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Combinatorial game theory monoids and their absolute restrictions: a survey

ALFIE M. DAVIES, URBAN LARSSON,
REBECCA MILLEY, RICHARD J. NOWAKOWSKI,
CARLOS P. SANTOS AND AARON N. SIEGEL

The classic order relation in Combinatorial Game Theory asserts that, given a winning convention, a game is greater than or equal to another whenever Left can exchange the second for the first in any disjunctive sum, and she can do this regardless of the other component, without incurring any disadvantage. In the early developments of Combinatorial Game Theory, the "other component" encompassed any possible game form, and a rich normal play theory was built by Berlekamp, Conway, and Guy (1976–1982), via a celebrated *local* comparison procedure. It turns out that the normal play convention is a lucky case, and recent research on other conventions therefore often restricts the ranges of games to various subclasses. In the case of misère play, it is possible to obtain partially ordered monoids with more structure by imposing restrictions. The same is true in scoring play. Furthermore, Absolute Combinatorial Game Theory was recently developed as a unifying tool for a local game comparison that generalizes the normal play findings, provided that the restricting set is *parental* (among a few other closure properties), meaning that any pair of finite, nonempty subsets of games from the restriction is permissible as sets of options for another game in the set. This survey aims to provide a concise overview of the current advancements in the study of these structures.

1. Background and purpose of the overview

Combinatorial games are two-player games with perfect information (no hidden information as in some card games) and no chance moves (no dice), where the players move alternately. When the current player has no more moves, the game

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ends and some given convention determines the result. Combinatorial Game Theory (CGT) is the branch of mathematics that studies combinatorial games.

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In the case of misère play, it is possible to obtain partially ordered monoids with more structure by imposing restrictions. The same is true in scoring play. Furthermore, Absolute Combinatorial Game Theory [Larsson et al. 2025b] was recently developed as a unifying tool for a local game comparison that generalizes the normal play findings, provided that the restriction is *parental* (among a few other closure properties), meaning that any pair of finite, nonempty subsets of games from the restriction is permissible as sets of options for another game in the set.[1] This survey aims to provide a concise overview of current advancements in the study of these structures.

We will be interested in short games and we assume familiarity with basic CGT concepts such as winning conventions, game forms, options, followers, outcomes, disjunctive sum, game inequality, game equivalence, game reductions, canonical forms, and so on. All of these concepts are presented and discussed in the classic references [Albert et al. 2007; Berlekamp et al. 1982a; 1982b; Conway 1976; Siegel 2013]. We intend for this to be a reasonably advanced survey. If you are a reader unfamiliar with CGT, acquiring knowledge of the fundamental concepts in the specialized literature will be necessary.

Under the normal play convention, it is well known that $G \succcurlyeq 0$ if and only if $G \in \mathscr{L} \cup \mathscr{P}$, and we will refer to this result as the *Fundamental Theorem of Normal Play* (FTNP). The ultimate reason for this theorem is that if Left has a winning strategy in a game X, then she also has it in $G + X$. She can use a "local response strategy", meaning that, in $G + X$, Left responds to Right's moves in

[1] There are almost as many naming conventions in this thrilling new study as there are coauthors; the two emphasized terms in this sentence have various other suggestive namings as follows: the term "local" is synonymous with *constructive*, *algorithmic*, *recursive*, *computable*, and *play* in various works within the literature, and the term "parental" is synonymous with *dicotic-closed* and *absolute*. In the near future, we hope for some agreement on various concepts. For now, we are satisfied to explain their contexts.

a component with a move in that same component, as if she were playing it in isolation. Under the normal play convention, Left will make the last move in both components, and, consequently, in the disjunctive sum as a whole. The FTNP is sufficient to prove that there is a perfect matching between the order relation and the outcome classes. That is: $G \succ 0$ if and only if $G \in \mathscr{L}$; $G = 0$ if and only if $G \in \mathscr{P}$; $G \parallel 0$ if and only if $G \in \mathscr{N}$; and $G \prec 0$ if and only if $G \in \mathscr{R}$. It is also well known that all game forms have inverses, meaning that game forms, together with the disjunctive sum, form a *group structure*.

By writing $-G = \{-G^{\mathcal{R}} \mid -G^{\mathcal{L}}\}$ to denote the conjugate of G, i.e., the game in which players' roles are reversed, it is straightforward to verify that $G - G \in \mathscr{P}$, which means that $G - G = 0$. This time, instead of the local response strategy, the argument employs an opposite strategy commonly referred to as the *Tweedledee–Tweedledum strategy*. Whenever a player makes a move in one component, the opponent plays the symmetrical move in the other component. This use of symmetry ensures the last move for the player who plays second. Even when considering other conventions, for the sake of simplicity in writing, we will continue to denote $\{-G^{\mathcal{R}} \mid -G^{\mathcal{L}}\}$ as $-G$, even though it may not be the inverse of G in other contexts. With the notion of conjugate in mind, the FTNP is also sufficient to provide an easy *local* way to compare G with H. To determine if $G \succcurlyeq H$, one simply needs to check if Left wins the game $G - H$ playing second. All of these concepts are specific to normal play, and this is important for what will follow. Local response strategies, Tweedledee–Tweedledum strategies, the matching between the order relation and the outcomes, and the group structure are all things that are lost in other conventions. Following the previous observations, regarding the normal play convention, the following list of facts has long been known and was first detailed in [Berlekamp et al. 1982a; 1982b; Conway 1976]. Some terminology, and the subdivision of item (2) into four parts is ours. We wish to highlight the idea of this subdivision, because it generalizes normal play to other settings.

(1) There is a *local* comparison procedure, which involves evaluating whether Left wins the game $G - H$ playing second. In other words, $G \succcurlyeq H$ if and only if
   (i) for each $G^R \in G^{\mathcal{R}}$, there is a $G^{RL} \in G^{L\mathcal{R}}$ such that $G^{RL} - H \succcurlyeq 0$ or there is an $H^R \in H^{\mathcal{R}}$ such that $G^R - H^R \succcurlyeq 0$, and
   (ii) for each $H^L \in H^{\mathcal{L}}$, there is a $G^L \in G^{\mathcal{L}}$ such that $G^L - H^L \succcurlyeq 0$ or there is an $H^{LR} \in H^{L\mathcal{R}}$ such that $G - H^{LR} \succcurlyeq 0$.

(2) There are four types of reductions:
   (i) Domination: If G is a game with two Left options $G^{L_1}, G^{L_2} \in G^{\mathcal{L}}$ and $G^{L_2} \succcurlyeq G^{L_1}$, then $G = \{G^{\mathcal{L}} \setminus \{G^{L_1}\} \mid G^{\mathcal{R}}\}$.

(ii) Nonatomic Reversibility: If G is a game with a Left option $G^L \in G^{\mathcal{L}}$ and there is a G^{LR} such that $G \succcurlyeq G^{LR}$, then, if $G^{LR\mathcal{L}}$ is nonempty, $G = \{G^{\mathcal{L}} \setminus \{G^L\}, G^{LR\mathcal{L}} \mid G^{\mathcal{R}}\}$.

(iii) Atomic Reversibility: If G is a game with a Left option $G^L \in G^{\mathcal{L}}$ and there is a G^{LR} such that $G \succcurlyeq G^{LR}$, then, if $G^{LR\mathcal{L}}$ is empty, $G = \{G^{\mathcal{L}} \setminus \{G^L\} \mid G^{\mathcal{R}}\}$.

(iv) Replacement by Zero: If $G = \{G^L \mid G^R\}$ is a game with a single move for each player, and both G^L and G^R are atomic reversible options, then $G = 0$.

Consider any game form under normal play. By exhaustively applying these reductions, in any order, the end product is unique, and is referred to as the game's *canonical form*.

(3) All games are invertible, and the inverse of a game G is its conjugate $-G$.

In normal play, the three latter reductions in item (2) merge into one, since the statement of nonatomic reversibility encompasses both atomic reversibility and replacement by zero. So, in that case, only the two reductions of domination and reversibility are mentioned in the literature.

In misère play, some items from the list fail or may require some modification. Nevertheless, when considering different classes of distinguishing games, it is possible to find monoids with some very interesting properties. In addition to the full universe of games ($\mathcal{M}$), the universes of dead-ending ($\mathcal{E}$) and dicotic ($\mathcal{D}$) games are notable examples.

Also, in scoring play, some items from the list fail. In this case, besides the full universe of games (or the Stewart universe, $\boldsymbol{S}$), notable examples include the universe of guaranteed scoring games ($\boldsymbol{Gs}$) and the Ettinger universe ($\boldsymbol{E}$). We will detail later the fact that the replacement by zero is always valid in classical nonscoring restrictions but may fail under the scoring play convention.

In this document, while adapting to literature, these universes are denoted by $\mathcal{M}$, $\mathcal{E}$, $\mathcal{D}$, $\boldsymbol{S}$, $\boldsymbol{Gs}$, and $\boldsymbol{E}$, respectively. It is important to emphasize that if, for example, a section pertains to $\mathcal{D}$, the symbol $\succcurlyeq$ will refer to the inequality defined in $\mathcal{D}$, avoiding the need to write $\succcurlyeq_{\mathcal{D}}$. This type of restriction is commonly referred to in the literature as *modulo* $\mathcal{D}$. In other words, if a section is about a specific universe $\mathcal{U}$, everything mentioned in that section will be modulo $\mathcal{U}$.

The concept of a *universe* of games is fundamental. A universe is a set (possibly a restriction) of games under a given convention that satisfies standard closure properties.

The idea of studying these restrictions originates from what we informally call *the waiting problem*. Consider misère play, and suppose that Left has no options in a game G, i.e., G is a *Left-end*. When playing first in misère play, Left wins G when played in isolation. But when played together with another game H, written as $G + H$, then there is the possibility that Left has to play to some $G + H^L$. Now, Right, who was *waiting* for the opportunity, can play in G, say to $G^R + H^L$. That move can lead to Left's defeat, with the moves that Left has to make in a follower of G being the decisive factor in her loss. The game $G = \{\varnothing \mid \mathbf{3}\}$, in which Left has no moves, but where Right can give Left three consecutive moves, is an example of a game that may cause this problem. In general, occurrences of the waiting problem reduce the richness of mathematical structure, resulting in fewer smaller equivalence classes of games born by a given birthday, and so on. Fortunately, some restrictions prevent the occurrence of this problem and, as a bonus, invite many recreational-play rulesets.

A game is a *dicot* if, in each subposition, either both players can move, or neither player can move. It is easy to observe that games such as $\{\varnothing \mid \mathbf{3}\}$ are not dicots, and consequently the waiting problem does not occur in a dicotic universe/restriction. A game is dead-ending if, whenever a player has no available move at a subposition, they have no move in any follower of that subposition. Again, it is easy to observe that games such as $\{\varnothing \mid \mathbf{3}\}$ are not dead-ending, and hence the waiting problem does not occur in a dead-ending universe/restriction. Of course, every dicot is also dead-ending, but the converse is not true. These classes eliminate the waiting problem, as the defining properties ensure that a player cannot play again in a component after running out of moves in it.

Regarding scoring play, the Ettinger universe is dicotic. In this universe, when a player runs out of moves in a component, the resulting score is some real number, and both players are left with no moves in that component. On the other hand, the universe of guaranteed scoring is analogous to a dead-ending universe from nonscoring theory. When a player runs out of moves in a component, if that component were played in isolation, a final score $s \in \mathbb{R}$ would be obtained. The guaranteed property assures that, with other components in play, the score of s in that component cannot become worse with respect to that player, even if play continues there. The waiting problem is again avoided.

Each of these restrictions has its algebraic structure, with larger equivalence classes than the corresponding full universe, and their analyses have been conducted over the years, as we will detail further below. Related to this type of research, an important event was the development of Absolute Combinatorial Game Theory [Larsson et al. 2025b]: a unifying additive theory for standard restrictions in CGT. The main result of this is so crucial for this survey that we will begin to detail it now, starting with some general definitions. Absolute theory encompasses all so-called *parental* (also called *dicotic-closed*) universes: any game form constructed with a pair of nonempty finite sets of elements from the universe, as Left and Right options respectively, is also an element of the universe.

Definition 1 (Maintenance). Let G and H be games in a universe $\mathcal{U}$. The pair (G, H) satisfies the maintenance, $\mathrm{Maint}(G, H)$, if

$$\forall G^R (\exists G^{RL} \text{ such that } G^{RL} \succcurlyeq_{\mathcal{U}} H \text{ or } \exists H^R \text{ such that } G^R \succcurlyeq_{\mathcal{U}} H^R)$$

and

$$\forall H^L (\exists H^{LR} \text{ such that } G \succcurlyeq_{\mathcal{U}} H^{LR} \text{ or } \exists G^L \text{ such that } G^L \succcurlyeq_{\mathcal{U}} H^L).$$

It is helpful to decompose the standard outcomes, to take an explicit note of the winner depending on who starts, and we write $\mathrm{o} = (\mathrm{o}_L, \mathrm{o}_R)$, with $\mathrm{o}_L, \mathrm{o}_R \in \{\mathrm{L}, \mathrm{R}\}$. so that for example, $\mathscr{L} = (\mathrm{L}, \mathrm{L})$, $\mathscr{N} = (\mathrm{L}, \mathrm{R})$, and so on. Here, the convention is the total order $\mathrm{L} > \mathrm{R}$. In absolute theory, we usually call Left-ends instead Left-atomic games (roughly, atoms can be adorned with a "score"). Here, we use the two terms interchangeably.

Definition 2 (Proviso). Let G and H be games in a universe $\mathcal{U}$. The pair (G, H) satisfies the proviso $\mathrm{Proviso}(G, H)$ if the following two items hold:

(i) if H is Left-atomic, then, for any Left-atomic X, $\mathrm{o}_L(G+X) \geqslant \mathrm{o}_L(H+X)$;

(ii) if G is Right-atomic, then, for any Right-atomic X, $\mathrm{o}_R(G+X) \geqslant \mathrm{o}_R(H+X)$.

Theorem 3 (Absolute Comparison). *Let $\mathcal{U}$ be a parental universe. Then $G \succcurlyeq_{\mathcal{U}} H$ if and only if* $\mathrm{Maint}(G, H)$ *and* $\mathrm{Proviso}(G, H)$.

The absolute comparison gives rise to four observations. Firstly, note that, in normal play, if $H^{\mathcal{L}}$ is empty, then the first item of the proviso is trivially satisfied, since $\mathrm{o}_L(H+X) = \mathrm{R}$. On the other hand, if $G^{\mathcal{R}}$ is empty, then the second item of the proviso is trivially satisfied, since $\mathrm{o}_R(G+X) = \mathrm{L}$. Consequently, in normal play, if $\mathrm{Maint}(G, H)$, then $G \geqslant H$. In other words, the exception never occurs, and the proviso is unnecessary.

The second observation is that the exception of the proviso usually occurs in restrictions under the misère and scoring play conventions. For example, in $\mathcal{D}$, $\mathrm{Maint}(\{0 \mid *\}, 0)$ holds but $\mathrm{Proviso}(\{0 \mid *\}, 0)$ does not. Therefore, the issue arises that the absolute proviso involves all possible Left-ends and all possible Right-ends, which are obviously infinite in number. Fortunately, recent research has revealed that it is often not necessary to test all Left-ends and Right-ends, but only a few relevant ones. We have a *local* comparison procedure whenever it is possible to consider only a finite number of ends. Such a procedure is of utmost importance both in practice and theory. Therefore, an "absolutely" crucial question is the following.

Question 1. Given a parental universe, how/when can the proviso be reduced to a finite number of tests?

The third observation once again involves Left-ends and Right-ends; somehow, handling the empty set of options is one of the most delicate tasks in CGT. The main reason why the proviso is necessary when the convention is not normal play is due to the fact that the absence of options can be advantageous for a player. Having no moves can be beneficial under the misère play convention, because it may signify victory, and having no moves in scoring play can imply obtaining a high score. Thus, issues related to *atomic reversibility* arise. Let us start by analyzing the reversibility condition. If $G \succcurlyeq G^{LR}$, then in a disjunctive sum $G+X$, Left never plays to G^L+X with the intention of responding to $G^{LR}+X^L$ from a Right move to some $G^{LR}+X$. This is because it is no worse to move directly to $G+X^L$, as the condition implies $G+X^L \succcurlyeq G^{LR}+X^L$. This means that Left only chooses G^L+X if she intends to respond locally, in a follower of G, in case Right responds to $G^{LR}+X$. This is what is almost always referred to in the specialized literature and is indeed the concept that underlies reversibility.

Yet there is another idea that is less frequently mentioned, but is equally important. It arises from the following question: "When should Left play to G^L+X, if G^L is an atomic reversible option, that is, if $G^{LR\mathcal{L}}=\varnothing$, for some $G^{LR} \preccurlyeq G$?" If $X^{\mathcal{L}}$ is not empty, then there is an $X^L \in X^{\mathcal{L}}$ such that $o_R(G^L+X) \preccurlyeq o_R(G+X^L)$. This is because

$$\begin{aligned} o_R(G^L+X) &\preccurlyeq o_L(G^{LR}+X) && \text{(arbitrary choice)}\\ &= o_R(G^{LR}+X^L) && \text{(best choice, } G^{LR} \text{ is a Left-end)}\\ &\preccurlyeq o_R(G+X^L) && \text{(reversibility condition).}\end{aligned}$$

In other words, in game practice Left rarely needs to opt for an atomic reversible option. She only chooses such an option in a disjunctive sum if all the other components are Left-ends. However, in these cases, the atomic reversible option may be the only winning move, and as a result, except in a few cases where it is a sole option, it cannot reverse out. Nevertheless, it may be possible to replace it with a simpler atomic reversible option that allows for obtaining a useful "canonical form". In summary, an interesting *choice* for the replacement may be made or, at the very least, a *method* of making that choice can be indicated. Hence, a second crucial question arises.

Question 2. Given a parental universe, how and when can we solve atomic reversibility?

The fourth observation concerns invertible game forms. For some restrictions, the approach was to start by proving the Conjugate Property, establishing that, if G is invertible, then its inverse is $-G$. With that, in some cases, a simple characterization of the invertible elements of the structure was also proved. Thus, a third crucial question is the following.

Question 3. Is it true that the elements of a given parental universe satisfy the *Conjugate Property*, meaning that the inverse of each invertible element is its conjugate? Is there a simple way to characterize the invertible elements of the universe?

In normal play there is a unique canonical form (after exhaustive reductions in any order). It turns out that this is a fairly unique situation and, in general, one does not a priori get a unique form after reductions. For further discussion on this topic, we recommend [Larsson et al. 2016], [Larsson et al. 2025a], and in particular [Siegel 2025]. See also Section 5.2.

The upcoming overview follows a straightforward logic. It will elucidate the findings and responses that have evolved throughout the course of research to tackle the three first mentioned questions, henceforth denoted as Q1, Q2, and Q3. A concise summary of this overview is provided in Table 1.

Before we proceed, it is important for the reader to be aware of certain nomenclature issues. Often, when embarking on mathematical research in a new subject, different names may arise for the same concepts. The subject covered in this survey is no exception. For "local comparison", at least four other terms have been used: "subordinate comparison", "recursive comparison", "play comparison" and "constructive comparison". For the terms "end", "Left-end" and "Right-end", the terms "atomic", "Left-atomic", and "Right-atomic" have also been used. "Nonatomic reversibility" and "atomic reversibility" have been referred to as "open reversibility" and "end-reversibility". "Maintenance" has also been mentioned as "common normal part". Instead of "parental universe", the term "dicotic closed universe" has been used. In fact, a universe was originally defined as a class of game forms satisfying option closure, disjunctive sum closure, and conjugate closure, and containing the terminal positions. Recently, it has been proposed that the word "universe" be used only for parental universes. In this survey, we have made agnostic choices for each of these concepts. It is natural that, as the theory develops, these choices will become more stabilized in the specialized literature.

2. The misère play convention

The *classical conventions* are normal play and misère play. As mentioned, normal play does not require any special treatment in terms of restrictions. This section concerns popular restrictions of misère play.

2.1. ***Full misère, $\mathcal{M}$.*** For a long time, partizan games in misère play were considered essentially intractable. Then, in 2007, Mesdal and Ottaway [2007] showed the following highly relevant theorem concerning the full universe $\mathcal{M}$ of misère play. Necessarily, this must be the starting point of this section.

Theorem 4. *If* G *and* H *are misère games such that* $H^{\mathcal{L}} = \varnothing$ *and* $G^{\mathcal{L}} \neq \varnothing$, *then* $G \not\succcurlyeq H$.

In brief, the theorem by Mesdal and Ottaway indicates that G can only be greater than or equal to a Left-end H if it is also a Left-end itself. Of course, by symmetry, adopting Right's perspective, we also have that if $G^{\mathcal{R}} = \varnothing$ and $H^{\mathcal{R}} \neq \varnothing$, then $G \not\succcurlyeq H$. This result implies the following corollary, which states that every nonterminal game form is distinct from zero.

Corollary 5. *Let* G *be a misère game. If* $G \not\cong \{\varnothing \mid \varnothing\}$, *then* $G \neq 0$.

Corollary 5 immediately answers Q3: the only invertible game form is the terminal $0 = \{\varnothing \mid \varnothing\}$. That is, $\mathcal{M}$ is a monoid with no invertible elements other than the identity (otherwise known as a *reduced* monoid). Naturally, $\mathcal{M}$ satisfies the Conjugate Property, but in a trivial and unenlightening way.

Another interesting consequence of Theorem 4 pertains to the proviso. The proviso is a fundamental concept used in Theorem 3, but this consequence was not mentioned, as absolute theory had not yet been developed. Consider a pair (G, H) where $H^{\mathcal{L}} = \varnothing \implies G^{\mathcal{L}} = \varnothing$ and $G^{\mathcal{R}} = \varnothing \implies H^{\mathcal{R}} = \varnothing$, meaning that H cannot be a Left-end without G being one, and G cannot be a Right-end without H being one. It is relatively easy to prove that, under the misère play convention, a pair meeting these conditions and satisfying the maintenance also satisfies the proviso. Therefore, the answer to Q1 also follows from Theorem 4; if (G, H) does not belong to this family of pairs, then inevitably $G \not\succcurlyeq H$. The proviso in $\mathcal{M}$ is as restrictive as it can be.

Proviso of $\mathcal{M}$:

(i) if $H^{\mathcal{L}} = \varnothing$ then $G^{\mathcal{L}} = \varnothing$;

(ii) if $G^{\mathcal{R}} = \varnothing$ then $H^{\mathcal{R}} = \varnothing$.

Siegel [2015] then determined the full mathematical structure of $\mathcal{M}$. We observe once again that Theorem 4 led to an understanding of the reductions in $\mathcal{M}$ and, consequently, to the establishment of canonical forms. The reason for this lies in the fact that it is easy to verify that atomic reversibility and replacement by zero are reductions that *never occur*, thus answering question Q2. The rationale behind this is that it would imply the reversibility criterion $G \succcurlyeq G^{LR}$ where $G^{LR\mathcal{L}}$ is empty, which is something that the aforementioned theorem indicates cannot happen (G^{LR} is a Left-end, and G is not). Therefore, just as in the case of the normal play convention, reversibility can be uniquely stated through the statement of nonatomic reversibility. In summary, in both normal play and misère play, reversibility has the same statement. However, in the former convention, *there can be options reversing out*, while in the latter, that case *never occurs*. The fact that the reductions are the same in both conventions, although for different reasons, may have led to initial misunderstandings and a delay in the mathematical development of other restrictions that require four reductions instead of two.

2.2. *Dicotic misère, $\mathcal{D}$.* The best way to start this section is by addressing Q1. This is because there are no dicots that are Left-ends or Right-ends, except for the terminal form $0 = \{\varnothing \mid \varnothing\}$. Therefore, the proviso reduces spectacularly.

Proviso of $\mathcal{D}$:

(i) if $H = 0$, then $\mathrm{o}_L(G) = \mathrm{L}$;

(ii) if $G = 0$, then $\mathrm{o}_R(H) = \mathrm{R}$.

Another statement for the proviso Proviso(G, H) is that $\mathrm{o}(G) \geqslant \mathrm{o}(H)$; see [Larsson et al. 2021]. As for the answer to Q1, the best reference is [Larsson et al. 2021].

The decisive breakthrough related to the study of the algebra of $\mathcal{D}$, particularly the answer to Q2, occurred with the publication of [Dorbec et al. 2015]. They prove that atomic reversibility has an independent status: an atomic reversible option should be replaced by $* = \{0 \mid 0\}$ if it is the only winning option when played in isolation. Typically, an option like that cannot be entirely removed (as in normal play); in endgames, it may be the only way to win. For example, in the form $\{0, * \mid *\}$, the Left option $*$ cannot reverse out, unlike in normal play. In misère play, when playing $\{0, * \mid *\}$ in isolation, $*$ is the only winning choice for Left. Note also that when the atomic reversible option is the sole option, it cannot reverse out without the form ceasing to be dicotic. The exception is the form $\{* \mid *\} = 0$, subject to the reduction replacement by zero. It is the only case in which the game, after the reduction, does not cease to be dicotic, as both options of the form reverse out simultaneously. It is this simultaneity that explains the particular nature of this reduction and the reason why it is highlighted from the others. By these concepts, useful canonical forms are easy to obtain.

Regarding Q3 and the nature of invertible elements, it was proven in [Larsson et al. 2025a] that $\mathcal{D}$ satisfies the Conjugate Property. With the help of this fact, it was further demonstrated in [Fisher et al. 2022] that a dicotic canonical form G is invertible if and only if all followers G' satisfy $G' - G'$ is no $\mathcal{P}$-position.

2.3. *Dead-ending misère, $\mathcal{E}$.* Regarding the algebraic structure of $\mathcal{E}$, Milley and Renault [2013] established a first fundamental result, that the ends are invertible, with their inverses being their conjugates. For $n > 0$, interesting particular cases are the forms $\boldsymbol{n} = \{\boldsymbol{n-1} \mid \varnothing\}$ and $\overline{\boldsymbol{n}} = \{\varnothing \mid \overline{\boldsymbol{n-1}}\}$, corresponding to situations where Left (Right) has to make n consecutive moves with no alternative at their disposal. Naturally, $\boldsymbol{0} = 0 = \{\varnothing \mid \varnothing\}$ is the identity (Figure 1 illustrates the game trees of some of these forms).

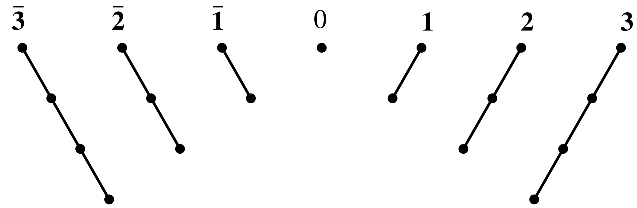

Figure 1. A player has n consecutive moves.

This subclass of forms behaves exactly like integers in normal play, corresponding to their canonical forms and constituting a natural group structure. Induction allows us to prove that if m and n are nonnegative, then the disjunctive sum of $\boldsymbol{m}$ with $\boldsymbol{n}$ is equivalent to $\boldsymbol{m+n}$. Also, for opposite signs, the disjunctive sum behaves as expected. For example, $\mathbf{3}+\bar{\mathbf{1}}=\mathbf{2}$ is a straightforward illustration; given the invertibility, we have

$$\mathbf{2}=\mathbf{2}+(\mathbf{1}+\bar{\mathbf{1}})=(\mathbf{2}+\mathbf{1})+\bar{\mathbf{1}}=\mathbf{3}+\bar{\mathbf{1}}.$$

However, it is important to note that, while well-behaved with respect to the disjunctive sum, these forms are all incomparable to each other. For example, there are situations where Left prefers **1**, and others where Left prefers **2**. Letting $X=\bar{\mathbf{1}}$, it happens to be the case that Left wins $\mathbf{1}+X$ and loses $\mathbf{2}+X$ playing first; if $X=\bar{\mathbf{1}}+\{\mathbf{2}\,|\,\bar{\mathbf{2}}\}$, Left loses $\mathbf{1}+X$ and wins $\mathbf{2}+X$.

A decisive breakthrough occurred in [Larsson et al. 2021] and [Larsson et al. 2025a]. In these works, the class of *waiting games* was introduced. Let us consider a disjunctive sum $G+X$ where X is a Left-end that is not a Right-end. When X is played in isolation, Right loses, because of the dead-ending property. But in a disjunctive sum with a game G, X may assist and provide Right with some control. Here is an example: if $X=\{\varnothing\,|\,\{\varnothing\,|\,0\},0\}$ and $G=*$, then Right wins playing first by moving to $*+0$; and if $G=\{*\,|\,*\}$, then Right wins by moving to $\{*\,|\,*\}+\{\varnothing\,|\,0\}$.

The key is to understand the best help that X can give Right. Incredibly, it is straightforward to identify which ends Right should prefer; they are precisely the ones where Right has two options at his disposal, either "removing them from the sum" (a move to 0) or "keeping them in the sum". The greater the depth, the longer Right can maintain these two possibilities. Thus, the greater the depth, the better it is for Right.

Once again, the base case is zero. For $n>0$, the waiting games $\boldsymbol{n}_0$ and $\bar{\boldsymbol{n}}_0$ are defined as $\boldsymbol{n}_0=\{(\boldsymbol{n}-\mathbf{1})_0,0\,|\,\varnothing\}$ and $\bar{\boldsymbol{n}}_0=\{\varnothing\,|\,\overline{(\boldsymbol{n}-\mathbf{1})}_0,0\}$.[2] Figure 2 illustrates

[2]Note that the definition is reversed in [Larsson et al. 2025a]. We propose the new convention in analogy with the "number of moves" idea.

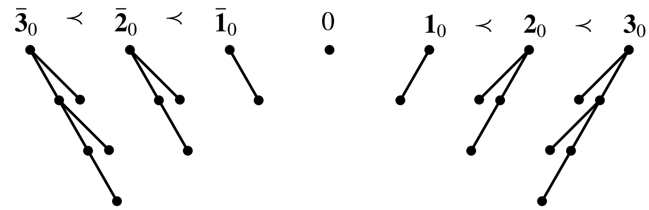

Figure 2. Waiting games.

the game trees of some waiting games. Note that, in normal play, these games are also the integers, but not in canonical form. This time, the class is not well behaved with respect to the disjunctive sum, as a sum like $\mathbf{2}_0 + \mathbf{3}_0$ does not equal $\mathbf{5}_0$. The very good news, however, is that the class is well behaved in terms of the order relation. Two fundamental facts are observed:

(1) If $n > m > 0$, then $\boldsymbol{n}_0 \succ \boldsymbol{m}_0$ and $\bar{\boldsymbol{n}}_0 \prec \bar{\boldsymbol{m}}_0$.

(2) $\boldsymbol{n}_0$ is the largest of the nonzero Right-ends born by day n (and $\bar{\boldsymbol{n}}_0$ is the smallest of the nonzero Left-ends born by day n).

These propositions are directly related to Q1. Consider the first item of the proviso, and the exception "Maint(G, H) holds, but not Proviso(G, H)". Then, by assumption, $H^{\mathcal{L}} = \varnothing$. In this case, $o_L(H + X) = \mathrm{L}$, for any Left-end X. Hence, one must investigate the possibility of $o_L(G + X) = \mathrm{R}$. Given the good behavior of the waiting games, with respect to the order relation, it is sufficient to check two cases: namely $o_L(G)$ and $o_L(G + \bar{\boldsymbol{n}}_0)$, where n is the rank of G. Suppose $o_L(G + \bar{\boldsymbol{n}}_0) = \mathrm{L}$. There cannot be any nonterminal Left-end X for which $o_L(G + X) = \mathrm{R}$. This is because $\bar{\boldsymbol{n}}_0$ concentrates all the "waiting power" that Right might need. In other words, the proviso of $\mathcal{E}$ is as follows.[3]

Proviso of $\mathcal{E}$:

(i) if $H^{\mathcal{L}} = \varnothing$, then $o_L(G) = o_L(G + \bar{\boldsymbol{n}}_0) = \mathrm{L}$, where n is the rank of G;

(ii) if $G^{\mathcal{R}} = \varnothing$, then $o_R(H) = o_R(H + \boldsymbol{m}_0) = \mathrm{R}$, where m is the rank of H.

It is not surprising that waiting games also have a role to play in answering Q2. In $\mathcal{D}$, atomic reversibility of a fundamental option is a replacement by $*$. Likewise, in $\mathcal{E}$, when G^L is a fundamental Left atomic reversible option, i.e., the only winning move in some endgames or a sole option that cannot reverse out without ceasing to have a dead-ending form, there is a minimum n for which G^L can be

[3] In English, unlike in other languages, the term "dead-end" is rooted in the idea of death. Since, in order to efficiently check the proviso, one must choose "perfect" ends that allow drawing conclusions for the infinity of other possible ends, the original name chosen for the waiting games was "perfect murders". However, this choice was not well received by the community because the joke is hard to follow.

replaced with $\{\varnothing \mid \bar{\boldsymbol{n}}_0\}$. Furthermore, in $\mathcal{E}$, $\{\bar{\mathbf{1}} \mid \mathbf{1}\} = 0$ since the options reverse out, but, in contrast to what happens in $\mathcal{D}$, this is not true for $\{* \mid *\}$.

Regarding Q3 and the nature of invertible elements, it was proven in [Larsson et al. 2025a] that $\mathcal{E}$ satisfies the Conjugate Property. With the help of this fact, it was further demonstrated in [Milley and Renault 2022] that a dead-ending canonical form G is invertible if and only if it is $\mathcal{P}$-free, i.e., if no follower is a $\mathcal{P}$-position.

3. The scoring play convention

In scoring play, there is no direct correspondence between the player who makes the last move and the outcome. When a player has no more moves, the game ends, but an evaluation of the outcome in terms of points is still required. Thus, knowing that a player has an empty set of options ($G^{\mathcal{L}} = \varnothing$ or $G^{\mathcal{R}} = \varnothing$) is insufficient; it is essential to include information about a final *score* to any empty sets of options. In this convention, to distinguish it from the classical conventions, it can be useful to adhere to other game brackets. Stewart [2011] introduces a notation based on triples. Later, Larsson et al. [2018b] propose an alternative notation with pairs instead of triples, which is the one we use in this document. Stated simply, the different empty sets are *adorned* with elements of $\mathbb{R}$. That information allows us to get the result when a player has no more moves. By using this notation, $1 = \langle \varnothing^1 \mid \varnothing^1 \rangle$ is a terminal game form with no options, corresponding to a final score of one point (for Left), whereas $\hat{1} = \langle 0 | \varnothing^0 \rangle$ is a game in which Left has a move to make without gaining or losing points. Here, the "hat" is an operator, which adorns each empty set of options (of a normal play game) with a "0" [Larsson et al. 2018b]. Moreover, $\langle \varnothing^2 | -3 \rangle$ is a game in which, if Left plays first, there are no available options, and she wins two points, but if Right plays first, he can make a final move, winning three points. This notation with adornments has practical advantages when we want to relate the classical theory to scoring theory.

3.1. ***Full scoring, S.*** When we play without restrictions, there is a clear link between $\mathcal{M}$ and $\boldsymbol{S}$. The answer to Q1 is similar; the proviso in $\boldsymbol{S}$ is as restrictive as it can be.

Proviso of $\boldsymbol{S}$:

(i) if $H^{\mathcal{L}} = \varnothing^{\ell}$ then $G^{\mathcal{L}} = \varnothing^{x}$, with $x \geqslant \ell$;

(ii) if $G^{\mathcal{R}} = \varnothing^{r}$ then $H^{\mathcal{R}} = \varnothing^{x}$, with $r \geqslant x$.

The main reference on the algebra of $\boldsymbol{S}$ is [Stewart 2011]. One of the main ideas to take away from this work is the fact that $\boldsymbol{S}$ has "little structure", with the author even needing to prove that there exist two forms, G and H, such that

$G \not\cong H$ and $G = H$, in other words, justifying the consideration of the quotient space. Just as it happens in $\mathcal{M}$, the proviso ensures that atomic reversibility and replacement by zero never occur; this answers Q2.

Furthermore, the only invertible elements are the trivial ones, with an empty game tree; for example, the inverse of $1 = \langle \varnothing^1 \,|\, \varnothing^1 \rangle$ is the game form $-1 = \langle \varnothing^{-1} \,|\, \varnothing^{-1} \rangle$. Stewart acknowledged this fact by writing:

> These are the only games that are invertible under scoring play, and any other nontrivial game cannot be inverted.

This result, analogous to Corollary 5, answers Q3.

3.2. ***Dicotic scoring, E.*** The main references on the algebra of $\boldsymbol{E}$ are [Ettinger 1996; 2000]. In particular, the first one is a visionary Ph.D. thesis; as we will see, the author realized some crucial ideas already in the 1990s, which were revisited some 30 years later.

In what follows, $\mathrm{Ls}(G)$ and $\mathrm{Rs}(G)$ (Left- and Right-score) refer to the optimal final score with Left and Right as starting player, respectively.[4] Ettinger studied dicotic scoring games without any restrictions on the final scores, allowing for *zugzwang* positions[5] such as $G = \langle \langle 1 \,|\, {-1} \rangle \,|\, 1 \rangle$. We will see that there is a link between $\mathcal{D}$ and $\boldsymbol{E}$.

Starting with Q1, already in the 1990s, Ettinger wrote in a very modern way:

> We now wish to explore the notion of equivalence. We wish to obtain a direct characterization of the $=_{\boldsymbol{E}}$ relation. That means that we want a criterion by which we can determine if $G =_{\boldsymbol{E}} H$ simply by examining G and H (and all hereditary elements if necessary) themselves. Recall that the definition of game equivalence is a universal statement over the entire universe.

He had a clear concern for what we today refer to as "local comparison". In his work, Ettinger demonstrated the intended proviso.

Proviso of $\boldsymbol{E}$:

(i) if $H^{\mathcal{L}} = \varnothing^{\ell}$, then $\mathrm{Ls}(G) \geqslant \ell$;

(ii) if $G^{\mathcal{R}} = \varnothing^{r}$, then $\mathrm{Rs}(H) \leqslant r$.

It is fascinating to analyze how Q2 was handled. Concerning the only situations in which a player opts for an atomic reversible option, Ettinger writes the

[4]These optimal scores should not be confused with the $\mathrm{LS}(G)$ and $\mathrm{RS}(G)$ of classical theory, which refer to the normal play stops.

[5]Zugzwang (from German "compulsion to move") refers to a situation where a player is placed at a disadvantage because they must make a move. In scoring-play, G is a zugzwang if $\mathrm{Ls}(G) < \mathrm{Rs}(H)$.

following ("test game" is what we refer to here as "distinguishing game"). And we assume here that $r \leqslant G$.

> Basically Left never benefits from playing from G to $\langle \dots \mid r, \dots \rangle$ because Right can immediately reverse to r (at least). Therefore it is always the case that Left may as well play elsewhere except when there is no elsewhere! Therefore Left only moves from G to $\langle \dots \mid r, \dots \rangle$ when the test game has reached an atom and this is the Left's unique best move.

Ettinger clarifies that the atomic reversible option can be replaced by any $\langle H|r\rangle$ to reduce G; he refrained from making a choice. It is clear that without making a choice, he could not obtain unique canonical forms, but he still indicated that he could do it, which is a very modern approach followed by some researchers in 2023, the time this text is being written. If he had chosen to replace it with $r + \hat{*} = \langle r\,|\,r\rangle$, the parallel with the atomic reversibility in $\mathcal{D}$ would be evident, and the canonical forms would be unique.

Furthermore, Ettinger realized the specificity that the analogue to replacement by zero has in scoring play. Consider a form like $G = \langle\langle 3\,|\,3\rangle\,|\,\langle 5\,|\,5\rangle\rangle$, where both options are atomic reversible options. It is clear that, unlike in normal play, this form cannot be replaced by zero, as Left will end the game with at least three points. On the other hand, it cannot be replaced by any score since $\mathrm{Ls}(G) < \mathrm{Rs}(G)$, and G is a zugzwang. Take note of what he writes about this.

> We will now introduce a very interesting set of games, which we will think of as *double-atoms* or *nonconstant constants*.

Both terms are of interesting analysis. The term "double-atoms" indicates that a game like G behaves as an asymmetrical terminal game like $\langle \varnothing^3\,|\,\varnothing^5\rangle$. The replacement would have to be made with a terminal form like this if such a form existed in the structure. The term "nonconstant constants" points to a certain type of avoidance property. These are forms that, given that the options are atomic reversible, are only considered in the endgames.

In summary, the universe $\boldsymbol{E}$ only accepts the following atomic replacement: if $G = \langle G^L\,|\,G^R\rangle$ is a game form with a single move for each player, both G^L and G^R are atomic reversible options, and $\mathrm{Ls}(G) = \mathrm{Rs}(G) = r$, then $G = r$.

Regarding Q3, Ettinger [1996] also includes a proof that $\boldsymbol{E}$ satisfies the Conjugate Property. Although it has not been done yet, we conjecture that there is a simple characterization of invertible elements, just as it was done for $\mathcal{D}$.

3.3. ***Guaranteed scoring, Gs.*** Stewart's Ph.D. thesis [2011] contains an analysis of scoring games almost without restrictions (only asymmetrical terminal forms like $\langle \varnothing^3\,|\,\varnothing^4\rangle$ were not included). This wide scope brought the possibility of games such as $\langle\varnothing^1|-1\rangle$. These type of games are hot, but without options for (at least) one of the players, a.k.a. *hot empty games*. Similar to misère play, Left wants to have the right to move, even knowing that she cannot. There are interesting mathematical implications of such games. Let us clarify first the different nature of the concepts of *n-scores* and *n-moves*. For example, a 1-score is the game $1=\langle\varnothing^1|\varnothing^1\rangle$, whose tree is empty; on the other hand, a 1-move is the game $\hat{1}=\langle\langle\varnothing^0|\varnothing^0\rangle|\varnothing^0\rangle$, whose tree is analogous to the tree of 1 in classical normal play (recall, a "hat" is used for game trees, where each empty set of options is adorned with a zero). In the second, Left does not win points, but the extra move may be very useful in presence of zugzwang components. The soul of scoring combinatorial game theory lies in the interplay between scores and moves.

Of course, the question arises as to whether the structure of normal play (denoted by $\boldsymbol{Np}$) is order-embedded in $\boldsymbol{S}$. In other words, if G and G' are normal play games, is it true that $\widehat{G}\succcurlyeq_{\boldsymbol{S}}\widehat{G}'$ if and only if $G\succcurlyeq_{\boldsymbol{Np}}G'$? The answer is negative: in $\boldsymbol{S}$, sometimes a player prefers to have moves, while at other times, they prefer not to. Let us suppose that X is the zugzwang $\langle-3|3\rangle$. In $\hat{1}+X$, playing first, Left wins (she can force Right to play in the zugzwang), but in $0+X$, playing first, Left loses (she has to play in the zugzwang). Let us suppose now that X is the hot empty game $\langle\varnothing^1|-1\rangle$. In $\hat{1}+X$, playing first, Left loses (she cannot access X when it is her turn to play), but in $0+X$, playing first, Left wins (it is her turn to play in X). In other words, zugzwangs promote normal play, while hot empty games do not. Since $\boldsymbol{S}$ includes both types of games, one could only expect a very complex algebra with small equivalence classes.

Larsson et al. [2016; 2018b] propose to instead analyze *guaranteed scoring games* ($\boldsymbol{Gs}$) in which all the atoms in $G^{\mathcal{R}}$ of $\langle\varnothing^\ell|G^{\mathcal{R}}\rangle$ are larger than or equal to ℓ (and the mirror concept for $\langle G^{\mathcal{L}}|\varnothing^r\rangle$). In this structure, there are no hot empty games, and so their harmful effect is dissolved; $\boldsymbol{Np}$ is order embedded in $\boldsymbol{Gs}$. In the guaranteed scoring universe it is never worse to be able to continue playing rather than having no moves. Furthermore, there are hundreds of recreational-play rulesets that satisfy the guaranteed property.

Starting with Q1, there is a link between $\mathcal{E}$ and $\boldsymbol{Gs}$. In $\mathcal{E}$, the waiting games are $\overline{\boldsymbol{n}}_0$ and $\boldsymbol{m}_0$, whereas in $\boldsymbol{Gs}$ they are $-\widehat{n}=\widehat{-n}$ and $\widehat{m}$, where n is the rank of G, and m is the rank of H.

Proviso of $\boldsymbol{Gs}$:

(i) if $H^{\mathcal{L}}=\varnothing^\ell$ then $\mathrm{Ls}(G-\widehat{n})\geqslant\ell$;

(ii) if $G^{\mathcal{R}}=\varnothing^r$ then $\mathrm{Rs}(H+\widehat{m})\leqslant r$;

The parallel remains when we consider Q2. When G^L is a fundamental Left-atomic reversible option, there is a minimum n for which this option can be replaced with $r-\widehat{n}$. Furthermore, since Larsson, Nowakowski and Santos

[Larsson et al. 2018b] include terminal games of the form $\langle \varnothing^{\ell} \,|\, \varnothing^{r} \rangle$ (with $\ell \leqslant r$), the replacement by $\langle \varnothing^{\ell} \,|\, \varnothing^{r} \rangle$ can be done when $\langle G^{L} \,|\, G^{R} \rangle$ is a game form with a single atomic reversible option for each player.

Regarding Q3, Larsson et al. [2018b] provide a proof that $\boldsymbol{Gs}$ satisfies the Conjugate Property. Although it has not been done yet, we conjecture (similar to $\mathcal{E}$) that there is a simple characterization of the invertible elements.

4. Summary table and some examples taken from game practice

Let us begin by summarizing the results from the previous sections in a table (Table 1). Then, via examples from the classical (nonscoring) setting, we illustrate how equivalence classes, and consequently, algebraic structures, change when we move from one ruleset or universe to another.

	Proviso	Atomic rev.	Repl. by termin.	Inv. elem.
$\boldsymbol{Np}$	Unnecessary	Atomic reversible options reverse out	Replacement by zero with no exceptions	All elements are invertible (group structure)
$\mathcal{M}$	(i) if $H^{\mathcal{L}} = \varnothing$ then $G^{\mathcal{L}} = \varnothing$ (ii) if $G^{\mathcal{R}} = \varnothing$ then $H^{\mathcal{R}} = \varnothing$	Never occurs	Never occurs	$\{\varnothing \,|\, \varnothing\}$ is the only invertible form
$\mathcal{D}$	(i) if $H = 0$ then $\mathrm{o}_L(G) = \mathrm{L}$ (ii) if $G = 0$ then $\mathrm{o}_R(H) = \mathrm{R}$	Fundamental atomic reversible options are replaced by $*$	Replacement by zero with no exceptions	Conjugate property is satisfied A canonical form G is invertible if and only if there is no $G' - G' \in \mathscr{P}$
$\mathcal{E}$	(i) if $H^{\mathcal{L}} = \varnothing$ then $o_L(G) = o_L(G + \bar{\boldsymbol{n}}_0) = \mathrm{L}$, where n is the rank of G (ii) if $G^{\mathcal{R}} = \varnothing$ then $o_R(H) = o_R(H + \boldsymbol{m}_0) = \mathrm{R}$, where m is the rank of H	Fundamental atomic reversible replaced by $\{\varnothing \,|\, \bar{\boldsymbol{n}}_0\}$, where n is minimal	Replacement by zero with no exceptions	Conjugate property is satisfied A canonical form G is invertible if and only if it is $\mathcal{P}$-free
$\boldsymbol{S}$	(i) if $H^{\mathcal{L}} = \varnothing^{\ell}$ then $G^{\mathcal{L}} = \varnothing^{x}$, with $x \geqslant \ell$ (ii) if $G^{\mathcal{R}} = \varnothing^{r}$ then $H^{\mathcal{R}} = \varnothing^{x}$, with $r \geqslant x$	Never occurs	Never occurs	Terminal game forms are the only invertible forms
$\boldsymbol{E}$	(i) if $H^{\mathcal{L}} = \varnothing^{\ell}$ then $\mathrm{Ls}(G) \geqslant \ell$ (ii) if $G^{\mathcal{R}} = \varnothing^{r}$ then $\mathrm{Rs}(H) \leqslant r$	Fundamental atomic reversible options are replaced by $\langle r \,|\, r \rangle$	Replacement by r can only be made if $\mathrm{Ls}(G) = \mathrm{Rs}(G) = r$	Conjugate property is satisfied Not yet characterized
$\boldsymbol{Gs}$	(i) if $H^{\mathcal{L}} = \varnothing^{\ell}$ then $\mathrm{Ls}(G - \widehat{n}) \geqslant \ell$ (ii) if $G^{\mathcal{R}} = \varnothing^{r}$ then $\mathrm{Rs}(H + \widehat{m}) \leqslant r$	Fundamental atomic reversible options are replaced by $r - \widehat{n}$, where n is minimal	Replacement by $\langle \varnothing^{\mathrm{Ls}(G)} \,|\, \varnothing^{\mathrm{Rs}(G)} \rangle$ with no exceptions	Conjugate property is satisfied Not yet characterized

Table 1. The rows are for the various universes discussed in this survey, and the columns are "Proviso", "Atomic reversibility", "Replacement by a terminal position", and "Invertible elements". Atomic reversibility is presented from Left's perspective.

Some rulesets are inherently dicotic. For example, if whenever one piece can capture a second, the reverse is an option for the opponent, we have a dicotic ruleset. CLOBBER, a ruleset invented in 2001 by Albert, Grossman, and Nowakowski, is an example of this. Players take turns by moving one of their pieces to an orthogonally adjacent opposing piece, removing that piece from the board. Figure 3 shows a replacement by zero in CLOBBER, using $\mathcal{D}$ as an appropriate algebraic structure in misère play. The position is indistinguishable from zero and can be thought of as if it were not on the board. It is noted that $*$ is an invertible element of $\mathcal{D}$.

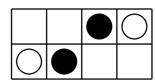

Figure 3. An illustration of $* + * = \{* \mid *\} =_{\mathcal{D}} 0$ in misère play CLOBBER.

Figure 4 shows a more complex position and its options. The literal form of the position is $\{* + *, \{* \mid 0\} \mid *, \{0, * \mid 0\}\}$. One of the options is the mentioned $* + * = 0$, and, consequently, the form can be reduced to $\{0, \{* \mid 0\} \mid *, \{0, * \mid 0\}\}$. On the other hand, the first Right option trivially dominates the second, and the form can be reduced to $\{0, \{* \mid 0\} \mid *\}$. Finally, it can be confirmed that the second Left option is atomic reversible modulo $\mathcal{D}$, and the canonical form of the position is $\{0, * \mid *\}$. This game value, for reasons beyond the scope of this text, is highly relevant in the algebra of $\mathcal{D}$; see [Larsson et al. 2018a].

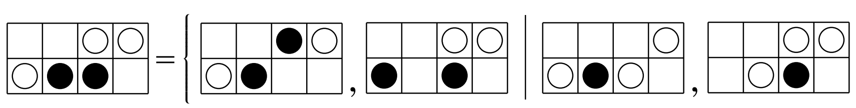

Figure 4. A position whose canonical form is $\{0, * \mid *\}$ (modulo $\mathcal{D}$).

Some rulesets are inherently dead-ending. For example, in placement games the cells available to a player on their turn are a subset of those available on the previous turn. DOMINEERING, a ruleset invented by Andersson in 1973, is an example of this. Players take turns by placing domino tiles on two adjacent empty cells. Left places tiles vertically, while Right places them horizontally. Figure 5 shows a DOMINEERING position which, just like the CLOBBER POSITION in Figure 3, has the literal form $\{* \mid *\}$. Under the misère play convention, $\mathcal{E}$ is the appropriate monoid.

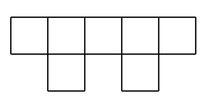

Figure 5. A DOMINEERING position whose literal form is $\{* \mid *\}$.

This time, the options are not atomic reversible, and the position is not indistinguishable from zero (the proviso in $\mathcal{E}$ fails). Figure 6 shows how $\{\, * \mid *\}$

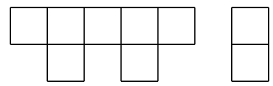

Figure 6. Right, playing first, loses $\{* \mid *\} + \mathbf{1}$.

can be distinguished from zero in $\mathcal{E}$. Naturally, two vertically adjacent empty cells constitute an $\mathscr{R}$-position. So, if $X = \mathbf{1}$, then Right playing first wins $0+X = 0+\mathbf{1}$. However, Right, playing first, loses $\{* \mid *\} + X = \{* \mid *\} + \mathbf{1}$; after a first move to $* + \mathbf{1}$, Left replies with $*$.

Let us now consider the position illustrated in Figure 7. Its literal form is $\{\bar{\mathbf{1}} \mid \mathbf{1}\}$, and the options are atomic reversible. In this case, the reduction of replacement by zero can be carried out, and the position is indistinguishable from zero.

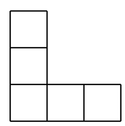

Figure 7. Replacement by zero in $\mathcal{E}$.

There are some rulesets with a wide variety of game forms. KONANE, a ruleset invented by the ancient Hawaiian Polynesians, is a good example. To move, a player's piece must jump over an opponent's piece into an empty cell. The jumped piece is captured and removed from the board. The player's piece cannot jump over pieces of the same color and cannot jump in a diagonal direction. However, that piece can jump over multiple pieces, although that is not required. Finally, the piece can only jump in one direction during each move. That means that moves that change direction are not allowed.

To further illustrate the fact that equivalence classes change when the restriction is altered, let us consider Figure 8. Played in isolation, the component X in the lower right corner is an $\mathscr{R}$-position (Right plays white stones). Note that all its Left options are Right-ends. Thus, playing first, Left loses $0 + X$. The game component in the lower left corner, just like the DOMINEERING position in Figure 7, is $\{\bar{\mathbf{1}} \mid \mathbf{1}\}$. However, playing first, Left wins $\{\bar{\mathbf{1}} \mid \mathbf{1}\} + X$. She starts by making a move to $\{\bar{\mathbf{1}} \mid \mathbf{1}\} + X^L$, capturing downwards with the topmost piece. Against that move, Right has to respond with $\mathbf{1} + X^L$. At that point, Left wins the game by capturing to the right with the piece she moved in the previous move, resulting in $\mathbf{1} + \bar{\mathbf{3}}$.

In other words, X distinguishes $\{\bar{\mathbf{1}} \mid \mathbf{1}\}$ from zero. In $\mathcal{M}$, unlike what happens in $\mathcal{E}$, the options of $\{\bar{\mathbf{1}} \mid \mathbf{1}\}$ are not atomic reversible. In fact, X is an example of a game that leads to the waiting problem discussed earlier. Left was "waiting" for

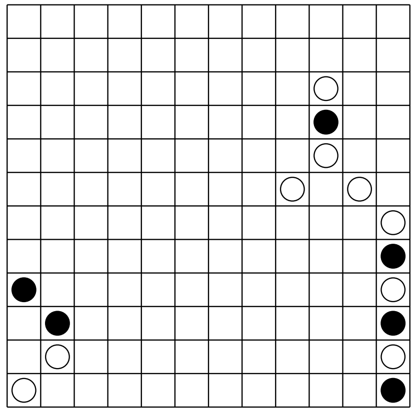

Figure 8. Left wins $\{\bar{\mathbf{1}} \mid \mathbf{1}\} + X$.

the opportunity to move to $\bar{\mathbf{3}}$ in a follower of X. She only had that opportunity because Right had to make a move in $\{\bar{\mathbf{1}} \mid \mathbf{1}\}$. That is what allowed for the "rebirth" of X after being a Right-end for some time. This type of effect does not occur in either $\mathcal{D}$ or $\mathcal{E}$.

5. Final remarks

It is common to conclude academic papers, especially surveys, with proposals for future work. In this text, almost all the highlighted subjects come with extremely interesting open problems. Given the abundance and high interest in almost all of them, we prefer to refrain from making a selection. Instead, we dedicate this final section to Milnor's universe, which is a nonparental scoring-play restriction, and to a brief discussion about the scope of the absolute theory.

5.1. ***The Milnor universe.*** The first mathematical approach to scoring-play is [Milnor 1953], a seminal paper that can be considered the precursor of CGT. Milnor restricted the studied universe to non-zugzwang dicotic forms ($\mathbb{M}$). That means that every dicotic form G in $\mathbb{M}$ satisfies $\mathrm{Ls}(G) \geqslant \mathrm{Rs}(G)$. By imposing this restriction, Milnor avoided many issues mentioned throughout this text. In fact, $\mathbb{M}$ is very well behaved, constituting a group structure. Following that approach, Hanner [1959] proved the existence of a mean value of a game G, a kind of "average value" for G. This result was very important, inspiring the modern temperature theory [Albert et al. 2007; Berlekamp et al. 1982a; 1982b; Conway 1976; Siegel 2013].

$\mathbb{M}$ was "revisited" in a disguised way much later. Let us return to classical CGT and consider the following ruleset: players play disjunctive sums with traditional game forms, but instead of ending in the usual way, the game ends when the values of the components are all numbers. At that point, the remaining numbers are added, and the sum determines the winner. If the sum is zero, it results in a *tie*. This change in the winning condition establishes a scoring ruleset, making infinitesimals *irrelevant*. In classical CGT, if $\mathrm{LS}(G) = \mathrm{RS}(G) = 0$, then G is an infinitesimal. In this alternative ruleset, infinitesimals result in a tie and should be considered equal to zero.

Moews [1991], Calistrate [1996] and Grossman and Siegel [2009] introduced the concept of *reduced canonical form*, which is, in a way, the mathematical approach that eliminates the infinitesimals from the normal play structure. A standard canonical form carries the information needed to determine $\mathrm{o}(G+X)$ for all X; the reduced canonical form carries only the information needed to determine $\mathrm{LS}(G+X)$ and $\mathrm{RS}(G+X)$ for all X. It preserves information about the stops of $G+X$ while discarding details about the number of moves it took to get there, such as the parity of that number.

It turns out that the ruleset we just mentioned describes Milnor's universe! There are no zugzwangs, as zugzwangs are numbers, and, in this setup, numbers are never touched. In other words, the mathematics of the reduced canonical forms is exactly the same as that of $\mathbb{M}$, which we will detail next. The development of the reduced canonical forms was motivated when certain rulesets where being analyzed. These included PARTIZAN SUBTRACTION$(1, 3 \,|\, 2, 3)$, where positions exhibit a very simple regularity in terms of reduced canonical forms, although this does not hold true for canonical forms. In these cases, it is good to "lose information" because complete information may be not necessary to find winning strategies. Although it was not the original intention, the knowledge about reduced canonical forms also led to the description of the structure of $\mathbb{M}$. Thus, the "revisit" was useful in two ways. The use of reduced canonical forms can be seen as the use of $\mathbb{M}$ instead of the classic normal play structure whenever the nature of the ruleset justifies it. The best reference is [Siegel 2013]. Although $\mathbb{M}$ is not an absolute universe, we have local comparison: namely, $G \succcurlyeq H$ if and only if $\mathrm{Rs}(G-H) \geqslant 0$.

Notice that maintenance is no longer necessary. The reductions are quite straightforward. If $\mathrm{Ls}(G) = \mathrm{Rs}(G) = r$, then G is simply equal to r (replacement by a terminal form). Otherwise, if G is hot, we proceed with nonatomic reversibility and domination. Atomic reversibility of a fundamental option never occurs, as for it to happen, G could not be hot. The issue of inverses does not even arise, as $\mathbb{M}$ is a group structure.

As a final observation, although the literal forms of $\mathbb{M}$ exist in $\boldsymbol{E}$, the order relation of $\mathbb{M}$ is not the same as the order of these forms when considering the order relation of $\boldsymbol{E}$. For example, $\langle 0 \,|\, 0\rangle = 0 \bmod \mathbb{M}$. However, we have $\langle 0|0\rangle \neq 0 \bmod \boldsymbol{E}$; Left, playing first, wins $\langle 0|0\rangle + \langle -3|3\rangle$, but loses $0 + \langle -3|3\rangle$.

Recalling the pioneering aspects presented in [Ettinger 1996], this fact was noted in the following manner.

> However, when we pass from $\mathbb{M}$ to $\boldsymbol{E}$ the comparison relations change in an essential way. [...] Evidently the introduction of games with negative incentives to the universe of games changes the equivalence classes. [...] Note that $\langle 0 \mid 0 \rangle \neq_{\boldsymbol{E}} 0$ in $\boldsymbol{E}$, where $=_{\boldsymbol{E}}$ is given by relativizing the standard definition of equivalence to $\boldsymbol{E}$. This is due to the presence of games with negative incentive in $\boldsymbol{E}$.

5.2. ***How comprehensive is Absolute Combinatorial Game Theory?*** Absolute Combinatorial Game Theory (Theorem 3) has made a significant contribution to much of the research presented in this survey. Table 1 shows seven parental universes that fall within its scope. Of course, the question arose as to whether that scope includes only a finite set of parental universes or if there exists an infinite number of parental universes. The answer to this question has been found: Santos et al. [2025] prove that, in misère play, there are infinitely many parental universes. This is achieved by recursively expanding $\mathcal{D}$ (finding parental universes between $\mathcal{D}$ and $\mathcal{E}$) and by recursively expanding $\mathcal{E}$ (finding parental universes between $\mathcal{E}$ and $\mathcal{M}$). Regarding dead-ending universes, i.e., parental universes between $\mathcal{D}$ and $\mathcal{E}$, Siegel [2025] proves a set of remarkable results:

(1) The dead-ends have a *multiversal* structure, meaning that their partial order and their simplest forms do not depend on the universe $\mathcal{U}$ being considered.

(2) There are uncountably many dead-ending universes $\mathcal{D}(\mathcal{A})$.

(3) One can expand universes with augmented games where the atomic reversible options are replaced by abstract objects Σ^L and Σ^R, and this approach has the advantage of avoiding arbitrary choices and canonical forms being defined "by convention" with dispensable edges (in the sense that there are no disjunctive sums where they make the difference between winning or losing). Furthermore, there are unique simplest forms in these structures.

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research@alfied.xyz *Memorial University of Newfoundland, Corner Brook, NL, Canada*

larsson@iitb.ac.in *Department of Industrial Engineering and Operations Research, Indian Institute of Technology Bombay, Mumbai, India*

rebeccamilley@mun.ca *Grenfell Campus, Memorial University of Newfoundland, Corner Brook, NL, Canada*

r.nowakowski@dal.ca *Mathematics and Statistics, Dalhousie University, Halifax, NS, Canada*

cmf.santos@fct.unl.pt *Center for Mathematics and Applications (NovaMath), Universidade Nova de Lisboa Campus de Caparica, Lisboa, Portugal*

aaron.n.siegel@gmail.com *San Francisco, CA, United States*