Games of No Chance 6
MSRI Publications
Volume **71**, 2025

Preface

URBAN LARSSON

It has been a great pleasure to edit this volume. Assistant editors have been Svenja Huntemann, Craig Tennenhouse, Simon Rubinstein Salzedo, Kyle Burke and Johan Wästlund. Without their timely inputs, this volume would not have come about in its totality.

A future of GoNC. The intention is that after this volume, *Games of No Chance* shall become an online open access (overlay) journal; this would be the last book in the MSRI Publications series. Anyone interested in joining the process of making this a reality is welcome to contact me at larsson@iitb.ac.in or urban031@gmail.com. We will continue to accept submissions, while figuring out the details of our forthcoming combinatorial games journal.

Acknowledgements. I would like to add our greatest appreciation to Silvio Levy and the teams at Cambridge University Press and MSRI (now the Simons Laufer Mathematical Sciences Institute, SLMath) over the years. I would also like to thank Richard J. Nowakowski, who edited the first four GoNC volumes, setting a solid foundation that has inspired much brilliant CGT research over the years.

Let us review the main scientific directions in this book, before discussing the articles in order of their appearance.

Overview

Theory building. In this volume, normal play theory [1; 2; 3; 8] is generalized in two directions via Absolute CGT and Affine normal play. Absolute theory finds that for so-called parental universes of games — scoring, misère and so on — if taking "options of options" result in a maintenance of game inequality, and a certain proviso is satisfied, then normal play theory implies that this inequality holds, irrespective of the original winning convention. The parental property (a.k.a. dicotic closure) has become central in much subsequent research: any two non-empty subsets of games contributes a game in the universe. Affine theory incorporates checks, entailing moves and more; in the impartial affine setting [6]

MSC2020: 91A46.
Keywords: combinatorial game theory.

Sprague–Grundy theory continues to hold with the addition of exactly one extra element, called the moon (adapted from *Winning Ways*). Here we solve the partizan setting, and unsurprisingly, complications arise. The first one being that many games equivalent to the moon modulo impartial, become distinguishable modulo the full universe. However, by being able to control intricate check sequences, Santos et al. find a constructive way to compare all partizan games. The setting is illustrated with many examples from the recreational ruleset Atari Go, a derivation of the classical game of Go, where the first capture wins.

More on absolute. When the absolute theory was developed, there were a handful of universes of games that satisfy dicotic closure, primarily dicotic and dead-ending misère play and dicotic and guaranteed scoring play. Two papers in this volume improve the situation further. Namely they define infinitely many misère universes, in fact Siegel shows that there are uncountably many absolute universes between the dicots and dead-ending ones. Santos et al. are able to also expand the space between dead-ending and full misère to an infinite number of universes. The first two papers on absolute theory do not develop reduction theorems such as domination (easy) and reversibility (harder). Reduction theorems for guaranteed scoring were known before this volume [5], and the same holds for dicotic misère [4]; the crux is to understand atomic reversibility in the respective case. Here we complement with the dead-ending misère reduction theorems. In addition we survey the recent development of these types of results both in scoring and misère play with an emphasis on aspects of absolute theory. As a special treat, we include a study of all equivalence classes of dead-ending games born by day 2.

Additive connections. The classical connection game of Hex belongs to the class of monotone set coloring games. Recently an additive theory was developed for such games [7]. Here Demer et al. prove that all games for which pass moves can be included without affecting equivalence classes are realizable as monotone set coloring games.

Nim values at large. Three interesting and diverse papers revolve around the classical Sprague–Grundy theory. Friedman studies the distribution of nim-values for generic impartial games, namely those arising from k-regular multi-graphs. He shows, via an iterative procedure, that the limiting distribution converges to a stationary distribution that only depends on k. Abrams et al. study generalizations of the nim-sum via an arbitrary nonnegative seed. They arrive at doubly infinite arithmetically periodic arrays with many interesting properties. Memgames are one-heap nim-type games with move restrictions imposed by a move memory; here partially solved in terms of nim-values. Solutions vary from regular, to fractal and chaotic where certain nim-values reveal surprising immortal behavior.

Wythoff Nim variations. Via a lexicographic ordering of the heaps in a multi heap variation of Wythoff Nim, where addition of tokens is a possibility, the P-positions form a disjoint cover of the positive integers. In yet another variation, Wythoff's Queen can bounce off the borders. Thereby the P-positions can sometimes be described by morphic sequences, and the theorem-prover Walnut can be employed. In m-Modular Wythoff there are more diagonal type moves, namely it suffices that the difference is congruent to zero modulo m. Remarkably the P-positions will be subsets of the classical Wythoff Nim.

Rigorous recreation. In 1983, the Chess periodical EG published a summary of a letter from Julius Telesin outlining how a king, a bishop and a knight can checkmate a lonely king on an arbitrarily large chessboard. Wästlund contributes a rigorous justification of the same. Coin moving puzzles are popular in recreational mathematics. Moves are restricted so that a coin can be placed only in a position that is adjacent to at least two other coins; by starting from one configuration the goal is to reach another. A few years ago Demain et al. specified which of these games are solvable on a triangular grid. Here Galliot et al. extend their work on square grids, and correct some early findings that include extra coins. When has the losing player something to do? In the game of Candy Nim, Mani et al. answer how to construct games such that the loser takes the largest possible number of candies. And moreover they bound the winner's treat in an arbitrary P position.

Complexity issues. Node-Kayles, Snort and Col are distance games played on a graph; a player colors a node, given that none of its defined neighbors has a given color. Here it is demonstrated that the general class is PSPACE-hard. Very little is known on the generic complexity of partizan normal play games modulo equivalence. In spite of having efficient reduction theorems, the bounds become enormous already for games born by day 4. Suetsugu substantially improve those bounds.

Surveys as such. Richman bidding combinatorial games have a history in these volumes; Kant et al. contributes a survey on the same. Flammenkamp's thesis "Subtraction Spielen mit Lange Perioden" (1997) is written in the German language, which might have impeded the spread of the good word. Through ingenious experimental work he provides evidence of exponential period lengths of our classical subtraction games. Our survey on subtraction games (written in English) highlights this development. As usual we provide an updated "Unsolved Problems in Combinatorial Games", by Nowakowski. The fast development of absolute additive theory, and in some cases reduction theorems, commend a survey, and we abide.

Organization of the volume

Surveys

1. *Combinatorial Game Theory monoids and their absolute restrictions: a survey*, by Alfie M. Davies, Urban Larsson, Rebecca Milley, Richard J. Nowakowski, Carlos P. Santos and Aaron N. Siegel

The classic order relation in Combinatorial Game Theory asserts that, given a winning convention, a game is greater than or equal to another whenever Left can exchange the second for the first in any disjunctive sum, and she can do this regardless of the other component, without incurring any disadvantage. In the early developments of Combinatorial Game Theory, the "other component" encompassed any possible game form, and a rich normal play theory was built by Berlekamp, Conway, and Guy (1976–1982), via a celebrated local comparison procedure. It turns out that the normal play convention is a lucky case, and recent research on other conventions therefore often restricts the ranges of games to various subclasses. In the case of misère play, it is possible to obtain partially ordered monoids with more structure by imposing restrictions. The same is true in scoring play. Furthermore, Absolute Combinatorial Game Theory was recently developed as a unifying tool for a local game comparison that generalizes the normal play findings, provided that the restricting set is parental (among a few other closure properties), meaning that any pair of finite, non-empty subsets of games from the restriction is permissible as sets of options for another game in the set. This survey aims to provide a concise overview of the current advancements in the study of these structures.

2. *A brief conversation about subtraction games*, by Urban Larsson and Indrajit Saha

In this survey we revisit finite subtraction, one-heap subtraction games on finite rulesets. The main purpose is to give a general overview of the development, and specifically to draw attention to Flammenkamp's thesis (1997), where he, contrary to other studies, experimentally observes exponential eventual period length of the outcomes, for a carefully selected subclass of games. In addition, we contribute an appendix on finite excluded subtraction by Suetsugu.

3. *Survey on Richman bidding combinatorial games*, by Prem Kant and Urban Larsson

In this survey, we explore the literature around Richman bidding in both its continuous and discrete forms. Our primary objective is to highlight recent advancements in discrete Richman bidding, which, in the normal play setting, generalizes the classical alternating play to infinitely many game families.

4. *Unsolved problems in combinatorial games*, by Richard J. Nowakowski

During the recent development of Combinatorial Game Theory, many more problems have been suggested than solved. Here is a collection of problems that many people have found interesting. Included is a discussion of recent developments in these areas. This collection was started by Richard Guy in 1991, and has been updated in each *Games of No Chance* volume.

Theory building

5. *Absolute combinatorial game theory*, by Urban Larsson, Richard J. Nowakowski and Carlos P. Santos

We propose a unifying additive theory for standard conventions in Combinatorial Game Theory, including normal play, misère and scoringplay, studied by Berlekamp, Conway, Dorbec, Ettinger, Guy, Larsson, Milley, Neto, Nowakowski, Renault, Santos, Siegel, Sopena, Stewart (1976-2019), and others. A game universe is a set of games that satisfies some standard closure properties. Here, we reveal when the fundamental game comparison problem, "Is $G \geq H$?", simplifies to a constructive "local" solution, which generalizes Conway's foundational result in ONAG (1976) for normal-play games. This happens in a broad and general fashion whenever a given game universe is absolute. Games in an absolute universe satisfy two properties, dubbed parentality and saturation, and we prove that the latter is implied by the former. Parentality means that any pair of non-empty finite sets of games is admissible as options, and saturation means that, given any game, the first player can be favored in a disjunctive sum. Game comparison is at the core of combinatorial game theory, and for example efficiency of potential reduction theorems rely on a local comparison. We distinguish between three levels of game comparison; superordinate (global), basic (semi-constructive) and subordinate (local) comparison. In proofs, a sometimes tedious challenge faces a researcher in CGT: in order to disprove an inequality, an explicit distinguishing game might be required. Here, we explain how this job becomes obsolete whenever a universe is absolute. Namely, it suffices to see if a pair of games satisfies a certain Proviso together with a Maintenance of an inequality.

6. *Affine normal play*, by Urban Larsson, Richard J. Nowakowski and Carlos P. Santos

There are many combinatorial games in which a move can terminate the game, such as a checkmate in chess. These moves give rise to diverse situations that fall outside the scope of the classical normal play structure. To analyze these games, an algebraic extension is necessary, including infinities as elements. In this work, affine normal play, the algebraic structure resulting from that extension, is analyzed. We prove that it is possible to compare two affine games using only their forms. Furthermore, affine games can still be reduced, although the reduced forms are not unique. We establish that the classical normal play is order-embedded in the extended structure, constituting its substructure of invertible elements. Additionally, as in classical theory, affine games born by day n form a lattice with respect to the partial order of games.

More on absolute

7. *On the general dead-ending universe of partizan games*, by Aaron N. Siegel

The universe E of dead-ending partizan games has emerged as an important structure in the study of misere play. Here we attempt a systematic investigation of the structure of E and its subuniverses. We begin by showing that the dead-ends exhibit a rich "absolute" structure, in the sense that they behave identically in any universe in which they appear.

We will use this result to construct an uncountable family of dead-ending universes and show that they collectively admit an uncountable family of distinct comparison relations. We will then show that whenever the ends of a dead ending universe U are computable, then there is a constructive test for comparison modulo U. Finally, we propose a new type of generalized simplest form that works for arbitrary universes (including universes that are not dead-ending), and that is computable whenever comparison modulo U is computable. In particular, this gives a complete constructive theory for UE with computable ends. This theory has been implemented in cgsuite as a proof of concept. As an application of these results, we will characterize the universe generated by misere Domineering, and we will compute the misere simplest forms of 2xn Domineering rectangles for small values of n.

8. *Infinitely many absolute universes*, by Urban Larsson, Richard J. Nowakowski, Carlos P. Santos

Absolute combinatorial game theory was recently developed as a unifying tool for constructive/local game comparison. The theory concerns parental universes (a.k.a. dicot closure) of combinatorial games; standard closure properties are satisfied and each pair of non-empty sets of forms of the universe makes a form of the universe. Here we prove that there is an infinite number of absolute misère universes, by recursively expanding the dicot misère universe and the dead-ending universe. On the other hand, we prove that normal-play has exactly two absolute universes, namely the full space, and the universe of all-small games.

9. *Reversibility, canonical form and invertibility in dead-ending misère play*, by Urban Larsson, Rebecca Milley, Richard J. Nowakowski, Gabriel Renault, Carlos P. Santos

In normal play combinatorial game theory, there is a slick reduction of game forms via domination and reversibility, which yields a unique reduced game form, dubbed the canonical form or simply the game value. In misère play, the situation is much more varied and complex. In restricted misère play (Plambeck and Siegel 2008), where the definition of inequality is weakened, domination is in analogy with normal play, but reversibility is not: in particular, if a Left option is reversible through a position with no Left option, then the reversible Left option cannot always be removed (Siegel 2015). Dorbec et al. (2015) found a modified reversibility reduction to give unique reduced forms for dicot games. We present a set of reductions for reversible options in *dead-ending* games (Milley et al. 2013). We prove that the reduced forms are unique with respect to our choice of reduction. We use uniqueness of reduced forms to prove that dead-ending, dicotic, and impartial restrictions have the conjugate property: if a game has an inverse, then it is the conjugate, i.e., the game where players have swapped roles.

10. *Dead-ending day-2 games under misère play*, by Aaron Dwyer, Rebecca Milley, Michael Willette

Combinatorial games do not exhibit the same algebraic structure in misère play as they do in normal play; e.g., no nonzero game has an additive inverse under misère play. Recently, misère research has considered 'restricted' play, where games can be equal or comparable modulo a subset (universe) of games, even if they are not in general. One universe well-suited for misère analysis is the set of dead-ending games: games with the property that if a specific player cannot move at some point in the game, then that player will never again be able to move. Dead-ending games have many nice properties: some, including normal-play numbers, are invertible "modulo dead-ending", there is an easy test for inequality, and there are reductions that give unique reduced forms. We apply recent results for inequalities and game simplification to find the unique, reduced dead-ending games born by day 2, and determine which of these are invertible modulo dead-ending games.

Additive connections

11. *All passable games are realizable as monotone set coloring games*, by Eric Demer, Peter Selinger and Kyle Wang

The class of passable games was recently introduced by Selinger as a class of combinatorial games that are suitable for modeling monotone set coloring games such as Hex. In a monotone set coloring game, the players alternately color the cells of a board with their respective color, and the winner is determined by a monotone function of the final position. It is easy to see that every monotone set coloring game is a passable combinatorial game. We prove the converse: every passable game is realizable, up to equivalence, as a monotone set coloring game.

Nim values at large

12. *Values of generic impartial combinatorial games*, by Eric Friedman

We introduce the study of "typical" combinatorial games, where a specific game is chosen at random from a large set of games. We study one such class of games, those arising from a k-regular multigraph, and show that the limiting distribution of (Sprague–Grundy) values converges to a stationary distribution which only depends on k. We provide an iterative procedure for computing this distribution and prove several high probability results for finite plays. Our work provides some initial steps towards formalizing the "renormalization approach" to combinatorial games which has proven effective at describing the properties of several classic combinatorial games but as yet is nonrigorous in most applications. In addition, our results may provide insights into properties of complex combinatorial games that have so far resisted formal analyses.

13. *A family of Nim-like arrays: stabilization*, by Lowell Abrams and Dena S. Cowen-Morton

In previous work, the authors and others constructed a family of Nim-like arrays using the operation of Nim addition composed with the operation of sequential compound. Every row of these arrays is a permutation of the natural numbers that, for large enough values, is arithmetically periodic. In this work, we regularize, or "stabilize" these arrays row by row, so that each row becomes both doubly infinite and everywhere arithmetically periodic. We study basic properties of these stabilized arrays, in particular showing that various interesting properties of the original arrays continue to hold for their stabilized counterparts. We then show that the row-permutations of the stabilized arrays are in fact affine permutations of the integers, and thus the groups generated by these permutations are themselves groups of affine permutations. To give a taste of the complexity of these groups, we analyze an illustrative example highlighting the structure of a subgroup of the multiplication group for one particular array of the family, and examine its Cayley graph.

14. *Memgames*, by Urban Larsson, Simon Rubinstein-Salzedo and Aaron N. Siegel

Memgames are heap games in which the play constraints on a given heap H are determined by the immediately preceding move on H. We analyze three related memgames, which we call Mem, and , that have simple, parameterless definitions but that nonetheless exhibit intricate and surprising nim value structures. The paper concludes with a long list of open questions and intriguing directions for further research.

Rigorous recreation

15. *The bishop and knight checkmate on a large chessboard*, by Johan Wästlund

In 1983, the chess periodical EG published a summary of a letter from Julius Telesin outlining how a king, a bishop and a knight can checkmate a lonely king on an arbitrarily large chessboard. The Telesin checkmating procedure doesn't seem to be widely known, and the published account left out a number of details. We describe the method precisely and show that it works against every defense. We also discuss the open question of the asymptotics of the largest distance to mate on a large square board.

16. *An update on the coin-moving game on the square grid*, by Florian Galliot, Sylvain Gravier and Isabelle Sivignon

This paper extends the work started in 2002 by Demaine, Demaine and Verill (DDV) on coin-moving puzzles. These puzzles have a long history in the recreational literature, but were first systematically analyzed by DDV, who gave a full characterization of the solvable puzzles on the triangular grid and a partial characterization of the solvable puzzles on the square grid. This article specifically extends the study of the game on the square grid. Notably, DDV's result on puzzles with two "extra coins" is shown to be overly broad: this paper provides counterexamples as well as a revised version of this theorem. A new method for solving puzzles with two extra coins is then presented, which covers some cases where the aforementioned theorem does not apply. Puzzles with just one extra coin seem even more complicated, and are only touched upon by DDV. This paper delves deeper, studying a class of such puzzles that may be considered equivalent to a game of "poking" coins. Within this class, some cases are considered that are amenable to analysis.

17. *P play in Candy Nim*, by Nitya Mani, Rajiv Nelkanti, Simon Rubinstein-Salzedo and Alex Tholen

Candy Nim is a variant of Nim in which both players aim to take the last candy in a game of Nim, with the added simultaneous secondary goal of taking as many candies as possible. We give bounds on the number of candies the first and second players obtain in 3-pile P positions as well as strategies that are provably optimal for some families of such games. We also show how to construct a game with N candies such that the loser takes the largest possible number of candies and bound the number of candies the winner can take in an arbitrary P position with N total candies.

Complexity issues

18. *Keeping your distance is hard*, by Kyle Burke, Silvia Heubach, Melissa Huggan and Svenja Huntemann

We study the computational complexity of distance games, a class of combinatorial games played on graphs. A move consists of placing a coloured token on an unoccupied vertex subject to it not being at certain distances to already occupied vertices. The last player to move wins. Well-known examples of distance games are Node-Kayles, Snort, and Col, whose complexities were shown to be PSPACE-hard. We show that many more distance games are also PSPACE-hard.

19. *Improving upper and lower bounds of the number of games born by day 4*, by Koki Suetsugu

In combinatorial game theory, the lower and upper bounds of the number of games born by day 4 have been recognized as $3.0 \cdot 10^{12}$ and 10^{434}, respectively. We improve the lower bound to $10^{28.2}$ and the upper bound to $4.0 \cdot 10^{184}$.

Wythoff variations

20. *K-Pile Wythoff Games*, by Aviezri S. Fraenkel and David Klein

An important aspect of the classic Wythoff game is that its P-positions form a disjoint cover of the positive integers by two sequences. Though generalizations of Wythoff to $K > 2$ piles abound, we believe that the generalization presented here is the first where the P-positions form a disjoint cover of the positive integers by $K > 3$ sequences. To achieve this we add a novel ingredient, we allow pile sizes to increase. This leads, inter alia, to games with infinitely many subpositions, yet every such game ends with no remaining tokens, due to a lexicographic order imposed on the moves. We refer to this game as Wytlex(K).

21. *Corner the empress*, by Robbert Fokkink, Gerard Francis Ortega and Dan Rust

Wythoff Nim, a.k.a. Corner the Lady, is a classic combinatorial game. A Queen is placed on an infinite chess board and two players take alternate turns, moving the Queen closer to the corner. The first player that corners the Queen wins. What happens if the Queen gets superior powers and is able to step off the diagonal or bounce against a side? We study the intriguing patterns that emerge from such games. In particular, we are interested in games in which the P-positions can be described by morphic sequences. We use the theorem-prover `Walnut` to prove some of our results.

22. *m-Modular Wythoff*, by Tanya Khovanova and Nelson Niu

We introduce a variant of Wythoff's game that we call m-modular Wythoff's game. In the original Wythoff's game, players can take a positive number of tokens from one pile, or they can take a positive number of tokens from both piles if the number of tokens they take from the first pile is equal to the number of tokens they take from the second. In our variant, we weaken this equality condition to one of equivalence modulo m. We characterize the P-positions of our m-modular variant as a finite subset of the P-positions of the known P-positions of the original Wythoff's game.

References

[1] M. H. Albert, R. J. Nowakowski, and D. Wolfe, *Lessons in Play*: *an Introduction to Combinatorial Game Theory*, A K Peters, 2007. Zbl

[2] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays* (*four volumes*), 2nd ed., A K Peters, 2001. MR Zbl

[3] J. H. Conway, *On Numbers and Games*, London Math. Soc. Monogr. **6**, Academic Press, 1976. MR Zbl

[4] P. Dorbec, G. Renault, A. N. Siegel, and E. Sopena, "Dicots, and a taxonomic ranking for misère games", *J. Combin. Theory Ser. A* **130** (2015), 42–63. MR Zbl

[5] U. Larsson, R. J. Nowakowski, J. P. Neto, and C. P. Santos, "Guaranteed scoring games", *Electron. J. Combin.* **23**:3 (2016), art. id. 3.27. MR Zbl

[6] U. Larsson, R. J. Nowakowski, and C. P. Santos, "Impartial games with entailing moves", *Integers* **21B** (2021), art. id. A17. MR Zbl

[7] P. Selinger, "On the combinatorial value of Hex positions", *Integers* **22** (2022), art. id. G3. MR Zbl

[8] A. N. Siegel, *Combinatorial Game Theory*, Graduate Studies in Mathematics **146**, Amer. Math. Soc., Providence, RI, 2013. MR Zbl

larsson@iitb.ac.in

Dept. Industrial Engineering and Operations Research
IIT Bombay, Indian Institute of Technology Bombay,
Powai, Mumbai, Maharashtra, Mumbai 400076, India