Games of No Chance 5
MSRI Publications
Volume **70**, 2017

Geometric analysis of a generalized Wythoff game

ERIC FRIEDMAN, SCOTT M. GARRABRANT,
ILONA K. PHIPPS-MORGAN,
A. S. LANDSBERG AND URBAN LARSSON

Combinatorial 2-player games can be studied from different perspectives. Traditionally the goal has been to acquire a perfect strategy, and to this purpose an efficient procedure (polynomial in succinct input size) is required. However, most combinatorial games are intrinsically hard to analyze; success is limited to a small number of games with predominant "mathematical structure". The classical games of Nim (Bouton 1901) and Wythoff Nim (Wythoff 1907) are easy to analyze rigorously, but already seemingly modest variants, like (p, q)-GDWN (Larsson 2012, 2014), appear to withstand log-polynomial descriptions. Therefore, development of new methods is highly desirable.

Here, we use methods from physics, such as *renormalization*, in an attempt to understand the larger geometry of a game's P-positions (safe positions for the Previous player), rather than their exact configurations (Friedman et al. 2007, 2009). By studying evolution diagrams of a general class of *linear extensions* of Nim, Wythoff Nim and GDWN, we observe that P-positions often distribute uniformly along lines (a.k.a. P-beams), visually separated from the move lines. Given a fundamental hypothesis, a *filling property* which generalizes directly from Wythoff Nim, we formulate natural equations on the slopes and densities of P-positions along these lines; here, a key innovation, a *reorganization* model, guides us in selecting the relevant rules (move lines). The exceptional case of the symmetric (p, q)-GDWN is interesting, because of observed quasi-log repetitive fluctuations, and these games have defied all previous analysis.

1. Introduction to the class Linear Nimhoff

In this paper we study a class of combinatorial games using renormalization-based techniques from physics in combination with computer simulations. This approach leads to a probabilistic geometric analysis of the underlying structure

MSC2010: 91A46.
Keywords: combinatorial game, GDWN, Nim, Wythoff Nim, renormalization, reorganization.

and behavior of a game. A number of interesting features are revealed, including observations of quasi log-periodic fluctuations. Our class of games, dubbed *Linear Nimhoff*, is a generalization of the classical Wythoff's game [Wythoff 1907] and the more recent GDWN [Larsson 2012a].

The *renormalization* approach to games involves:

1) Identifying broad, overall patterns in games (on a course grained level) by focusing on their scaling, asymptotic, and/or global, probabilistic behavior;
2) Analyzing these patterns using scaling/course-graining techniques derived from self-consistency conditions;
3) A novel method for this paper, a *reorganization* model, which filters out game rules that do not contribute to the broad overall patterns.

Two of the best-known impartial combinatorial games are Nim and Wythoff's modification of Nim, a.k.a Wythoff's game. In two-pile Nim, players alternate in removing tokens from a pile of their choosing, with the player who removes the last token declared the winner. Wythoff's game is an extension of two-pile Nim in which players, in addition to being able to remove tokens from a selected pile (as in Nim), also have the option of removing the same number of tokens from both piles simultaneously. Both games can be trivially but conveniently recast in terms of a marker moving on a semi-infinite, two-dimensional integer grid. The marker's (x, y) positional coordinates indicate the current number of tokens in each pile; the lower left corner $(0, 0)$ of the grid represents the game's terminal position. In Nim, the marker can slide leftwards or downwards; in Wythoff's game, a diagonal move (down and to the left) is also allowed. Both of these games have been well studied and are considered completely "solved" in the sense that a complete specification of the P- and N-positions[1] in these games is possible (see, e.g., [Bouton 1901; Wythoff 1907; Berlekamp et al. 1982]).

In *Linear Nimhoff*, the game marker can move not only leftwards, downwards, and diagonally, but along other prescribed directions as well. As we shall see, this seemingly simple extension leads to nontrivial *geometric* structures associated with the P-positions of the game. Mostly, we find that the P-positions in Linear Nimhoff lie along certain diffuse *lines*, and otherwise, in some specific cases exhibiting various quasi-log-periodic behavior within diverging *P-beams* [Larsson 2012a]. We show how the slopes and densities of these lines (or in the case of beams, the mean values of slopes and densities) can be computed via a semi-heuristic geometrical technique (adapted from renormalization models in physics) first described in [Friedman et al. 2007; Friedman et al. 2009].

[1]In keeping with standard terminology, a P-position in a combinatorial game is a winning position for the **P**revious player (i.e, the player who just moved to that position); an N-position is a winning position for the **N**ext player to move.

The results rely on a probabilistic description of the games' underlying structure, and in view of recent results on a symmetric restriction GDWN [Larsson 2012a; Larsson 2014], we will see that this assumption should be relaxed in some cases. Therefore we define classes of games, guided by the observed behaviour of their outcomes.

(i) The *strict class* of Linear Nimhoff will consist of games for which the P-positions satisfy a probabilistic description along lines (each probabilistic line is described unambiguously by a slope and a density).

(ii) The *nonprobabilistic class* will contain any other Linear Nimhoff game.

The main purpose of this paper is to explain the probabilistic geometry of games in the strict class. For games in the nonprobabilistic class, we argue that some general behavior will carry over, in spite of apparent fluctuations gradually transforming the probabilistic geometry. Experimental data show that, in most cases, if the probabilistic geometry breaks up into something else, then the new patterns will satisfy some *quasi log-periodic fluctuations* centred in the values obtained by the strict class computations.

(iii) The *QLPF class* (pronounced culpif) is a subset of the nonprobabilistic class (ii). It contains games for which the P-positions (P-beams) follow some quasi log-periodic fluctuations. An approximate scaling factor can be computed, by experimental data, often indicating log-periodic regions void of P-positions, which heuristically shows why the intersection of this class with the strict class (i) is empty.

(iv) The class of *relaxed Linear Nimhoff* is as QLPF, but where a relaxed geometric assumption (allowing for forbidden regions to separate fluctuated P-beams instead of probabilistic P-lines) includes games with log-periodic fluctuations to the class in (i). In particular, the mean values of the densities and slopes of the P-beams will satisfy the equations as obtained in (i).

The type of visual fluctuations in relaxed Linear Nimhoff can be very hard to describe; we rather suggest an empirical classification of those games. By generalizing the outcomes of games, by counting the number of P-positions as options (a method adapted from Blocking the Queen games [Cook et al. 2015]), we obtain more accurate predictions. It remains an open problem to determine if the classes (i) and (iv) are nonempty. Through many experiments, we believe that classes (iii) and (iv) are the same, so we often identify QLPF with relaxed Linear Nimhoff, although we are not yet aware of any method to prove this.

1.1. *Game rules.* Linear Nimhoff is an impartial combinatorial game played by moving a marker along positions on a semi-infinite, two-dimensional integer grid along given half-lines. A *position* $X = (x, y)$ of a game is a vector in $\Re^2$

whose coordinates are nonnegative integers; the set of all positions constitutes the *position space* of the game.

A *rule* $r = (a, b)$ is a vector in $\Re^2$ other than $(0, 0)$, whose coordinates are nonnegative integers. The *ruleset* is a set $R = \{r_i = (a_i, b_i), \text{ for } i = 1 \dots n)\}$ of rules, for some $n \geqslant 2$; by assumption, the ruleset of Linear Nimhoff always includes the Nim-rules $(a_1, b_1) = (1, 0)$ and $(a_n, b_n) = (0, 1)$. The ruleset designates the legal moves in the game: from a position X, one can select any rule $r \in R$ and move along this vector to any *valid* position $X - kr$, for k a positive integer. That is, from a general position $X = (x, y)$, a player can move to any position in the set $\{(x - ka, y - kb) : k \in \mathbb{Z}_{>0}, (a, b) \in R, x - ka \geq 0, y - kb \geq 0\}$. The class GDWN [Larsson 2012a] is a restriction of the general class Linear Nimhoff in that $(a, b) \in R$ implies also $(b, a) \in R$. The class (p, q)-GDWN consists of all games of the form $\{(1, 0), (q, p), (1, 1), (p, q), (0, 1)\}$, $0 < p < q$ integers.

A player loses the game when no legal moves remain available, as occurs when the position $(0, 0)$ is reached.

Since the rules imply that there are no infinite sequences of moves, every position in Linear Nimhoff can be uniquely characterized as being a P-position or an N-position. The set of all P-positions in the game is denoted $\mathcal{P}$, and the set of all N-positions is $\mathcal{N}$.

Figure 1 illustrates the allowed moves for one instantiation of the game (under ruleset $R = \{(1, 0), (2, 1), (3, 3), (0, 1)\}$). Observe that two-pile Nim and Wythoff Nim constitute special cases of Linear Nimhoff, with rulesets $\{(1, 0), (0, 1)\}$ and $\{(1, 0), (1, 1), (0, 1)\}$, respectively.

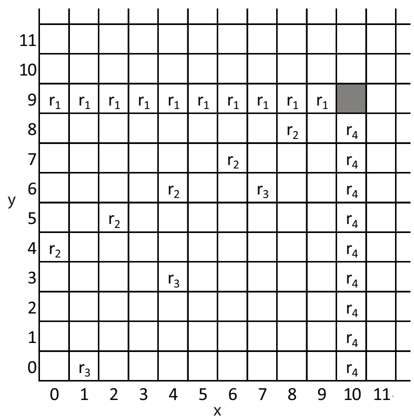

Figure 1. Legal moves in Linear Nimhoff. The figure illustrates the positions available to a player from the starting position shown (filled square) under rules $\{r_1 = (1, 0), r_2 = (2, 1), r_3 = (3, 3), r_4 = (0, 1)\}$.

2. Observations on the P-Positions of Linear Nimhoff

Figure 2 depicts the locations of the P-positions in Linear Nimhoff for a variety of different rulesets that we believe belong to the strict class.

For reference, Figures 2a,b illustrate the special cases of two-pile Nim and Wythoff's game, respectively. In Nim, the P-positions lie along the main diagonal, while in Wythoff's game the P-positions lie near two lines passing through the origin with slopes $\phi^{-1} = \frac{-1+\sqrt{5}}{2}$ and $\phi = \frac{1+\sqrt{5}}{2}$. The remainder of Figure 2 illustrates what occurs for more general cases in the strict class. In Figure 3, we show an instance of the class (iv) — the game (3, 5)-GDWN; that is the ruleset $R = \{(1, 0), (5, 3), (1, 1), (3, 5), (0, 1)\}$, indicating a quasi-log-periodic behavior.

The computer simulations reveal the following:

(1) In a typical game of Linear Nimhoff in the strict class, the P-positions approximate certain lines passing through the origin (see Figures 2c–f). By experimental data [Larsson 2012a], some of these "lines" will tend to diverge, but apparently never beyond certain bounds; see also Section 5, where the subclass GDWN of Linear Nimhoff is discussed. Therefore, and similar to this

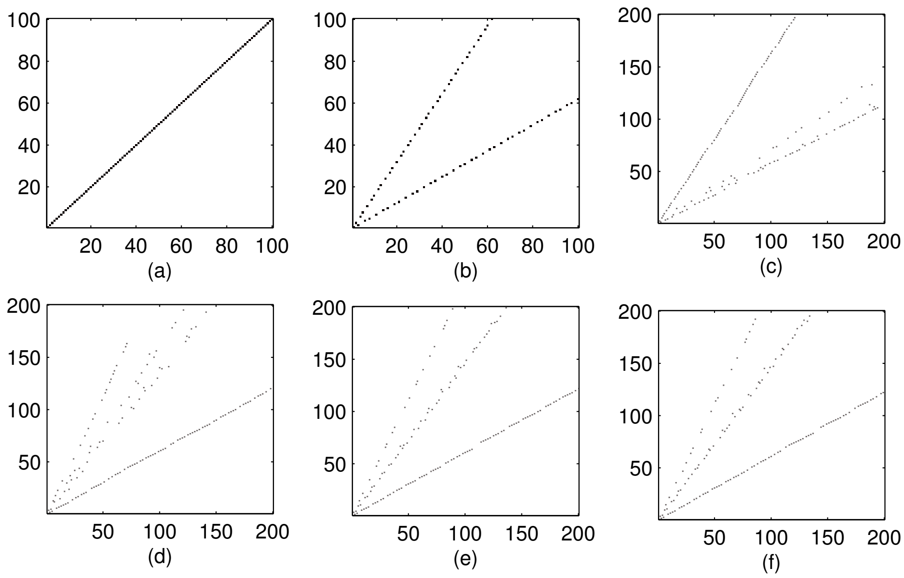

Figure 2. The locations of the P-positions (in black) in the x-y plane for various rule sets in Linear Nimhoff. (a) $R = \{(1, 0), (0, 1)\}$ (i.e., two-pile Nim); (b) $R = \{(1, 0), (1, 1), (0, 1)\}$ (i.e., Wythoff's game); (c) $R = \{(1, 0), (3, 2), (1, 1), (0, 1)\}$; (d) $R = \{(1, 0), (1, 1), (2, 3), (1, 2), (0, 1)\}$; (e) $R = \{(1,0), (1,1), (1,2), (0,1)\}$; (f) $R = \{(1,0), (1,1), (1,2), (1,8), (0,1)\}$.

Figure 3. The location of P-beams (in white in this picture) of (3, 5)-GDWN for $x \leqslant 32600$, $y \leqslant 32600$; pixels have been coarsened for better visibility.

reference, we refer to these lines as either *P-lines* or *P-beams*, depending on conjectured behavior as in the strict class (i) or the QLPF class (iii), respectively.

In many cases, though, the P-positions exhibit very modest scatter about the P-lines; when computation is extended beyond a few hundred, the width of the scatter remains small in relation to the scale of the P-line. This includes some GDWN games, such as (1, 2)-GDWN and (2, 3)-GDWN [Larsson 2014]. Certain (p, q)-GDWN games appear to belong to the "diverging beams" class rather than the "small scatter along lines" class (we define this class in Section 5.1). In the case of the strict class, we assume that, along any given line, the proportion of the number of P-positions (below given x-coordinates) does not vary beyond certain bounds; the distribution is more or less uniform on large spatial scales (i.e., density variations are purely local). In QLPF it turns out that we instead compute mean values of the density and slope of a P-beam, see also Section 5, which appears to coincide with the slope and density of an imagined P-line, but instead representing a "centre" of a P-beam.

In most observed examples, if the rules of a Linear Nimhoff game are not symmetric then the P-positions are "uniformly distributed", and the suggested notion would be "P-line", but we have found an exception: for the ruleset $R = \{(1, 0), (5, 3), (8, 5), (1, 1), (3, 5), (0, 1)\}$, the fluctuations from the game $R = \{(1, 0), (5, 3), (1, 1), (3, 5), (0, 1)\}$ survive the introduction of the new rule, $r' = (8, 5)$. On the other hand, there are many symmetric rulesets where "P-line" appears to be the correct notion; see Section 6.1.

(2) The number of P-lines which appear in the figures (for games in the strict class) as well as their slopes and densities, depends nontrivially on the particular ruleset under consideration. For games such as Nim, Wythoff's Nim, (1, 2)-GDWN and some more GDWN games [Larsson 2012a], the number of P-lines

present equals the *number-of-rules-minus-one*: there are n P-lines and there are $n+1$ rules in the ruleset. Suppose that R is in the strict class, and let $\Delta(R) = \#R - \#$(P-lines associated with R). A comparison of Figures 2e and f illustrates that the addition of a new rule to a ruleset does not always create a new P-line. For (p, q)-GDWN games it is conjectured [Larsson 2012a] that the geometric behavior is as for Wythoff's game, unless (p, q) is of a very special form, as will be defined in Section 5.1: we get $\Delta = 1$ in the latter case, but $\Delta = 3$ in the former case. As we will see, the estimates of slopes and densities of P-lines in the strict class requires $\Delta = 1$. If this is not the case, then we provide an algorithm to reduce the number of rules, described as a novel *reorganization* model in Section 4.

3. Analysis of Linear Nimhoff

The intention of this section is to characterize the overall geometric structure of the P-positions in the strict class of Linear Nimhoff. Towards this end, we will forego determining the precise locations of P-positions in favour of a more global geometrical description that quantifies the number, slopes, and densities of the P-lines. The motivation behind this approach is as follows: recent work [Zeilberger 2001; Zeilberger 2004; Friedman et al. 2007; Friedman et al. 2009] suggests that some combinatorial games — including some that are presumed to be computationally "hard" — may display certain regularities which are manifest in the underlying geometric structure of the game's P-positions.

In particular, such games may simultaneously display both order (in the sense of a regular underlying geometry) and disorder (in the form of scatter about this regular structure). While the disordered component (i.e., scatter) is believed to be associated with a game's complexity and may resist analytical treatment, the ordered component may be tractable to analysis and yield critical new insights into the game [Friedman et al. 2007; Friedman et al. 2009]. For this reason, in this section, the overall goal is to calculate the number, slopes, and densities of the P-lines in the strict class of Linear Nimhoff, where scatter along lines is the most prominent feature.

A position X in Linear Nimhoff is called a *parent* of position Y, if a player can move from X to Y in one turn, and Y a *child* of X — in standard terminology the children are the "options". Given a rule $r \in R$, we define X to be a *parent under* r of Y, and Y to be a *child under* r of X, if one can move from X to Y using r. The *slope* $s(r)$ of a rule $r = (a, b)$ in Linear Nimhoff is denoted by $s(r) = b/a$.

3.1. ***Forbidden regions.*** In this section, we build a rigorous model for the analysis of Linear Nimhoff. We locally extend the ruleset R, to allow for any nonempty set of rules, so that for example R can be void of nim-type rules, or it may contain instead of (0, 1) a rule such as $(0, k)$, any $k > 0$. The reason for this relaxation of R is that the scatter-along-lines type of geometry also appear frequently in this bigger class.

For $\alpha, \beta \in \Re \cup \{\infty\}$, with $0 \leqslant \alpha < \beta$, then $F_{\alpha,\beta} = \{(x, y) : \alpha \leqslant y/x \leqslant \beta\} \subset \Re^2$ is a *forbidden region* if $F \cap P$ is finite, and the region is *sharp* if, for any $\epsilon > 0$, there are infinitely many P-positions (x, y) with ratios y/x in each of the intervals $(\alpha - \epsilon, \beta)$ and $(\alpha, \beta + \epsilon)$, or $\alpha = 0$, $\beta = \infty$ respectively.

Let $B \subset P(R)$. Sometimes this set is a P-beam, or perhaps even a P-line. Let $\alpha = \liminf y/x \leqslant \limsup y/x = \beta$, for $(x, y) \in B$. Then B is a *P-line* if $\alpha = \beta$. In general, if $\alpha \leqslant \beta$, then B is a *P-beam* if it contains no forbidden region. Thus we generalize notation and let $\Delta(R) = \#R - \#$P-beams.

Hypothesis 1. For each forbidden region F, there exists a single rule $r \in R$ by which it is possible to move from almost all positions in F to a P-position; in other words, almost all N-positions within a given forbidden region are parents of P-positions under the *same* rule r.

For example in the game of Nim, there are precisely two sharp forbidden regions, $F_{0,1}$ and $F_{1,\infty}$ respectively. Note that, given Hypothesis 1, the positions in a forbidden region may have moves to P-positions via other rules as well. This is exemplified in the middle forbidden region of Wythoff Nim $F_{\phi^{-1},\phi}$, where each position of the form (x, b_n), $|b_n - x| < n$ or (b_n, y), $|b_n - y| < n$ has not only a diagonal move, but also a Nim-type move to a P-position.[2]

Consider $r \in R$. We write $F' = F'(r)$ to denote almost all positions in a forbidden region F, satisfying Hypothesis 1; if $\forall x \in F'\ \exists y \in P : y + kr = x$, for some positive integer k, we say that the rule *r fills the forbidden region F* (with N-positions).

Lemma 2. *Consider a ruleset R and suppose that r fills the forbidden region $F_{\alpha,\beta}(r)$. If F is sharp and Hypothesis 1 holds, then $\alpha < s(r) < \beta$.*

Proof. Suppose that $s(r) = \beta + \delta$, for some $\delta > 0$. By Hypothesis 1, $\exists y \in P$: $y + kr = x$, $\forall x \in F \cap N$. Then, $\forall (u, v) : \beta < v/u < \beta + \delta$, $\exists k' \in \mathbb{Z}_{>0}$, $x \in F \cap N$: $x + k'r = (u, v)$, which gives $y + (k + k')r = (u, v)$, and so there is a move from (u, v) to the P-position y. But, if $F_{\alpha,\beta}(r)$ is sharp, there are infinitely many P-positions of the form (u, v), so $\delta \leqslant 0$. The lower bound is analogous. □

[2]The ideas presented here are even more general. We may, for example, exclude one or both Nim-type moves; in the game $R = \{(1, 1)\}$, the set of P-positions is $\{(0, x), (x, 0) : x \geqslant 0\}$. There are more simple examples of R, for which we can justify the value of the definition of forbidden regions. Take for example the ruleset $R = \{(x, 2x), (2x, x)\}$. The reader may check that the P-positions are precisely the positions of the forms $(0, x)$, $(x, 0)$ or $(2x, 2x)$, so that we have two sharp forbidden regions, again $F_{0,1}$ and $F_{1,\infty}$, but for a different reason than that of Nim. Note that, in each example mentioned in this paragraph, the various sets $F \cap P$ are empty.

See also Figure 8 (the picture to the left) in Section 5. From Lemma 2 it follows that no two sharp forbidden regions can be filled by the same game rule.

Observation 3. Let $F(r)$ and $G(r')$ be two distinct sharp forbidden regions with fill rules r and r' respectively. Then $r \neq r'$.

Lemma 4. *Given Hypothesis 1, then* $\Delta(R) \geqslant 1$.[3]

Proof. By Observation 3, there can be at most one sharp forbidden region per rule. Recall $\Delta(R) = \#R - \#$(P-beams associated with R). Each sharp forbidden region is bounded by a P-beam on each side, except for the case of nim-type moves in R, in which case there is only one P-beam on one of the sides. □

3.2. *Equivalence classes.* In Wythoff's game, the observation that there is a P-position in every row, column, and diagonal proves crucial to its analysis. A similar situation exists in Linear Nimhoff except we require more general terms than rows, columns, and diagonals. For this purpose, we will use equivalence classes to define sets with similar properties.

Consider some rule r. We define an equivalence relation $\sim_r$ as follows: for positions X and Y, $X \sim_r Y$ if either Y is a child under r of X, Y is a parent under r of X, or Y equals X. The equivalence classes under $\sim_r$ are sets of colinear points lying along lines with the same slope as r. Let C_r be the set of all equivalence classes under $\sim_r$. For example, $C_{(1,0)}$ is the set of all rows in the game's two-dimensional position space, $C_{(0,1)}$ is the set of all columns, and $C_{(1,1)}$ is the set of all diagonals with slope 1.

In Wythoff's game, the statement that there is exactly one P-position in every diagonal is equivalent to the statement that every set in $C_{(1,1)}$ contains exactly one P-position. We generalize this idea, which suffices to prove a strengthening of Lemma 2.

Theorem 5. *Let $r \in R$ and suppose $F = F_{\alpha,\beta}$ is a forbidden region such that $\alpha < s(r) < \beta$. Given Hypothesis 1, the rule r fills F' if and only if almost all sets in C_r contain exactly one P-position.*

Proof. Suppose that the rule r fills F'. Then each position $X \in F' \cap N$ has a P-position as a child under r. Since $r \in R$, by definition of a P-position, no equivalence class in C_r contains more than one P-position. Since $F \cap P$ is finite, the forward implication follows.

Conversely, assume there exists exactly one P-position in almost all sets in C_r. Then almost all positions are equivalent under $\sim_r$ to a P-position. All but finitely many positions in F are N-positions. Let f be an arbitrary N-position in F. By Lemma 2, all of its parents under r are also in F, so all of them are also

[3] If R does not contain both nim-type move vectors, then $\Delta(R) \geqslant 0$ and $\Delta(R) \geqslant -1$ respectively one or zero nim-type moves.

N-positions. Thus, except for finitely many positions in F, f has a child under r which is a P-position. This is true for all f in F', so the rule r fills F'. □

As a consequence of Lemma 2 we noted that no single game rule r can fill two distinct forbidden regions. We strengthen this in the following assumption:

Hypothesis 6. Let R be a ruleset for which $|R| = n + 1$. Then there are n P-beams and $n + 1$ forbidden regions associated with R. That is, $\Delta(R) = 1$.

3.3. ***The slopes and densities of P-lines in the strict class.*** For the strict class of Linear Nimhoff, we impose the following restriction to the family of rulesets:

Hypothesis 7. Let $B \subset P(R)$. If B is a P-beam, then B is a P-line.

The key to understanding the properties of the P-lines in the strict class of Linear Nimhoff lies within an analysis of the forbidden regions. Associated with each forbidden region are various constraints, which collectively can be solved to yield quantitative predictions for the slopes and densities of the game's P-lines. Here we describe how these constraints are constructed. Consider a game of Linear Nimhoff with ruleset R whose P-positions lie within n P-lines. Label these n P-lines $l_1, l_2, \ldots, l_n$ in order of increasing slope, where

$$m_i = \lim_{(x,y)\in l_i} \frac{y}{x}$$

denotes the *slope* of line l_i. Thus, designating the set of all n P-lines as $L = \{l_1, l_2, \cdots, l_n\}$, we have $\forall\, l_i, l_j \in L,\ i < j \Rightarrow m_i < m_j$. These n P-lines divide the plane into $n+1$ forbidden regions, which are filled under $n+1$ distinct rules. Let $R = \{r_1 = (1, 0), r_2 = (a_2, b_2), \cdots, r_{n+1} = (a_{n+1}, b_{n+1}), r_{n+1} = (0, 1)\}$ be a set of $n+1$ fill rules, labeled by increasing slope, i.e. $\forall\, r_i, r_j \in R,\ i < j \Rightarrow s(r_i) < s(r_j)$. Recall that R always contains rules $r_1 = (1, 0)$ and $r_{n+1} = (0, 1)$, since these two rules are responsible for filling the forbidden regions bordering the x and y axes, respectively.

Hypothesis 8. For any P-line $\ell_i = \{(x_n, y_n)\}$, the projected density along the x-axis $\lambda_i = \lim n/x_n$ exists, for (x_n) increasing.

Denote by c_r^+ and c_r^- those classes in C_r, for which membership of $(0, y)$ implies $y > 0$ and $y < 0$ respectively (y rational).

Lemma 9. *If ℓ is a P-line with slope greater than $s(r)$, then $c_r^- \cap \ell = \varnothing$. If ℓ is a P-line with slope smaller than $s(r)$, then $c_r^+ \cap \ell = \varnothing$.*

Proof. The proof is a geometric argument displayed in Figure 4. □

Therefore, P-lines with slopes greater than $s(r)$ only contribute P-positions to sets in C_r that lie along lines with positive y-intercepts, while P-lines with

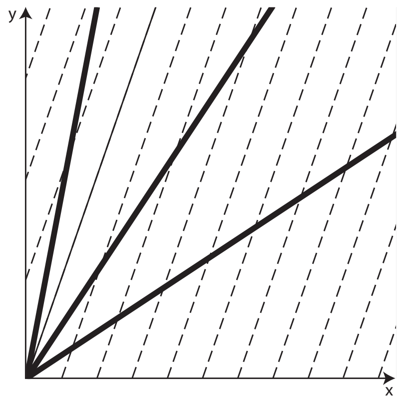

Figure 4. In this depiction, dashed lines represent the equivalence classes comprising C_r; the heavy solid lines are P-lines; the thin solid line is the line of slope $s(r)$ passing through the origin.

slopes less than $s(r)$ only contribute P-positions to sets in C_r that lie along lines with positive x-intercepts.

As we describe next, by dividing up C_r into two parts in this manner, each rule in R (save for rules $(1, 0)$ and $(0, 1)$) will give rise to two geometric constraints in the form of algebraic equations. Solving these equations yields analytical values for the slopes and densities of the P-lines. Numerical simulations of Linear Nimhoff under different rulesets show full agreement with these predicted values (to within numerical uncertainty). We prove the following theorem.

Theorem 10. *Suppose that a ruleset R, with $n+1$ rules, satisfies Hypotheses 1-8. Then the following system of equations holds*:

$$\sum_{j=1}^{i-1} \frac{\lambda_j}{b_i - m_j a_i} = 1 \;\; \forall\, i \in \{2, 3, \ldots, n+1\}, \tag{1}$$

$$\sum_{j=i}^{n} \frac{\lambda_j}{m_j a_i - b_i} = 1 \;\; \forall\, i \in \{1, 2, \ldots, n\}. \tag{2}$$

Proof. To find the first half of the constraints, let $r_i = (a_i, b_i)$ be an arbitrary element in R other than $r_1 = (1, 0)$. Combining all the P-positions within P-lines with slope less than $s(r_i)$, by Theorem 5, we get exactly one P-position in (almost) every set in C_{r_i} that lies along a line with positive x-intercept.

The P-line l_j with slope $m_j < s(r_i)$ and density (per unit x) λ_j will on average contribute P-positions to a fraction

$$\frac{\lambda_j}{b_i - m_j a_i}$$

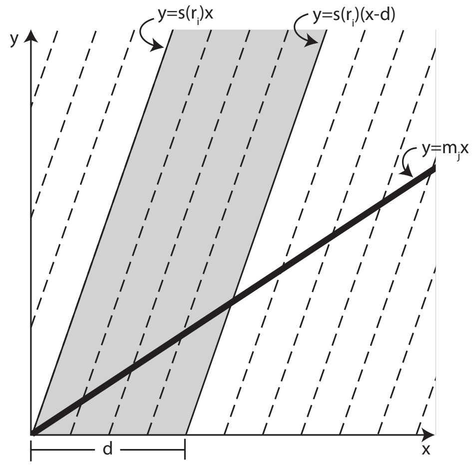

Figure 5. Fractional contribution of P-positions by a P-line.

of the sets in C_{r_i} that lie along lines with positive x-intercepts. Namely, the heavy solid line in Figure 5 represents a P-line with slope m_j and density (per unit x) λ_j; the dashed lines represent the equivalence classes associated with rule $r_i = (a_i, b_i)$, with slope $s(r_i) = b_i/a_i$. Two of these equivalence classes (depicted as thin solid lines in the figure) have been distinguished, and the area between them shaded. Observe that the section of the P-line intersecting the shaded region has horizontal extent $s(r_i)d/(s(r_i) - m_j)$, and hence the expected number of P-positions along this segment of the P-line is $\lambda_j s(r_i)d/(s(r_i)-m_j)$. Note also that the total number of equivalence classes which intersect this segment of the P-line is db_i, since the horizontal spacing between adjacent equivalence classes is $1/b_i$. Thus, the P-line contributes P-positions to a fraction $\lambda_j s(r_i)/b_i/(s(r_i) - m_j)$ of equivalence classes. Substituting $s(r_i) = b_i/a_i$ yields the desired ratio $\lambda_j/(b_i - m_j a_i)$.

Because there is exactly one P-position in almost every set in C_{r_i}, the sum of these fractions over all P-lines with slope less than $s(r_i)$ must equal 1. As a result of their ordered labeling, the P-lines with slope less than $s(r_i)$ are exactly l_1 through l_{i-1}. This yields the equation

$$\sum_{j=1}^{i-1} \frac{\lambda_j}{b_i - m_j a_i} = 1.$$

This equality holds for all r_i in R except for $r_1 = (1, 0)$. The rule $(1, 0)$ is excluded because there are no P-lines with slopes less than $s(1, 0)$, and there are no sets in $C_{(1,0)}$ that lie along lines with positive x-intercepts.

Similarly, to find the second half of the constraints, let $r_i = (a_i, b_i)$ be some arbitrary element in R' other than $r_{n+1} = (0, 1)$. Combining all the P-lines with slope greater than $s(r_i)$, we get exactly one P-position in almost every set in C_{r_i} that lies along a line with positive y-intercept. Geometry shows that the P-line l_j with slope $m_j > s(r_i)$ and density (per unit x) λ_j will contribute P-positions to a fraction

$$\frac{\lambda_j}{m_j a_i - b_i}$$

of the sets in C_{r_i} that lie along lines with positive y-intercepts. Because there is exactly one P-position in every set in C_{r_i}, the sum of these fractions over all P-lines with slope greater than $s(r_i)$ must equal 1. As a result of their ordered labeling, the P-lines with slope greater than $s(r_i)$ are exactly l_i through l_n. This yields the equation

$$\sum_{j=i}^{n} \frac{\lambda_j}{m_j a_i - b_i} = 1.$$

This equality holds for all r_i in R except for $r_{n+1} = (0, 1)$. The rule $(0, 1)$ is excluded because there are no P-lines with slopes greater than $s(0, 1)$, and there are no sets in $C_{(0,1)}$ that lie along lines with positive y-intercepts. □

4. Finding P-lines when $\Delta > 1$: a reorganization model

We wish to refine the renormalization approach in the previous sections, and to this purpose we introduce a more dynamic *reorganization* model, which allows us to temporarily relax Hypothesis 6. It describes a reorganization of P-positions within given P-lines if and only if we "adjoin" a nonfill rule (defined in the next paragraph) to a given ruleset. This model is consistent with computer simulations, as well as with results and conjectures in previous work [Larsson 2012a] (the previous results concern a sometimes trivial reorganization where locations of P-positions stay fixed).

In the preceding section, it was shown how to compute the slopes and densities of the P-lines in the strict class of Linear Nimhoff assuming the game's *fill rules* are known, that is whenever $\Delta = 1$, and in this case we make an assumption by saying $\text{fill}(R) = R$. Since this work does not concern the precise location of P-positions, but rather the precise asymptotics of density and slopes of lines, we ignore local influence of any subset of rules, as long as the overall geometry is preserved, and so in the computation of densities and slopes, the rules that are not fill rules must be ignored (to assure $\Delta = 1$). Here we suggest a recursive algorithm for determining which rules in a ruleset R are fill rules.

The simplest case, say R^0, has two rules, namely $(1, 0)$ and $(0, 1)$. Here, the game is Nim, and the P-positions lie along a single P-line of slope and density both equal to 1. Hence in this case, $\text{fill}(R^0) = R^0$. Now, consider $R^0 \cup \{(2, 1)\}$.

It is easy to prove (by induction) that the P-positions are the same as Nim, so the adjoined rule (2, 1) is redundant to the geometry of the game; it is clearly not a fill rule, because $\mathrm{fill}(R^0 \cup \{(2, 1)\}) = R^0$. If we on the other hand adjoin the diagonal Wythoff type move, we know that $\mathrm{fill}(R^0 \cup \{(1, 1)\}) = R^0 \cup \{(1, 1)\}$. Next consider the game $R = R_0 \cup \{(2, 1), (1, 1)\}$. Here we have again $\Delta = 1$ (see Section 5.1), so no rule is redundant to the geometry of the game. If we want to compute the number of P-lines in R recursively, the order of adjoining moves is clearly important; we must begin by adjoining the rule (1, 1) to R_0. If we start with (2, 1), the procedure would instead point towards Wythoff's game, which is wrong. Yet, another example is the game $R = R_0 \cup \{(2, 2), (1, 1)\}$. This game is clearly Wythoff Nim, but if we were to at first adjoin (2, 2) to the rules of Nim, then this move should be included, since it splits Nim's P-positions. However the game $R_0 \cup \{(2, 2)\}$ is not Wythoff Nim, and $\Delta(R) = 2$, and the rule (2, 2) is irrelevant (since it is included in (1, 1)). We therefore suggest the rule (a, b) be tested for inclusion before the rule (a', b') if $a + b \leqslant a' + b'$ (in case of multiples we may instead assume they have been removed before starting the algorithm). We next describe the iteration step.

Let $r = (a, b)$ be an element of R other than $(1, 0)$ and $(0, 1)$, and let $R \setminus \{r\}$ be a set containing $n - 1$ rules (a_i, b_i) for which $a + b \geqslant c + d$ for all $(c, d) \in R$. Assume we know the slopes and densities of the P-lines for the game with ruleset $R \setminus \{r\}$. Let L be the set of P-lines for the game $R \setminus \{r\}$, labeled l_1 to l_n in order of increasing slope, and let i be the number of P-lines with slope less than $s(r)$. Consider the inequalities

$$\sum_{j=1}^{i} \frac{\lambda_j}{b - m_j a} \leq 1, \qquad \sum_{j=i+1}^{n} \frac{\lambda_j}{m_j a - b} \leq 1. \tag{3}$$

The first (second) inequality holds if there is, on average, at most one P-position per set in C_r lying along a line with positive x-intercept (y-intercept). If one of the P-lines has slope equal to $s(r)$, the second inequality will contain a division by zero, and should be treated as not holding.

If both inequalities are satisfied, in the strict case, when r is adjoined to the ruleset $R \setminus \{r\}$, the P-positions can reposition themselves within the same P-lines such that each set in C_r contains at most one P-position. Therefore, in this model, the slopes and densities of the P-lines for the game with ruleset R will be the *same* as those for the game with the simpler ruleset $R \setminus \{r\}$ (which, by this recursive argument, was presumed to have been previously studied).

In contrast, if one or both of the inequalities is not satisfied, then if r is added to the ruleset, the overall geometry of the P-lines must change in order for there to be at most one P-position in each set in C_r.

By this reorganization model, the elements of R can therefore be assumed in bijective correspondence to the forbidden regions, which allows us to calculate the slopes and densities of the P-lines using Theorem 10.

4.1. ***The general class*** $\boldsymbol{R = \{(1, 0), r, (0, 1)\}}$**.** As an illustration, we now use the above methods to solve the general class of Wythoff-like games with rulesets of the form $R = \{(1, 0), (a, b), (0, 1)\}$. The first step is to determine which rules have an effect on the geometry. Here, we need only concern ourselves with rule $r = (a, b)$, since $(1, 0)$ and $(0, 1)$ are always fill rules. To test whether r affects the geometry, we must consider the simpler game with ruleset $R \setminus \{r\}$. For this game, the ruleset is $\{(1, 0), (0, 1)\}$, which is equivalent to Nim and is known to have a single P-line with slope and density (per unit x) both equal to 1. Next, we must count the number, q, of P-lines with slope less than $s(r)$. If $a < b$, then $s(r) = b/a > 1$, which implies that the one P-line has slope less than $s(r)$ and thus $q = 1$. Otherwise, $s(r) = b/a \leq 1$, which implies that the one P-line has slope greater than or equal to $s(r)$ and $q = 0$. In the first case where $a < b$, the two inequalities (3) from Section 4 are

$$\sum_{j=1}^{1} \frac{\lambda_j}{b - m_j a} = \frac{1}{b - a} \leqslant 1, \tag{4}$$

and

$$\sum_{j=2}^{1} \frac{\lambda_j}{m_j a - b} = 0 \leqslant 1. \tag{5}$$

Because $a < b$ and both a and b are integers, $b - a \geq 1$, which implies that $1/(b - a) \leq 1$ and hence both inequalities are satisfied. Therefore, $r = (a, b)$ will have no effect on the overall geometry, and the game with ruleset $R = \{(1, 0), (a, b), (0, 1)\}$ with $a < b$ will still have a single P-line with slope and density (per unit x) both equal to 1, just as in two-pile Nim. Similarly, if $b < a$, then $q = 0$ and the two inequalities are

$$\sum_{j=1}^{0} \frac{\lambda_j}{b - m_j a} = 0 \leq 1, \tag{6}$$

and

$$\sum_{j=1}^{1} \frac{\lambda_j}{m_j a - b} = \frac{1}{a - b} \leq 1. \tag{7}$$

Since $b < a$ and both a and b are integers, $a - b \geq 1$, which implies that $1/(a - b) \leq 1$, and thus both inequalities are satisfied. Therefore, we conclude that $r = (a, b)$ will have no effect on the overall geometry, and the game with ruleset $R = \{(0, 1), (a, b), (1, 0)\}$ with $b < a$ will still have a single P-line with slope and density (per unit x) both equal to 1.

All that remains is the case in which $a = b$. In this case, $s(r) = 1$, which is the same as the slope of the single P-line in the game with ruleset $\{(0, 1), (1, 0)\}$. As a result, the second inequality contains a division by zero and is treated as not holding. Therefore, the rule (a, a) *will* have an effect on the overall geometry of the P-lines and is in R'. This means that $R' = \{(0, 1), (a, a), (1, 0)\}$. Applying the methods of Section 3.3 — see (1) and (2) — we find that this game will have two P-lines satisfying

$$\frac{\lambda_1}{a - m_1 a} = 1, \quad \lambda_1 + \lambda_2 = 1, \quad \frac{\lambda_2}{m_2 a - a} = 1, \quad \frac{\lambda_1}{m_1} + \frac{\lambda_2}{m_2} = 1.$$

Solving this system of four equations yields predictions for the slopes and densities of the P-lines:

$$\begin{aligned} m_1 &= \frac{-1 + \sqrt{1 + 4a^2}}{2a}, & \lambda_1 &= \frac{1 + 2a - \sqrt{1 + 4a^2}}{2}, \\ m_2 &= \frac{1 + \sqrt{1 + 4a^2}}{2a}, & \lambda_2 &= \frac{1 - 2a + \sqrt{1 + 4a^2}}{2}. \end{aligned}$$

For the special case of Wythoff's Game where $r = (1, 1)$, these predictions yield the standard result. The prediction for the case $r = (a, a)$ for other values of a is consistent with [Connell 1959].

5. The class (p, q)-GDWN

Through several experiments, the QLPF class has been observed in Linear Nimhoff; early fluctuations appear to stabilize to a quasi-log-periodic behavior within each P-beam. So far, almost every such observation is contained in a proper subclass of Generalized Diagonal Wythoff Nim (GDWN) [Larsson 2012a; Larsson 2014]. The class GDWN simplifies the equations, because the rules are symmetric $(p, q) \in R$ if and only if $(q, p) \in R$, and so the P-positions are also symmetric, (x, y) is a P-position if and only if (y, x) is also. The games with only one additional symmetric rule,

$$R = \{(1, 0), (q, p), (1, 1), (p, q), (0, 1)\},$$

where $p < q$ are positive integers, have attained most attention so far, and this general class is also dubbed (p, q)-GDWN. This is where we mostly observed QLPF games; more specifically they appear when $(p, q) \notin \{(1, 2), (2, 3)\}$ is either a Wythoff pair or a dual Wythoff pair. The first few such pairs are displayed in Tables 1 and 2, respectively.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n	1	3	4	6	8	9	11	12	14	16	17	19	21	22
B_n	2	5	7	10	13	15	18	20	23	26	28	31	34	36

Table 1. The first few Wythoff pairs (A_n, B_n).

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n+1	2	4	5	7	9	10	12	13	15	17	18	20	22	23
B_n+1	3	6	8	11	14	16	19	21	24	27	29	32	35	37

Table 2. The first few dual Wythoff pairs (A_n+1, B_n+1).

The Wythoff pairs are of the form $(A_n, B_n) = (\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor)$ whereas the dual Wythoff pairs are of the form $(A_n+1, B_n+1) = (\lfloor \phi n + 1 \rfloor, \lfloor \phi^2 n + 1 \rfloor)$, for some $n > 0$. Here, we collect these pairs as the *WdW-pairs*, and we denote the set $\Omega = \{(p, q) : (p, q) \text{ is a WdW-pair}\}$. Note that, viewed as an increasing sequence of pairs of integers, Ω is the total order

$$\Omega = \{(1, 2), (2, 3), (3, 5), (4, 6), (4, 7), (5, 8), (6, 10), (7, 11), \dots\},$$

with alternating entries from the two sequences.

The first case where "P-lines" gets distorted, creating fluctuated P-beams, is the game (3, 5)-GDWN; (3, 5) is the third WdW-pair, and it appears that fluctuations are not possible for the pairs (1, 2) and (2, 3), because the outer P-beams are too stable — they are generated greedily [Larsson 2012a; Larsson 2014]. The QLPF behavior is discussed in Figures 6, 7, 8 and 9 for the game of (3, 5)-GDWN. Previous work focused on "the split", the existence of a forbidden region between the beams (rather than the mean slopes of the P-beams) and obtained two decimal conjectures for the bounds of the forbidden region, namely ≈ 1.74 and ≈ 1.57, as can also be extrapolated from Figure 3. Our new computations give rather the mean of the slopes (of course we obtain a much higher precision in the new estimates), and they confirm the previous observations: ≈ 1.760145300 and ≈ 1.537962520; see also Figure 9.

In general for (p, q)-GDWN, it is convenient to note that $m_1 = 1/m_4$, $m_2 = 1/m_3$, $m_1 = \lambda_1/\lambda_4$ and $m_2 = \lambda_2/\lambda_3$. Thus, to compute the conjectured mean slopes of P-beams, it suffices to solve the system of densities

$$1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4,$$

$$\lambda_1 = p - \frac{q\lambda_1}{\lambda_4}, \quad 1 = \frac{\lambda_1\lambda_4}{\lambda_4 - \lambda_1} + \frac{\lambda_2\lambda_3}{\lambda_3 - \lambda_2}, \quad 1 = \frac{\lambda_2\lambda_3}{q\lambda_2 - p\lambda_3} + \frac{\lambda_2\lambda_3}{q\lambda_3 - p\lambda_2} + \frac{\lambda_1\lambda_4}{q\lambda_4 - p\lambda_1}.$$

Figure 6. Left: P-positions of (3, 5)-GDWN for $x \leqslant 32600$, $y \leqslant 32600$. Right: P-positions of (3, 5)-GDWN for $x \leqslant 47800$, $y \leqslant 47800$; by an experimental ≈ 1.478 scaling, one may conjecture a geometric "log-invariance" of P-positions.

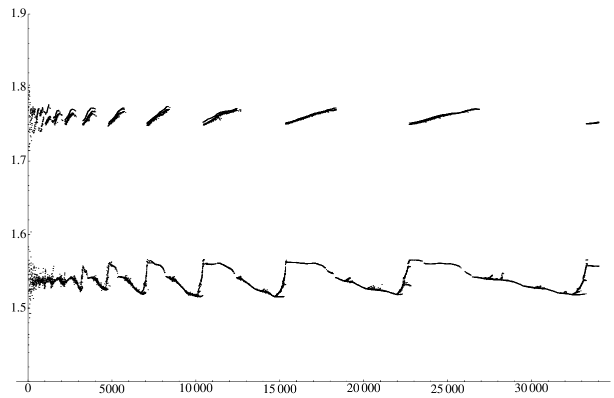

Figure 7. The game (3, 5)-GDWN: the ratios $y/x > 1$ whenever (x, y) is a P-position, for $x \leqslant 35000$.

A reasonable conjecture is that, if a quasi log-periodic behavior has not started to appear within a few thousand multiples of the move rule, then the P-beam will be a P-line with bounded, perhaps $o(\log x)$, scatter; see figures in [Larsson 2012a] for the games (1,2)-GDWN and (2,3)-GDWN.[4] [5]

For several cases, the distribution along the x-axis appears to be nonuniform, when computations are extended beyond a few hundreds; see below figures

[4]By rigorous methods a split of P-beams is demonstrated for the two cases $(p, q) = (1, 2)$ and (2, 3) [Larsson 2014], but the method did not suffice to obtain any precision to the extent of that split. This work is the first significant progress on that subject.

[5]For Maharaja Nim [Larsson et al. 2013], the type of scatter has a structure which is possible to express via a *Dictionary* on algebraic words. The scatter in Linear Nimhoff appears more random, and indeed the methods in Maharaja Nim depend on the big restriction of (1, 2)-GDWN where only Knight type moves are adjoined to Wythoff Nim.

Figure 8. The fill-rule properties of the rules (0, 1), left, (1, 1), middle, and (3, 5), right, respectively visualized for the game (3, 5)-GDWN, parents of the P-positions under the respective fill rule are displayed in black for $x \leqslant 10000$, $y \leqslant 10000$.

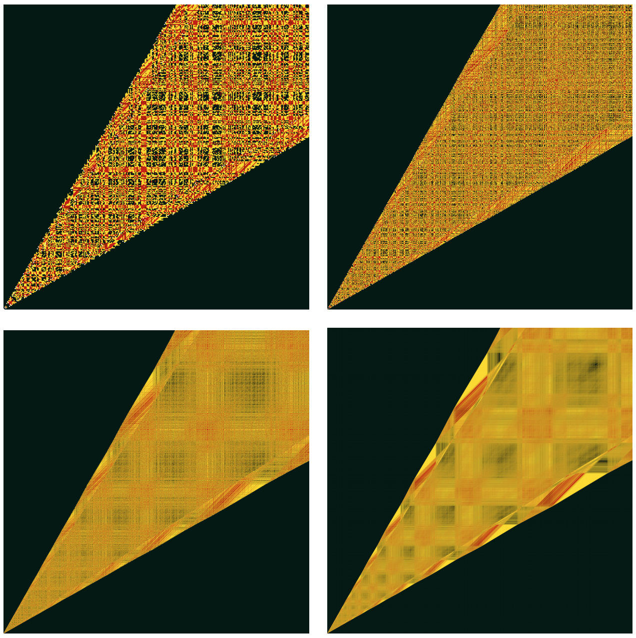

Figure 9. Initial fluctuations appear to stabilize to a quasi-log-periodic geometry in the game (3, 5)-GDWN. The first picture consists of a few hundred pixels (on each axis), and the last picture displays $\sim 20000 \times 20000$ pixels. The coloring scheme in the picture is that white pixels are P-positions, and the remaining colors are N-positions. Except for possibly the first picture, only the patterns of N-positions are visible (unless zooming in). Black means here that the pixel sees only one P-position (viewing along the rules of the game), yellow means the pixels detects two P-positions, and for red it detects three (or more) P-positions.

concerning the game (3, 5)-GDWN, which is the “first” game observed — the only (p, q)-GDWN games in the strict class, with $(p, q) \in \Omega$, appear for $(p, q) = (1, 2)$ and $(p, q) = (2, 3)$, where P-positions distribute uniformly along the P-lines. Nevertheless, by adapting the mean values to the slopes and densities, again the experimental data agree with the values computed in [Larsson 2012a], up to three-digit precision.

The properties of forbidden regions and fill rules appear to still govern the overall behavior, but the assumption of a uniform distribution along the x-axis does not hold any longer. A reasonable guess is that there is uniform behavior from the point of view of each filling rule r (rather than along the x-axis), which then stabilizes the patterns to a sufficient degree (but we do not develop this idea further here).

The slopes of the upper P-lines of (1,2)-GDWN satisfy the pair of fourth degree equations

$$(z-1)\left(\frac{1}{2z-1}-\frac{1}{z-2}\right)=\frac{w^2-1}{2w-1}, \quad \frac{z^2-1}{z}=(w-1)\frac{-w^2+2w+2}{w},$$

which show that the hypothesis of this work is consistent with previous work, and moreover they provide a huge improvement in numerical precision of the previously conjectured upper slopes; previously four digits, and now $w \approx 2.24772558355773557$ and $z \approx 1.47779977527220012$ respectively (and the densities of the four P-lines, in order of increasing slope, are ≈ 0.11021167, 0.25912616, 0.382936586 and 0.24772558).

5.1. ***A linear Nimhoff relaxation of the Wythoff/dual Wythoff pair conjecture.*** In this section, we show that the inequalities (3) from Section 4 are consistent with an asymetric relaxation of the conjecture that (p, q)-GDWN splits if and only if $(p, q) \in \Omega$ [Larsson 2012a].

Theorem 11. *Consider the ruleset R = Wythoff Nim and a pair of positive integers (p, q). The proposition $(p, q) \in \Omega$ if and only if* $\mathrm{fill}(R \cup (p, q)) = R \cup (p, q)$ *is consistent with the reorganization model using the inequalities* (3).

Proof. By adjoining the vector (p, q) to Wythoff Nim, we get two cases for the inequalities:

$$\sum_{j=1}^{1}\frac{\lambda_j}{q-m_j p}\le 1, \quad \sum_{j=2}^{2}\frac{\lambda_j}{m_j p-q}\le 1, \tag{8}$$

or

$$\sum_{j=1}^{2}\frac{\lambda_j}{q-m_j p}\le 1, \quad \sum_{j=3}^{2}\frac{\lambda_j}{m_j p-q}\le 1, \tag{9}$$

because $m_1 = \phi^{-1}$ and $m_2 = \phi$. Also $\lambda_1 = \phi^{-2}$ and $\lambda_2 = \phi^{-1}$, and where the cases are $1 < q/p < \phi$ and $\phi < q/p$ respectively.

Case 1. The inequalities are $\dfrac{\phi^{-2}}{q-\phi^{-1}p} \leqslant 1$ and $\dfrac{\phi^{-1}}{\phi p-q} \leqslant 1$, by (8). Hence

$$\phi^{-1} \leqslant \phi q - p \quad \text{and} \quad \phi^{-1} \leqslant \phi p - q \tag{10}$$

if and only if there is no new split of P-lines when adjoining the move vector (p, q), with $1 < q/p < \phi$, to Wythoff Nim. The first inequality in (10) is trivially true, so it suffices to verify the second. The dual Wythoff pairs are of the form $(p, q) = (2, 3), (4, 6), (5, 8), \ldots$ and the ratios are smaller than ϕ. In this case, the conjecture is that there is a split (an inequality does not hold). But, for example $(p, q) = (4, 5)$ is not a dual Wythoff pair, so it should not split (inequalities hold). Obviously the first inequality holds since $q > p$. For the second inequality we compute

$$\phi^{-1} \leqslant \phi\lfloor\phi n + 1\rfloor - \lfloor\phi^2 n + 1\rfloor$$

if and only if

$$\begin{aligned}
\phi &\leqslant \phi\lfloor\phi n + 1\rfloor - \lfloor\phi^2 n\rfloor,\\
0 &\leqslant \phi\lfloor\phi n\rfloor - \lfloor\phi n\rfloor - n,\\
0 &\leqslant \phi^{-1}\lfloor\phi n\rfloor - n,\\
\phi n &\leqslant \lfloor\phi n\rfloor,
\end{aligned}$$

which is false, because ϕ is irrational.

For the other direction, suppose that $q/p < \phi$, but (p, q) is not a dual Wythoff pair. Then, by complementarity, either

$$1 < p = \lfloor\phi n + 1\rfloor$$

with $2 < q < \lfloor\phi^2 n + 1\rfloor$, or

$$1 < p = \lfloor\phi^2 n + 1\rfloor$$

with $2 < q \leqslant \lfloor\phi^2 n + 1\rfloor + \lfloor\phi n\rfloor$.

In either case, we must prove that the second inequality in (10) holds. In the first case, it suffices to justify

$$\begin{aligned}
\phi^{-1} &\leqslant \phi\lfloor\phi n + 1\rfloor - \lfloor\phi^2 n\rfloor,\\
-1 &\leqslant \phi\lfloor\phi n\rfloor - \lfloor\phi n\rfloor - n,\\
n - 1 &\leqslant \phi\lfloor\phi n\rfloor - \lfloor\phi n\rfloor,\\
n - 1 &\leqslant \phi^{-1}\lfloor\phi n\rfloor,\\
\phi(n-1) &\leqslant \lfloor\phi n\rfloor.
\end{aligned}$$

In the second case, we justify

$$\begin{aligned}
\phi^{-1} &\leqslant \phi\lfloor\phi^2 n+1\rfloor - \lfloor\phi^2 n+1\rfloor - \lfloor\phi n\rfloor,\\
\phi^{-1} &\leqslant \phi^{-1}\lfloor\phi^2 n+1\rfloor - \lfloor\phi n\rfloor,\\
0 &\leqslant \lfloor\phi^2 n\rfloor - \phi\lfloor\phi n\rfloor,\\
0 &\leqslant n - \phi^{-1}\lfloor\phi n\rfloor,\\
\phi n &\geqslant \lfloor\phi n\rfloor.
\end{aligned}$$

Case 2. The inequalities are

$$\frac{\phi^{-2}}{q-\phi^{-1}p} + \frac{\phi^{-1}}{q-\phi p} \leqslant 1, \tag{11}$$

by the first inequality in (9) and $0 \leqslant 1$ by the second. We wish to prove that (p,q) is a Wythoff pair if and only if there is no new split of P-lines when adjoining the move vector (p,q) with $\phi < q/p$ to Wythoff Nim.

The inequality (11) is equivalent with

$$1 \leqslant q - p - \frac{qp}{q-p}\phi^{-3}. \tag{12}$$

By letting $(p,q) = (\phi n, \phi^2 n)$ be a Wythoff pair, we get

$$1 \leqslant n - \frac{\lfloor\phi^2 n\rfloor\lfloor\phi n\rfloor}{n}\phi^{-3},$$

which simplifies to

$$(n^2-n)\phi^3 \geqslant \lfloor\phi^2 n\rfloor\lfloor\phi n\rfloor \geqslant (\phi n-1)(\phi^2 n-1) = \phi^3 n^2 - \phi^2 n - \phi n + 1.$$

This holds if and only if $0 \geqslant 1$. Hence, the hypothesis of a new P-line is consistent with the conjecture for (p,q)-GDWN in this case. Next, we must check that if (p,q) with $q/p > \phi$ is not a Wythoff pair, then inequality (12) holds. In case $p = \lfloor\phi n\rfloor$, then the problem reduces to justifying the inequality

$$(m^2-m)\phi^3 \geqslant \lfloor\phi n + m\rfloor\lfloor\phi n\rfloor$$

for any $m > n$. We get

$$\begin{aligned}
(m^2-m)\phi^3 &\geqslant (\phi n+m)\phi n,\\
m^2\phi^3 &\geqslant \phi^2 n^2 + \phi m n + m\phi^3,\\
m^2\phi^2 + m^2\phi &\geqslant \phi^2(m^2-(m+n)(m-n)) + \phi(m^2-(m-n)m) + m\phi^3,\\
m(\phi^2+\phi) &\leqslant \phi^2(m+n)(m-n) + \phi(m-n)m,
\end{aligned}$$

which holds because $m > n$. Hence, no new P-line is introduced if $q/p > \phi$ and (p,q) is not a Wythoff pair. □

The games in GDWN have symmetric rulesets, and so, to justify that conjectures in previous work is consistent with the reorganization model, it suffices to prove that introducing the move vector (q, p), to the Linear Nimhoff game $R = \{(1, 0), (1, 1), (p, q), (0, 1)\}$, $(p, q) \in Q$, introduces a fourth P-line (in the case $(p, q) \leqslant (2, 3)$) or otherwise P-beam. Because computational experiments have shown that P-beams are not always P-lines, a general proof could be a bit more demanding. Perhaps the model will even be refuted in some cases of fluctuating P-beams, although experimental results point towards that the model holds also in this interesting QLPF-case.

In the other cases, it is known from [Larsson 2012a] that if $(p, q) \notin Q$, then $\mathrm{fill}(R) = R \setminus \{(p, q), (q, p)\}$. This means that the slopes and densities of any such game are identical to Wythoff's game. The individual locations of those P-positions can be completely different from those of Wythoff Nim, if $q/p > \phi$, as is illustrated in [loc. cit.] for (2,4)-GDWN and (7,12)-GDWN. If $q/p < \phi$, then in fact the P-positions are identical [loc. cit.] (which holds also for Linear Nimhoff games of the form $\{(1, 0), (1, 1), (p, q), (0, 1)\}$). The "scatter along a P-line" can vary hugely; for example for (7, 12)-GDWN, around x-coordinate 40000, only the first digit in the ratio y/x for an "upper" P-position ≈ 1.6 has been experimentally confirmed.

Using terminology in this study, Theorem 11 can be restated as follows.

Corollary 12. *If $R = \{(1, 0), (1, 1), (p, q), (0, 1)\}$ is in the strict class, then there are three P-lines for the rule set R if and only if $(p, q) \in Q$.*

In this case, we believe that Linear Nimhoff is in the strict class, but in going from one adjoined move of Wythoff Nim to two adjoined moves, we restate a conjecture from [Larsson 2012a]. Given the three P-lines from Corollary 12, we can justify numerically for the first few games that our reorganization model corresponds to the conjectures, even though several of these games are believed to be QLPF-games, with an even more interesting behavior.

Conjecture 13 [Larsson 2012a]. *Consider $R = \{(1, 0), (q, p), (1, 1), (p, q), (0, 1)\}$. There are four P-beams (P-lines in case of $(p, q) = (1, 2)$ or $(2, 3)$) for the move set R if and only if $(p, q) \in Q$. Otherwise there are two P-lines of slopes and densities as for Wythoff's game. In case of P-beams, this class of games belongs to the QLPF class of relaxed Linear Nimhoff.*

By Corollary 12, we know that $\mathrm{fill}(R) = R = \{(1, 0), (1, 1), (p, q), (0, 1)\}$ whenever $(p, q) \in Q$, and so there are three P-lines. Experimentally it seems that $m_1 \approx \phi^{-1}$ and $\lambda_1 \approx \phi^{-2}$ as for Wythoff Nim. Thus it is a delicate matter to introduce the new rule (q, p). If the approximate values are in fact equalities, then for the Wythoff pairs, we get one set of inequalities, and for the dual Wythoff pairs, we get another set. We display first the equations for the three conjectured

P-lines for the game $\{(1,0),(1,1),(p,q),(0,1)\}$. For the P-lines with positive x-intercept, we get exactly one P-position in every set in C_{r_i} that lies along a line with positive x-intercept:

$$\sum_{j=1}^{i-1} \frac{\lambda_j}{b_i - m_j a_i} = 1 \quad \forall i \in \{2,3,4\}, \tag{13}$$

For those with positive y-intercept,

$$\sum_{j=i}^{n} \frac{\lambda_j}{m_j a_i - b_i} = 1 \quad \forall i \in \{1,2,3\}.$$

Altogether,

$$\frac{\lambda_1}{1-m_1} = 1, \qquad \frac{\lambda_1}{q-m_1 p} + \frac{\lambda_2}{q-m_2 p} = 1, \qquad \lambda_1+\lambda_2+\lambda_3 = 1,$$

and

$$\frac{\lambda_1}{m_1} + \frac{\lambda_2}{m_2} + \frac{\lambda_3}{m_3} = 1, \qquad \frac{\lambda_2}{m_2-1} + \frac{\lambda_3}{m_3-1} = 1, \qquad \frac{\lambda_3}{m_3 p - q} = 1.$$

Computer explorations give that the slope m_3 is greater (smaller) than p/q if (p,q) is a Wythoff pair (dual Wythoff pair). (Note that the slope $0 < p/q < 1$.)

Thus, in the case of a *Wythoff pair*, to justify that the new rule (q,p) is part of the rule set, it suffices to show that the second inequality does not hold, i.e., that

$$\frac{\lambda_1}{m_1 p - q} + \frac{\lambda_2}{m_2 p - q} + \frac{\lambda_3}{m_3 p - q} > 1. \tag{14}$$

(Since there is no P-line below the rule (p,q), the first inequality is trivially satisfied.)

In the case of a *dual Wythoff pair*, to justify that the new rule (q,p) is part of the rule set, and since there is exactly one P-line below the rule (p,q), we must show that one of the inequalities does not hold, i.e., that

$$\frac{\lambda_1}{q-m_1 p} > 1 \quad \text{or} \quad \frac{\lambda_2}{m_2 p - q} + \frac{\lambda_3}{m_3 p - q} > 1. \tag{15}$$

We have verified the inequalities (14) and (15) numerically, for a few initial (p,q) pairs, but we omit further details. Instead we propose the following problem.

Problem 14. Prove the analogue of Theorem 11 in this setting; in particular, the occurrence of a new P-line for the cases $(p,q) = (1,2)$ and $(2,3)$, where it has been conjectured that the games are in the strict class.

6. A scheme built on observed reflections in fluctuation

A subclass of the (p, q)-GDWN games exhibits quasi-log-periodic fluctuations, summarized in Tables 3 and 4, where "?" means that it is uncertain whether a visual inspection indicates an integer number of "half" log-periods.

Let the numbers $a^{-1} < 1$ and $a > 1$ denote the mean slopes of the outer P-beams, whereas $b^{-1} < 1$ and $b > 1$ are the mean slopes of the inner P-beams, respectively. If a is the slope of the 1st line (the mean slope of the top P-beam), the log-period of bouncing from the the fourth line along (0, 1) to the 1st line and back along (1, 0) is $\log a - \log a^{-1} = 2\log a$. The log-period of bouncing from the fourth line to the third line along (1, 1), then back to the fourth line along (1, 0) is $\log \frac{a-1}{b-1}$.

Analogously, we study

$$\xi_{\mathrm{a}}(a, b) = \frac{2\log a}{\log(a-1) - \log(b-1)}, \quad \xi_{\mathrm{b}}(a, b) = \frac{2\log b}{\log(a-1) - \log(b-1)},$$

$$\xi_{\mathrm{ab}}(a, b) = \frac{\log b - \log a^{-1}}{\log(a-1) - \log(b-1)}, \quad \xi_{\mathrm{ba}}(a, b) = \frac{\log a - \log b^{-1}}{\log(a-1) - \log(b-1)},$$

in Tables 3 and 4, looking for integers whenever possible; the entry (a^{-1}, a) corresponds to the visible number of log-periods for ξ_{a}, (b^{-1}, b) to ξ_{b}, (a^{-1}, b) to ξ_{ab}, and (b^{-1}, a) to ξ_{ba}. The type of bouncing is displayed in Figure 10, where the white dotted box corresponds to the $2\log a$ bounce, and the red-black zig-zag pattern in the lower (7, 4)-forbidden sector corresponds to the $\log \frac{a-1}{b-1}$ bounce. The blue box is analogous, but here the relation of bounce is measured between the first and third P-beam. In both these cases, we visually detect four periods. Sometimes the visual inspection indicates that we should rather count the number

a^{-1}	–	a, 3 or b, 2.5	b, 4	a, 6.5	a, 7?	b, 8.5	a, 11	b, 10
b^{-1}	–	a, 2.5	a, 4	$a, 6 \approx b, 6$	?	b, 8	b, 10	a, 10
p	1	3	4	6	8	9	11	12
q	2	5	7	10	13	15	18	20

Table 3. Geometric behavior for the Wythoff pairs.

a^{-1}	–	a, 3	a, 5	b, 6.5	?	a, 9	a, 11	a, 13
b^{-1}	–	b, 2.5	b, 4	b, 6	b, 6?	b, 8	b, 10	b, 12
p	2	4	5	7	9	10	12	13
q	3	6	8	11	14	16	19	21

Table 4. Geometric behavior for the dual Wythoff pairs.

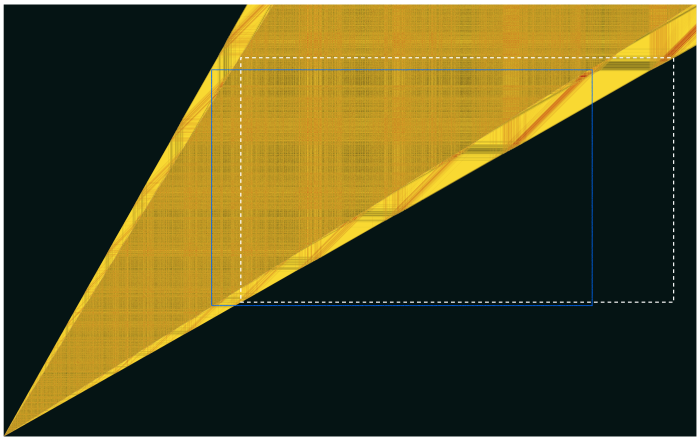

Figure 10. Visual interpretation of $\xi_{\mathrm{ba}}(a, b)$ (blue) and $\xi_{\mathrm{ab}}(a, b)$ (white dashed) for the game (4, 7)-GDWN.

of "half" log-periods (here "half" means either a bounce with the inner or outer P-beam). We leave it as an open problem to justify these integer of half integer approximations in terms of the game rules.

Note that the log-ratio for $\xi_{\mathrm{a}}(a, b)$ appears to approximate the lower sequence of the Wythoff pairs. Tables 5 and 6 seem to indicate that each one of the four log ratios could contribute to explain the "bouncing" between P-beams, and how

(1, 2)	[2.247725584, 1.477799775, 1.687531557]
(3, 5)	[1.760145300, 1.537962520, 3.270827298]
(4, 7)	[1.768998972, 1.601914235, 4.656921040]
(6, 10)	[1.697534360, 1.589805226, 6.308758458]
(8, 13)	[1.662070300, 1.582782860, 7.966064987]
(9, 15)	[1.678278787, 1.607189817, 9.353017596]
(11, 18)	[1.656365788, 1.598892313, 11.01365251]
(12, 20)	[1.668937168, 1.615891660, 12.39876038]
(14, 23)	[1.653187304, 1.608112999, 14.06105137]

Table 5. Wythoff pairs (p, q), [slope a, slope b, $\xi_{\mathrm{a}}(a, b)$] (initial 9 digits).

(2, 3)	[1.739269208, 1.408430574, 1.865590884]
(4, 6)	[1.638930839, 1.482391155, 3.515816330]
(5, 8)	[1.675293133, 1.547469192, 4.917904008]
(7, 11)	[1.640358498, 1.550853934, 6.574385950]
(9, 14)	[1.621326573, 1.552514257, 8.234019700]
(10, 16)	[1.640881924, 1.578215903, 9.625732990]

Table 6. Some dual Wythoff pairs (p, q), [slope a, slope b, $\xi_{\mathrm{a}}(a, b)$].

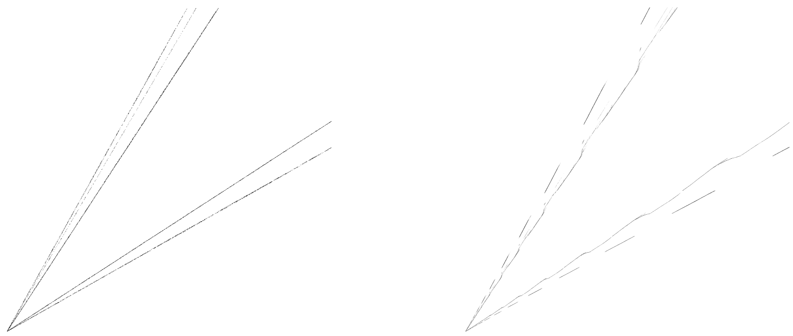

Figure 11. Initial P-positions for two one-rule-extensions of (3, 5)-GDWN. To the left, the adjoined rule is $r = (4, 7)$, and to the right the adjoined rule is $r = (5, 8)$.

the fluctuations become stable.

The log-ratio $\xi(a, b) > 2$ gives fluctuations. Is this a requirement for fluctuations to become permanent?

Tables 5 and 6 display bounces between the P-beams, which indicates mostly integer log-ratio, or otherwise half of integer log-ratio.

In Figure 11, we display a variation in the behavior of adjoining a new rule to the game of (3, 5)-GDWN. In the left-most picture, the rule (4, 7) has been adjoined. The behavior satisfies the reorganization model and a new P-line has occured. To the right, it appears that the new rule, which is (5, 8), is not quite able to change the quasi-log periodic behavior of (3, 5)-GDWN; neither is it clear whether a new P-line (or P-beam) appears (according to the reorganization model, a new P-beam should appear).

6.1. ***Questions.*** Experimental data [Larsson 2010] give uniformly distributed P-positions along P-lines also for the games

$$\{(1,2),(2,3)\}\text{-GDWN} = \{(1,0),(2,1),(3,2),(1,1),(2,3),(1,2),(0,1)\},$$

$\{(1,2),(2,3),(3,5)\}$-GDWN and $\{(1,2),(2,3),(3,5),(5,8)\}$-GDWN. For each of these games a new Fibonacci-type pair has been adjoined, and data shows that this gives birth to two new symmetric P-lines.

For the game $\{(1,2),(2,3),(3,5),(5,8),(8,13)\}$-GDWN, however, computations do not seem to indicate a new split; only the same number of P-beams as for the game $\{(1,2),(2,3),(3,5),(5,8)\}$-GDWN can be distinguished (although the P-positions are clearly different). These types of questions relate to the inequalities (3), and in this spirit one would also like to justify the hypothesis, from an arXiv preprint version of [Larsson 2012a], that the game $\{(x, y) : x, y \leqslant 5\}$ has exactly five (uniform) upper P-lines, and similar problems.

7. Discussion

We remark here that the game of Linear Nimhoff is itself a special case of some other general games that have been studied in the literature — "vector subtraction games" [Golomb 1966] a.k.a. "invariant games" [Duchêne et al. 2010; Larsson et al. 2011; Larsson 2012b], "vector addition games" [Larsson et al. 2013], and the "n-vectors game" [Duchêne et al. 2009]. The special restrictions on Linear Nimhoff give rise to unique features not apparent in these more general games. Another related class of games is "Nimhoff" [Fraenkel et al. 1991]; they focus on games between Nim and Wythoff. Thus, this class differs from ours in that they are not concerned with linear rules (our class is the most general on what we regard as linear extensions of 2-pile Nim). Another difference is that the paper [Fraenkel et al. 1991] concerns structures of Grundy-values of games (bridging the complexity class between Nim and Wythoff's game). In this paper we restrict attention to the patterns of P-positions (corresponding to Grundy-value 0).

In this work we have characterized the overall geometric structure of the P-positions in the game of Linear Nimhoff. More specifically, our analysis has produced highly accurate quantitative predictions about the number, slopes, and densities of the P-lines observed in the game, predictions which have been subsequently borne out of numerical simulations. Unlike standard game-theoretic techniques commonly used to analyze combinatorial games, the methodology employed here offers a probabilistic/geometric description, rather than an exact, deterministic specification, of the locations of the P-positions. The virtue of this approach is that it has broad explanatory powers and allows one to tackle more complex games for which standard deterministic methods have failed. It is usually believed that such methods need be nonrigorous, but here we build a rigorous model, and instead leave the questions of existence for future study.

Generalized classes of games that include Linear Nimhoff have been defined previously in the literature, but Linear Nimhoff itself has not been extensively analyzed, and its restricted structure gives rise to certain features not readily apparent in these more general games.

Linear Nimhoff can also be seen as a specific form of the more general "n-vectors game", introduced in [Duchêne et al. 2009]. The n-vectors game is defined the same way as Linear Nimhoff, with three distinctions: (i) the vectors in the n-vectors game can exist in any vector space of the form $\Re^p$, (ii) the coordinates of the vectors need not be integers, and (iii) a player can only move to a position that can be expressed as a sum of nonnegative multiples of the vectors in the ruleset. Because in Linear Nimhoff $(1, 0)$ and $(0, 1)$ are always included in the ruleset (by definition), it follows that every position in Linear Nimhoff can be trivially expressed as a sum of nonnegative integer multiples of these two rules, and hence the third distinctive feature of the n-vectors game becomes irrelevant. So Linear Nimhoff with ruleset R is equivalent to the n-vectors game played with the vectors in R which includes rules $(1, 0)$ and $(0, 1)$. We note also that through appropriate changes in bases, it is possible to recast some other n-vectors games into Linear Nimhoff form.

In other games, log-periodicity in P-position density first conjectured using heuristic analysis was later formally proven [Garrabrant et al. 2013].

Since Wythoff's game provides a model for self-organization according to Phyllotaxis [Kappraff et al. 1998], one might want to consider the apparent self-organization in these new games as forms of generalized Phyllotaxis, where the fill rule property plays an analogous significant rule as for Wythoff's game.

Acknowledgements

Garrabrant was supported in part by a W. M. Keck Foundation Summer Research Fellowship; Phipps-Morgan was supported in part through an NSF Division of Undergraduate Education STEP 1 grant. Larsson was supported in part by the Killam Trust.

References

[Bouton 1901] C. L. Bouton, Nim, a game with a complete mathematical theory. *Ann. of Math.* **3** (1901–1902), 35–39.

[Berlekamp et al. 1982] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy, *Winning ways for your mathematical plays*. Academic Press, Inc., 1982.

[Connell 1959] Ian G. Connell, A generalization of Wythoff's game. *Canad. Math. Bull.* **2** (1959), 181–190.

[Cook et al. 2015] M. Cook, U. Larsson, and T. Neary, A cellular automaton for blocking queen games, *Cellular Automata and Discrete Complex Systems*, 21st IFIP WG 1.5 International Workshop, Automata 2015, Turku, Finland, June 8–10, Proceedings, J. Kari (ed.), LNCS 9099 (2015), 71–84.

[Duchêne et al. 2009] E. Duchêne and S. Gravier, Geometrical extensions of Wythoff's game. *Discrete Math.* **309**:11 (2009), 3595–3608.

[Duchêne et al. 2010] E. Duchêne and M. Rigo, Invariant games. *Theoret. Comput. Sci.* **411**:34-36 (2010), 3169–3180.

[Friedman et al. 2007] E. J. Friedman and A. S. Landsberg, Nonlinear dynamics in combinatorial games: renormalizing Chomp. *CHAOS* **17** (2007), 023117.

[Friedman et al. 2009] E. J. Friedman and A. S. Landsberg, On the geometry of combinatorial games: a renormalization approach, in: M. H. Albert and R. J. Nowakowski (eds.), *Games of No Chance* 3 (MSRI series), Cambridge University Press (2009).

[Fraenkel et al. 1991] A. S. Fraenkel and M. Lorberbom, Nimhoff games. *J. Combin. Theory Ser. A* **58** (1991), 1–25.

[Garrabrant et al. 2013] S. M. Garrabrant, E. J. Friedman, and A. S. Landsberg, Cofinite induced subgraphs of impartial combinatorial games: an analysis of CIS-Nim. *INTEGERS* **13** (2013), #G2.

[Golomb 1966] S. W. Golomb, A mathematical investigation of games of "take-away". *J. Combinatorial Theory* **1** (1966), 443–458.

[Kappraff et al. 1998] J. Kappraff, D. Blackmore, G. Adamson, Phyllotaxis as a dynamical system: a study in number (Section 6 Wythoff's Game). Chapter 17 in the book Symmetry in Plants, R. V. Jean and D. Barabé (eds.), Series in Mathematical Biology and Medicine, Vol. 4 (1998), World Scientific.

[Larsson et al. 2011] U. Larsson, P. Hegarty, and A. S. Fraenkel, Invariant and dual subtraction games resolving the Duchêne–Rigo conjecture. *Theoret. Comput. Sci.* **412** (2011), 729–735.

[Larsson 2010] U. Larsson, preprint of: A generalized diagonal Wythoff Nim. http://arxiv.org/abs/1005.1555.

[Larsson 2012a] U. Larsson, A generalized diagonal Wythoff Nim. *INTEGERS* **12** (2012), #G02.

[Larsson 2012b] U. Larsson, The $\star$-operator and invariant subtraction games, *Theoret. Comput. Sci.* **422** (2012), 52–58.

[Larsson 2014] U. Larsson, Wythoff Nim extensions and splitting sequences. *J. Integer Seq.* **17** (2014), Article 14.5.7.

[Larsson et al. 2013] U. Larsson and J. Wästlund, From heaps of matches to the limits of computability. *Electron. J. Combin.* **20** (2013), P41.

[Wythoff 1907] W. A. Wythoff, A modification of the game of Nim. *Nieuw Arch. Wisk.* **7** (1907), 199–202.

[Zeilberger 2001] D. Zeilberger, Three-rowed Chomp. *Adv. Appl. Math.* **26** (2001), 168.

[Zeilberger 2004] D. Zeilberger, Chomp, recurrences, and chaos. *J. Differ. Equations* **10** (2004), 1281.

ejf@icsi.berkeley.edu — *ICSI and Department of EECS, UC Berkeley, Berkeley, CA, United States*

scott@garrabrant.com — *W. M. Keck Science Department, Claremont McKenna, Pitzer, and Scripps Colleges, Claremont, CA, United States*

ilona.k.pm@gmail.com — *W. M. Keck Science Department, Claremont McKenna, Pitzer, and Scripps Colleges, Claremont, CA, United States*

alandsberg22@gmail.com — *W. M. Keck Science Department, Claremont McKenna, Pitzer, and Scripps Colleges, Claremont, CA, United States*

urban031@gmail.com — *Department of Industrial Engineering and Management, Technion - Israel Institute of Technology, Haifa, Israel*