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Survey on Richman bidding combinatorial games

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In this survey, we explore the literature around Richman bidding in both its continuous and discrete forms. Our primary objective is to highlight recent advancements in discrete Richman bidding, which, in the normal play setting, generalizes the classical alternating play to infinitely many game families.

1. Introduction

Consider a game position of TOPPLING DOMINOES as in Figure 1, with two red pieces and a blue piece between them. In this game, the players, Red and Blue, at their turn, topple red and blue dominoes, respectively, in either direction. The winner is determined by the normal play convention: a player who cannot move loses. In the alternating play convention, Red wins irrespective of the starting player; indeed, if Red is "Left", who is positive, the well-known game value is $\frac{1}{2} > 0$.

Now consider a variation where the move order is instead decided by bidding. Both players start with a preallocated budget. Before each move, they bid any nonnegative integer amount from their budget. The highest bidder wins the bid, transfers the winning amount to the other player, and makes a move in the game. One of the players holds a tie-breaking marker to resolve any ties. If there is a tie in the bids, the player with the tie-breaking marker wins the bid, and transfers the winning amount together with the marker, and makes a move in the game. The marker holder may optionally include the marker with their bid; if they win, they transfer the winning bid and the marker, even without a tie. If a player wins the bid but cannot make a move, the game ends, and that player loses. This is a bidding-normal-play situation. We assume that the bids are sealed.

Let us consider a specific budget allocation to the TOPPLING DOMINOES position in Figure 1. Suppose Blue has ₹1 with the tie-breaking marker, while Red has no money. Assume, in their first bidding round, Blue goes all in by bidding ₹1 along with the marker, and Red bids ₹0. Blue wins the bid, transfers ₹1 and the marker to Red, and topples the blue domino towards the left. With a

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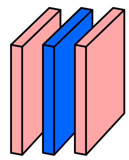

Figure 1. A TOPPLING DOMINOES position (with two red pieces and a blue piece between them).

single standing red domino, Red bids ₹1 with the tie-breaking marker, and Blue bids ₹0. Red wins the bid, transfers ₹1 and the marker back to Blue and topples the last red domino. With no dominoes left, both players bid ₹0 to avoid making a move in the game. Blue wins the bid by transferring the marker but cannot make a move, thus losing the game. Could Blue have bid differently to change the outcome? The answer is "no". Even if Red always bids ₹0, she will win this game. Later in Section 3, we will see that this is true for all such games.

Now consider the disjunctive sum domino position as in Figure 2. In alternating normal play, Red wins this disjunctive sum regardless of the starting player; indeed, if player Red is "Left", then the alternating play game value is $\frac{1}{2}$ ⧾$_1 = \{0|1\} + \big\{0|\{0|-1\}\big\} > \frac{1}{2} > 0$.

But what about bidding play? Let us again use the budget allocation where Blue has ₹1 with the tie-breaking marker, and Red has no money. Suppose that Blue first bids ₹0 with the tie-breaking marker and then bids ₹1 in the second round to secure two consecutive moves in the game. He plays both moves in the second component, ensuring that a single blue domino remains in this component. Both players then bid ₹0. Red wins the bid by transferring the marker and topples the leftmost domino of the first component towards the right, ending that component. Blue bids ₹0 again. If Red bids ₹1, she wins the bid but cannot move in the game and loses. If Red bids ₹0, Blue wins the bid by transferring the tie-breaking marker and topples the last blue domino left in the second component. As in the previous example, both players bid ₹0 to avoid winning the last bid of the game. Red wins the bid by transferring the marker but cannot move in the game, so she loses. For all other choices by Red, Blue still wins the game. Hence, the established alternating play theory is not applicable here.

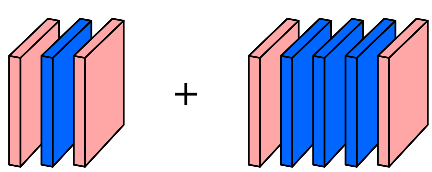

Figure 2. A TOPPLING DOMINOES disjunctive sum position.

The authors together with Rai and Upasany [7; 8] extend the alternating normal play theory to include bidding play. This is an important development because it dramatically enriches both "auction theories" and "combinatorial game theories" (CGT). Standard theorems in the CGT literature now become reference points for further studies in the bidding setup.

Outline. The purpose of this survey is to study the development of *Richman bidding combinatorial games*, as exemplified in the introduction. In Section 2, we review the development of both the continuous and the discrete models [6; 9; 10; 11; 12; 13]. In Section 3, we discuss how the discrete bidding convention, as explained in the above sample games, recently entered the theory of combinatorial games [4; 5]. The papers [7; 8] define "normal play" in the bidding setup and develop classical concepts such as *outcomes*, *disjunctive sum*, and *constructive game comparison*.

2. Richman bidding

Let us explore a game scenario involving two players, Left and Right, based on a directed graph as in Figure 3. A token, X, is placed on the top vertex. From each vertex, both players can move the token along the directed edges. Among these vertices, two are labelled ℓ and r. Left aims to bring the token to ℓ, while Right aims to bring the token to r. These games are referred to as *double-reachability games* since each player has a different target. Let us consider *random turn play* where, at each turn, a spinner, equally likely to direct Left or Right, determines the next mover. It is easy to calculate that in this game, Left wins with a probability of $\frac{3}{4}$, and Right wins with a probability of $\frac{1}{4}$.

Instead of randomly deciding the next mover, David Richman introduces, in the mid 1980s, bidding as a mechanism for players to secure the next move in the double-reachability game.[1] In this framework, both players start with a preallocated amount of money. Players simultaneously bid any nonnegative real number from their allocated resources to gain the move; the higher bidder transfers the bid amount to the other player and makes a move in the finite directed graph D. The directed graph D contains two labelled vertices, ℓ and r, with paths from every other vertex leading to either of them. Similar to the previously discussed random turn play of double-reachability game, each player can slide the token along an arc in the directed graph. Since the money is paid only back and forth, for convenience, let us assume a total budget, $\mathrm{TB} = 1$. In case of a tie in bidding, a fair coin toss determines the winner. In this Richman game, the objective remains to reach a predefined winning target.

[1] After his tragic passing, this concept was further developed and published by Lazarus et al. in [10; 11].

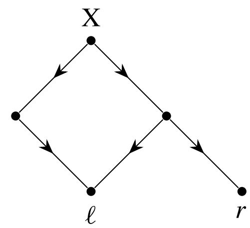

Figure 3. A directed game graph.

Lazarus et al. in [10] show that in the Richman game, given the initial budget partition and the token's location at vertex v, there exists an optimal deterministic strategy for one of the players to secure victory. Specifically, there exists a unique rational-valued threshold function $R(v)$ (also known as the Richman cost function), such that if Left's share of the money exceeds $R(v)$, she secures victory; if it is less, then Right wins. Remarkably, the method of conducting bids does not affect this outcome. Moreover, for acyclic digraphs, they also found a constructive approach to determine the Richman cost function.

Furthermore, Richman got a surprising connection with the random turn game.

Theorem 1 [10]. *Suppose that*, *in the digraph D*, *the probability of Left winning the random turn game from the vertex v is $r(v)$. Then*

$$R(v) = 1 - r(v).$$

For example, for the bidding variation of the game in Figure 3, the Richman cost function is $R(v) = \frac{1}{4}$. That is, if Right's initial budget is more than $\frac{3}{4}$, he can ensure his win, and if Left's budget exceeds $\frac{1}{4}$, she can force a victory.

Extending this work, Lazarus et al. [11] show, for infinite "double-reachability bidding games", the existence of a nontrivial Richman interval $(1 - R(v), R(v))$. If Left's share of the money exceeds $R(v)$, she can force a win; if it is less than $1 - R(v)$, then Right can force a win, and in all other cases, the game is a draw. In this case, the Richman cost $R(v)$ is not necessarily a rational number; instead, it can be any real number between 0 and 1.

Avni et al. [2] study infinite-duration games on directed graphs using the same Richman bidding setup. Motivated by various real-life applications, they focus on *parity games* and *mean-payoff games*. In parity games, the vertices of the directed graph are labelled, and a player wins if the parity of the highest index visited infinitely often is odd. In mean-payoff games, vertices have associated weights, and the payoff is the long-run average of the accumulated weights of the visited vertices. Player 1 aims to maximize this payoff and wins the game if it is ≥ 0, while Player 2 aims to minimize it. They show that parity games are linearly reducible to reachability games, and thus a threshold budget exists.

For mean-payoff games, they first determine the mean-payoff value within its strongly connected components. Building on this, they construct a reachability game to show the existence of a threshold budget for the original mean-payoff game. For strongly connected mean-payoff games, they show that the mean-payoff value does not depend on the initial budget. Furthermore, this value is equal to the mean-payoff value of the random turn mean-payoff game, where, the next mover is selected uniformly at random on each turn.

Richman's theory is elegant and reveals surprising connections with random turn games, but it considers bidding with real numbers, which is not suitable for recreational play. Additionally, their study is limited to symmetric games, where both players can move from each position. Motivated by these limitations, Mike Develin and Sam Payne discretize Richman's bidding process in [6] and develop the theory for partizan games. They consider a game G played on a coloured directed graph, with red and blue edges representing the moves of Left and Right, respectively. The terminal nodes are red, blue or uncoloured. Red and blue nodes denote winning positions for Left and Right, respectively, and the uncoloured nodes denote ties. There is a token at one of the vertices, and a move by any player consists of sliding the token along their corresponding coloured edge. The game ends at a terminal node, or if one player is unable to move, in which case this player loses. Similar to Richman's theory, both players start the game with a preallocated amount of money, and their sum constitutes the total budget TB. Players bid simultaneously to make a move in the game; the bid winner pays the winning amount to the other player and, unlike in Richman's theory, can also force the other player to make a move. The idea here is to model the *zugzwang* position, where no player wants to make a move in the game. To address tie situations, they introduce a tie-breaking marker. At the beginning of the game, one player starts with the marker. In case of a tie, the marker holder can either claim victory by giving away the marker or keep the marker, allowing the opponent to win. In contrast to the introductory example, here players always want to win the bid. With this discrete bidding notion, they establish that having the tie-breaking marker is advantageous but not worth more than a standard chip. Analogous to the results of Richman's theory, they prove the existence of a threshold amount for Left to win the game. This threshold amount depends only on the game G and the total budget TB. Consequently, the precise manner of bidding does not impact the theoretical outcome. As an illustration, they study the game of TIC-TAC-TOE in their discrete Richman bidding model. For all TB $\in \mathbb{N}$, they determine the threshold amount that Left requires to win the game, along with her optimal strategy. Independently, Theodore Hwa has calculated the solution of TIC-TAC-TOE in the continuous Richman setting [6]. He finds that the Richman cost of the initial position for this game is $\frac{133}{256}$. Note that the Richman cost is greater than $\frac{1}{2}$

even though the game is symmetric because a draw is also a possible outcome. Interestingly, the optimal moves of players in the continuous Richman setting are also optimal in the discrete Richman bidding model, except when $\mathrm{TB} = 5$.

In 2019, Larsson and Wästlund extend Richman's work with continuous bidding to include partizan games [9]. Building on Richman's continuous bidding framework, they introduce negative bids to handle *zugzwang* positions. The game setup is the same as in [6], but the bidding method contrasts with theirs. Here, bidding is via an open scheme, where the player who made the last move places the next bid. When one player bids, the other player can either accept the bid (taking the bid amount and allowing the bidder to make the next move) or reject the bid (paying the same amount and making the move). Similar to the Richman interval for symmetric infinite directed graphs, they show the existence of a nontrivial interval for partizan games with finitely many positions. When turns are determined randomly, the lower and upper bounds of this interval represent the maximum probability that Left can force a win and avoid a loss, respectively. Additionally, when turns are determined by their bidding rule, the lower and upper values of the same interval represent the minimum amount Right needs to avoid losing and to force a win, respectively. They identify a CHESS position with such a nontrivial interval and show that in a three-piece CHESS scenario (two kings and one additional piece), the interval collapses to a single value for all such positions, despite its partizan nature. Furthermore, in CHESS, they identify *zugzwang* positions where the optimal strategy involves placing a negative bid.

In the traditional setting, Richman games are zero-sum. In this view, Meir et al. [12] extend this theory to general sum games, where terminal positions are assigned with payoffs for both players. They study combinatorial games where both players have the same move options under both discrete and continuous bidding. Bidding is simultaneous, and the bid winner makes a move in the game. In the discrete setting, all ties are resolved in favour of one of the players. With this setup, they show the existence of pure strategy subgame perfect equilibrium (PSPE). Interestingly, they find that there can be multiple PSPEs, but they form a meet-semilattice with a unique minimum, referred to as the *Bottom Equilibrium*. Due to the presence of multiple PSPEs, simultaneous bidding becomes crucial. For binary games (where each node has at most two children), they show that, for all sufficiently large total budgets, the Bottom Equilibrium exhibits monotonicity and Pareto-optimality. However, for nonbinary games, they provide an example where these properties do not hold. Additionally, for continuous bidding, they show the existence of PSPE.

In [13], Larsson et al. introduce Richman bidding to scoring play via the ruleset CUMULATIVE SUBTRACTION, a variation of the classical normal play SUBTRACTION GAMES. In this game, there is a finite heap of pebbles and a subtraction set $S \subset \mathbb{N}$. A player, on their turn, picks a number from the set S and removes that many pebbles from the heap. The game ends when the number of pebbles in the heap is smaller than the minimum number in the set S. The final score is the total number of pebbles collected by Left during play minus the total number collected by Right; if the result is positive, Left wins; if it is negative, Right wins, and otherwise, the game is a tie. They introduce bidding in this game and call the new game BIDDING CUMULATIVE SUBTRACTION (BCS). The bidding process is similar to that in [6], with the key difference that the winner of the bid must make a move in the game. In case of a tie, the player with the tie-breaking marker wins the bid and transfers the marker. In this setup, alternating play becomes a special case with a total budget of zero. For any fixed total budget TB, if the BCS with the subtraction set S satisfies some standard axioms of monotonicity and marker worth, they prove the existence of a unique equilibrium for all budget partitions. Specifically, for a fixed heap size, they show that if the subtraction set $S = \{1\}$, then BCS has a unique equilibrium. Moreover, these equilibrium outcomes are eventually periodic with a period of 2, with increasing heap sizes.

3. Bidding normal play in a disjunctive sum

Now, we follow up on the TOPPLING DOMINOES example introduced earlier and study normal play combinatorial games [4; 5] with discrete Richman bidding. We explain how the Richman bidding convention in [8] generalizes the standard CGT *outcomes* and the *disjunctive sum* theories.

In [8], we observe that the tie-breaking rule of [13] does not give a pure strategy equilibrium. To illustrate this, consider the case where $\mathrm{TB} = 2$ and the game is $* = \{0|0\}$ under the bidding convention of [13]. Suppose Left has a budget of ₹1 along with the marker. Both players have the terminal game as their only option, so they do not want to hold the marker in the next position. Consequently, for the first bid, Left aims for a tie, while Right prefers a strict win of the bid by either player. That is, Left prefers the bidding pair $(1, 1)$ or $(0, 0)$, while Right prefers $(1, 0)$ or $(0, 1)$. Hence, there is no pure strategy equilibrium. Moreover, this situation also violates the standard normal play convention where “last move wins” is the same as “cannot move loses”. For instance, if the bidding pair is $(1, 0)$, Left gets the last move but loses the game because she keeps the marker.

In [8], we modify the tie-breaking rule of [13] by allowing the marker holder to explicitly announce the marker along with their bid. With this additional rule, the marker can be transferred to the other player even without a tie. To set this up, consider a total budget $\mathrm{TB} \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$. We define $\mathcal{B} = \{0, \dots, \mathrm{TB}, \widehat{0}, \dots, \widehat{\mathrm{TB}}\}$, the set of all feasible budgets, where $\widehat{\cdot}$ indicates

Bids	0	1
0	L	R
1	R	L
$\widehat{0}$	L	R
$\widehat{1}$	L	L

Table 1. The table displays the winner using our bidding convention in [8].

that Left holds the tie-breaking marker. We consider only those game forms G which are finite and contain no cycles, denoted by $\mathbb{G}$. A *bidding game* is a game form G together with the total budget TB, denoted by (TB, G). An instance of a bidding game is a triple $(\text{TB}, G, \widetilde{p})$, where $\widetilde{p} \in \mathcal{B}$ denotes Left's part of the budget. If TB is understood, we write $(G, \widetilde{p})$. With this setup, we show the existence of a unique deterministic outcome of the bidding game. Moreover, this additional rule establishes that "last move wins" is equivalent to "cannot move loses", which was not achievable with the bidding rule of [13]. Furthermore, this is still a generalization of alternating play when $\text{TB} = 0$.

To illustrate this, consider the same example where $\text{TB} = 2$ and the game is $* = \{0|0\}$ with Left's budget of ₹1 together with the marker. With our bidding convention, Left has four bidding alternatives to start with, as displayed in Table 1. The final winner is displayed for each case, depending on whether Right bids ₹0 or ₹1. In Table 1 observe that row 4 is better for Left as compared to other rows, regardless of Right's choice of move, so Left will choose row 4, which is to start by bidding $\widehat{1}$. In this case, she gets the last move and wins the game irrespective of Right's choice of move.

Theorem 2 [8]. *Consider the bidding convention where the tie-breaking marker may be included in a bid. For any game* $(\text{TB}, G, \widetilde{p})$, *there is a pure strategy subgame perfect equilibrium*, *computed by standard backward induction.*

By the existence of PSPE, we may now drop the assumption of simultaneous bidding. We refer to the pure subgame perfect equilibrium of a game $(\text{TB}, G, \widetilde{p})$ as the "partial outcome", $o(G, \widetilde{p}) \in \{\text{L}, \text{R}\}$, where by convention the total order of the results is $\text{L} > \text{R}$. The *outcome* of the bidding game (TB, G) is $o(G) = o_{\text{TB}}(G)$, defined via the 2(TB+1)-tuple of partial outcomes as

$$o(G) = \big(o(G, \widehat{\text{TB}}), \dots, o(G, \hat{0}), o(G, \text{TB}), \dots, o(G, 0)\big).$$

Next, for a fixed TB, we define the outcome relation $\geqslant$. If G and H are games, then $o(G) \geqslant o(H)$ if $o(G, \widetilde{p}) \geqslant o(H, \widetilde{p})$ for all $\widetilde{p} \in \mathcal{B}$. The outcome relation induces the natural partial order of outcomes.

Similar to [6], in our bidding setup, the tie-breaking marker is advantageous but not worth more than a standard chip. We define an outcome to be feasible if it satisfies, for all games G and all budget partitions,

- $o(G, \widetilde{p}) \leqslant o(G, \widetilde{p+1})$ for fixed marker holder (outcome monotonicity), and
- $o(G, \widehat{p}) \leqslant o(G, p+1)$ (marker worth).

We prove the existence of a game form for each such feasible outcome class.

This creates the base to delve deeper into the study of combinatorial games with a bidding setup. In [7], we extend our study of individual combinatorial games with a bidding setup to include games in the disjunctive sum. We next define several key definitions. The disjunctive sum of the game forms $G = \{G^{\mathcal{L}} | G^{\mathcal{R}}\}$ and $H = \{H^{\mathcal{L}} | H^{\mathcal{R}}\}$ is defined recursively as

$$G + H = \{G + H^{\mathcal{L}}, G^{\mathcal{L}} + H \mid G + H^{\mathcal{R}}, G^{\mathcal{R}} + H\},$$

where $G + H^{\mathcal{L}} = \{G + H^L : H^L \in H^{\mathcal{L}}\}$ when $H^{\mathcal{L}} \neq \varnothing$, and otherwise the set is not defined and omitted. The other terms, $G^{\mathcal{L}} + H$, $G + H^{\mathcal{R}}$ and $G^{\mathcal{R}} + H$, are defined in a similar way.

Fix a TB and consider games $G, H \in \mathbb{G}$. Then $G \geqslant H$ if, for all games X, $o(G + X) \geqslant o(H + X)$. The relation $\geqslant$ induces a partial order on $\mathbb{G}$. Moreover, it leads to game equality, $G = H$ if $G \geqslant H$ and $H \geqslant G$ and also to the $G > H$ if $G \geqslant H$ but $H \not\geqslant G$.

We find that the 0 value is the unique identity in this bidding setup. Therefore, we define a game G as *invertible* if there exists a game G' such that $G + G' = 0$.

A game G is a *number* if for all G^L and for all G^R, $G^L < G < G^R$, and all options are numbers. We prove that in a number game G, an optimal bidding strategy for both players is to bid 0 at each follower.

Next, we develop a constructive comparison test for bidding games.

Theorem 3 (Constructive Comparison [7]). *Consider a game form $G \in \mathbb{G}$ and any total budget. Suppose that in the game $(G, 0)$, an optimal bidding strategy by Left is to bid 0 at each follower. Then $G \geqslant 0$ if and only if $o(G, 0) = \mathrm{L}$.*

We list some immediate consequences of this result.

Corollary 4 (Constructive Comparison Tests [7]). *Consider any bidding game* (TB, G).

(1) *If $o(G, 0) = \mathrm{L}$, with a Left 0-bid strategy, then $G \geqslant 0$.*

(2) *If $o(G, \widehat{\mathrm{TB}}) = \mathrm{R}$, with a Right 0-bid strategy, then $G \leqslant 0$.*

(3) *If $o(G, 0) = \mathrm{L}$, with a Left 0-bid strategy, and $o(G, \widehat{\mathrm{TB}}) = \mathrm{L}$, then $G > 0$.*

(4) *If $o(G, \widehat{\mathrm{TB}}) = \mathrm{R}$, with a Right 0-bid strategy, and $o(G, 0) = \mathrm{R}$, then $G < 0$.*

(5) *If $o(G, 0) = \mathrm{R}$ and $o(G, \widehat{\mathrm{TB}}) = \mathrm{L}$, then G is incomparable with 0.*

By using Constructive Comparison Tests, we find that the existence of a "game inverse" is not obvious in general. For example, the game $* = \{0|0\}$ fails to have an inverse if $\mathrm{TB} > 0$. Thus, our bidding game monoids are not groups when $\mathrm{TB} > 0$. Nevertheless, for any $\mathrm{TB} \geqslant 0$, we prove that the numbers constitute a group structure with subgroups integers and dyadic rationals. For $\mathrm{TB} > 0$, although the game $*$ does not have an inverse, it continues to be an infinitesimal.

4. A discussion on bidding rules

Richman bidding in combinatorial games appears in both continuous and discrete forms. This concept originates with Richman's work published in 1996, where he introduces continuous bidding (using real numbers) as a mechanism to determine the next mover in a combinatorial game [10]. Remarkably, Berlekamp also introduces a similar auction in [3] to determine the mean and the temperature [4] of a normal play game position. Both these works appear in the first volume of Games of No Chance in 1996.

From a recreational perspective, continuous bidding is impractical. To address this issue, Mike Develin and Sam Payne [6] introduce discrete Richman bidding in 2010. In their bidding rule, the winner of the bid decides the next mover in the game; there are instances where the bid winner may prefer the other player to make a move. However, from the perspective of normal play, their bidding rule creates a discrepancy where "last move wins" is not equivalent to "cannot move loses", an analogy that is foundational to alternating play combinatorial games. Recent research has explored various bidding rules, particularly regarding the tie-breaking mechanism, but none resolve this discrepancy.

The bidding rule in the scoring play paper [13] is similar to that of [6], with the main difference being that the winner of the bid has to move in the game (there are no zugzwangs). Subsequently, [8] further modifies the bidding rule from [13]. In addition to the scoring play bidding rule, now the marker holder may give away the marker by including it with the bid, even in the case of a strict win. With this modification, "the last move wins" becomes equivalent to "cannot move loses". This rule also implies PSPE, so that standard backward induction techniques apply. This creates a strong foundation to develop the theory for a more general playing order in a combinatorial game. We generalize the concept of outcomes via a disjunctive sum of games to a constructive game comparison.

While [8] focuses on combinatorial games in general, there has been substantial research on games on graphs considering different bidding mechanisms, such as Poorman and Taxman, with both continuous and discrete bidding. Guy Avni and Thomas A. Henzinger have compiled this research comprehensively in another survey [1].

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