge$. For another example, in dicot misère, $\uparrow \,\|_{\mathbb{D}}\, \mathbf{0}$. This follows by noting that in normal-play, $\uparrow \succ_{\mathbb{N}_{\mathbb{P}}} \mathbf{0}$, so (by the Maintenance part; see also Theorem 37), we cannot have "$\preccurlyeq$", but in dicot misère, R $= o_R(\uparrow) <_{\mathbb{D}} o_R(\mathbf{0}) =$ L.

Thus here Theorem 4 simplifies game comparison a lot, because it is no longer required to find a distinguishing game "X" to separate games (see, e.g., [8], where that technique has been used for dicot misère).

This signals the usefulness of the absolute theorem. However, while the intuition of the Maintenance should be clear, it might take a bit longer to capture the essence of the Proviso. The Proviso has the flavor of a certain *waiting problem*, which is connected to the machinery of atomic games, and the universe of so-called *dead-ending* misère-play games [23] serves a perfect fit.

2.2. *No waiting at a dead-end.* Dead-ending gives a nice structure to miséreplay. Suppose that Left has no options in the game G (i.e., G is Left-atomic), but when played together with another game H, written $G + H$, then Left can play to, say, $G + H^L$. Now, if Right plays in the game G to, say, $G^R + H^L$, then possibly Left had forced play in the "G" component, which worsens her situation; this could happen unless G is dead-ending. More specifically, suppose that Right can win only by opening up the G component to give Left new moves; he is able to *wait* while Left plays first in some other component. If G is dead-ending, such waiting is not meaningful for Right, since Left could never be forced to play in a follower of the Left-atomic G. Thus, we see an example of how a restriction of the set of games may simplify analysis of games. The *waiting problem* concerns such situations, where Right, by waiting for a Left move in H, can open up further Left play in the G component. If such threats do not exist, Left may ignore the G component and instead focus all efforts on the H component.

Let us illustrate this important idea, which inspired this work, with a more

concrete example. When G is Left-atomic (Right-atomic) we typically write $G = \{\varnothing | G^{\mathcal{R}}\}$ ($G = \{G^{\mathcal{L}} | \varnothing\}$).[7]

2.3. *A concrete example of the waiting problem.* Milley and Renault [23] first noted the importance of the waiting problem via Left- and Right-atomic games in pairwise comparison of dead-ending games. The restrictions before this were dicot universes, and Milnor [24] was the first to observe that the restriction of the dicot scoring-play universe, where it is never bad to play first, allows for local game comparison, similar to normal-play.[8] But, similarly to the setting for impartial games, Milnor's setting is too restricted to fit within the absolute umbrella.

Consider the literal form game $G = \{\varnothing | \{\mathbf{0} | \varnothing\}\}$ (Figure 1) in a misère-play disjunctive sum with the game $*$, written $G + *$, and note, first of all, that G is not dead-ending. If the game G is played alone, Left wins, playing first, but she loses if Right starts: this is a "hot Left-atomic game", where both players want to start. But, on the other hand, the empty set of options $G^{\mathcal{L}} = \varnothing$ is not good for Left, played in the context of the disjunctive sum $G + *$. Namely, playing first, Left has to move to G, and then Right moves to the game $\{\mathbf{0} | \varnothing\}$, where Left loses. The presence of $*$ allowed Right to *wait* for his opportunity to play in G, where Left's play is bound to lose.

Observe that "hot-atomic" games do not exist in normal-play. In normal-play the worst thing that can happen, in any situation, is to run out of move options; it is well known [7] that this cannot be good, in any context. This is not just a subtle difference between conventions. Indeed, in normal-play a large number of moves is never bad for a player, but in misère-play this is usually bad (but not always).[9]

This discussion illustrates a usual difficulty in proofs in combinatorial game theory. Milley and Renault's dead-ending restriction achieves some interesting results because, as we pointed out, the potential power of waiting for new

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[7]Later, in Section 3.1, inspired by guaranteed scoring-play, we will adorn the empty sets of options, to allow for absolute generalizations.

[8]In fact today we know that Milnor's universe is the same as a subset of the normal-play games, namely the reduced canonical form of normal-play games, games without infinitesimals.

[9]Normal-play has a unique so-called *zugzwang* (where no player wants to start), namely the neutral element $\mathbf{0}$, but misère-play has infinitely many zugzwangs, and in particular each game of the form $\{\boldsymbol{\ell} | \boldsymbol{r}\}$ is a zugzwang, where $\boldsymbol{\ell}$ is $\ell > 0$ moves for Left, and $\boldsymbol{r}$ is $-r > 0$ moves for Right. Consider the game $\mathbf{100}$. Left has 100 moves, whereas Right cannot move. This seems like a terrible game for Left, in the misère-play convention, but note that played together with the huge zugzwang $\{\mathbf{1000} | -\mathbf{1000}\}$, then Left wins the game $\mathbf{100} + \{\mathbf{1000} | -\mathbf{1000}\}$, because Right will have to open the huge zugzwang, no matter who starts. However, since in normal-play, the only zugzwang is $\mathbf{0}$, this type of situation cannot happen. See the recent manuscript [18] for more on this topic, and an introduction to the relevant concept of a "perfect murder", in the context of dead-ending games.
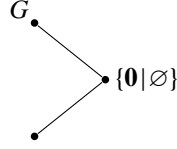
Figure 1. The game $G = \{\varnothing \mid \{\mathbf{0} \mid \varnothing\}\}$.

opportunities is dismantled in their universe. Indeed, if G in Figure 1 were instead the dead-ending $\{\varnothing \mid \mathbf{0}\}$, then Right's potential "waiting" in $G + *$ would be useless, since it would lead to swift Left win. Let us give one more example on the waiting problem, which illustrates how dead-ending misère differs from the full universe of misère-play, denoted by $\mathbb{M}$, by using explicitly the Proviso from Theorem 4.

Example 6 (Dead-ending [18; 23] vs. Full Misère [31]). Consider the literal form games $\mathbf{1} = \{\mathbf{0} \mid \varnothing\}$ and $\bar{\mathbf{1}} = \{\varnothing \mid \mathbf{0}\}$. In the universe of dead-ending misère-play, similar to normal-play, $\mathbf{1} + \bar{\mathbf{1}} =_{\mathbb{E}} \mathbf{0}$, whereas in full misère-play, $\mathbf{1} + \bar{\mathbf{1}} \parallel_{\mathbb{M}} \mathbf{0}$. The Maintenance, which does not distinguish between the universes, holds. Thus, the remaining tests are in the Proviso. For dead-ending misère, it suffices to argue that $\mathrm{L} = o_L(\mathbf{1} + \bar{\mathbf{1}} + X) = o_L(\mathbf{0} + X)$. And this holds, since X is Left-atomic dead-ending: Right cannot open up any move sequence for Left in the X component, when Left plays to $\bar{\mathbf{1}} + X$. Moreover, $\mathrm{L} = o_L(\mathbf{0} + X)$, since X is Left-atomic. The arguments for o_R are symmetric, since now X is Right-atomic dead-ending.

   On the other hand, for full misère, we may choose the Left-atomic distinguishing game $X = \{\varnothing \mid \{\mathbf{2} \mid \varnothing\}\}$, which indeed opens up two new horrible moves for Left, to obtain $\mathrm{R} = o_L(\mathbf{1} + \bar{\mathbf{1}} + X) < o_L(\mathbf{0} + X) = \mathrm{L}$. Analogously, we may get $\mathrm{L} = o_R(\mathbf{1} + \bar{\mathbf{1}} + X) > o_R(\mathbf{0} + X) = \mathrm{R}$, by picking the Right-atomic distinguishing game $X = \{\{\varnothing \mid \bar{\mathbf{2}}\} \mid \varnothing\}$. Hence the confusion in terms of full misère. (There is a local justification of this by using Theorem 40.)

   Thus, we have seen the Proviso in action, and the meaning of the Left- and Right-atomic games in the Proviso should have become clearer. But, again, what is a reasonable explanation of the interaction between the Proviso and the Maintenance? This should be addressed before we plunge into the somewhat extensive and technical definitions and proofs.

   The intuition is that all absolute game conventions behave in a similar fashion, with respect to maintenance of an inequality, playing in the individual games G and H alone as expressed in the Maintenance, except possibly at the endgame. This is easier to see whenever $H = \mathbf{0}$, because then Maintenance reduces to that for all G^R, there is a G^{RL} such that $G^{RL} \succcurlyeq \mathbf{0}$. In this way Left may ignore

the X component, as long as Right plays in G. And analysis of a move in the X component may be induced by induction, unless X is atomic. Whenever $H \neq \mathbf{0}$, by lifting the framework from normal-play, then a Right move in the H component may be interpreted as a conjugate of H^L, even if H is not invertible. The Proviso, on the other hand, interprets the individual universes, and tests if the inequality still holds for atomic X from the specified universe's point of view.

2.4. *Guaranteed scoring-play*. The main ideas presented so far are universal, in the sense that they do not depend on the aforementioned winning conventions. This work sprung out of studies in *scoring combinatorial games*. The major novelty of a scoring universe is that for all games, each empty set of options has an adorned terminal score. Stewart [33] studied the full universe of scoring combinatorial games and noticed that a special type of games are problematic, namely the so-called "hot-atomic" games.

Suppose that Left cannot move in a game component, say G, i.e., G is Left-atomic, and that she prefers to make no move there again. Suppose that Right is able to wait while Left moves somewhere else, and then he finds a move to a G^R in a way that opens up new move options for Left inside G^R, and that those new options worsen her terminal score. This type of situation is analogous to our example that distinguishes the values in full misère to those of the dead-ending restriction. But for scoring games the analogue to dead-ending is the restriction called *guaranteed scoring-play* [15], denoted by $\mathbb{G}$s, with a similar effect with respect to the waiting problem. This is achieved, not by restricting the available moves, in case of atomic games, but instead by restricting the possible terminal scores. The *guaranteed property* disallows hot-atomic games, at each stage of play, i.e., if a player cannot move in a game component, then the score cannot get worse for this player in this component, even if play would continue there at some later point, by the other player opening up new play possibilities.

First a note on change of terminology: in purely move-oriented games, such as normal- and misère-play, the game **1** denotes one move for Left (as in Example 6). In a scoring universe, the number of available moves is still important, but the central concept is the score. Therefore it makes sense to instead let the game **1** denote the purely atomic game, with no options for either player, and terminal score 1, independently of who is to move. Similarly, **2** is the terminal game where Left is rewarded 2 points independently of who is to play, and so on.

Let us look at the interplay of the Proviso with the Maintenance in guaranteed scoring-play. The first issue is how to interpret the Proviso in this universe. By the guaranteed property, the worst thing that can happen for Left, with respect to a Left-atomic game X, is that Right can control the moves in X freely, and that the terminal scores are no better than the terminal score of X alone, when

Left starts. By the guaranteed property, then the terminal score in X is the same, independently of the sequence of play, and without loss of generality, we may assume it be 0. Moreover, the issue of Right having maximal control of the moves in X translates to, again by the guaranteed property, that Right can pass whenever he wishes, and that Left is forced to move only in say G. Therefore [17] the Proviso simplifies to $\underline{o}_L(G) \geqslant \underline{o}_L(H)$ and $\bar{o}_R(G) \geqslant \bar{o}_R(H)$, where the underline (overline) denotes that Right (Left) can pass, as an extra option at any stage of play, even as a final move. Thus such modified Right- and Left-outcomes, may be thought of as *waiting-protected* results in optimal play, with respect to a given starting player. In Section 6.2, we will develop this idea for the misère dead-ending universe as well.

Example 7 (Guaranteed Scoring-play). In the guaranteed scoring universe, we have $\{\mathbf{1} \mid \{\mathbf{1} \mid \mathbf{0}\}\} \succcurlyeq \mathbf{1}$. This is justified by noting that the Maintenance is trivially satisfied if Left starts, and if Right starts, then $\{\mathbf{1} \mid \mathbf{0}\}$ is answered by Left moving to $\mathbf{1}$ which satisfies the Maintenance. The Proviso is satisfied because $\underline{o}_L(\{\mathbf{1} \mid \{\mathbf{1} \mid \mathbf{0}\}\}) = \underline{o}_L(\mathbf{1}) = 1$ and $\bar{o}_R(\{\mathbf{1} \mid \{\mathbf{1} \mid \mathbf{0}\}\}) = \bar{o}_R(\mathbf{1}) = 1$; in particular, Right cannot access the game $\mathbf{0}$, since Left would not use the passing advantage. In contrast, $\{\mathbf{1} \mid \mathbf{2}\} \not\succcurlyeq \mathbf{1}$ since the Maintenance is not satisfied; Left has no response to Right's move to $\mathbf{2}$. This is analogous to $* \not\succcurlyeq \mathbf{0}$ in normal-play. Also, $\{\{\mathbf{0} \mid \mathbf{2}\} \mid \{\mathbf{1} \mid \mathbf{0}\}\} \not\succcurlyeq \mathbf{1}$, because the Proviso does not hold, since Right's waiting-protection gives

$$\underline{o}_L(\{\{\mathbf{0} \mid \mathbf{2}\} \mid \{\mathbf{1} \mid \mathbf{0}\}\}) = 0 < 1 = \underline{o}_L(\mathbf{1}).$$

Observe that, in this case, the Maintenance holds. But it is irrelevant. Guaranteed scoring-play includes elements of normal-play, but is much richer.

By these examples, we now hope that the reader is sufficiently motivated and prepared to dive into the notation and terminology necessary to unify move-oriented classical and scoring-play conventions, and exploit their common features.

3. Combinatorial game spaces and absolute universes

This section has six subsections, so let us begin by outlining the content. Since all games are zero-sum, it is convenient to think of each terminal situation by using an associated "score", which may depend on whose turn it is to move. The *space* of all such games is defined in Section 3.1, followed by addition and conjugates of games. Sections 3.1 and 3.2 concern exclusively the *form* of a game, which concerns the "how-to play". In the latter section we axiomatize the concept of a universe of games, together with parentality. The evaluation of a terminal score is developed in Section 3.3, together with the notion of a combinatorial game

space, followed by the definition of the partial outcome functions in Section 3.4, which thus addresses the "why-to play". In Section 3.5, we define outcome saturation together with the notion of an absolute universe of games, and prove that absolute is a consequence of parentality, i.e., "only form matters". and in Section 3.6, we conclude with the definition of superordinate order of games, and prove that the order is compatible with the disjunctive sum operator.

3.1. *Additive game forms.* Let $\mathcal{A} = (\mathcal{A}, +)$ be a totally ordered, additive abelian group. A terminal position will be of the form $\{\varnothing^\ell | \varnothing^r\}$, where $\ell, r \in \mathcal{A}$. The intuition, adapted from scoring game theory [15], is that if Left is to move, then the game is finished, and the "score" is ℓ, and similarly for Right, where the "score" would be r. In general, if G is a game with no Left options then we write $G^\mathcal{L} = \varnothing^\ell$ for some $\ell \in \mathcal{A}$ and if Right has no options then we write $G^\mathcal{R} = \varnothing^r$ for some $r \in \mathcal{A}$.[10]

We refer to $\varnothing^a$ as an *atom* and $a \in \mathcal{A}$ as the *adorn*. A position for which at least one of the players does not have a move is called *atomic*, and if Left (Right) does not have a move is called *Left- (Right-)atomic*. A *purely atomic* position is both Left- and Right-atomic. Let us make the identification $\boldsymbol{a} = \{\varnothing^a | \varnothing^a\}$ for any $a \in \mathcal{A}$. For example, $\boldsymbol{0} = \{\varnothing^0 | \varnothing^0\}$ where 0 is the neutral element of $\mathcal{A}$.

We separate the form of a game, which defines how to play it without incentive, from how to play it well; a space of game forms never concerns any evaluation of results, such as a win-loss situation, which is rather the content of Sections 3.3 and 3.4.

Definition 8 (Free Space of Game Forms). Let $\mathcal{A}$ be a totally ordered abelian group, and let $\Omega_0 = \{\{\varnothing^\ell | \varnothing^r\} \mid \ell, r \in \mathcal{A}\}$. For $n > 0$, let Ω_n be the set of game forms with finite sets of options in Ω_{n-1}, including Left- and/or Right-atomic games. The set of game forms of *birthday n* is $\Omega_n \setminus \Omega_{n-1}$.[11] Let $\Omega = \bigcup_{n \geqslant 0} \Omega_n$. Then $\Omega = (\Omega, \mathcal{A})$ is a *free space* of game forms.

Recreational combinatorial games often decompose into independent game components as play progresses, e.g., DOMINEERING, CLOBBER, KONANE and AMAZONS. The current player plays in exactly one such component. This is one of the motivations for the *disjunctive sum* operator. The other motivation is that we may add any couple of games within the same convention. We give the

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[10]Here, the empty set is *not* a game, and therefore the notation should not be confused with standard set theory, where the empty set can be an element in a set. The empty game in our notation is $\{\varnothing^\ell | \varnothing^r\}$ and never $\varnothing$. In our approach, the empty game carries information, and this information has to be interpreted before we are able to compare games in a given class.

[11]The notion of birthday here is on the literal form of a game. In the literature, sometimes "birthday" concerns the game tree after reduction. Siegel [30] uses the term formal birthday for the literal form birthday.

explicit definition in terms of the adorns. Here, and elsewhere, an expression of the type $G^{\mathcal{L}} + H$ denotes the set of games of the form $G^L + H$, $G^L \in G^{\mathcal{L}}$, and this notion is only defined if $G^{\mathcal{L}}$ is nonatomic.

Definition 9. Consider a free space $(\Omega, \mathcal{A})$, and a pair of games $G, H \in (\Omega, \mathcal{A})$. The disjunctive sum of G and H is given by

$$G + H = \begin{cases} \{\varnothing^{\ell_1+\ell_2} | \varnothing^{r_1+r_2}\} \text{ if } G = \{\varnothing^{\ell_1}|\varnothing^{r_1}\} \text{ and } H = \{\varnothing^{\ell_2}|\varnothing^{r_2}\}, \\ \{\varnothing^{\ell_1+\ell_2} | G^{\mathcal{R}} + H, G + H^{\mathcal{R}}\} \text{ if } G = \{\varnothing^{\ell_1}|G^{\mathcal{R}}\}, H = \{\varnothing^{\ell_2}|H^{\mathcal{R}}\}, \\ \quad \text{and at least one of } G^{\mathcal{R}} \text{ and } H^{\mathcal{R}} \text{ nonatomic}, \\ \{G^{\mathcal{L}} + H, G + H^{\mathcal{L}} | \varnothing^{r_1+r_2}\} \text{ if } G = \{G^{\mathcal{L}}|\varnothing^{r_1}\}, H = \{H^{\mathcal{L}}|\varnothing^{r_2}\}, \\ \quad \text{and at least one of } G^{\mathcal{L}} \text{ and } H^{\mathcal{L}} \text{ nonatomic}, \\ \{G^{\mathcal{L}} + H, G + H^{\mathcal{L}} | G^{\mathcal{R}} + H, G + H^{\mathcal{R}}\} \text{ otherwise.} \end{cases}$$

Let $((\Omega, \mathcal{A}), +)$ denote the induced free semigroup.

Note that if $G, H \in (\Omega, \mathcal{A})$, a free space, then, by Definition 9, $G + H \in (\Omega, \mathcal{A})$.[12] Since our structure is associative, it is a commutative semigroup. The proof of these facts is analogous to that given for guaranteed scoring games [15, Theorem 7, page 6]; commutativity is easy to see by Definition 9, since $\mathcal{A}$ is abelian.

The game $\mathbf{0} = \{\varnothing^0|\varnothing^0\}$ is in $(\Omega, \mathcal{A})$ for any $\mathcal{A}$. In any circumstance, it should be the case that $G + \mathbf{0}$ and G are identical. Since $\mathbf{0}$ represents the empty game, there is no move, and the "score" is 0 regardless of whose turn it is. We prove it to give an example of "induction on options", and also to establish the neutral element of our structure so that the semigroup is in fact a monoid. We will write $G \equiv H$ if G and H are identical, that is, their literal forms have the same followers.

Lemma 10. *Let* $G \in ((\Omega, \mathcal{A}), +)$. *Then* $G + \mathbf{0} \equiv G$.

Proof. If G is purely atomic, then $G = \{\varnothing^{\ell} | \varnothing^{r}\}$ for some $\ell, r \in \mathcal{A}$. Then $G + \mathbf{0} = \{\varnothing^{\ell}|\varnothing^{r}\} + \{\varnothing^0|\varnothing^0\} = \{\varnothing^{\ell}|\varnothing^{r}\}$, by Definition 9.

Let $G = \{G^{\mathcal{L}}|G^{\mathcal{R}}\}$ be of birthday at least one. By the definition of disjunctive sum, if $G = \{\varnothing^{\ell}|G^{\mathcal{R}}\}$ then $G + \mathbf{0} = \{\varnothing^{\ell}|G^{\mathcal{R}}\} + \{\varnothing^0|\varnothing^0\} = \{\varnothing^{\ell}|G^{\mathcal{R}} + \mathbf{0}\}$, if $G = \{G^{\mathcal{L}}|\varnothing^{r}\}$ then $G + \mathbf{0} = \{G^{\mathcal{L}}|\varnothing^{r}\} + \{\varnothing^0|\varnothing^0\} = \{G^{\mathcal{L}} + \mathbf{0}|\varnothing^{r}\}$, and if $G = \{G^{\mathcal{L}}|G^{\mathcal{R}}\}$ then $G + \mathbf{0} = \{G^{\mathcal{L}}|G^{\mathcal{R}}\} + \{\varnothing^0|\varnothing^0\} = \{G^{\mathcal{L}} + \mathbf{0}|G^{\mathcal{R}} + \mathbf{0}\}$.

By induction, we have that $G^{\mathcal{L}} + \mathbf{0} \equiv G^{\mathcal{L}}$ and also $G^{\mathcal{R}} + \mathbf{0} \equiv G^{\mathcal{R}}$. Therefore $G \equiv G + \mathbf{0}$. □

In Section 3.6 we define a partial order on this monoid.

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[12]We use the symbol "+" for both addition in $(\mathcal{A}, +)$ and for the disjunctive sum $(\Omega, +)$, defined by the surrounding context; then, a correct, but too heavy, notation would be $((\Omega, (\mathcal{A}, +)), +)$, where the parentheses determine the respective meanings of "+".

The *conjugate* of a game denotes another game where the only difference is that Left and Right have "switched roles".

Definition 11. The conjugate of $G \in \Omega$ is

$$\overleftrightarrow{G} = \begin{cases} \{\varnothing^{-r} | \varnothing^{-\ell}\} & \text{if } G = \{\varnothing^\ell | \varnothing^r\} \text{ for some } \ell, r \in \mathcal{A}, \\ \{\overleftrightarrow{G^\mathcal{R}} | \varnothing^{-\ell}\} & \text{if } G = \{\varnothing^\ell | G^\mathcal{R}\} \text{ for some } \ell \in \mathcal{A}, \\ \{\varnothing^{-r} | \overleftrightarrow{G^\mathcal{L}}\} & \text{if } G = \{G^\mathcal{L} | \varnothing^r\} \text{ for some } r \in \mathcal{A}, \\ \{\overleftrightarrow{G^\mathcal{R}} | \overleftrightarrow{G^\mathcal{L}}\} & \text{otherwise,} \end{cases}$$

where $\overleftrightarrow{G^\mathcal{L}}$ denotes the set of games $\overleftrightarrow{G^L}$, for $G^L \in G^\mathcal{L}$, and similarly for $G^\mathcal{R}$.

By the recursive definition of the free space $(\Omega, \mathcal{A})$, each combinatorial game space is closed under conjugation. In normal-play, the games form an ordered group and each game G has an additive inverse, appropriately called $-G$, and $-G = \overleftrightarrow{G}$. However, there are other spaces of games, for example scoring and misère games, where $\overleftrightarrow{G}$ is not necessarily $-G$ (see [22]).

3.2. *Universes of games and parentality.* Many results in the literature concern strictly smaller structures than a free space. In dicot scoring-play the classical examples are Milnor's and Hanner's nonnegative incentive games [24], and later Ettinger studied all dicot scoring games [9]. Johnson [14] studies dicot scoring games with length of play of fixed parity. The normal-play theory started with impartial rulesets that do not distinguish between the players [4], and another classical normal-play restriction concerns all-small (dicot) games [3]. Dorbec et al. study dicot misère-play [8], whereas Milley and Renault study dead-ending misère-play [23]. Plambeck's sets of impartial games generated via disjunctive sums on single rulesets [26] are universes, although not absolute. (As we will see in Section 3.5, all except [4; 14; 24; 26] fit our theory.)

Definition 12 (Universe). A universe of games, $((\mathbb{U}, \mathcal{A}), +)$ satisfies $(\mathbb{U}, \mathcal{A}) \subseteq (\Omega, \mathcal{A})$, for some free monoid $((\Omega, \mathcal{A}), +)$, with

(1) $\{\varnothing^a | \varnothing^a\} \in \mathbb{U}$ for all $a \in \mathcal{A}$;

(2) (options closure) if $G \in \mathbb{U}$ and H is an option of G then $H \in \mathbb{U}$;

(3) (disjunctive sum closure) if $G, H \in \mathbb{U}$ then $G + H \in \mathbb{U}$;

(4) (conjugate closure) if $G \in \mathbb{U}$ then $\overleftrightarrow{G} \in \mathbb{U}$.

Note that the neutral element, associativity and commutativity are carried over from the free space $(\Omega, \mathcal{A})$, and so each universe $\mathbb{U} = ((\mathbb{U}, \mathcal{A})+)$ is also a commutative monoid.

Studied sets of games are often closed under taking options (a.k.a. hereditary closure), disjunctive sum (additive closure), and conjugation (flip the game board). The next property is somewhat less common.

Definition 13 (Parentality). A universe $\mathbb{U}$ of combinatorial games is parental if for any pair of finite nonempty sets of games $\mathcal{G}, \mathcal{H} \subset \mathbb{U}$, $\{\mathcal{G}|\mathcal{H}\} \in \mathbb{U}$.

Not all universes are parental. For example, no universe of impartial games is parental; the restriction that both players have the same options is disqualified by the generality of the sets $\mathcal{G}$ and $\mathcal{H}$ in Definition 13. No universe without $\mathscr{P}$-positions, or of the form of Milnor's positional games, is parental; the restriction that it is never bad to start is disqualified by the generality of the sets $\mathcal{G}$ and $\mathcal{H}$. Each free space is parental by definition. Similarly, given any free space, the nonrestricted dicot and dead-ending universes are parental.

3.3. *Evaluation maps relating to standard game spaces.* So far, only the "form" of a game has been considered. In this section, we elaborate on the incentive to play.

Definition 14 (Evaluation Map). The result of a terminal game is evaluated via one of two order-preserving maps $\nu_L, \nu_R : \mathcal{A} \to \mathcal{S}$, depending on who is to move; if Left (Right) is to move in a Left-atomic (Right-atomic) game, with adorn $\ell \in \mathcal{A}$ ($r \in \mathcal{A}$), then the result is $\nu_L(\ell)$ ($\nu_R(r)$).

In play Left (Right) seeks to maximize (minimize) the result. Indeed, we extend these evaluation maps to optimal play outcome functions in Definition 18.[13]

Let us first gather what we have got so far, by combining a free space with an evaluation function, in the notion of a combinatorial game space.

Definition 15 (Combinatorial Game Space). A combinatorial game space, abbreviated Ω, is the free commutative monoid $((\Omega, \mathcal{A}), +)$, together with a totally ordered set $\mathcal{S}$ and two order-preserving evaluation maps $\nu_L : \mathcal{A} \to \mathcal{S}$ and $\nu_R : \mathcal{A} \to \mathcal{S}$. That is,

$$\Omega = ((\Omega, \mathcal{A}), \mathcal{S}, \nu_L, \nu_R, +),$$

where "+" is the disjunctive sum over the free space $(\Omega, \mathcal{A})$. If $|\mathcal{A}| > 1$ then $(\Omega, \mathcal{A})$ is a scoring-play space with $\nu(a) = \nu_L(a) = \nu_R(a)$ for all $a \in \mathcal{A}$.

Observation 16. In the literature, combinatorial games are normal-play, misère-play or scoring-play.

(1) *Normal-play* corresponds to the trivial group $\mathcal{A} = \{0\}$ and the set $\mathcal{S} = \{-1, +1\}$, together with the maps $\nu_L(0) = -1$, $\nu_R(0) = +1$,

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[13]Observe that we could not have directly set the adorns to +1 and -1 respectively, to model for example that Left wins or loses normal-play. Namely, this cannot model disjunctive sum play, where there is no concept of win/loss on individual game components. But luckily, this idea is instead neatly captured by the ν function. In scoring-play however, the "winner" is not the relevant concept, and again ν is flexible enough to instead capture the sum of the terminal component scores. See Lemma 20 for more on this topic.

(2) *Misère-play* corresponds to the trivial group $\mathcal{A} = \{0\}$ and the set $\mathcal{S} = \{-1, +1\}$, together with the maps $\nu_L(0) = +1$, $\nu_R(0) = -1$,

(3) *Scoring-play* corresponds to $\mathcal{A} = \mathcal{S} = \mathbb{R}$, with its natural order and addition, and with ν, the identity map.

A universe $\mathbb{U} \subseteq (\Omega, \mathcal{A})$ inherits the totally ordered set $\mathcal{S}$ and the two order-preserving evaluation maps $\nu_L : \mathcal{A} \to \mathcal{S}$ and $\nu_R : \mathcal{A} \to \mathcal{S}$ from its enclosing free space, which defines the structure $\mathbb{U} = ((\mathbb{U}, \mathcal{A}), \mathcal{S}, \nu_L, \nu_R, +)$.

Only "form" is required to define universes of games, but one mostly thinks of a universe of games in terms of an associated winning convention. These topics are expanded upon in the paper [19].

Remark 17. It is easy to imagine scoring-play games that do not satisfy our axioms. For example, the players score points in some board game, but the result depends on the final score difference modulo 3. We might have $\mathcal{A} = \{0, 1, 2\}$, where addition is modulo 3; $\mathcal{S} = \{-1, 0, +1\}$, $\nu(0) = 0$, $\nu(1) = +1$, and $\nu(2) = -1$, and so ν is not order-preserving.

3.4. *The outcome function.* The mappings of adorns in $\mathcal{A}$ to elements of $\mathcal{S}$, vial the evaluation maps ν_L and ν_R from Definition 15 is extended to generic positions via two recursively defined *partial outcome functions*. The definition is general enough to match the usual definitions of outcomes for several universes studied in literature, in the sense of providing a generalization of Observation 16 to any game G. The outcome functions are not sensitive to the specific universe, only to the game space.

Definition 18. Let Ω be a combinatorial game space, and consider given evaluation maps $\nu_L : \mathcal{A} \to \mathcal{S}$ and $\nu_R : \mathcal{A} \to \mathcal{S}$. The Left- and Right-outcome functions are $o_L : \Omega \to \mathcal{S}, o_R : \Omega \to \mathcal{S}$, where

$$o_L(G) \begin{cases} \nu_L(\ell) & \text{if } G = \{\varnothing^\ell | G^\mathcal{R}\}, \\ \max_L\{o_R(G^L)\} & \text{otherwise,} \end{cases}$$

$$o_R(G) \begin{cases} \nu_R(r) & \text{if } G = \{G^\mathcal{L} | \varnothing^r\}, \\ \min_R\{o_L(G^R)\} & \text{otherwise,} \end{cases}$$

where the $\max_L$ ($\min_R$) ranges over all Left (Right) options of the given game $G \in \Omega$. The outcome of G is the ordered pair of Left- and Right-outcomes $o(G) = (o_L(G), o_R(G))$.

A brief note on terminology: the function o_L is here called the Left-outcome, and this function concerns optimal play results, when player Left starts. Usually, when we refer to "outcome" in combinatorial game theory, the move-flag has not yet been assigned; hence this function is necessarily an ordered pair $o = (o_L, o_R)$.

The outcome function turns a universe of games into a partially ordered monoid, and this we show in Section 3.6.

Let $G \in \mathbb{U}$. From Definition 18 we have that $o_L(G) = v_i(\ell)$ and $o_R(G) = v_j(r)$ for some $\ell, r \in \mathcal{A}$, and $i, j \in \{L, R\}$. Therefore we may always assume that the set of results is $\mathcal{S} = \{v_L(a) : a \in \mathcal{A}\} \cup \{v_R(a) : a \in \mathcal{A}\}$. If the union of the images were smaller than the union of the codomains, we may restrict $\mathcal{S}$ to be the union of the images. With this convention, we get the following trivial observation.

Lemma 19. *If* $|\mathcal{A}| = 1$ *then* $|\mathcal{S}| \leqslant 2$.

Proof. If $|\mathcal{A}| = 1$ then $\mathcal{S}$ contains exactly $v_L(0)$ and $v_R(0)$, at most 2 values. □

Note that if $|\mathcal{S}| = 1$, then play is trivial.

If one component in a disjunctive sum is a purely atomic game, then the outcome functions are easy to calculate. We state them for Left-outcomes only.

Lemma 20. *Let* $G, H \in \mathbb{U}$, *a universe of combinatorial games. Let* $\boldsymbol{c} = \{\varnothing^c | \varnothing^c\}$.

- *If* $G = \{\varnothing^a | G^{\mathcal{R}}\}$ *and* $H = \{\varnothing^b | H^{\mathcal{R}}\}$, *then* $o_L(G + H) = v_L(a + b)$.
- *If* $o_L(G) = v_L(a)$, *then* $o_L(G + \boldsymbol{c}) = v_L(a + c)$.
- $o_L(G + \boldsymbol{c}) \geqslant o_L(H + \boldsymbol{c})$ *if and only if* $o_L(G) \geqslant o_L(H)$.

Proof. These follow directly from Definitions 9 and 18. □

3.5. *Saturation, parentality and absolute universes.* The *adjoint* of a game is similar to the conjugate. We will have use for it in Theorem 24, where we prove that parentality implies saturation in absolute universes. The following definition is a generalization of the standard concept from misère-play theories.

Definition 21 (Adjoint). Consider a universe $\mathbb{U}$ and a game $G \in \mathbb{U}$. Then the adjoint of G is the game

$$G^{\circ} = \begin{cases} \{-\boldsymbol{r} | -\boldsymbol{\ell}\} & \text{if } G = \{\varnothing^{\ell} | \varnothing^r\}, \\ \{G^{\mathcal{R}\circ} | -\boldsymbol{\ell}\} & \text{if } |G^{\mathcal{R}}| > 0 \text{ and } G^{\mathcal{L}} = \varnothing^{\ell}, \\ \{-\boldsymbol{r} | G^{\mathcal{L}\circ}\} & \text{if } |G^{\mathcal{L}}| > 0 \text{ and } G^{\mathcal{R}} = \varnothing^r, \\ \{G^{\mathcal{R}\circ} | G^{\mathcal{L}\circ}\} & \text{otherwise,} \end{cases}$$

where $\circ$ applied to a set operates on its elements.

In all universes studied in the literature, for any game G, it is possible to conceive a disjunctive sum $G + H$ where the first player has the advantage. Let $* = \{\mathbf{0} | \mathbf{0}\}$. For example, regarding normal-play, take $H = -G + *$; regarding misère-play, take $H = \{G^{\circ} | G^{\circ}\}$; regarding scoring-play, take $H = \{\boldsymbol{s} | -\boldsymbol{s}\}$, with a sufficiently large s, or in terms of the adjoint let $H = \{G^{\circ} + \mathbf{1} | G^{\circ} - \mathbf{1}\}$. The following definition formalizes this idea in terms of *saturation*.

Definition 22 (Outcome Saturation). A universe $\mathbb{U}$ of combinatorial games is saturated if, for all $G \in \mathbb{U}$, for any pair $x, y \in \mathcal{S}$, there is an $H \in \mathbb{U}$ such that $o_L(G + H) \geqslant x$ and $o_R(G + H) \leqslant y$.

Definition 23 (Absolute Universe). A universe is absolute if it is parental and saturated.

We will prove that absolute universes have rich algebraic structures, and the naming is inspired by *neutrality* with respect to "winning condition".[14]

It turns out that saturation is a consequence of parentality.

Theorem 24. *Parental universes are outcome saturated.*

Proof. Consider a parental universe $\mathbb{U} \subset \Omega$. First of all note that if $G \in \mathbb{U}$, then $\overleftrightarrow{G}, G^\circ \in \mathbb{U}$.

Consider the case of $\mathbb{U}$ scoring-play, and let $G \in \mathbb{U}$. Then the evaluation map is independent of player. (The result may of course depend on who is to play, as is the case for $o(\{\varnothing^\ell | \varnothing^r\}) = (v(\ell), v(r))$.) Consider given $x, y \in S$. Since v is order-preserving, there is an $\ell \in \mathcal{A}$ such that $v(\ell) \geqslant x$, and there is an $r \in \mathcal{A}$ such that $v(r) \leqslant y$. Let $H = \overleftrightarrow{G} + \{\boldsymbol{\ell} | \boldsymbol{r}\}$, and note that, since $\mathbb{U}$ is parental, $\{\boldsymbol{\ell} | \boldsymbol{r}\} \in \mathbb{U}$. If Left starts, she can play to $G + \overleftrightarrow{G} + \boldsymbol{\ell}$, and mimic all remaining moves. This proves that $o_L(G + H) \geqslant v(\ell) \geqslant x$. The proof of $o_R(G + H) \leqslant y$ is similar.

Next, we consider the case when $\mathcal{A} = \{0\}$ and $\mathcal{S} = \{-1, 1\}$. It suffices to prove that, for any $G \in \mathbb{U}$, there is an $H \in \mathbb{U}$ such that $o_L(G + H) = 1$ and $o_R(G + H) = -1$. Suppose first that $v_L(0) = 1$ and $v_R(0) = -1$, i.e., the misère-play convention. Take $H = G^\circ + *$, and note that $H \in \mathbb{U}$ since $\mathbb{U}$ is parental. Left can play to $G + G^\circ$, and mimic Right until he makes the last move. This shows that $o_L(G + H) = v_L(0) = 1$. Similarly, $H = G^\circ + *$ gives $o_R(G + H) = v_R(0) = -1$.

Next, suppose $v_L(0) = -1$ and $v_R(0) = 1$ (i.e., normal-play). Then instead take $H = \overleftrightarrow{G} + * \in \mathbb{U}$, by parentality. Left plays to $G + \overleftrightarrow{G}$ and mimics Right, until she takes the last move. This shows that $o_L(G + H) = v_R(0) = 1$, and similarly $o_R(G + H) = v_L(0) = -1$. $\qquad\square$

Note that for normal- and misère-play the proof gives that there is a game H that satisfies "next player wins". Similarly one can find H satisfying the other outcome classes.

Corollary 25. *Let $\mathbb{U}$ be any parental normal- or misère-play universe. For all $G \in \mathbb{U}$ and for any $x, y \in \{-1, 1\}$, there is an $H \in \mathbb{U}$ such that $o_L(G + H) = x$ and $o_R(G + H) = y$.*

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[14]Naming is inspired by "neutral" or "absolute geometry", a relaxation of Euclidean geometry, that provides an umbrella for many geometries. By analogy, in absolute geometry, the four first postulates together give a rich theory, and it is neutral with respect to the parallel postulate.

Proof. Modify the game H in the second part of the proof of Theorem 24. We give more details in Observation 45.   □

Hence, for normal- and misère-play, saturation has this stronger notion. For scoring-play, one may obtain a stronger notion in the following spirit.

Corollary 26. *A scoring-play universe satisfies for all $G \in \mathbb{U}$, for any given pair $x, y \in \mathcal{S}$, there is an $H \in \mathbb{U}$ such that $o_L(G + H) \geqslant x$ and $o_R(G + H) \leqslant y$, and there is a $K \in \mathbb{U}$ such that $o_L(G + K) \leqslant x$ and $o_R(G + K) \geqslant y$.*

Proof. This is a consequence of the first part of the proof of Theorem 24, but for the second part of the statement, use instead $H = G^\circ + *$.   □

The important corollary is the next one. It proves that only form matters, when it comes to the validation of whether a universe is absolute.

Corollary 27. *Any parental universe is absolute.*

Proof. This is a direct consequence of Theorem 24.   □

3.6. *Order in universe.* In general, two games G and H are *equal* if both players are indifferent to playing G or H in any "situation", which means in any disjunctive sum since we are discussing combinatorial games. This idea is usually extended to define a partial order on games.

Definition 28 (Superordinate Game Comparison). Let $\mathbb{U} \subset \Omega$ be a universe of combinatorial games. For $G, H \in \mathbb{U}$, $G \succcurlyeq H$ modulo $\mathbb{U}$, or $G \succcurlyeq_\mathbb{U} H$, if and only if $o_L(G + X) \geqslant o_L(H + X)$ and $o_R(G + X) \geqslant o_R(H + X)$, for all games $X \in \mathbb{U}$. And $G = H$ if $G \succcurlyeq H$ and $H \succcurlyeq G$.

Definition 28 can be extended to have the distinguishing games in a sub- or superset of $\mathbb{U}$. Suppose, for example, that $G = H$ are Milnor-type games. Then G, H could perhaps be distinguished in the general dicot universe, the guaranteed universe, or perhaps in the free space of all scoring combinatorial games, and in general, the relation between the games is not the same. The importance of "modulo $\mathbb{U}$" will be expanded upon in Section 6. When the context is clear, however, instead of the phrase "for $G, H \in \mathbb{U}$, $G \succcurlyeq H$ modulo $\mathbb{U}$", we often write simply "for $G, H \in \mathbb{U}$, $G \succcurlyeq H$".

The next result shows that the order is compatible with the disjunctive sum operator, so that a universe is in fact a partially ordered quotient monoid.

Theorem 29. *Let $G, H \in \mathbb{U}$. If $G \succcurlyeq H$, then for all $J \in \mathbb{U}$, $G + J \succcurlyeq H + J$.*

Proof. Consider any game $J \in \mathbb{U}$. Since $G \succcurlyeq H$, it follows that $o_L(G + (J + X)) \geqslant o_L(H + (J + X))$ for any $X \in \mathbb{U}$. Because disjunctive sum is associative this inequality is the same as $o_L((G + J) + X)) \geqslant o_L((H + J) + X)$. The same argument gives $o_R((G + J) + X) \geqslant o_R((H + J) + X)$, and thus, since X is arbitrary, this gives that $G + J \succcurlyeq H + J$.   □

It is possible to have quasi-identities such that $X \neq 0$ and $X + X = X$. Consider the universe of impartial games with the misère-play convention. It is well known that, for $X = *2 + *2 \neq 0$, we have $X + X = X$. The full equivalence is related to invertibility as follows; the proof is similar to that of Theorem 29.

Corollary 30. *Let $G, H \in \mathbb{U}$. Then, for all invertible $J \in \mathbb{U}$,*

$$G \gtrsim H \iff G + J \gtrsim H + J.$$

4. The main lemma

This section concerns lemmas for Theorems 35 and 4. They compensate a potential loss of group structure in an absolute universe.

Lemma 31. *Let $\mathbb{U}$ be an absolute universe, and suppose that $G, H \in \mathbb{U}$. If $G \not\gtrsim H$ then*

$$o_L(G + T) < o_L(H + T) \quad \text{for some } T \in \mathbb{U},$$
$$o_R(G + V) < o_R(H + V) \quad \text{for some } V \in \mathbb{U}.$$

Proof. Because $G \not\gtrsim H$, at least one of the two inequalities must hold. Without loss of generality, we may assume the first. Thus, let T be such that

$$o_1 = o_L(G + T) < o_L(H + T) = o_2,$$

and we have to find a $V \in \mathbb{U}$ such that

$$o_R(G + V) < o_R(H + V). \tag{1}$$

Since $\mathbb{U}$ is outcome saturated, for every $H^R \in H^{\mathcal{R}}$ there exists a game $H^{R'} = (H^R)' \in \mathbb{U}$ such that

$$o_2 \leqslant o_R(H^{R'} + H^R). \tag{2}$$

Let $H^{\mathcal{R}'} = \{H^{R'} : H^R \in H^{\mathcal{R}}\}$, possibly the empty set if H is Right-atomic. Since $\mathbf{0} \in \mathbb{U}$ (by definition of a universe) and $\mathbb{U}$ is parental, the game V satisfies

$$V = \{H^{\mathcal{R}'}, \mathbf{0} \mid T\} \in \mathbb{U}.\text{[15]}$$

We claim that

$$o_R(G + V) \leqslant o_1, \tag{3}$$

and

$$o_2 \leqslant o_R(H + V), \tag{4}$$

regardless of $H^{\mathcal{R}}$. In the game $G + V$, Right has a move in V to $G + T$, and so inequality (3) holds.

---

[15]If $H^{\mathcal{R}}$ is nonempty, then the Left option $\mathbf{0}$ can be omitted.

If Right's best move in $H+V$ is to $H+T$, then $o_L(H+T) = o_2$ implies (4). If Right's best move in $H+V$ is to H^R+V, then Left can respond to $H^R+H^{R'}$, and so, by (2), $o_2 \leqslant o_R(H^R+H^{R'}) \leqslant o_L(H^R+V) = o_R(H+V)$. Thus inequality (4) holds. $\qquad\square$

In normal-play universes it is automatically true that $G + \overleftrightarrow{G} = \mathbf{0}$ and that $G \succcurlyeq H$ is equivalent to $G + \overleftrightarrow{H} \succcurlyeq \mathbf{0}$. To get around the problem of noninvertible elements in the other absolute universes, we use the following concept, introduced independently in [9] for scoring games, and in [31] for misère games. It is defined for any universe.

Definition 32. Let $G, H \in \mathbb{U} \subseteq (\Omega, \mathcal{A})$. Then G is *downlinked* to H, denoted by $G \swarrow H$, if $o_L(G + T) < o_R(H + T)$ for some $T \in \mathbb{U}$.

Lemma 33 (Main Lemma). *Let $G, H \in \mathbb{U} \subseteq (\Omega, \mathcal{A})$. If $\mathbb{U}$ is absolute, then*

$$G \swarrow H \quad \text{if and only if} \quad \text{for all } G^L,\ G^L \not\succcurlyeq H \text{ and for all } H^R,\ G \not\succcurlyeq H^R.$$

Proof. If $|\mathcal{S}| \leqslant 1$ then there is nothing to prove so we may assume that $|\mathcal{S}| > 1$. If $|\mathcal{A}| = 1$, that is, $\mathcal{A} = \{0\}$, then, by Lemma 19, $|\mathcal{S}| = 2$, and so without loss of generality we may assume that $\mathcal{S} = \{-1, 1\}$. This gives one out of two cases, normal- or misère-play, as displayed in the examples of universes in Section 3, and the results are known; in particular, Siegel solved the case of misère-play [31, p. 274]. We include these cases for completeness, and to illustrate the core idea before plunging into the harder case for scoring-play. The forward direction ignores play convention.

($\Rightarrow$) Suppose $G \swarrow H$, that is, $o_L(G + T) < o_R(H + T)$ for some $T \in \mathbb{U}$. Then, for any G^L and any H^R,

$$o_R(G^L + T) \leqslant o_L(G + T) < o_R(H + T),$$

and therefore $G^L \not\succcurlyeq H$, and

$$o_L(G + T) < o_R(H + T) \leqslant o_L(H^R + T),$$

and therefore $G \not\succcurlyeq H^R$.

($\Leftarrow$) Suppose for all G^L, $G^L \not\succcurlyeq H$, and for all H^R, $G \not\succcurlyeq H^R$. Then, by Lemma 31, for each G^{L_i}, we have $X_i \in \mathbb{U}$ such that

$$o_L(G^{L_i} + X_i) < o_L(H + X_i), \tag{5}$$

and for each H^{R_j}, we have Y_j such that

$$o_R(G + Y_j) < o_R(H^{R_j} + Y_j). \tag{6}$$

Consider first the case of $\mathcal{A} = \{0\}$, and let $\mathcal{X} = \{X_i\}$ and $\mathcal{Y} = \{Y_i\}$ be the sets of all such X_i and Y_i.

We must prove that $G \,\diagup\, H$, i.e., we must find T such that $-1 = o_L(G + T) < o_R(H + T) = 1$, which may be interpreted as second player wins in either case. By symmetry it suffices to demonstrate one of these, say $-1 = o_L(G + T)$, for normal- and misère-play respectively.

Define

$$
T_{\mathrm{nor}} = \begin{cases}
\mathbf{0} & \text{if } G = H = \mathbf{0}, \\
\{\varnothing^0 \,|\, \overleftarrow{H^{\mathcal{L}}}\} & \text{if } G = \mathbf{0},\ H^{\mathcal{L}} \neq \varnothing,\ H^{\mathcal{R}} = \varnothing^0, \\
\{\overrightarrow{G^{\mathcal{R}}} \,|\, \varnothing^0\} & \text{if } H = \mathbf{0},\ G^{\mathcal{L}} = \varnothing^0,\ G^{\mathcal{R}} \neq \varnothing, \\
\{\mathcal{Y} \,|\, \mathcal{X}\} & \text{if } H^{\mathcal{L}} \neq \varnothing,\ G^{\mathcal{R}} \neq \varnothing,
\end{cases}
$$

$$
T_{\mathrm{mis}} = \begin{cases}
* & \text{if } G = H = \mathbf{0}, \\
\{\mathbf{0} \,|\, H^{\mathcal{L}\circ}\} & \text{if } G = \mathbf{0},\ H^{\mathcal{L}} \neq \varnothing,\ H^{\mathcal{R}} = \varnothing^0, \\
\{G^{\mathcal{R}\circ} \,|\, \mathbf{0}\} & \text{if } H = \mathbf{0},\ G^{\mathcal{L}} = \varnothing^0,\ G^{\mathcal{R}} \neq \varnothing, \\
\{\mathcal{Y} \,|\, \mathcal{X}\} & \text{if } H^{\mathcal{L}} \neq \varnothing,\ G^{\mathcal{R}} \neq \varnothing.
\end{cases}
$$

In the case of normal-play, we use T_{nor} and prove in each case that Right wins $G + T$ when Left starts. In the first two cases, Left loses immediately. In the third case, Right can mimic Left's moves until end of play and hence Left loses. For the fourth case, if Left plays to $G + Y_j$, then (6) implies that Left loses, and if Left plays to $G^{L_i} + T$, then Right can respond to $G^{L_i} + X_i$, and Left loses, by (5).

The argument for misère-play is almost identical, but use instead T_{mis}.

In the case of scoring-play, we use a *centralization* idea. Recall that, by Definition 15, for $a \in \mathcal{A}$, $v_L(a) = v_R(a)$. For all Left options of G, let $\boldsymbol{a}_i = \{\varnothing^{a_i} \,|\, \varnothing^{a_i}\}$ be such that $o_L(G^{L_i} + X_i) = v(a_i)$, and for all Right options of H, let $\boldsymbol{b}_j = \{\varnothing^{b_j} \,|\, \varnothing^{b_j}\}$ be such that $o_R(G + Y_j) = v(b_j)$.

Now, by Lemma 20, for each i,

$$
o_L(G^{L_i} + X_i) < o_L(H + X_i) \iff o_L(G^{L_i} + X_i - \boldsymbol{a}_i) < o_L(H + X_i - \boldsymbol{a}_i).
$$

For all i, let $X_i' = X_i - \boldsymbol{a}_i$, and for all j, let $Y_j' = Y_j - \boldsymbol{b}_j$. Let $\mathcal{X}' = \{X_i'\}$ and $\mathcal{Y}' = \{Y_i'\}$ be the sets of all such thus centralized X_i' and Y_i'.

We know that, for all i, $-\boldsymbol{a}_i$ exists, and that $X_i' \in \mathbb{U}$ because $\mathbb{U}$ is closed under disjunctive sum. Then, by Lemma 20, for all i,

$$
o_L(G^{L_i} + X_i') = v(0) < o_L(H + X_i'), \tag{7}
$$

where the inequality is by definition of X_i' and by (5). Analogously, by (6), for all j, we get

$$
o_R(G + Y_j') = v(0) < o_R(H^{R_j} + Y_j'). \tag{8}
$$

Pick any $t = \{\varnothing^t \mid \varnothing^t\}$ and $s = \{\varnothing^s \mid \varnothing^s\}$ with $t < s$. By outcome saturation, for each i and j, we can find $G^{R_i,(t)}$ and $H^{L_j,(s)}$ such that

$$o_R(G^{R_i,(t)} + G^{R_i}) \leqslant v(t) \tag{9}$$

and such that $o_L(H^{L_j,(s)} + H^{L_j}) \geqslant v(s)$. Also, let $G^{(0)}$ and $H^{(0)}$ be the games with $o_R(G^{(0)} + G) \leqslant v(0) \leqslant o_L(H^{(0)} + H)$. Define

$$T = \begin{cases} \left\{\{G^{\mathcal{R},(t)}, \mathbf{0} \mid t - \boldsymbol{\ell}\} \mid \{s - r \mid H^{\mathcal{L},(s)}, \mathbf{0}\}\right\} & \text{if } G^{\mathcal{L}} = \varnothing^\ell \text{ and } H^{\mathcal{R}} = \varnothing^r, \\ \{G^{(0)} \mid \mathcal{X}'\} & \text{if } G^{\mathcal{L}} \neq \varnothing \text{ and } H^{\mathcal{R}} = \varnothing^r, \\ \{\mathcal{Y}' \mid H^{(0)}\} & \text{if } G^{\mathcal{L}} = \varnothing^\ell \text{ and } H^{\mathcal{R}} \neq \varnothing, \\ \{\mathcal{Y}' \mid \mathcal{X}'\} & \text{if } G^{\mathcal{L}} \neq \varnothing \text{ and } H^{\mathcal{R}} \neq \varnothing. \end{cases}$$

Because of the parental property, $T \in \Omega$.

Now, we will argue that $G \diagup H$ via T. There are four cases, one for each case in the definition of T.

<u>Case 1</u>: In the first case, we claim that

$$o_L(G + T) = o_R\big(G + \{G^{\mathcal{R},(t)}, \mathbf{0} \mid t - \boldsymbol{\ell}\}\big) \leqslant v(t). \tag{10}$$

By the same argument, it follows that

$$v(s) \leqslant o_L(H + \{s - r \mid H^{L_j,(s)}, \mathbf{0}\}) = o_R(H + T).$$

By the definition of $t < s$, and since v is order-preserving, together this implies that $o_L(G + T) < o_R(H + T)$. To prove the claim, the first equality in (10) follows since there is no Left move in G. For the inequality, if Right moves in G, then Left has a response to $G^{R_i} + G^{R_i,(t)}$, with outcome as in (9). Right can play to $G + t - \boldsymbol{\ell}$, which proves this case, since $o_L(G + t - \boldsymbol{\ell}) = v(\ell + t - \ell) = v(t)$.

<u>Case 2</u>: In the second case,

$$o_L(G + T) \leqslant v(0) < o_L(H + X_i') = o_R(H + T).$$

The first equality follows, because if Left moves to $G + G^{(0)}$, then $o_R(G + G^{(0)}) \leqslant v(0)$. If Left moves to $G^{L_i} + T$, then Right has a response to $G^{L_i} + X_i'$, with $o_L(G^{L_i} + X_i') = v(0)$, by (7), so Left cannot do better than $v(0)$. The equality follows because Right does not have a move in H.

<u>Case 3</u>: This argument is in analogy with the second case.

<u>Case 4</u>: If Left has an option in G and Right has an option in H, the following diagram shows all the cases. Possible best play for Left in $G + T$ is indicated with the uppermost "or" (and similarly for Right's play in the game $H + T$ in the

bottom). We use the centralization to $v(0)$ to bound the desired Left-outcomes.

$$o_L(G+T)$$

$$= \qquad \text{``or''} \qquad =$$

$$o_R(G^{L_i}+T) \leqslant o_L(G^{L_i}+X_i') = v(0) \qquad o_R(G+Y_j') = v(0)$$

$$\text{``}<\text{''}$$

$$o_L(H^{R_k}+T) \qquad\qquad o_L(H+X_m')$$

$$= \qquad \text{``or''} \qquad =$$

$$o_R(H+T)$$

We claim that, by construction, the Left-outcomes below the dotted line reside *above* $v(0)$. Namely, by (8), a (perhaps suboptimal) Left move in T gives

$$o_L(H^{R_k}+T) \geqslant o_R(H^{R_k}+Y_k') > v(0).$$

Moreover, by (7), for all m, $o_L(H+X_m') > v(0)$. For the leftmost inequality, Right played a (perhaps suboptimal) move in the T component. $\qquad\square$

The next lemma concerns an elementary rephrasing of the downlinked idea for any universe of games.

Lemma 34. *Let G, $H \in \mathbb{U} \subseteq (\Omega, \mathcal{A})$. Then $G \succcurlyeq H$ implies for all H^L, $G \not\lhd H^L$, and for all G^R, $G^R \not\lhd H$.*

Proof. For all $T \in \mathbb{U}$, for any H^L, we have

$$o_L(G+T) \geqslant o_L(H+T) \geqslant o_R(H^L+T).$$

Therefore, we cannot have $o_L(G+T) < o_R(H^L+T)$.

For all $T \in \mathbb{U}$, for any G^R, we have

$$o_L(G^R+T) \geqslant o_R(G+T) \geqslant o_R(H+T).$$

Therefore, we cannot have $o_L(G^R+T) < o_R(H+T)$. $\qquad\square$

5. The main result

We summarize the results in the previous section by proving the Maintenance of the Absolute Fundamental Theorem (Theorem 4). It says that, in any absolute universe, if Left prefers a game G before H, then she has a response to maintain this preference for any starting move by player Right, where we think of a move in the H component as a move on its conjugate, by the other player.

Theorem 35 (Maintenance). *Suppose that* $\mathbb{U} \subseteq (\Omega, \mathcal{A})$ *is an absolute universe of combinatorial games and let* $G, H \in \mathbb{U}$. *If* $G \succcurlyeq H$, *then*:

- *For all* G^R, *there is a* H^R *such that* $G^R \succcurlyeq H^R$ *or there is a* G^{RL} *such that* $G^{RL} \succcurlyeq H$.

- *For all* H^L, *there is a* G^L *such that* $G^L \succcurlyeq H^L$ *or there is a* H^{LR} *such that* $G \succcurlyeq H^{LR}$.

Proof. By Lemma 34, G is not downlinked to any H^L and no G^R is downlinked to H.

Therefore, Lemma 33 gives that, for all H^L, there is a G^L such that $G^L \succcurlyeq H^L$ or there is a H^{LR} such that $G \succcurlyeq H^{LR}$, and for all G^R, there is a H^R such that $G^R \succcurlyeq H^R$ or there is a G^{RL} such that $G^{RL} \succcurlyeq H$.   □

Next, we restate and prove the main result, the Absolute Fundamental Theorem, Theorem 4, of semilocal game comparison, also called basic order or play-comparison [16; 17], for absolute universes.

Theorem 4 (Basic Game Comparison). *Suppose* $\mathbb{U} \subseteq (\Omega, \mathcal{A})$ *is an absolute universe of combinatorial games and let* $G, H \in \mathbb{U}$. *Then* $G \succcurlyeq H$ *if and only if the following two conditions hold*:

Proviso: $o_L(G + X) \geqslant o_L(H + X)$ *for all Left-atomic* $X \in \mathbb{U}$ *and* $o_R(G + X) \geqslant o_R(H + X)$ *for all Right-atomic* $X \in \mathbb{U}$.

Maintenance: *For all* G^R, *there is an* H^R *such that* $G^R \succcurlyeq H^R$, *or there is a* G^{RL} *such that* $G^{RL} \succcurlyeq H$, *and for all* H^L, *there is a* G^L *such that* $G^L \succcurlyeq H^L$, *or there is an* H^{LR} *such that* $G \succcurlyeq H^{LR}$.

Proof. ($\Rightarrow$) $G \succcurlyeq H$ implies the Maintenance by Theorem 35. The Proviso also holds because, if not, it directly contradicts $G \succcurlyeq H$.

($\Leftarrow$) Assume the Proviso and the Maintenance, and suppose that $G \nsucccurlyeq H$. The last happens because $o_L(G + X) < o_L(H + X)$ or $o_R(G + X) < o_R(H + X)$ for some X. Consider an X of smallest birthday in those conditions, and without loss of generality, assume $o_L(G + X) < o_L(H + X)$. Left starts disjunctive sum $H + X$. We have three cases:

(1) The game X cannot be Left-atomic since this is in contradiction with the assumption of Proviso.

(2) Suppose that Left's optimal move is in the H component, i.e., $o_L(H + X) = o_R(H^L + X)$. Because of the Maintenance, we have that either there exists a G^L such that $G^L \succcurlyeq H^L$ or there exists a H^{LR} such that $G \succcurlyeq H^{LR}$. If the first holds,

$$o_L(G + X) \geqslant o_R(G^L + X) \geqslant o_R(H^L + X) = o_L(H + X).$$

If the second holds,

$$o_L(G + X) \geqslant o_L(H^{LR} + X) \geqslant o_R(H^L + X) = o_L(H + X).$$

Both contradict the assumption $o_L(G + X) < o_L(H + X)$.

(3) Suppose $o_L(H + X) = o_R(H + X^L)$. By the smallest birthday assumption, $o_R(G + X^L) \geqslant o_R(H + X^L)$. Therefore,

$$o_L(G + X) \geqslant o_R(G + X^L) \geqslant o_R(H + X^L) = o_L(H + X).$$

Once again, this contradicts the assumption $o_L(G + X) < o_L(H + X)$.

Hence, we have shown that $G \succcurlyeq H$. $\qquad\square$

It may seem at a first glance that the "for all atomic X" part of the Proviso is still a burden. But, we will see in the next section, in Theorem 40, that the Proviso will often be much simplified, in a shift from basic to subordinate order.

6. Absolute universes and their provisos

In Section 6.1, we continue investigating how normal-play is central to our theory, via an embedding of universes. Then, in Section 6.2, we study how the basic game comparison, from the last section, translates to a given absolute universe, to obtain a subordinate and constructive level of game comparison.

Recall that we compare games modulo some universe $\mathbb{U}$. In this section we will develop tools for interchanging comparison universes, while keeping the game trees identical. In this study we settle the problem: how does comparison in normal-play compare with comparison in other absolute universes? We sometimes index the set of adorns via $\mathcal{A} = \mathcal{A}_{\mathbb{U}}$ to clarify that the set of adorns belong to a given universe $\mathbb{U} \subseteq \Omega$. For comparison in a universe $\mathbb{U}$, we abbreviate $G \succcurlyeq H$ modulo $\mathbb{U}$ by $G \succcurlyeq_{\mathbb{U}} H$.

6.1. A normal-play order-embedding. In normal-play, X atomic implies that X is a number. By the number avoidance theorem [7], "do not play in a number, unless all components are numbers", the Proviso reduces to "$o_L(G) \geqslant o_L(H)$ and $o_R(G) \geqslant o_R(H)$", and this follows by induction on the Maintenance. Hence, as we already noted in the Second Fundamental Theorem, in this case, the proviso can be removed.

Although intuitively clear, we have not yet formally introduced the concept of "winning", but of course, for example in normal-play, Left wins G independently of who starts whenever $o(G) = (+1, +1)$.[16] The reason for this apparent negligence is that the various abstract game comparisons do not require

---

[16]An order-preserving "winning function" can be defined to make the jump from outcomes to winning consistent with our theory.

an explicit definition of winning. In normal- and misère-play win-loss is the natural notion, but a scoring-play universe cannot in general be reduced to a naïve win-loss situation, although in a recreational playing the terminal score is often reinterpreted as win-draw-loss. In theory, it is more convenient to apply a minimax algorithm on the terminal scores. As we have seen, normal- and misère-play can also be conveniently recast in this manner via the ν function.

Moreover, we may relax a game in any universe to view only its game tree, and then play it using the normal-play convention. Formally, we use a map on the group of adorns, where each adorn becomes the trivial element; given a game $G \in \mathbb{U}$, for each atom $\varnothing^a$ in each atomic follower of G, let $\varnothing^a \to \varnothing^0$. Denote this normal-play projection of $G \in \mathbb{U}$ by $G_{\mathbb{Np}} \in \mathbb{Np}$, and we may omit the subscript when the context is clear.

Further universal aspects of normal-play are revealed in the following observation and theorems.

Observation 36. We revisit Example 5, which concerns dicot misère. In both normal- and misère-play, let $* = \{\mathbf{0} \,|\, \mathbf{0}\}$, $\uparrow = \{\mathbf{0} \,|\, *\}$, and $\curlywedge = \{\mathbf{0}, * \,|\, *\}$ be the games where these are the literal forms (no reductions by domination or reversibility) There is an *order-preserving* map but not an order-embedding of misère-play into normal-play. For instance, consider $\mathbb{D}$ the dicot games with misère-play convention. Then $\curlywedge \succcurlyeq_{\mathbb{D}} \mathbf{0}$, and similarly $\curlywedge =_{\mathbb{Np}} \uparrow \succcurlyeq_{\mathbb{Np}} \mathbf{0}$. But the opposite is not true: $\uparrow \succcurlyeq_{\mathbb{Np}} \mathbf{0}$, but $\uparrow \,\|_{\mathbb{D}}\, \mathbf{0}$.

Theorem 37 (Normal-play Order-preserving Map). *Let $G, H \in \mathbb{U}$, an absolute universe. If $G \succcurlyeq_{\mathbb{U}} H$ then $G \succcurlyeq_{\mathbb{Np}} H$.*

Proof. Apply the Maintenance for the universe $\mathbb{U}$ as a strategy for Left to win $G_{\mathbb{Np}} - H_{\mathbb{Np}}$ playing second. □

However, $G \succ_{\mathbb{U}} H$ does not imply $G \succ_{\mathbb{Np}} H$; see Example 5.

We can prove a little more, by looking beyond Observation 36, and instead adapt ideas from guaranteed scoring games [15; 16; 17].

Let $G \in \mathbb{Np}$, and let $\mathbb{U} \subseteq \Omega$ be a universe of combinatorial games. Define the *normal-play mapping* $\zeta : \mathbb{Np} \hookrightarrow \mathbb{U}$ as $\zeta(G) \equiv G$, where $\mathbf{0} \in \mathcal{A}_{\mathbb{Np}}$ becomes instead $\mathbf{0} \in \mathcal{A}_{\mathbb{U}}$, and as before "$\equiv$" denotes identical literal forms. The game tree and all adorns are identical, but play convention and thus the surrounding context changes, i.e., the interpretation of "$\mathbf{0}$" is reflected in the maps ν_L and ν_R for the respective universes. Observe that in the game $\zeta(G)$, any terminal score is 0, and so played alone this game is trivial. But in disjunction with another scoring game, it could have a major influence on the outcome. For example, if $H \in \mathbb{U}$ is a zugzwang, then the normal-play game $G = \mathbf{1}$ gives a beneficial situation for Left in the game $\zeta(G) + H$.

In normal-play if the game value is an integer n, it denotes "n moves for Left" if $n \geqslant 0$ and otherwise it denotes "n moves for Right". If we let $\mathbb{U}$ be a scoring-play universe, then n normal-play moves map to n scoring-play moves (for the same player), indicated by $\zeta(n) = \overline{n}$ if $n \geqslant 0$, and $\zeta(n) = \underline{n}$ if $n \leqslant 0$, as not to confuse with the other type of "numbers" in scoring-play, that is, the "scores" $\boldsymbol{a} = \{\varnothing^a \mid \varnothing^a\}$.

For a game G, let

$$\mathrm{adorn}_L(G) = \big\{\ell \mid \{\varnothing^\ell \mid G^{\mathcal{R}}\} \text{ is a follower of } G\big\},$$

$$\mathrm{adorn}_R(G) = \big\{r \mid \{G^{\mathcal{L}} \mid \varnothing^r\} \text{ is a follower of } G\big\}$$

denote sets of adorns in atomic followers of G, and let

$$\mathrm{adorn}(G) = \mathrm{adorn}_L(G) \cup \mathrm{adorn}_R(G).$$

Definition 38 (Horror Vacui). A universe $\mathbb{U} \subseteq (\Omega, \mathcal{A})$ is horror vacui[17] if for all games of the forms $X = \{X^{\mathcal{L}} \mid \varnothing^r\}$, for $i \in \{L, R\}$,

$$v_i(\max(\mathrm{adorn}_i(G)) \leqslant v_R(r),$$

and for all games of the form $X = \{\varnothing^\ell \mid X^{\mathcal{R}}\}$, for $i \in \{L, R\}$,

$$v_i(\min(\mathrm{adorn}_i(G)) \geqslant v_L(\ell).$$

Intuitively, if a player is faced with a position in which they have no move, they would rather "pass" and have the other player continue than have the game finish. Of course, in normal-play, the worst possible "option" is the empty set. In misère-play the "worst" would be an infinite string of moves, but such games go beyond this study. The *horror vacui* property captures the worst of normal-play, and in such horror vacui universes, we will show that we have an order-embedding to normal-play. By definition, each normal-play universe is horror vacui, but no misère-play universe is horror vacui. Guaranteed scoring-play is the archetype of a non-normal-play horror-vacui universe.

Theorem 39 (Normal-play Order-embedding). *Suppose that $\mathbb{U} \subseteq (\Omega, \mathcal{A})$ is an absolute and horror vacui universe. Then the normal-play mapping $\zeta : \mathbb{N}\mathbb{p} \hookrightarrow \mathbb{U}$ is an order-embedding.*

Proof. Theorem 37 holds in particular for any game $G \in \mathbb{U}$, an absolute universe, where all adorns are 0s. Thus, it suffices to prove that $G \succcurlyeq_{\mathbb{N}\mathbb{p}} H$ implies $\zeta(G) \succcurlyeq_{\mathbb{U}} \zeta(H)$.

First, suppose that Left has no option in $\zeta(H) + X$, where $X \in \mathbb{U}$. This occurs when $X = \{\varnothing^x \mid X^{\mathcal{R}}\}$ and $H = \{\varnothing^0 \mid H^{\mathcal{R}}\}$. In this case, $o_L(\zeta(H) + X) = v_L(x)$, with evaluation in the universe $\mathbb{U}$. Now $\zeta(G) + X = \zeta(G) + \{\varnothing^x \mid X^{\mathcal{R}}\}$; if Left has

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[17]"Nature abhors a vacuum" is a postulate attributed to Aristotle.

a move in G, then Right could never improve his prospects by playing in the X component, because $\mathbb{U}$ is horror vacui (Definition 38). If Left cannot play in G the game is over and $o_L(\zeta(G) + X) = v_L(x)$. In both cases, $o_L(\zeta(G) + X) \geqslant v_L(x) = o_L(\zeta(H) + X)$.

Now assume Left has a move in $\zeta(H) + X$, and with evaluation in $\mathbb{U}$, suppose first that there is a Left move, X^L, such that $o_L(\zeta(H) + X) = o_R(\zeta(H) + X^L)$. Then, by induction, $o_R(\zeta(H) + X^L) \leqslant o_R(\zeta(G) + X^L)$ and $o_R(\zeta(G) + X^L) \leqslant o_L(\zeta(G) + X)$, which yields $o_L(\zeta(H) + X) \leqslant o_L(\zeta(G) + X)$.

The remaining case is that there is a Left move, H^L, with $o_L(\zeta(H) + X) = o_R(\zeta(H^L) + X)$.[18] In $\mathbb{Np}$, $G \succcurlyeq H$, i.e., $G - H \succcurlyeq_{\mathbb{Np}} \mathbf{0}$, and so Left has a winning move in $G - H^L$. There are two possibilities, either $G^L - H^L \succcurlyeq_{\mathbb{Np}} \mathbf{0}$ for some G^L or $G - H^{LR} \succcurlyeq_{\mathbb{Np}} \mathbf{0}$ for some H^{LR}. If the first occurs, then $G^L \succcurlyeq_{\mathbb{Np}} H^L$ and, by induction, $o_R(\zeta(G^L) + X) \geqslant o_R(\zeta(H^L) + X)$, which gives the inequalities

$$o_L(\zeta(H) + X) = o_R(\zeta(H^L) + X) \leqslant o_R(\zeta(G^L) + X) \leqslant o_L(\zeta(G) + X).$$

If $G - H^{LR} \succcurlyeq_{\mathbb{Np}} \mathbf{0}$ occurs, then, by induction, $o_L(\zeta(G) + X) \geqslant o_L(\zeta(H^{LR}) + X)$. We also have $o_L(\zeta(H^{LR}) + X) \geqslant o_R(\zeta(H^L) + X)$ and since we are assuming that $o_L(\zeta(H) + X) = o_R(\zeta(H^L) + X)$, we can conclude that $o_L(\zeta(G) + X) \geqslant o_L(\zeta(H) + X)$. $\qquad\square$

6.2. *Interpretation of the provisos in different universes.* Absolute universes as considered in the literature have unique interpretations of the Proviso on the subordinate level. For each scoring-play universe, we have $\mathcal{A} = \mathbb{R}$.

(1) In *normal-play*, $\mathbb{Np}$, the Proviso is implied by Maintenance.

(2) In *dicot misère-play*, $\mathbb{D}$, the cases X Left-atomic or X Right-atomic individually imply $X \equiv \mathbf{0}$. Consequently, requiring that $o_L(G + X) \geqslant o_L(H + X)$ (X Left-atomic) and $o_R(G + X) \geqslant o_R(H + X)$ (X Right-atomic) is the same as requiring $o_L(G) \geqslant o_L(H)$ and $o_R(G) \geqslant o_R(H)$.

(3) The case of *dead-ending misère-play*, $\mathbb{E}$, has some similarities with guaranteed scoring-play below, and we define the set of waiting-protected games, adapted from [18].[19] The Proviso simplifies to waiting-protected outcomes, defined by $\underline{o}_L(G) = \min\{o_L(G + \underline{n}) : n \geqslant 0\}$ and $\bar{o}_R(G) = \max\{o_R(G + \bar{n}) : n \geqslant 0\}$, where $\underline{\mathbf{0}} = \mathbf{0}$, and for $n \geqslant 1$, $\underline{n}$ is the game where Right can either eliminate the game, or play to $\underline{n-1}$. The analogous concept holds for $\bar{n}$ and Left. We use that $\underline{o}_L(G) \geqslant \underline{o}_L(H)$ and $\bar{o}_R(G) \geqslant \bar{o}_R(H)$ together with the Maintenance imply the Proviso (by induction). Recall Example 6.

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[18]We may identify $\zeta(H^L) = \zeta(H)^L$.

[19]Waiting-protected games are called "perfect murder" games in [18].

(4) The structure of the *free misère space*, $\mathbb{M}$, is not as rich as that of $\mathbb{Np}$. For example, $G = \mathbf{0}$ implies that $G \equiv \mathbf{0}$, the empty tree. Suppose that $G^{\mathcal{L}} \neq \varnothing^0$. Then, by a mimic-strategy, Left loses playing first in the game $G + \{\varnothing^0 \mid G^{\mathcal{L}^\circ}\}$, where $G^{\mathcal{L}^\circ}$ is the set of all adjoints of $G^L \in G^{\mathcal{L}}$. This proves that $G \not\equiv \mathbf{0}$, a next player winning position. For the definition of adjoint, see Definition 44, adapted from [30, Chapter V.6]). Siegel [31] proved that we can still have local comparison.

(5) For *dicot scoring-play*, $\mathbb{Ds}$, Ettinger [9] proved local comparison for scores, i.e., "Is $G \succcurlyeq \{\varnothing^a \mid \varnothing^a\}$?". The cases X Left-atomic and X Right-atomic imply $X = \{\varnothing^a \mid \varnothing^a\}$ (or more generally $X = \{\varnothing^\ell \mid \varnothing^r\}$) and hence the Proviso only needs to consider alternating play, similar to dicot misère.

(6) *Guaranteed scoring-play* [15], $\mathbb{Gs}$, has a straightforward translation of the Proviso, somewhat simpler than for dead-end misère, and a normal-play order-embedding. The last is explained by the fact that the "guaranteed property" *is* the horror vacui property in Definition 38. The Proviso simplifies to the *waiting-protected Left- and Right-scores*, defined by $\underline{o}_L(G) = \min\{o_L(G + \underline{n}) : n \geqslant 0\}$ and $\bar{o}_R(G) = \max\{o_R(G + \bar{n}) : n \geqslant 0\}$,[20] where $\underline{0} = \mathbf{0}$, and for $n \geqslant 1$, $\underline{n}$ is the game where Right can play to $\underline{n-1}$. That is, $\underline{o}_L(G)$ is the best score Left can achieve if Right, and only Right, is allowed to pass as many times as he wishes (as discussed in Section 2.4). Similarly, $\bar{o}_R(G)$ is the best score that Right can achieve if Left is allowed to pass. We use that $\underline{o}_L(G) \geqslant \underline{o}_L(H)$ and $\bar{o}_R(G) \geqslant \bar{o}_R(H)$ together with the Maintenance imply the Proviso (by induction). Recall Example 7.

(7) For *free scoring space* [33], $\mathbb{S}$, the situation is similar to free misère-play, but with infinitely many terminal scores. A similar argument to that in item (3) gives that $G = \{\varnothing^\ell \mid \varnothing^r\}$ if and only if $G \equiv \{\varnothing^\ell \mid \varnothing^r\}$.

We get the following result concerning the details of a local (or subordinate) Proviso of the aforementioned absolute universes. Note that the pass-allowed outcomes depend on the given universe, dead-end or scoring, respectively.

Theorem 40 (Subordinate Game Comparison). *Let $G, H \in \mathbb{U} \subseteq (\Omega, \mathcal{A})$, an absolute universe. Then $G \succcurlyeq_{\mathbb{U}} H$ if the Maintenance holds and $\mathbb{U}$ is*

(1) *normal-play*;

(2) *dicot misère-play, and $o(G) \geqslant o(H)$*;

(3) *dead-ending misère-play, and $\underline{o}_L(G) \geqslant \underline{o}_L(H)$ and $\bar{o}_R(G) \geqslant \bar{o}_R(H)$*;

(4) *free misère-play, and $H^{\mathcal{L}} = \varnothing^0 \Rightarrow G^{\mathcal{L}} = \varnothing^0$ and $G^{\mathcal{R}} = \varnothing^0 \Rightarrow H^{\mathcal{R}} = \varnothing^0$*;

(5) *dicot scoring-play, and $o(G) \geqslant o(H)$*;

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[20] Waiting protected is called pass-allowed in [15].

(6) *guaranteed scoring-play, and* $\underline{o}_L(G) \geqslant \underline{o}_L(H)$ *and* $\bar{o}_R(G) \geqslant \bar{o}_R(H)$;

(7) *free scoring-play, and* $H^{\mathcal{L}} = \varnothing^a \Rightarrow G^{\mathcal{L}} = \varnothing^b$ *with* $b \geqslant a$, *and* $G^{\mathcal{R}} = \varnothing^a \Rightarrow$ $H^{\mathcal{R}} = \varnothing^b$ *with* $a \geqslant b$.

Proof. All items except for (7) are explained in the list preceding the theorem. Let us give the proof of (7). We are assuming Maintenance and wish to prove that $H^{\mathcal{L}} = \varnothing^a \Rightarrow G^{\mathcal{L}} = \varnothing^b$ with $b \geqslant a$, and $G^{\mathcal{R}} = \varnothing^a \Rightarrow H^{\mathcal{R}} = \varnothing^b$ with $a \geqslant b$, is equivalent to

(a) $o_L(G + X) \geqslant o_L(H + X)$ for all Left-atomic $X \in \Omega$;

(b) $o_R(G + X) \geqslant o_R(H + X)$ for all Right-atomic $X \in \Omega$.

($\Rightarrow$) Assume $H^{\mathcal{L}} = \varnothing^a$ and $G^{\mathcal{L}} = \varnothing^b$, with $b \geqslant a$. Then Left cannot move in both $G + X$ and $H + X$, where X is Left atomic. By the assumption $b \geqslant a$, item (a) holds by additivity of adorns. Analogously (b) holds.

($\Leftarrow$) Given the Proviso, i.e., items (a) and (b), and Maintenance, we wish to show that $H^{\mathcal{L}} = \varnothing^a \Rightarrow G^{\mathcal{L}} = \varnothing^b$, with $b \geqslant a$. Assume $H^{\mathcal{L}} = \varnothing^a$. Thus, $o_L(H + X) = a + x$ if X Left-atomic, with $X^{\mathcal{L}} = \varnothing^x$. Suppose that $|G^{\mathcal{L}}| > 0$. Then we must find a Left-atomic X such that $o_L(G + X) < a + x$, where $X^{\mathcal{L}} = \varnothing^x$. Let $X = \{\varnothing^x | \overset{\leftrightarrow}{G^{\mathcal{L}}}\}$, with $x = 1 - a$, and so $o_L(G + X) \leqslant 0 < a + x$. Therefore $|G^{\mathcal{L}}| = 0$, and the proviso implies $b \geqslant a$. $\square$

7. Discussion of outcome saturation

Consider a universe $\mathbb{U}$ and a game $G \in \mathbb{U}$. Recall the definition of conjugate and adjoint. The conjugate of G is

$$\overset{\leftrightarrow}{G} = \begin{cases} \{\varnothing^{-r} | \varnothing^{-\ell}\} & \text{if } G = \{\varnothing^{\ell} | \varnothing^r\} \text{ for some } \ell, r \in \mathcal{A}, \\ \{\overset{\leftrightarrow}{G^{\mathcal{R}}} | \varnothing^{-\ell}\} & \text{if } G = \{\varnothing^{\ell} | G^{\mathcal{R}}\} \text{ for some } \ell \in \mathcal{A}, \\ \{\varnothing^{-r} | \overset{\leftrightarrow}{G^{\mathcal{L}}}\} & \text{if } G = \{G^{\mathcal{L}} | \varnothing^r\} \text{ for some } r \in \mathcal{A}, \\ \{\overset{\leftrightarrow}{G^{\mathcal{R}}} | \overset{\leftrightarrow}{G^{\mathcal{L}}}\} & \text{otherwise.} \end{cases}$$

The adjoint of G is

$$G^{\circ} = \begin{cases} \{-\boldsymbol{\ell} | -\boldsymbol{r}\} & \text{if } G = \{\varnothing^{\ell} | \varnothing^r\}, \\ \{G^{\mathcal{R}\circ} | -\boldsymbol{\ell}\} & \text{if } |G^{\mathcal{R}}| > 0 \text{ and } G^{\mathcal{L}} = \varnothing^{\ell}, \\ \{-\boldsymbol{r} | G^{\mathcal{L}\circ}\} & \text{if } |G^{\mathcal{L}}| > 0 \text{ and } G^{\mathcal{R}} = \varnothing^r, \\ \{G^{\mathcal{R}\circ} | G^{\mathcal{L}\circ}\} & \text{otherwise.} \end{cases}$$

The adjoint has an analogous property to the conjugate. By using the evaluation function, ν, we may unify the classical settings of normal- and misère-play with that of scoring-play etc. Typically, the adjoint is used in misère-play, whereas

the conjugate is used in normal-play. As we will see, either may be useful in scoring-play. The following lemma is used implicitly in the proof of Theorem 24.

Lemma 41. *Consider a universe* $\mathbb{U}$ *and a game* $G \in \mathbb{U}$. *Then* $o_L(G + G^\circ) \leqslant \nu_R(0)$ *and* $o_L(G + \overleftrightarrow{G}) \leqslant \nu_L(0)$.

Proof. Right can mimic each Left move. Then use additivity of the adorns. In the first case, Right is the current player at the terminal position. In the second case, this player is Left. $\square$

For any scoring universe $\mathbb{U}$, with $G \in \mathbb{U}$ and $\ell, r \in \mathbb{R}$,

$$o_L(G + G^\circ + \{\varnothing^\ell | \varnothing^r\}) \leqslant \nu(r) \quad \text{and} \quad o_L(G + \overleftrightarrow{G} + \{\varnothing^\ell | \varnothing^r\}) \leqslant \nu(\ell).$$

Observation 42. For any scoring universe $\mathbb{U}$, with $G \in \mathbb{U}$ and $\ell, r \in \mathbb{R} = \mathcal{A}$,

$$o_L(G + G^\circ + \{\varnothing^\ell | \varnothing^r\}) \leqslant r \quad \text{and} \quad o_L(G + \overleftrightarrow{G} + \{\varnothing^\ell | \varnothing^r\}) \leqslant \ell.$$

Similarly,

$$o_R(G + G^\circ + \{\varnothing^\ell | \varnothing^r\}) \geqslant \ell \quad \text{and} \quad o_R(G + \overleftrightarrow{G} + \{\varnothing^\ell | \varnothing^r\}) \geqslant r.$$

The inequalities can be reversed, if we make a small change to the "adjoint", which is the variant we use in the proof of the main lemma in this paper. This is the variant that induces the definition of (scoring-play) saturation.[21]

Observation 43. For any scoring universe $\mathbb{U}$, for any $G \in \mathbb{U}$, $x \in \mathbb{R}$,

$$o_L(G + G^\circ + \{\boldsymbol{\ell} | \cdot\}) \geqslant \boldsymbol{\ell} \quad \text{and} \quad o_L(G + \overleftrightarrow{G} + \{\boldsymbol{\ell} | \cdot\}) \geqslant \boldsymbol{\ell}.$$

Similarly,

$$o_R(G + G^\circ + \{\cdot | \boldsymbol{r}\}) \leqslant \boldsymbol{r} \quad \text{and} \quad o_R(G + \overleftrightarrow{G} + \{\cdot | \boldsymbol{r}\}) \leqslant \boldsymbol{r}.$$

We review the misère-play outcome saturation, and normal-play is analogous, but instead of the adjoint it uses the conjugate.

Definition 44 (Misère-play Adjoint [30]). Let $\mathbb{U}$ be a misère-play universe. The *adjoint* of $G \in \mathbb{U}$ is the game G° given by

$$G^\circ = \begin{cases} \{\mathbf{0} | \mathbf{0}\} & \text{if } G = \mathbf{0}, \\ \{G^{\mathcal{R}^\circ} | \mathbf{0}\} & \text{if } G \neq \mathbf{0} \text{ and } G \text{ is Left-atomic}, \\ \{\mathbf{0} | G^{\mathcal{L}^\circ}\} & \text{if } G \neq \mathbf{0} \text{ and } G \text{ is Right-atomic}, \\ \{G^{\mathcal{R}^\circ} | G^{\mathcal{L}^\circ}\} & \text{otherwise}, \end{cases}$$

where $G^{\mathcal{R}^\circ}$ is the set of games of the form G^{R°.

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[21]For the classical winning conditions (normal- and misère-play) saturation is usually the stronger notion: for any game G, and any outcome x, there is a game H such that $o(G + H) = x$.

Note that the adjoint $G^\circ \in \mathbb{U}$ if $G \in \mathbb{U}$, and it is never atomic. The following observation is well known.

Observation 45 (Misère-play Saturation). Given any game G in misère-play, using the adjoint G°, we get that $G + G^\circ$ is a $\mathscr{P}$-position, that is, $o(G + G^\circ) = (-1, +1)$.[22]

Given any misère-play game G, we need to find games G_L, G_R, G_N and G_P such that each outcome class is represented, that is, such that $o(G + G_i) = i$. We have already shown this for $G_P = G^\circ$. For G_N, we take $G_N = \{G^\circ | G^\circ\}$; the first player to move in $G + G_N$ can move to $G + G^\circ$ and win. Hence $o(G + G_N) = \mathscr{N}$. Consider the game $G_L = \{G^{\mathcal{R}\circ}, G^\circ | G_N\}$ (where we assume there is a G^R and otherwise omit that part). We determine the outcome of $G + \{G^{\mathcal{R}\circ}, G^\circ | G_N\}$. If Left begins she can play to $G + G^\circ$ and win. If Right begins by playing in the G-component, then Left can respond to $G^R + G^{R\circ}$ and win because this is a $\mathscr{P}$-position. Otherwise Right plays to $G + G_N$, which is an (already proved) $\mathscr{N}$-position, so Left wins in either case. Hence $o(G + G_L) = \mathscr{L}$, and an analogous argument proves that $o(G + G_R) = \mathscr{R}$.

8. Future directions and open problems

Let us mention a couple of related problems.

• Are there infinitely many absolute universes [19]?

• Prove reduction theorems, via domination and reversibility, and discuss the possibility of an absolute "canonical form" game value. For what choice of "smallest" do we obtain uniqueness? Observe that for normal-play no choice is required, because uniqueness is automatic. For the guaranteed and dead-ending universes a choice of "smallest" form is necessary.

• Normal-play games satisfy a nice categorical structure. It is demonstrated in [16] that certain aspects of this structure generalize to absolute universes. In that paper it is also shown that absolute game comparison has an analogue play-comparison to normal-play games, but where the given Proviso induces the termination of a game. See [16] for open problems in related areas.

• Prove mean value theorems that generalize the classical settings of Hanner (with respect to "scores") and Conway (with respect to "moves"). In particular, is it possible to have a combined mean value with respect to "scores" and "moves" in various scoring-play settings, say for example in guaranteed scoring-play?

• Is there an absolute theory for loopy (cyclic) games?

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[22]Here, we use the classical notation of outcomes relating to the standard winning convention.

References

[1] M. H. Albert, R. J. Nowakowski, and D. Wolfe, *Lessons in Play*: *an Introduction to Combinatorial Game Theory*, A K Peters, 2007. Zbl

[2] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays*, *II*: *Games in Particular*, Academic Press, 1982. MR Zbl

[3] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays*, *I*, 2nd ed., A K Peters, 2001. Zbl

[4] C. L. Bouton, "Nim, a game with a complete mathematical theory", *Ann. of Math.* (2) **3**:1-4 (1901/02), 35–39. MR Zbl

[5] J. H. Conway, *On Numbers and Games*, London Math. Soc. Monogr. **6**, Academic Press, 1976. MR Zbl

[6] J. H. Conway, "Loopy games", *Ann. Discrete Math.* **3** (1978), 55–74. MR Zbl

[7] J. H. Conway, *On Numbers and Games*, 2nd ed., A K Peters, 2001. MR Zbl

[8] P. Dorbec, G. Renault, A. N. Siegel, and E. Sopena, "Dicots, and a taxonomic ranking for misère games", *J. Combin. Theory Ser. A* **130** (2015), 42–63. MR Zbl

[9] J. M. Ettinger, *Topics in combinatorial games*, Ph.D. thesis, University of Wisconsin-Madison, 1996, https://www.proquest.com/docview/304286281.

[10] J. M. Ettinger, "A metric for positional games", *Theoret. Comput. Sci.* **230**:1-2 (2000), 207–219. MR Zbl

[11] A. S. Fraenkel and Y. Yesha, "The generalized Sprague–Grundy function and its invariance under certain mappings", *J. Combin. Theory Ser. A* **43**:2 (1986), 165–177. MR Zbl

[12] P. M. Grundy, "Mathematics and games", *Eureka* **1939**:2 (1939), 6–8.

[13] P. M. Grundy and C. A. B. Smith, "Disjunctive games with the last player losing", *Proc. Cambridge Philos. Soc.* **52** (1956), 527–533. MR Zbl

[14] W. Johnson, "The combinatorial game theory of well-tempered scoring games", *Internat. J. Game Theory* **43**:2 (2014), 415–438. MR Zbl

[15] U. Larsson, R. J. Nowakowski, J. P. Neto, and C. P. Santos, "Guaranteed scoring games", *Electron. J. Combin.* **23**:3 (2016), art. id. 3.27. MR Zbl

[16] U. Larsson, R. J. Nowakowski, and C. P. Santos, "Game comparison through play", *Theoret. Comput. Sci.* **725** (2018), 52–63. MR Zbl

[17] U. Larsson, R. J. Nowakowski, and C. P. Santos, "Games with guaranteed scores and waiting moves", *Internat. J. Game Theory* **47**:2 (2018), 653–671. MR Zbl

[18] U. Larsson, R. Milley, R. Nowakowski, G. Renault, and C. P. Santos, "Reversibility, canonical form, and invertibility in dead-ending misère play", pp. 245–266 in *Games of No Chance* 6, edited by U. Larsson, Math. Sci. Res. Inst. Publ. **71**, Cambridge Univ. Press, 2025.

[19] U. Larsson, R. J. Nowakowski, and C. P. Santos, "Infinitely many absolute universes", pp. 229–244 in *Games of No Chance* 6, edited by U. Larsson, Math. Sci. Res. Inst. Publ. **71**, Cambridge Univ. Press, 2025.

[20] N. A. McKay, "Canonical forms of uptimals", *Theoret. Comput. Sci.* **412**:52 (2011), 7122–7132. MR Zbl

[21] G. A. Mesdal and P. Ottaway, "Simplification of partizan games in misère play", *Integers* **7** (2007), art. id. G06. MR Zbl

[22] R. Milley, "Partizan Kayles and misère invertibility", *Integers* **15** (2015), art. id. G3. MR Zbl

[23] R. Milley and G. Renault, "Dead ends in misère play: the misère monoid of canonical numbers", *Discrete Math.* **313**:20 (2013), 2223–2231. MR Zbl

[24] J. Milnor, "Sums of positional games", pp. 291–301 in *Contributions to the Theory of Games, II*, edited by H. W. Kuhn and A. W. Tucker, Ann. of Math. Stud. **28**, Princeton Univ. Press, 1953. MR Zbl

[25] T. E. Plambeck, "Taming the wild in impartial combinatorial games", *Integers* **5**:1 (2005), art. id. G5. MR Zbl

[26] T. E. Plambeck, "Advances in losing", pp. 57–90 in *Games of No Chance* 3, edited by M. H. Albert and R. J. Nowakowski, Math. Sci. Res. Inst. Publ. **56**, Cambridge Univ. Press, 2009. MR Zbl

[27] T. E. Plambeck and A. N. Siegel, "Misère quotients for impartial games", *J. Combin. Theory Ser. A* **115**:4 (2008), 593–622. MR Zbl

[28] A. N. Siegel, "Coping with cycles", pp. 91–124 in *Games of No Chance* 3, edited by M. H. Albert and R. J. Nowakowski, Math. Sci. Res. Inst. Publ. **56**, Cambridge Univ. Press, 2009. MR Zbl

[29] A. N. Siegel, "New results in loopy games", pp. 215–232 in *Games of No Chance* 3, edited by M. H. Albert and R. J. Nowakowski, Math. Sci. Res. Inst. Publ. **56**, Cambridge Univ. Press, 2009. MR Zbl

[30] A. N. Siegel, *Combinatorial Game Theory*, Graduate Studies in Mathematics **146**, Amer. Math. Soc., Providence, RI, 2013. MR Zbl

[31] A. N. Siegel, "Misère canonical forms of partizan games", pp. 225–239 in *Games of No Chance* 4, edited by R. J. Nowakowski, Math. Sci. Res. Inst. Publ. **63**, Cambridge Univ. Press, 2015. MR Zbl

[32] R. P. Sprague, "Über mathematische Kampfspiele", *Tôhoku Math. J.* **41** (1935), 438–444. Zbl

[33] F. Stewart, "Scoring play combinatorial games", pp. 447–467 in *Games of No Chance* 5, edited by U. Larsson, Math. Sci. Res. Inst. Publ. **70**, Cambridge Univ. Press, 2019. MR Zbl

larsson@iitb.ac.in   Department of Industrial Engineering and Operations Research, Indian Institute of Technology Bombay, Mumbai, India

r.nowakowski@dal.ca   Mathematics and Statistics, Dalhousie University, Halifax, NS, Canada

cmf.santos@fct.unl.pt   Center for Mathematics and Applications (NovaMath), Universidade Nova de Lisboa Campus de Caparica, Lisbon, Portugal