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An historical tour of binary and tours

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Recreational mathematics has an old and honourable history. We illustrate that history and perhaps a bit of the utility of recreational mathematics by discussing a number of recreations involving binary representations and paths on graphs.†

Leibniz's binary arithmetics

In the seventeenth century, Francis Bacon used binary 5-tuples as a code, but binary arithmetic as we currently understand it — doing actual arithmetic with binary numbers rather than just using binary representations — starts with Leibniz about 1679, though he didn't publicize it until the late 1600s. He heard about the Fu-Hsi ordering of the I-Ching hexagrams from Jesuit missionaries in China in 1701 and wrote a good deal about it thereafter (see Figures 1 and 2 and p. 219).

Figure 1. The title page of Leibniz's booklet [Leibniz 1734] explaining binary notation to a nobleman shows a medallion he created, later borrowed by the Stadtsparkasse of Hanover to honor Leibniz himself.

† This is an elaboration of notes for a talk at the First European Congress of Mathematics, Paris, July 1992, and given several times since. This material has been edited for *Games of No Chance* 5 by Urban Larsson (urban031@gmail.com). In a few occurences when uncertainty about historical

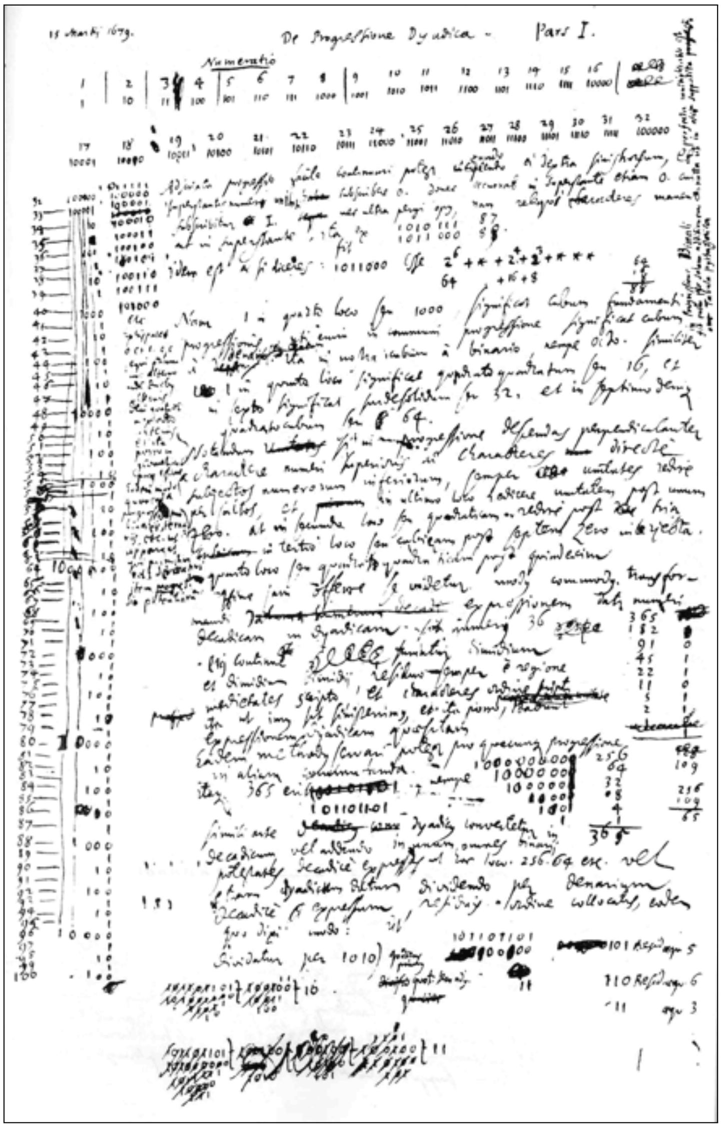


Figure 2. Leibniz's first writing on binary arithmetic, dated 11 March 1679.

However, Leibniz was anticipated by Thomas Harriot, 1604, who did not publish, and by John Napier, whose *Rabdologiæ* of 1617 gave binary arithmetic as far as computing square roots, but this seems to have been ignored.

references remained in the author's Word file, question marks have been erased in this version, following also suggestions from referees. The editor advanced this decision for readability, and would be greatful for any comments in the future to complement or assist the numerous findings in this work.

But binary ideas go much further back. Some simple counting systems are more or less base 2 and there are many instances of duality in nature — hands, sexes, etc. But we are interested in material that is somewhat more mathematical.

Binary multiplication

The earliest implicit use of binary representations occurs in ancient Egyptian mathematics. Figure 3 is Problem 30 of the Rhind Mathematical Papyrus, ca. 1700 B.C.E., computing $\left(\frac{2}{3}+\frac{1}{10}\right)\times 13$. The problem is to solve $\left(\frac{2}{3}+\frac{1}{10}\right)x = 10$, which is being done by false position, using $x = 13$ as a trial.

Because of their complicated notation for numbers, especially fractions, they multiplied by repeatedly doubling, then adding the appropriate terms. For instance, to multiply a number by 13, they computed successively the double, the quadruple and the octuple of the number, then added the number to its quadruple and its octuple. This is also known as Russian peasant multiplication and was in use in Russia until the twentieth century. It was sufficiently common that duplication and mediation (halving) were reckoned among the basic rules of arithmetic in the middle ages and could be found in arithmetic books until the seventeenth century. It uses the fact that every integer is a sum of distinct powers of two, which is the same as saying that every integer has a binary representation.

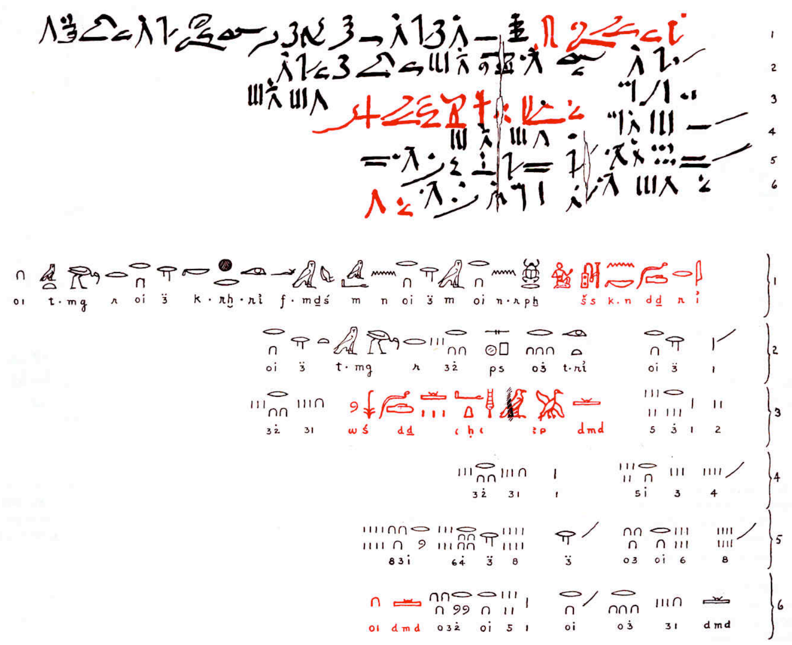


Figure 3. Problem 30 from the Rhind Mathematical Papyrus of ca. 1700 B.C.E.

Binary weights

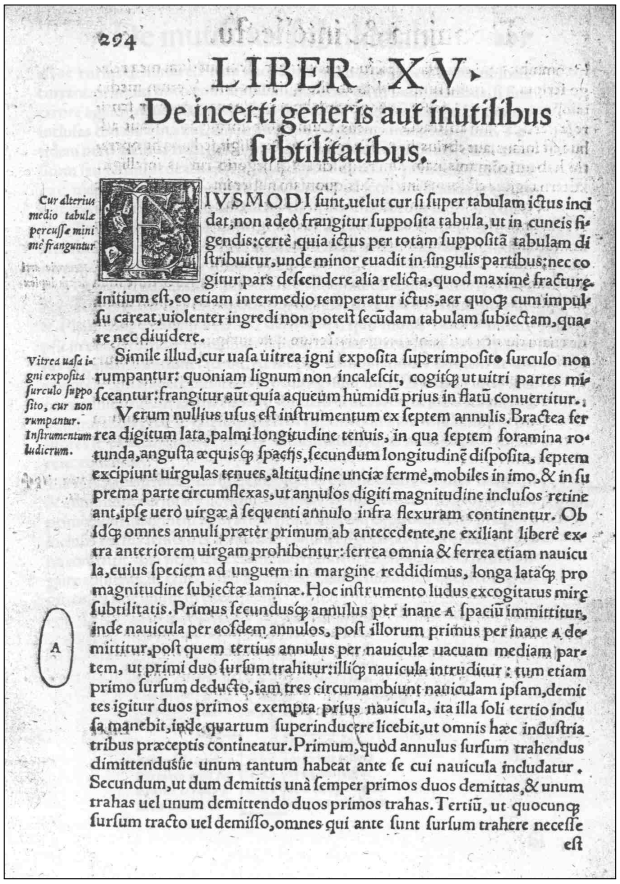
A somewhat more explicit use of binary representation occurs in Arabic sources about the eleventh century. This is the use of weights $1, 2, 4, \ldots$ to make all integer weights on a scale. (Actually, if we allow putting weights into the other side, one can use weights $1, 3, 9, \ldots$) This is often known as Bachet's weights problem, but it already appears in Fibonacci and several other European books before Bachet.

Chinese rings and Gray code

At about the same time, the puzzle known as the Chinese Rings (Figure 4) appears in China, though tradition attributes it back to the semi-legendary Hung Ming of about 200 C.E.

One Oriental name, *Lau kák ch'a*, translates as "delay-guest-instrument".

Figure 4. Four examples of Chinese Rings. From p. 107 of [Slocum and Botermans 1986].



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LIBER XV.

De incerti generis aut inutilibus ſubtilitatibus.

Cur alterius medio tabulæ percuſſæ minimè franguntur

EIVSMODI ſunt, uelut cur ſi ſuper tabulam ictus incidat, non adeò frangitur ſuppoſita tabula, ut in cuneis figendis: certè quia ictus per totam ſuppoſitā tabulam diſtribuitur, unde minor euadit in ſingulis partibus: nec cogitur pars deſcendere alia relicta, quod maximè fracturæ initium eſt, eo etiam intermedio temperatur ictus, aer quoq; cum impulſu careat, uiolenter ingredi non poteſt ſecūdam tabulam ſubiectam, quare nec diuidere.

Vitrea uaſa igni expoſita ſurculo ſuppoſito, cur non rumpantur.

Simile illud, cur uaſa uitrea igni expoſita ſuperimpoſito ſurculo non rumpantur: quoniam lignum non incaleſcit, cogitq; ut uitri partes miſceantur: frangitur aūt quia aqueum humidū prius in flatū conuertitur.

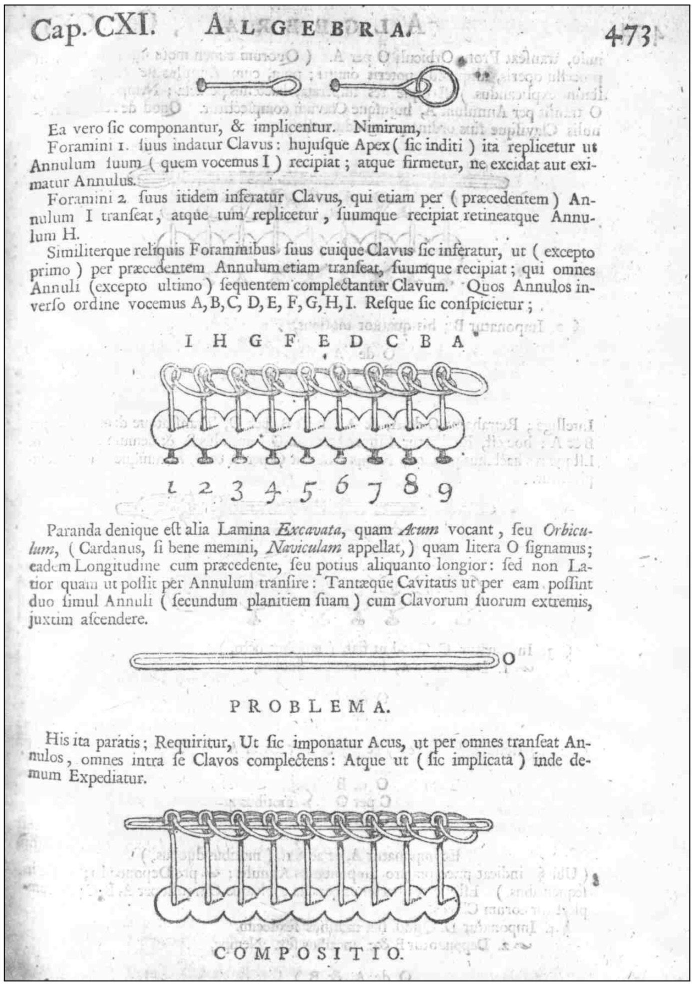
Inſtrumentum ludicrum.

Verum nullius uſus eſt inſtrumentum ex ſeptem annulis. Bractea ferrea digitum lata, palmi longitudine tenuis, in qua ſeptem foramina rotunda, anguſta æquisq; ſpacijs, ſecundum longitudinē diſpoſita, ſeptem excipiunt uirgulas tenues, altitudine unciæ fermè, mobiles in imo, & in ſuprema parte circumflexas, ut annulos digiti magnitudine incluſos retineant, ipſæ uerò uirgæ à ſequenti annulo infra flexuram continentur. Obidq; omnes annuli præter primum ab antecedente, ne exiliant liberè extra anteriorem uirgam prohibentur: ferrea omnia & ferrea etiam nauicula, cuius ſpeciem ad unguem in margine reddidimus, longa lataq; pro magnitudine ſubiectæ laminæ. Hoc inſtrumento ludus excogitatus miræ ſubtilitatis. Primus ſecundusq; annulus per inane A ſpaciū immittitur, inde nauicula per eoſdem annulos, poſt illorum primus per inane A demittitur, poſt quem tertius annulus per nauiculæ uacuam mediam partem, ut primi duo ſurſum trahitur: illiq; nauicula intruditur: tum etiam primo ſurſum deducto, iam tres circumambiunt nauiculam ipſam, demittes igitur duos primos exempta prius nauicula, ita illa ſoli tertio incluſa manebit, inde quartum ſuperinducere licebit, ut omnis hæc induſtria tribus præceptis contineatur. Primum, quòd annulus ſurſum trahendus dimittendusue unum tantum habeat ante ſe cui nauicula includatur. Secundum, ut dum demittis unà ſemper primos duos demittas, & unum trahas uel unum demittendo duos primos trahas. Tertiū, ut quocunq; ſurſum tracto uel demiſſo, omnes qui ante ſunt ſurſum trahere neceſſe eſt

Figure 5. Early description of the Chinese rings, from [Cardan 1550].

It is first known in Europe in 1550 when Cardan describes and illustrates it in his *De subtilitatis*; see Figure 5. It has recently been recognised that Luca Pacioli, in his unpublished manuscript *De viribus quantitatis* of about 1500, describes the puzzle, with no diagram. Wallis explains it quite clearly in his *De algebra tractatus* of 1693 (Figure 6). An old English phrase for it is “tiring irons” or “tarrying irons”, and these words are recorded in the Oxford English Dictionary as far back as 1601. Descriptive or picturesque names in various languages include *Chinese rings*, *Chainese rings*, *Cardan’s rings*, *Ryou-kaik-tjyo*, *Lau kák ch’a*, *Kau tsz’ lin wain* (‘Nine-connected-rings’), *Tiring* or *Tarrying irons*, *Baguenaudier*, *Meleda*, *Zauberkette*, *Magische Ringspiel*, *Nürnberger Tand*, *Grillenspiel*, *Zankeisen*, *Nodi d’anelli*.

It is difficult to describe the puzzle and how to do it, and even finding good images is hard. Suffice it to say that it has some number, n, of rings attached to


Cap. CXI. ALGEBRA. 473

Ea vero ſic componantur, & implicentur. Nimirum,

Foramini 1. ſuus indatur Clavus: hujuſque Apex (ſic inditi) ita replicetur ut Annulum ſuum (quem vocemus I) recipiat; atque firmetur, ne excidat aut eximatur Annulus.

Foramini 2. ſuus itidem inferatur Clavus, qui etiam per (præcedentem) Annulum I tranſeat, atque tum replicetur, ſuumque recipiat retineatque Annulum H.

Similiterque reliquis Foraminibus ſuus cuique Clavus ſic inferatur, ut (excepto primo) per præcedentem Annulum etiam tranſeat, ſuumque recipiat; qui omnes Annuli (excepto ultimo) ſequentem complectantur Clavum. Quos Annulos inverſo ordine vocemus A, B, C, D, E, F, G, H, I. Reſque ſic conſpicietur;

I H G F E D C B A

1 2 3 4 5 6 7 8 9

Paranda denique eſt alia Lamina *Excavata*, quam *Acum* vocant, ſeu *Orbiculum*, (Cardanus, ſi bene memini, *Naviculam* appellat,) quam litera O ſignamus; eadem Longitudine cum præcedente, ſeu potius aliquanto longior: ſed non Latior quam ut poſſit per Annulum tranſire: Tantæque Cavitatis ut per eam poſſint duo ſimul Annuli (ſecundum planitiem ſuam) cum Clavorum ſuorum extremis, juxtim aſcendere.

O

PROBLEMA.

His ita paratis; Requiritur, Ut ſic imponatur Acus, ut per omnes tranſeat Annulos, omnes intra ſe Clavos complectens: Atque ut (ſic implicata) inde demum Expediatur.

COMPOSITIO.

Figure 6. First illustrated discussion of Chinese rings, from [Wallis 1685].

a bar by wires and systematically looped over a second bar in such a way that one can take off or put on only one ring at a time, which is either the end ring or the next to the last ring on the bar. Figure 7 shows the solution for four rings.

If we represent a state of the puzzle as a sequence of ons and offs, or better, 1s and 0s, then each position is a binary number and the movement changes such a binary number to another one which differs in just one place, which is either the last place or the place next to the last 1. Figure 8 (left) shows this for five rings, from Afriat, p. 31, with the binary patterns written in the image by me. Looking at the sequence of moves, Figure 8 (right) as it appears, one sees that starting from the position with all rings off is more interesting and the pattern

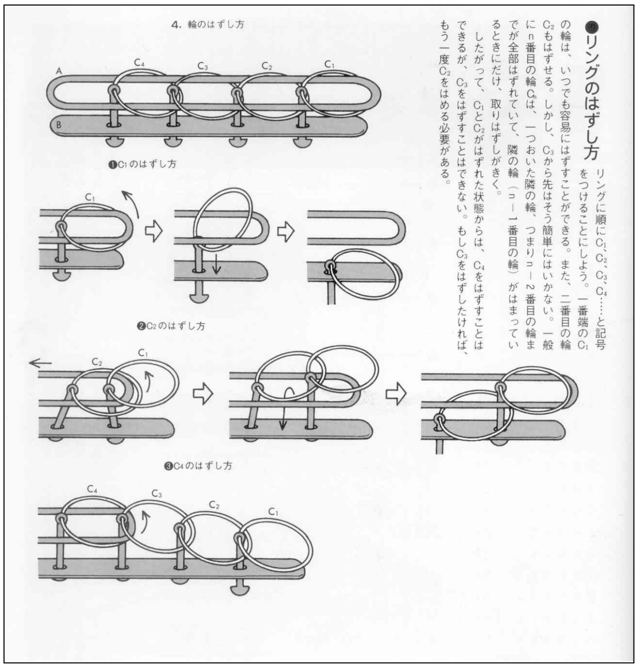


Figure 7. A 4-ring solution, from [Takagi 1982], p. 205.

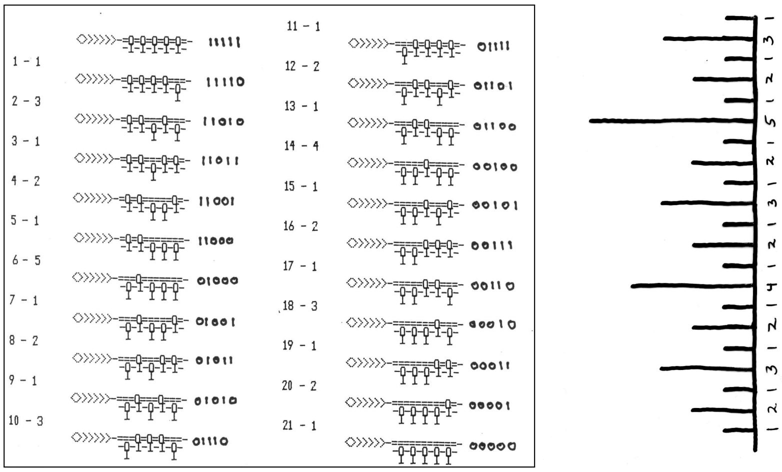


Figure 8. A 5-ring solution, from [Afriat 1982], with the corresponding Gray Code (right).

of moves is: 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, . . . (Flip the picture, view Figure 8 from the side.)

You may recognize this pattern, especially if you are English or American and hence use rulers marked in halves, quarters, eighths, ... Figure 9 is a picture of

Figure 9. A genuine left-handed ruler.

a Left-Hand Ruler, obtainable from left-handed shops! If you are metric, then you may not be so familiar with this.

Consequently the binary number for the k-th position is not the binary representation of k. With five rings, the first few binary numbers are 00000, 00001, 00011, 00010, 00110, 00111, 00101, 00100, 01100, ... See Figure 8 (right). (Note that it would be easier to have numbered the rings 0, 1, 2, ... in the preceding discussion, so the sequence of moves is: 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, ...)

This pattern was rediscovered by Frank Gray in 1947 and patented by Bell Labs in 1953! See below for more history. Generalizations of such patterns are now called Gray codes, but we will call the present pattern the Gray Code.

Proposition 1 below is easy to obtain, but not as well known as it should be. (I first found it in 1970, but not clearly formulated.)

Let $n = \sum b_{n,i} 2^i$ be the binary expansion of the integer n. Let $C = C_n$ be the number of consecutive 1s at the end of this binary expansion. We can also say that C_n is the largest i such that the i-th bit changes when we go from n to $n+1$ or that C_n is the power of 2 which exactly divides $n+1$, i.e. it is the number of 0s at the end of $n+1$. $C_n + 1$ gives the number of the ring moved at step $n+1$, if we start at one. Let G_n be the number of the n-th Gray Code value and let $\sum g_{n,i} 2^i$ be its binary expansion. Then G_n has its C_n-th bit changed to produce G_{n+1}. Thus $|G_{n+1} - G_n| = 2^C$. By careful counting, one can show that

$$g_{n,i} \equiv b_{n,i} - b_{n,i+1} \equiv b_{n,i} + b_{n,i+1} \equiv b_{n',i+1}, \tag{1}$$

where $n' = n + 2^i$.

In other words, we can find the binary form of G_n as follows. Write n in binary as $\sum b_{n,i} 2^i$. Shift it one place to the right, throwing away the rightmost bit; this produces the binary expression of $\lfloor n/2 \rfloor$ (with $\lfloor\ \rfloor$ = floor function). Then add the binary expressions for n and $\lfloor n/2 \rfloor$, but without carrying, which we denote by $\oplus$. This establishes the following.

Proposition 1. *Let $B(k)$ be the binary representation of k and let $G(k)$ be the k-th binary word in the Gray Code. Then $G(k) = B(k) \oplus B(\lfloor k/2 \rfloor)$. That is, we shift the binary representation of k to the right, losing the end digit, and do a Boolean addition (also known as addition mod 2, exclusive or, XOR) with $B(k)$.*

n	Binary	C	$C+1$	Gray	G
0	00000	0	1	00000	0
1	00001	1	2	00001	1
2	00010	0	1	00011	3
3	00011	2	3	00010	2
4	00100	0	1	00110	6
5	00101	1	2	00111	7
6	00110	0	1	00101	5
7	00111	3	4	00100	4
8	01000	0	1	01100	12
9	01001	1	2	01101	13
10	01010	0	1	01111	15
11	01011	2	3	01110	14
12	01100	0	1	01010	10
13	01101	1	2	01011	11
14	01110	0	1	01001	9
15	01111	4	5	01000	8
16	10000	0	1	11000	24
17	10001	1	2	11001	25
18	10010	0	1	11011	27
19	10011	2	3	11010	26
20	10100	0	1	11110	30
21	10101	1	2	11111	31

Figure 10. The relation between binary and Gray coding.

For example:

- For $k = 7$, we have $B(7) = 00111$, $B(\lfloor 7/2 \rfloor) = B(3) = 00011$, and $G(7) = 00111 \oplus 00011 = 00100 = 4$.
- For $k = 15$, we have $B(15) = 01111$, $B(\lfloor 15/2 \rfloor) = B(7) = 00111$, and $G(15) = 01111 \oplus 00111 = 01000 = 8$.

For the inverse process, we have

$$b_{n,j} = \sum_{i \geq j} g_{n,i}.$$

Otherwise stated, we have to sum all the shifts of G. That is, the binary value corresponding to a Gray code G is given by $G \oplus \lfloor G/2 \rfloor \oplus \lfloor G/4 \rfloor \oplus \lfloor G/8 \rfloor \oplus \cdots$

E.g., if $G = 15$, we have that the binary value of 15 is 1111, so we compute $1111 \oplus 0111 \oplus 0011 \oplus 0001 = 1010 = 10$, i.e. $G(10) = 15$.

Note that $n = 101010\ldots$ gives $G(n) = 111111\ldots$, which is the desired endpoint of the Chinese Rings process and the Rings are solved in n steps.

Suppose we have r rings. By considering the cases when r is odd and when it is even, we can determine n. In both cases, we can write $n = \lfloor \frac{2}{3} 2^r \rfloor$.

This and other basic results were (first) developed by Louis A. Gros, a notary of Lyon about whom little is known, in a very rare pamphlet titled *Théorie du Baguenodier*, Lyon, 1872. I have not been able to see this yet; a photocopy was made and given to the Radcliffe Science Library at Oxford by Afriat, but the library could not find it when I asked for it.

The function $G(k)$ can be viewed as a permutation of the binary n tuples — indeed of the binary numbers in $[2^{n-1}, 2^n - 1]$. Francis Clarke at Swansea has made some study of this permutation; in particular, he has determined its order.

In 1880, Lucas published a report that Dr. O.-J. Broch, former Minister and President of the Royal Norwegian Commission at the Universal Exposition of 1878, recently told him that country people in Norway still used the rings to close their chest and sacks. No one has ever confirmed this fact and in 1904 a Norwegian ethnographer said he had never heard of it.

Looking at the sequence of 5-tuples given above, we see that the right-hand triples are a sequence which goes through every binary triple and ends adjacent to where it started. If we depict a binary triple as a point on the 3-cube, we have a circuit along the edges of this cube which goes through every vertex just once and returns to its starting point. See Figure 11 (left).

Such circuits are generally known as Hamiltonian circuits, for reasons to be seen shortly, and the Hamiltonian circuits on the n-cube are the generalized Gray codes.

Exercise. Show that every Hamiltonian circuit on the 3-cube is equivalent to this circuit. (This is no longer true in higher dimensions.)

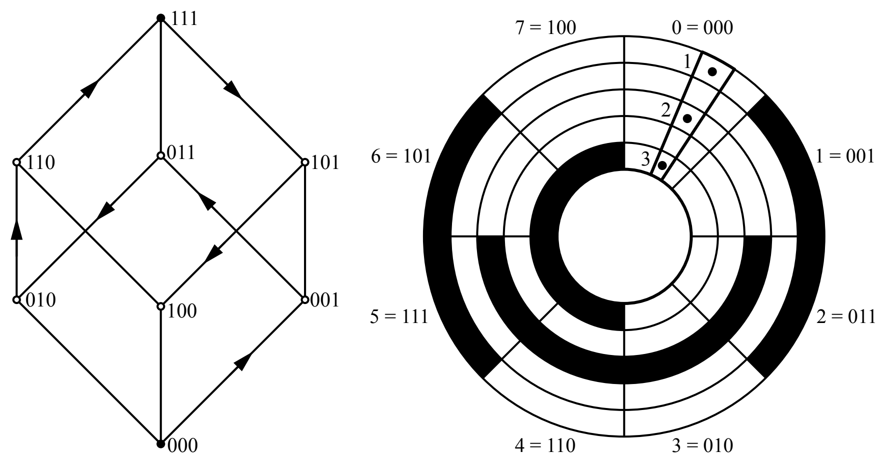


Figure 11. Left: a 3-cube showing Gray Code. Right: a switch, using Gray Code to minimize errors.

In 1947, Frank Gray of Bell Labs was using binary representations for coding and he found that a certain kind of error was minimized if the codings for adjacent numbers differed in just one binary bit. That is, he wanted to use a coding which was a Hamiltonian circuit on the n-cube.

Figure 11 (right) shows a switch for detecting the position of a central axis which has an arm containing three contact points marked $1, 2, 3$. The black areas are at some voltage and the white areas are insulation. As shown all three contacts have no voltage, so the signal on the arm is 000. (Alternatively, one can have the arm contacts having voltage and the black areas as detectors.) As we turn, the signal becomes 001, then 011, 010, ... When the arm is at one of the transition angles, it is possible for a contact to register 0 instead of 1 or vice versa. The Gray coding guarantees that there is never more than one contact point that can be in error and that the two possible signals give adjacent positions, so the effect of such an error is minimized. If one uses the ordinary binary coding, one can have all three contacts being subject to error and the position of the arm is completely undetermined.

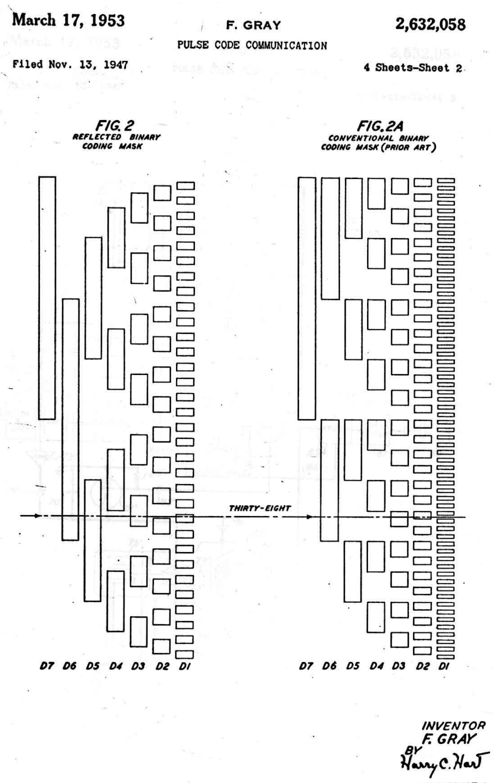


Figure 12. Patent for the Gray Code.

Gray constructed his coding precisely by the pattern of the Chinese Rings, though he also went on to consider other Hamiltonian circuits on the n-cube. Bell Labs patented the idea in 1953 (Figure 12), but I believe the patent was cancelled as part of an anti-trust settlement.

The number of Hamiltonian circuits or Gray codes on the n-cube remains an interesting problem, which leads to the question of when two circuits are equivalent. If we take both the symmetries of the circuit (i.e. starting at any point in either direction) and the symmetries of the cube as equivalences, the determination of the number of distinct circuits requires careful computation. For $n = 3$, there is essentially only one circuit and there are nine for the 4-cube. But already for the 5-cube, the counting which has been done failed to consider all the symmetries, so

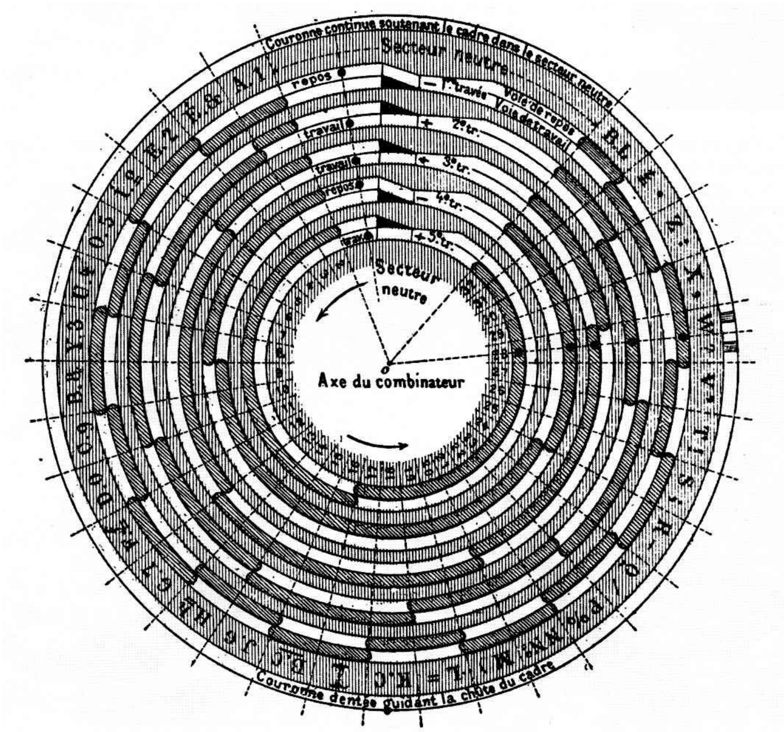


Figure 13. Baudot's use of the Gray Code in the 1870s. From [Heath 1972], p. 82.

we do not know how many inequivalent circuits there are, but it will be somewhere in the order of a million.

Surprisingly, the idea of a switch using the Gray code was discovered by the Belgian telegraph engineer Jean Émile Baudot (1845–1903) — the eponym of baud — in the 1870s, and utilised in his printing telegraphs. See Figure 13. I have also read that Stibitz used the same code in 1941, and Jack Good also recognised the pattern at Bletchley Park, 1943.

I-Ching hexagrams

Trying to keep in historical order, the ordinary binary representations of integers 0 to 63 occur in the 64 hexagrams of the I-Ching; Figure 14 (left) shows some objects decorated with the simpler 8 trigrams. The book itself goes back to ca. 600 B.C.E., but had the hexagrams in a traditional order attributed to King Wan, which has no known mathematical structure. Figure 15 (top) shows the 64 hexagrams in Wan's order, and Figure 14 (right) shows the fu-hsi ordering, which occurs in many places. The fu-hsi ordering only dates from the eleventh century and the binary pattern is clearly seen in the associated "segregation

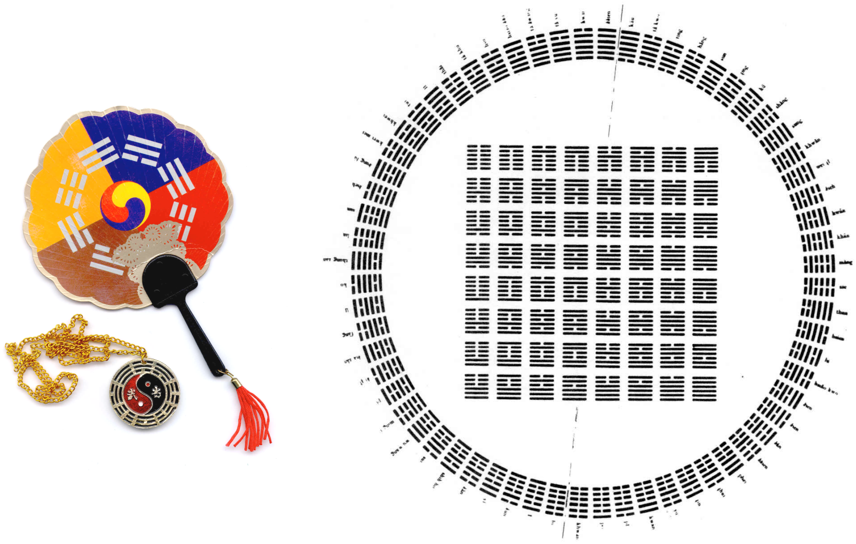
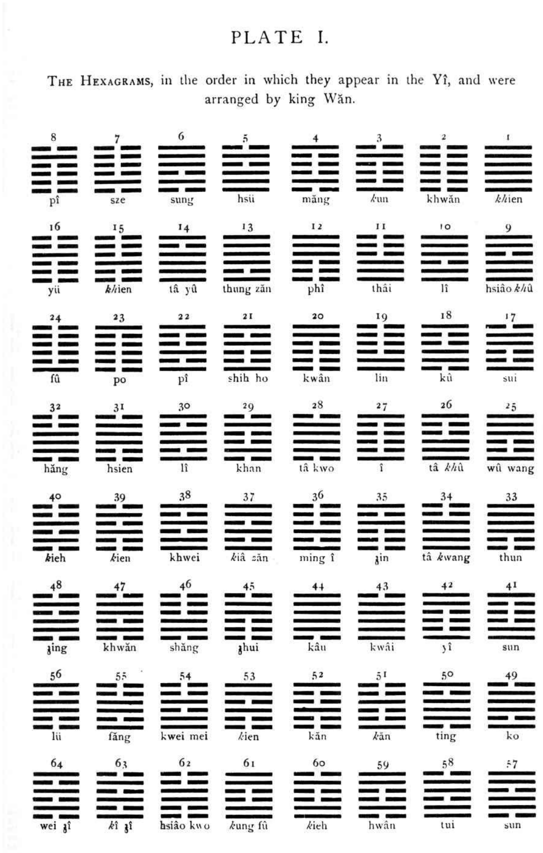


Figure 14. Left: I-Ching trigrams make a popular pattern. Right: the Fu-Hsi ordering (Yoshida Mitsuyoshi, *Jinkōki*, 1627; see [Sato 2013]).

table" (Figure 15, bottom), also occurring in many places. Leibniz learned about the later order from the Jesuit missionary Joachim Bouvet in 1701 and this inspired him to write a great deal more on binary arithmetic. He even produced a theological analogy that God was the One who created Everything out of Nothing; this pleased him so much that he had a special medal made to commemorate it (Figure 1, left). He expected this idea would convert the Emperor of China to Christianity.

Binary divination

Another explicit use of binary occurs in divination cards. These are 6 cards on which numbers are written. The subject, assumed to be less than 64 years old, says which cards his age occurs on and you add up the first numbers on these cards to determine his age (Figure 16, left). The first known European appearance of these is in the unpublished manuscript *De viribus quantitatis* by Luca Pacioli, about 1500. However, I have seen a Japanese description of two versions of the idea called Magic Cards and Magic Picture which says they have been known since at least the fourteenth century. Figure 16, right, shows versions from the early seventeenth century.



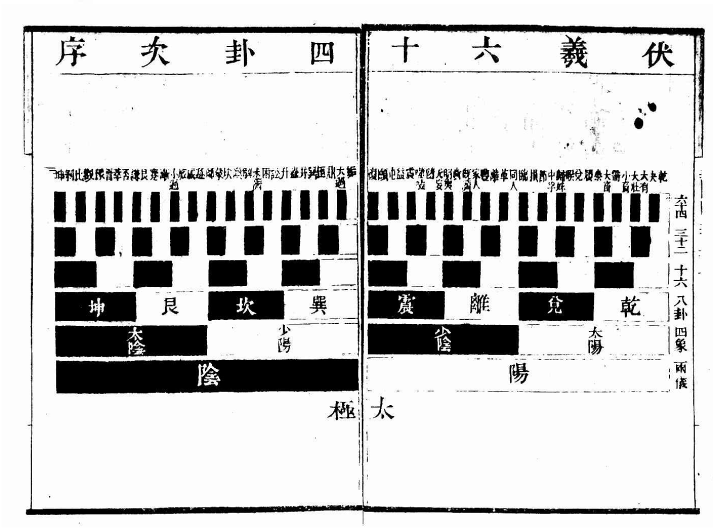
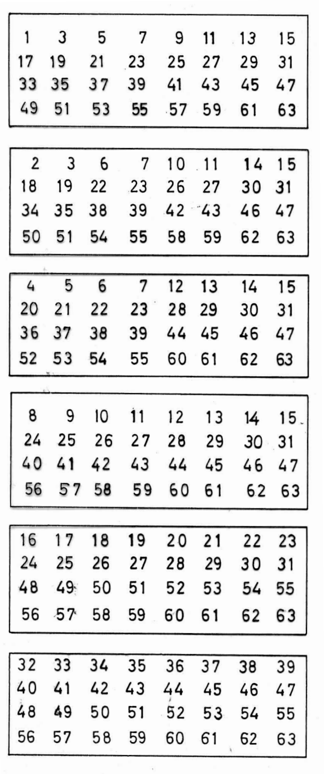


Figure 15. Top: the non-mathematical ordering of Wan [Legge 1899]. Bottom: the "segregation table" in Zhu Xi's *Zhouyi benyi* (twelfth century C.E.), reproduced in Hu Wei's *Yitu minghian* (1706).



1	3	5	7	9	11	13	15
17	19	21	23	25	27	29	31
33	35	37	39	41	43	45	47
49	51	53	55	57	59	61	63

2	3	6	7	10	11	14	15
18	19	22	23	26	27	30	31
34	35	38	39	42	43	46	47
50	51	54	55	58	59	62	63

4	5	6	7	12	13	14	15
20	21	22	23	28	29	30	31
36	37	38	39	44	45	46	47
52	53	54	55	60	61	62	63

8	9	10	11	12	13	14	15
24	25	26	27	28	29	30	31
40	41	42	43	44	45	46	47
56	57	58	59	60	61	62	63

16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31
48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63

32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47
48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63

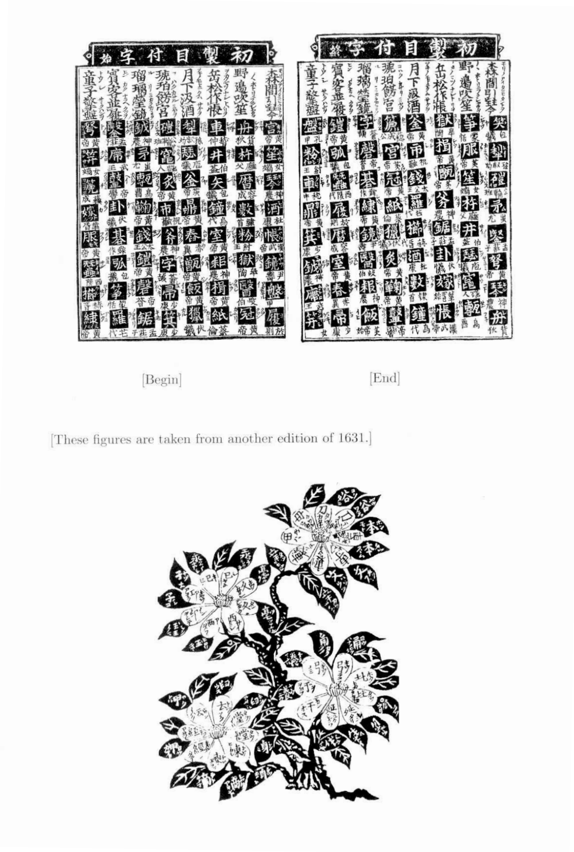


Figure 16. Left: divination cards. Right: magic cards and a magic picture from the *Jinkōki* of Yoshida. (Note by editor: For more information about this early seventeenth-century work see [Sato 2013].)

Knight's tours

Now we go backwards a bit in time to find the beginning of our other main idea: Hamiltonian circuits and paths. The oldest Hamiltonian circuits and paths are knight's tours and paths on the chessboard. In the mathematical literature, knight's tours first appear in the 1723 edition of Ozanam's *Récréations mathématiques et physiques* (see Figure 17) and are first studied systematically by Euler (1759) and Vandermonde (1771). Euler describes it as "a curious question which does not submit to any analysis".

But if we examine the chess literature, we immediately discover tours going back to the dawn of chess. Murray's *History of Chess* describes a half-board path in the *Kāvyālaṅkāra* of Rudraṭa, c900 (Figure 18, left). These are given in poetic forms in the shapes of "wheel, sword, club, bow, spear, trident, and

I. *de M. de Montmort.*

1	38	31	44	3	46	29	42
32	35	2	39	30	43	4	47
37	8	33	26	45	6	41	28
34	25	36	7	40	27	48	5
9	60	17	56	11	52	19	50
24	57	10	63	18	49	12	53
61	16	59	22	55	14	51	20
58	23	62	15	64	21	54	13

II. *de M. Moivre.*

34	49	22	11	36	39	24	1
21	10	35	50	23	12	37	40
48	33	62	57	38	25	2	13
9	20	51	54	63	60	41	26
32	47	58	61	56	53	14	3
19	8	55	52	59	64	27	42
46	31	6	17	44	29	4	15
7	18	45	30	5	16	43	28

III. *de M. de Mairan.*

40	9	26	53	42	7	64	29
25	52	41	8	27	30	43	6
10	39	24	57	54	63	28	31
23	56	51	60	1	44	5	62
50	11	38	55	58	61	32	45
37	22	59	48	19	2	15	4
12	49	20	35	14	17	46	33
21	36	13	18	47	34	3	16

Figure 17. Knight's tours from [Ozanam 1723], pp. 261–262.

plough, which are to be read according to the chessboard squares of the chariot [= rook], horse [= knight], elephant [= bishop]." The poet placed syllables in the cells of a half chessboard so that it reads the same straight across as when following a piece's path. With help from the commentator Nami, of 1069, the rook's and knight's path's are reconstructed, and are given in Murray. Both are readily extended to full board paths, but not tours, by placing a second copy of

1	30	9	20	3	24	11	26
16	19	2	29	10	27	4	23
31	8	17	14	21	6	25	12
18	15	32	7	28	13	22	5

32	35	30	25	8	5	50	55
29	24	33	36	51	56	7	4
34	31	26	9	6	49	54	57
23	28	37	12	1	52	3	48
38	13	22	27	10	47	58	53
19	16	11	64	61	2	43	46
14	39	18	21	44	41	62	59
17	20	15	40	63	60	45	42

Figure 18. Rudrata, earliest half board path (left). Earliest knight's tour (right). From [Murray 1913], pp. 54 and 336.

the half board beneath the given copy and seeing that the first cell of the second board is connected to the last cell of the first board.

The next oldest known versions, which are full-board tours, appear in *Kitâb ash-shaṭranj mimma'l-lafahu'l-'Adli waṣ-Ṣûlî wa ghair-huma* (Book of the Chess; extracts from the works of al 'Adlî, aûlî and others), by an unknown author, copied by Abû Isḥâq Ibrâhîm ibn al Mubârak ibn 'Alî al Mudhahhab al Baghdâdî, in 1141. Murray gives two distinct tours. The solution of the first is a numbered diagram, Figure 18 (right), but the second is "solved" four times by acrostic poems, where the initial letters of the lines give the tour in an algebraic notation. There are also a knight/bishop tour and a knight/queen tour, where moves of the two types alternate.

A natural question arises: how many knight's tours are there? A little trial soon sends you to smaller boards, where two investigators found 9862 knight's tours on the 6×6 board in the 1970s.

This enumeration accounted for the symmetry group of a circuit, which is D_{36}, by taking a corner as the starting cell and one of the two cells adjacent to the corner as the second cell of the circuit. However, I don't believe anyone has examined these circuits to see how many have various symmetries of the board and thus to determine the number of inequivalent circuits. On the 8×8 board, some 75,000 tours were found having the same first 35 moves! In 1975, I made some crude estimates and predicted there are $10^{23\pm3}$ tours on the 8×8 board.

Martin Loebbing and Ingo Wegener, in "The number of knight's tours equals 33,439,123,484,294 – counting with binary decision diagrams" *Electronic J. Combinatorics* **3** (1996), R5, gives a somewhat vague description of a method for counting knight's tours — they speak of directed knight's tours, but it is not clear if they have properly accounted for the symmetries of a tour or of the board. Several people immediately pointed out that the number is incorrect because it has to be divisible by four. Two comments have appeared (ibid.) — on 15 May 1996, the authors admitted this and said they would redo the problem, but they have submitted no further comment as of Jan 2001. On 18 Feb 1997, Brendan McKay announced that he had done the computation another way and found 13,267,364,410,532.

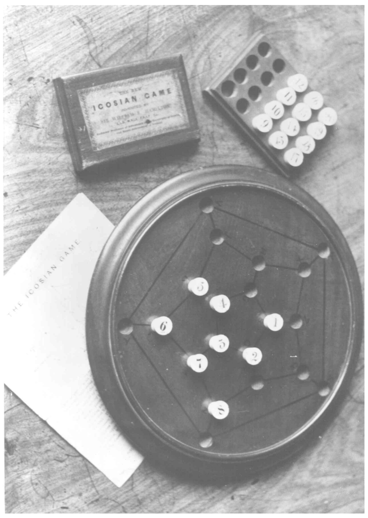
In view of the difference between these values and my 1975 estimate, it might be worth explaining my reasoning. In 1964, Duby found 75,000 tours with the same first 35 moves. The average valence for a knight on an 8×8 board is 5.25, but one cannot exit from a cell in the same direction as one entered, so we might estimate the number of ways that the first 35 moves can be made as $4.2535 = 9.9 \times 1021$. Multiplying by 75,000 then gives 7.4×1026. I think I assumed that some of the first moves had already been made, e.g. we only allow one move from the starting cell, and factored by 8 for the symmetries of the square, to get 2.2×1025. I can't find my original calculations, and I find the estimate 1025 in later papers, so I suppose I tried to reduce the effect of the 4.2535 some more. In retrospect, I had no knowledge of how many of these had already been tried. If about half of all moves from a cell had already been tried before any circuit was found, then the estimate would be more like $2.2534 \times 75,000 = 7.1 \times 1016$. If we divide the given number of circuits by 75,000 and take the 34th root, we get an average valence of 1.78 remaining, far less than I would have guessed.

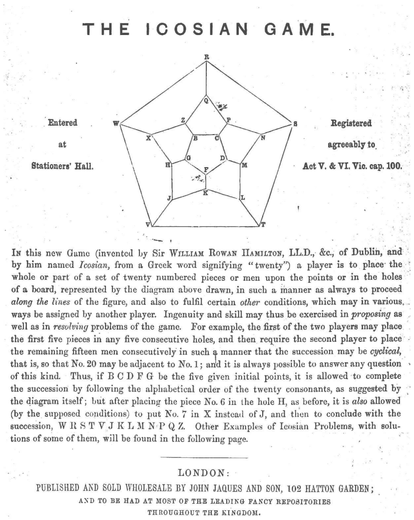
I am grateful to Don Knuth for this reference. Neither he nor I expected to ever see this number calculated!

The Icosian Game

In the 1850s, Kirkman observed that there was only one inequivalent Hamiltonian circuit on the dodecahedron. Hamilton developed this idea into a board game called The Icosian Game (after the 20 vertices) and he also developed the mathematics into the first description of a group by means of generators and relations. See Figure 19.

Only one example of the board for The Icosian Game was known until recently. (Three more have been found in the last decade or so.) Lucas and Ahrens, writing about the turn of the century, describe a solid version of the game and Ahrens even gives an address of where to buy one, but no examples were known until about 2000.





THE ICOSIAN GAME.

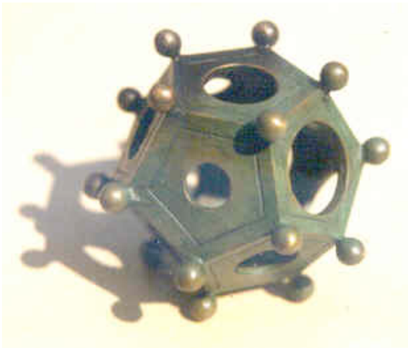
Entered at Stationers' Hall.

Registered agreeably to Act V. & VI. Vic. cap. 100.

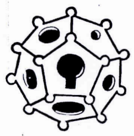
In this new Game (invented by Sir William Rowan Hamilton, LL.D., &c., of Dublin, and by him named *Icosian*, from a Greek word signifying "twenty") a player is to place the whole or part of a set of twenty numbered pieces or men upon the points or in the holes of a board, represented by the diagram above drawn, in such a manner as always to proceed *along the lines* of the figure, and also to fulfil certain *other* conditions, which may in various ways be assigned by another player. Ingenuity and skill may thus be exercised in *proposing* as well as in *resolving* problems of the game. For example, the first of the two players may place the first five pieces in any five consecutive holes, and then require the second player to place the remaining fifteen men consecutively in such a manner that the succession may be *cyclical*, that is, so that No. 20 may be adjacent to No. 1; and it is always possible to answer any question of this kind. Thus, if B C D F G be the five given initial points, it is allowed to complete the succession by following the alphabetical order of the twenty consonants, as suggested by the diagram itself; but after placing the piece No. 6 in the hole H, as before, it is *also* allowed (by the supposed conditions) to put No. 7 in X instead of J, and then to conclude with the succession, W R S T V J K L M N P Q Z. Other Examples of Icosian Problems, with solutions of some of them, will be found in the following page.

LONDON:
PUBLISHED AND SOLD WHOLESALE BY JOHN JAQUES AND SON, 102 HATTON GARDEN;
AND TO BE HAD AT MOST OF THE LEADING FANCY REPOSITORIES
THROUGHOUT THE KINGDOM.

Figure 19. The Icosian Game: an exemplar in the Royal Irish Academy, from Robin Wilson, and its instructions.



Take, for instance, the object illustrated below this paragraph. For the sake of those who find pleasure in testing their wits, I ask: What is it?—hastening to add that archæologists themselves have yet to agree on an answer.



THE MYSTERIOUS DODECAHEDRON WITH THE PENTAGONAL ENDS.

Judging by its external appearance as shown in the picture, it is a bronze object shaped like a pentadodecahedron. Round openings of various sizes are found in the center of each face. The interior of the object is hollow. All specimens of this artifact have been found north of the Alps, which indicates a Roman origin.

One interpreter sees this mysterious thing as a mere toy; another as a die used in games of chance; a third as a model used in teaching the measurement of cylindrical bodies; a fourth as a candleholder.

What is it?

Since this book was first published, I have received over a hundred answers to this question from both experts and laymen all over the world. The experts' explanations tend to be quite authoritative in tone, though they contradict each other. The most probable solution—though far from established—is that we have here a musical instrument.

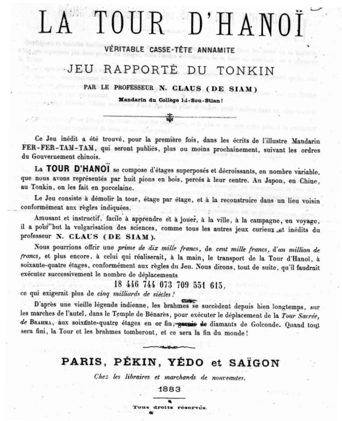
Figure 20. The Roman Dodecahedron and C. W. Ceram's discussion of it (*Gods, graves, and scholars*, 2nd ed., Gollancz, London, p. 25).

On the other hand, there are about 100 examples of Roman bronze hollow dodecahedra with knobs at the corners which look exactly like solid versions of the Icosian Game; Figure 20 shows a photo of a facsimile. Archaeologists are mystified as to what these objects are; there are hundreds on conjectures in the archaeological literature, now including mine that they might be early versions of Hamilton's game.

The Tower of Hanoi

The greatest of French recreational mathematicians was François-Édouard-Anatole Lucas (1842–1891), who died at the height of his powers from blood poisoning caused by a scratch from a plate dropped at a scientific banquet. In 1883, he brought out his Tower of Hanoi, which I will presume is known to all of you (see Figure 21). The story about the 64 discs in Benares appears in the original literature and was so widely spread that it appeared as truth in Robert Ripley's *Believe It Or Not!* (Figure 22).

When I gave this talk in 1993, Jean Brette, of the Palais de la Découverte in Paris, told me that there was an original example of the Tower of Hanoi in the Conservatoire National des Arts et Métiers – Musée National des Techniques. At the time, it was closed for refurbishment, but some years later, I wrote to ask about this and Elisabeth Lefevre sent details and photocopies of the box, instruction sheet and some other material. The bottom of the box has an ink



LA TOUR D'HANOÏ

VÉRITABLE CASSE-TÊTE ANNAMITE

JEU RAPPORTÉ DU TONKIN

PAR LE PROFESSEUR N. CLAUS (DE SIAM)

Mandarin du Collège Li-Sou-Stian!

Ce Jeu inédit a été trouvé, pour la première fois, dans les écrits de l'illustre Mandarin FER-FER-TAM-TAM, qui seront publiés, plus ou moins prochainement, suivant les ordres du Gouvernement chinois.

La **TOUR D'HANOÏ** se compose d'étages superposés et décroissants, en nombre variable, que nous avons représentés par huit pions en bois, percés à leur centre. Au Japon, en Chine, au Tonkin, on les fait en porcelaine.

Le Jeu consiste à démolir la tour, étage par étage, et à la reconstruire dans un lieu voisin conformément aux règles indiquées.

Amusant et instructif, facile à apprendre et à jouer, à la ville, à la campagne, en voyage, il a pour but la vulgarisation des sciences, comme tous les autres jeux curieux et inédits du professeur **N. CLAUS (DE SIAM)**.

Nous pourrions offrir une *prime de dix mille francs*, de *cent mille francs*, d'*un million de francs*, et plus encore, à celui qui réaliserait, à la main, le transport de la Tour d'Hanoï, à soixante-quatre étages, conformément aux règles du Jeu. Nous dirons, tout de suite, qu'il faudrait exécuter successivement le nombre de déplacements

18 446 744 073 709 551 615,

ce qui exigerait plus de *cinq milliards de siècles!*

D'après une vieille légende indienne, les brahmes se succèdent depuis bien longtemps, sur les marches de l'autel, dans le Temple de Bénarès, pour exécuter le déplacement de la *Tour Sacrée, de* BRAHMA, aux soixante-quatre étages en or fin, gemmis de diamants de Golconde. Quand tout sera fini, la Tour et les brahmes tomberont, et ce sera la fin du monde!

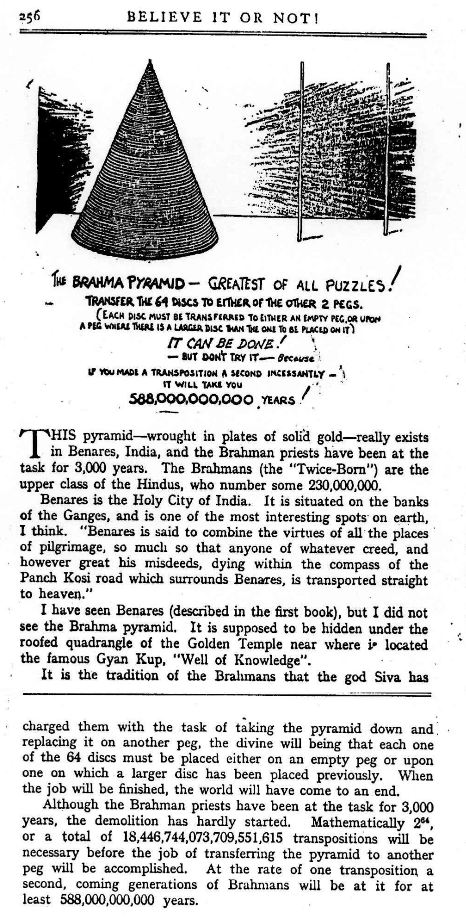
PARIS, PÉKIN, YÉDO et SAÏGON

Chez les libraires et marchands de nouveautés.

1883

Tous droits réservés.

Figure 21. The original box cover and instructions (1883) for the Tower of Hanoi, where the game's introduction is attributed to N. Claus (de Siam), a pseudonym of Lucas (d'Amiens).



256 BELIEVE IT OR NOT!

THE BRAHMA PYRAMID — GREATEST OF ALL PUZZLES!
TRANSFER THE 64 DISCS TO EITHER OF THE OTHER 2 PEGS.
(EACH DISC MUST BE TRANSFERRED TO EITHER AN EMPTY PEG, OR UPON A PEG WHERE THERE IS A LARGER DISC THAN THE ONE TO BE PLACED ON IT)
IT CAN BE DONE!
— BUT DON'T TRY IT — Because
IF YOU MADE A TRANSPOSITION A SECOND INCESSANTLY — IT WILL TAKE YOU
588,000,000,000 YEARS!

THIS pyramid—wrought in plates of solid gold—really exists in Benares, India, and the Brahman priests have been at the task for 3,000 years. The Brahmans (the "Twice-Born") are the upper class of the Hindus, who number some 230,000,000.

Benares is the Holy City of India. It is situated on the banks of the Ganges, and is one of the most interesting spots on earth, I think. "Benares is said to combine the virtues of all the places of pilgrimage, so much so that anyone of whatever creed, and however great his misdeeds, dying within the compass of the Panch Kosi road which surrounds Benares, is transported straight to heaven."

I have seen Benares (described in the first book), but I did not see the Brahma pyramid. It is supposed to be hidden under the roofed quadrangle of the Golden Temple near where is located the famous Gyan Kup, "Well of Knowledge".

It is the tradition of the Brahmans that the god Siva has charged them with the task of taking the pyramid down and replacing it on another peg, the divine will being that each one of the 64 discs must be placed either on an empty peg or upon one on which a larger disc has been placed previously. When the job will be finished, the world will have come to an end.

Although the Brahman priests have been at the task for 3,000 years, the demolition has hardly started. Mathematically 2^{64}, or a total of 18,446,744,073,709,551,615 transpositions will be necessary before the job of transferring the pyramid to another peg will be accomplished. At the rate of one transposition a second, coming generations of Brahmans will be at it for at least 588,000,000,000 years.

Figure 22. From Robert Ripley's *Believe It Or Not!*, book 2 (Simon & Schuster, 1931, and other editions).

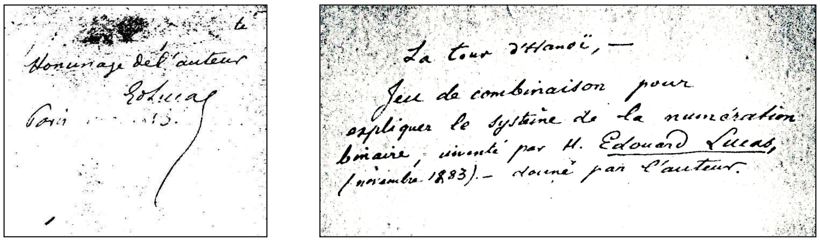


Figure 23. The bottom of the original Tower of Hanoi box (left); the inside of the lid of the box (right).

inscription: Hommage de l'auteur / Ed Lucas / Paris 1888 (Figure 23, left). The date is not clearly legible on the photocopy, but is known from the Museum's records. Inside the cover, apparently in the same hand (that is, in Lucas's writing), is an ink inscription (Figure 23, right); it translates to "The Tower of Hanoi: a combination game to explain the binary numbering system. Invented by Mr. Edouard Lucas, November 1883. Present of the author." These comments are very important historically in that Lucas never publicly admitted to inventing the game!

Despite its age, the Tower of Hanoi continues to surprise. In the late 1980s, I observed that the discs can be placed on the pegs in 3^n ways and wondered if any position was more difficult to obtain from the initial position than the position with all discs on another peg. In fact there is not, but there are 2^n positions which are just as difficult. The following analysis is based on work I did then, but this was improved by seeing the approach used by Daniele Parisse: "The Tower of Hanoi and the Stern-Brocot array", PhD Thesis at Fakultät für Mathematik, Ludwig-Maximilians-Universität München, 1997, under the direction of Andreas Hinz (an amended version was printed). In late 2000 and early 2001, I used this material as part of "The history of some combinatorial recreational problems", a chapter for *History of Combinatorics*, edited by Robin J. Wilson. In so doing, I found I needed some extra results; they eventually turned out to be pretty straightforward, but little of my original 1993 organization remains! The following is the revised material.

The first article on the puzzle, [Longchamps 1883], showed that it takes $2^n - 1$ moves to solve the problem when there are n discs. There has been some question as to whether this or any other early discussion actually showed that this number of moves is minimal, but if M_n is the minimal number of moves for n discs, then the basic argument clearly leads to

$$M_n = M_{n-1} + 1 + M_{n-1}. \tag{2}$$

In actually carrying out the minimal solution, one finds that the sequence of discs moved is precisely the same as the sequence of rings moved in the Chinese Rings: 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, ..., though there is not an analogous binary representation for the positions. Further each disc always moves through the pegs in the same cyclic order, with alternate discs moving in alternate directions. The Tower of Hanoi corresponds to a Hamiltonian circuit on the n-dimensional cube, while the Chinese Rings corresponds to a Hamiltonian path from one corner to a diametrically opposite corner — in both cases the route is the very particular one that we have called the Gray Code. This seems to have first been observed by Crowe [1956] (see also [Gardner 1957]). Because of the recursive nature of the solution, it has been a popular problem for testing programming techniques in recent years.

A position means a legal arrangement of the discs, i.e., no disc is on a smaller one. The order of discs on a peg is then determinate and we only have to say which discs are on which peg. Let us call the position, with all discs on one peg, an initial or perfect position.

To analyse my 1980s question of how many positions are maximally difficult, let $A(n, d)$ be the number of positions with n discs requiring d moves to obtain from the initial position. All the positions with the n-th disc still in its initial place can be viewed as positions on the first $n-1$ discs and so $A(n, d) = A(n-1, d)$ for $d < 2^{n-1}$. If the n-th disc has moved, there are two positions it can get to in $2^{n-1} - 1$ moves and the other discs are in a pile of $n-1$ at the other non-initial position. Hence $A(n, 2^{n-1} - 1 + d) = 2A(n-1, d)$, for $d < 2^{n-1}$.

If $0 \le d < 2^n$ and we write d in its binary representation: $d = \sum d_i 2^i$ and set $S(d) = \sum d_i$ — i.e., $S(d)$ is the number of ones in the binary representation of d — then one readily sees there are $2^{S(d)}$ positions that require d moves to achieve. The average value δ of d turns out to be precisely $\frac{2}{3}$ of the maximum number, $2^n - 1$, of moves. Andreas Hinz [1989a; 1992] found this same result and further determined the average number of moves between any two positions is asymptotic to $(466/885)2^n$.

The basic idea of Hinz is to examine the graph of positions connected by legal moves. This graph was already formulated by Scorer, Grundy and Smith [Scorer et al. 1944], though they didn't proceed as far as Hinz. This graph is in a triangular array with 2^n points along each edge. It is constructed recursively and it is difficult to relate a given arrangement of the discs to its location in the graph. I have reformulated this and determined how to relate a position to its point in the graph. This process also yields the above results and some others. This takes a little notation. Recall we use $\oplus$ for Boolean addition or "exclusive or" and we will use $\odot$ for Boolean multiplication or "and". We count discs and pegs starting with 0.

The Disc-Peg Incidence matrix A is defined by: $A(i, j) = 1$ if disc i is on peg j, and $A(i, j) = 0$ otherwise. We will do two examples. If discs 0 and 1 are on peg 1 and disc 2 is on peg 0, then the matrix is the first shown below, while if discs 0 and 1 are on peg 1, disc 2 is on peg 0, and disc 3 is on peg 2, then the matrix is the second shown.

		peg 0 1 2	peg 0 1 2
	disc		
$A =$	0	0 1 0	0 1 0
	1	0 1 0	0 1 0
	2	1 0 0	1 0 0
	3		0 0 1

It is easier to view this matrix by its columns or its rows. The three columns give three binary n-tuples x_0, x_1, x_2 which indicate which discs are on the pegs. These "peg contents n-tuples" describe a partition of the discs, so $x_i \odot x_j = 00\ldots0$ for $i \neq j$ and $\sum x_i = 11\ldots1$. We can also interpret x_j as the integers given by the binary n-tuples, i.e. as $x_j = \sum_i A(i, j)2^i$ and we denote this vector as x. In our examples, we get $x = (4, 3, 0)$ and $x = (4, 3, 8)$. The fact that $\sum x_i = 2^n - 1$ tells us that we could plot the positions with triangular coordinates.

However, not all the points satisfying this condition correspond to a partition of the discs and adjacent points in the resulting plot are not connected by legal moves, so this does not lead to a useful graph. (One can see a connection with the Chinese Rings if one thinks of a Chinese Rings position as a pair of binary n-tuples, one recording the rings on the bar and the other recording the rings off the bar. Then these two n-tuples correspond to a partition of the rings.)

If we consider the rows of A, we can define the "disc location n tuples" or "disc location vector" as $p = (p_i)$, where $p_i = j$ if $A(i, j) = 1$, i.e. if disc i is on peg j. In our examples, we get $p = (1, 1, 0)$ and $p = (1, 1, 0, 2)$. Thus $p \in (\mathbb{Z}_3)^n$ and all such points occur, i. e. the puzzle has 3^n positions. Since each peg is adjacent to both others, $\mathbb{Z}_3$ behaves like a 3-point circle; when $n = 2$, we have a 9-point torus. We will tend to identify a position with its disc location vector. It is straightforward, but a bit messy, to find the mappings between these two descriptions and to describe the legal moves in each case.

Now we need some more notation.

Let $d(p, q)$ be the shortest number of moves between positions p and q. Since moves are reversible, we have $d(p, q) = d(q, p)$ and $d(\cdot, \cdot)$ is easily seen to be a metric.

Let $\boldsymbol{i}$ be the disc location vector $(i, i, \ldots, i)$, i.e., the perfect or initial position with all discs on peg i.

Let $i \circ j = -(i + j) \pmod 3$, so $i \circ j = i$ if $i = j$; $i \circ j \notin \{i, j\}$ if $i \neq j$.

We want to determine $d_i = d(\boldsymbol{i}, p)$ for some position p. Though the process is reversible, it seems easier to describe it as starting from $\boldsymbol{i}$. The largest disc, numbered $n-1$ since we start counting with zero, wants to go onto peg p_{n-1}.

If $p_{n-1} = j \neq i$, then we have to move the first $n-1$ discs from peg i onto the other disc, namely disc $i \circ j$. This takes $2^{n-1}-1$ moves. Then we move disc $n-1$ onto peg j and we have reduced the problem by one disc, using 2^{n-1} moves. But our pile of $n-1$ discs is now on peg $i \circ p_{n-1}$, so our reduced situation starts from this peg and the roles of pegs $i = i \circ p_{n-1} \circ p_{n-1}$ and $i \circ j = i \circ p_{n-1}$ have been interchanged.

If $p_{n-1} = i$, then we don't have to carry out the 2^{n-1} moves and the reduced situation still starts from peg $i = i \circ p_{n-1}$.

This establishes the following.

Proposition 2. *For a position p, the value of $d = d_i = d(i, p)$ is determined by the following process.*

```
t = i
d = 0
FOR k = n − 1 TO 0 STEP −1
      IF p_k ≠ t THEN d = d + 2^k
      t = t ∘ p_k
NEXT k
```

In our examples, the three distances are $d_0 = 0+2+1 = 3$, $d_1 = 4+2+1 = 7$, $d_3 = 4+0+0 = 4$ for the first and $d_0 = 8+4+2+1 = 15$, $d_1 = 8+0+2+1 = 11$, $d_3 = 0+4+0+0 = 4$ for the second.

Now consider computing d_0, d_1, d_2 in parallel as we will generally do. Observe that when (a, b, c) is a permutation of (0, 1, 2), then so is $(a \circ p_k, b \circ p_k, c \circ p_k)$. Hence the three t values in the algorithm, which are originally (0, 1, 2), always remain a permutation of (0, 1, 2) at each stage. Hence we see:

Corollary 3. *Each binary place has the value* 1 *twice in the binary expansions of d_0, d_1, d_2.*

If we let $D_k(a)$ be the k-th digit of the binary representation of a, we can express the result of Corollary 3 as

$$\sum_i D_k(d_i) = 2 \text{ for each } k. \tag{3}$$

Hence we also have:

Corollary 4. $\sum_i d_i = 2(2^n - 1)$.

Since the situation is symmetric in the pegs, we readily see:

Corollary 5. *The average value of d_i over all p is $\frac{2}{3}(2^n - 1)$.*

Now considering the calculation of d_0, d_1, d_2 in parallel, we see that p uniquely determines (d_0, d_1, d_2). For if $q \neq p$, then the first place, counting down from $n-1$, where the vectors differ will give a different binary digit in two of the distances.

Proposition 6. *The set of positions in the Tower of Hanoi with n discs is in one-to-one correspondence with the set of triples of binary n-tuples*, (d_0, d_1, d_2), *satisfying* (3).

Proof. Proposition 2 gives a mapping from the set of positions to the distance triples and Corollary 3 says these triples satisfy (3). The above discussion shows the mapping is one-to-one. But there are precisely 3^n such triples and we already know there are 3^n positions of the puzzle, so the mapping must also be onto. □

One can determine $p = (p_k)$ from a triple (d_0, d_1, d_2) satisfying (3) by the following.

```
t(0) = 0; t(1) = 1; t(2) = 2
FOR k = n − 1 TO 0 STEP −1
    FOR i = 0 TO 2
        IF D_k(d_i) = 0 THEN p_k = t(i)
    NEXT i
    FOR i = 0 TO 2
        t(i) = t(i) ∘ p_k
    NEXT i
NEXT k
```

We have that $d_0 + d_1 + d_2$ adds to a constant, $2(2^n - 1)$, which leads us to think of using triangular coordinates, but the sum is twice what it ought to be. But this is what happens when we take distances to the corners rather than the usual distances to the edges. This suggests that the natural coordinates are the complementary distances $d'_i = (2^n - 1) - d_i$. In our examples, these are $(4, 0, 3)$ and $(0, 4, 11)$. Then (3) becomes

$$\sum_i D_k(d'_i) = 1, \text{ for each } k. \tag{4}$$

Thus $\sum_i d'_i = 2^n - 1$, so the (d'_i) can be used as triangular coordinates for a graph of the positions within a triangle of edge $2^n - 1$, i.e. having 2^n points along each edge. Proposition 6 tells us which points in the triangle are legal positions of the puzzle.

In triangular coordinates, two points are adjacent if and only if they differ by one in two coordinates. E.g. $(0, 0, 3)$ is adjacent to $(0, 1, 2)$. The truth of this for our (d'_i) implies the same relationship for the (d_i). We want to see that two positions in the Tower of Hanoi differ by just one move if and only if the corresponding points are adjacent in our triangular graph.

To see this, we consider the possible moves from a position p. From any position, there are at most three moves, of two types.

1) The smallest disc can be moved from its peg to either of the other pegs. These moves can be made from any position.
2) The second smallest of the uppermost discs, i.e. the smaller of the uppermost discs on the pegs which the smallest disc is not on, can be moved from its peg to the other such peg. An empty peg acts as if it had an infinitely large and immovable disc on it, but when there are two empty pegs, there is no move of this type. That is, there are only two moves, of type 1, from the perfect positions.

Consider a move of the first type from a position p to a position q and its effect on the distances d_i. The calculations will be identical until the final stage when one will be added to one distance and subtracted from another. The same holds for the coordinates d_i' and so p and q will be adjacent in our triangular graph.

Now what happens if we make a move of the second type? If the second smallest uppermost disc is the k-th, then the k discs $0, 1, \ldots, k-1$ are all on the same peg. From the symmetry of the situation, let us assume position p has discs $0, 1, \ldots, k-1$ on peg 0 and disc k on peg 1 and we want to move it to peg 2 to obtain position q. In computing the distances for p and q, everything is identical for $n-1, n-2, \ldots, k+1$. We can ignore these discs – the only effect is that this permutes the t values in the algorithms, but we only need to examine the set of distances.

Using the known result (2) that it takes $2^k - 1$ moves to move the first k discs from one peg to another, we see that for position p, the distances are these:

for position p:	for position q:
$d_0 = 2^k - 1 + 1 + 2^k - 1 = 2^{k+1} - 1,$	$d_0 = 2^k - 1 + 1 + 2^k - 1 = 2^{k+1} - 1,$
$d_1 = 2^k - 1,$	$d_1 = 1 + 2^k - 1 = 2^k,$
$d_2 = 1 + 2^k - 1 = 2^k,$	$d_2 = 2^k - 1.$

So we see that p and q are adjacent in our triangular graph.

So every pair of positions differing by one move in the Tower of Hanoi corresponds to an adjacent pair of points in our triangular graph. But a point in the triangular graph has: 2 adjacent points if it is at a corner; 4 adjacent points if it is in the interior of an edge; 6 adjacent points otherwise. The three corners are well behaved; both adjacent points are one move from the corner. For all other points I claim that only three of the adjacent points satisfy (4). Renumber the pegs so that $D_0(d'_0) = 1$, i.e. disc 0 is on peg 0. Suppose that the first zero value in the binary representation of d_0 is the k-th digit, i.e. $D_k(d'_0) = 0$, but $D_j(d'_0) = 1$ for $j = 0, 1, \dots, k-1$. Renumber so that $D_k(d'_1) = 1$. Then the binary representations have the forms

$$d'_0 = \dots 011 \dots 1,$$
$$d'_1 = \dots 100 \dots 0,$$
$$d'_2 = \dots 000 \dots 0.$$

It is clear that we can subtract one from d'_0 and add it to either of the other coordinates while preserving (4). Adding one to d'_0 gives us $\dots 100 \dots 0$ and (4) can only hold if we subtract one from d'_1, getting $\dots 011 \dots 1$. If d'_0 is not changed, then we have to add one to either of the other two distances and this gives an end digit of one and so (4) does not hold. Hence the only situations where p and q are adjacent points in our triangular graph are those corresponding to moves in the Tower of Hanoi. This completes the proof of the following.

Theorem 7. *The graph of positions in the Tower of Hanoi with n discs and with adjacency between positions one move apart, is isomorphic to the graph of triples of binary n tuples* (d'_0, d'_1, d'_2) *satisfying* (4) *considered as triangular coordinates in a triangle of edge length* 2^{n-1} *and with adjacency being adjacency in the lattice.*

Figure 24 shows a Tower of Hanoi diagram for $n = 3$, with some disc location vectors and our first example plotted. This picture was described by Scorer, Grundy and Smith [Scorer et al. 1944] by a different process, which I illustrate for the passage from 1 to 2 discs. For 1 disc, there are three positions in a triangle; Figure 25 (left). For 2 discs, we get this repeated three times, once for each peg that the second disc is on; Figure 25 (middle). We place these copies at the corners of a triangle and then reflect each small triangle about the axis through the corner of the big triangle; Figure 25 (right). This reflection is the geometric process associated with the $i \circ j$ operation above and corresponds to the fact that

$p = (1, 1, 0)$
$x = (4, 3, 0)$
$d = (3, 7, 4)$
$d' = (4, 0, 3)$

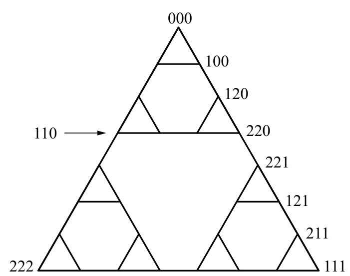


Figure 24. A Tower of Hanoi diagram for $n = 3$.

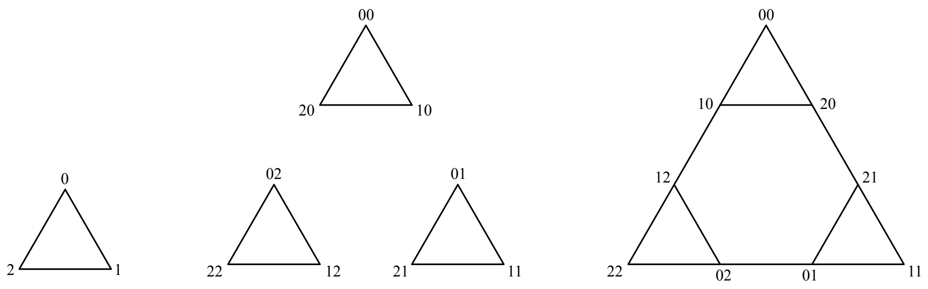


Figure 25. The passage from 1 to 2 discs, from [Scorer et al. 1944].

when we start at i and we want to move a disc from i to j, with $i \neq j$, we have first to move all the previous discs to peg $i \circ j$, so the problem is reduced to a smaller problem, but starting on the different peg $i \circ j$.

Scorer, Grundy and Smith noted that every position, except the perfect ones, has three moves from it, one of each of the kinds discussed above and that the mapping of triples $x = (x_0, x_1, x_2)$ to triples $d' = (d'_0, d'_1, d'_2)$ has order 2. The distances d and the coordinates d'_i really tell us all we could want to know about a position; Figure 24 show the next stage of the graph and the point $p = (1, 1, 0)$ on it). From the symmetry of the triangular pattern, some results can be deduced from properties of the triangle — e.g., δ is the distance of the centroid from a vertex and the average value of d'_i over all p is the distance of the centroid from an edge, i.e., $\delta/2$.

When we think of the Chinese Rings positions as pairs of binary n tuples, each position has either one or two moves and a move consists of shifting a bit from one n-tuple to another according to certain rules. In the Tower of Hanoi, each position is a triple of binary n-tuples and has two moves of the same sort, but generally has a more complex move, though this shifts one from one coordinate to another. The Gray Code permits us to recognise adjacent positions in the Chinese Rings; the triangular coordinates do the same for the Tower of Hanoi. I still feel that the analysis is not quite satisfactory in that there is no formula for the Tower of Hanoi analogous to $G(k) = B(k) \oplus B(\lfloor k/2 \rfloor)$. The standard problem of moving from one perfect position to another, say from 0 to 1, corresponds to moving along the edge $d_2 = 0$ of our triangular graph and the point with coordinates $(2^n - 1 - k, k, 0)$ is the k-th point on the solution path. If we are given the point number, k, we can recreate p by the method after Proposition 6. Early methods of doing this seem rather more complex; see [Hinz 1989a].[1]

[1] Editor's note: for a comprehensive survey of similar topics, see also [Hinz et al. 2013].

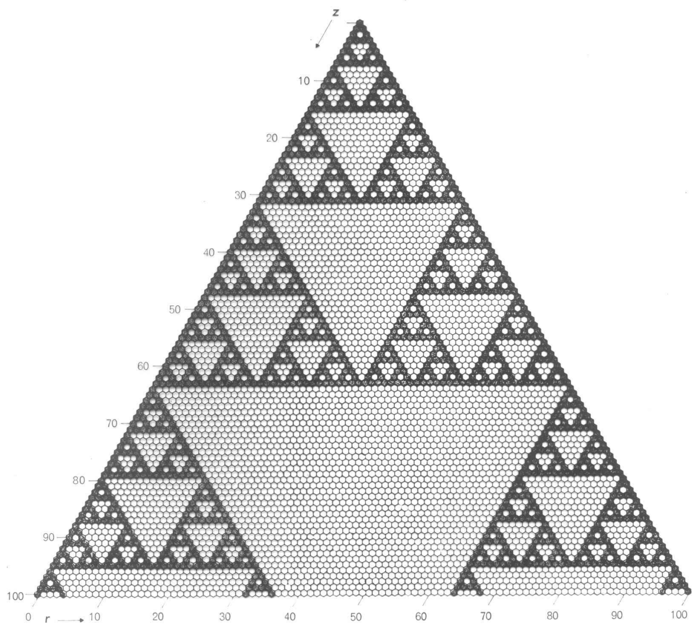


Figure 26. Odd values in Pascal's Triangle. From Siegfried Rösch, *Farbenlehre, auf die Mathematik angewandt Studien am Pascalschen Dreieck*; palette Nr. 15 (Spring 1964), Sandoz AG, Basel.

Related ideas

Ian Stewart, and perhaps others, have pointed out that the triangular pattern arising is the same as the pattern of odd binomial coefficients (BC) in the rows $0, 1, \ldots, 2^n - 1$ of Pascal's triangle. This is an easy consequence of the result that $\mathrm{BC}(m, k)$ is odd if and only if $m + (m - k)$ has no carries when done in binary. If we have peg 0 at the top of our triangle, then $\mathrm{BC}(m, k)$ is located at the point $((2^n - 1) - m, k, m - k)$ on the triangular graph and is a legal position if and only if $k \odot m - k = 0$. This also allows us to deduce the number of positions in the m-th row as I did before. This pattern is also the n-th stage in the construction of the fractal called Sierpiński's Gasket (Figure 26). Hinz showed that the average distance between points in these patterns satisfies a fifth-order recurrence and is asymptotic to $466/885$ of the maximal length.

Hinz also considered arrangements of the discs which were not in correct order and asked how many moves were needed to get to a correct order.

Donald Knuth told me about the problem where we imagine the three pegs in a line and one can only move to an adjacent peg. That is, one cannot move from one end to the other. If we want to move the whole pile from one end to

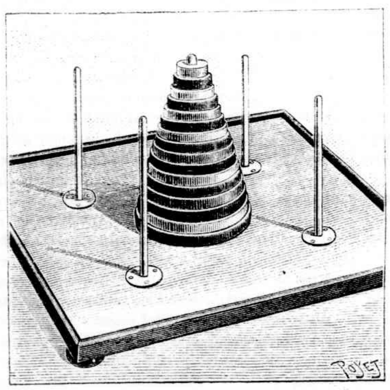


Fig. 5. — Nouvelle Tour d'Hanoi.

No. of Disks	Pegs					
1	1	1	1	1	1	1
2	3	3	3	3	3	3
3	7	5	5	5	5	5
4	15	9	7	7	7	7
5	31	13	11	9	9	9
6	63	17	15	13	11	11
7	127	25	19	17	15	13
8	255	33	23	21	19	17
9	511	41	27	25	23	21
10	1023	49	31	29	27	25

Figure 27. Variant Towers of Hanoi, from Lucas ("Nouveaux jeux scientifiques de M. Édouard Lucas", *La Nature* **17** (1889), 301–303), with a list of best solutions (Joe Celko, Puzzle Column: "Mutants of Hanoi", *Abacus* **1**:3 (1984), 54–57).

the other end, then this variation has a remarkable connection with Hamiltonian circuits which I will leave as an exercise! One can also consider the problem with all movements having to go in the same direction.

There are also unsolved problems. Suppose we have four (or more) pegs (Figures 27, left). Then we can carry out the transfer more easily. Some study reveals a fairly natural method but there are many ways to carry out the transfer in the same number of moves. Hinz has also investigated this and found that this allows one to transfer 64 discs to another peg, in less than six hours using four pegs, compared to some 5×10^9 centuries when using three pegs. No one has yet come up with a proof that this method is really minimal and Knuth suggests that it may be impossible because of the many different ways which give the minimal number of moves. Some best known results are shown on the Figure 27, right.[2]

A number of variations of the Chinese Rings have been devised, several in recent years. In 1891, George E. Everett of Grand Island, Nebraska, obtained a UK patent for the Loony Loop; Figure 28 (left). I have not found a US patent on this. It appeared about 1900 in English puzzle boxes as the Canoe Puzzle. A number of topological variations have appeared more recently. A Hungarian example of the 1980s was called Bogi; Figure 28 (right) is the instructions. Two

[2] We thank one of our reviewers for the following remark: Here the problem on the lower bounds of the number of moves for the k-peg Tower of Hanoi ($k \geq 4$) is unsolved. But in recent years, there have been breakthroughs and it seems to be solved completely for the four-peg case: T. Bousch, "La quatrième tour de Hanoï", *Bull. Belg. Math. Soc. Simon Stevin* **21**:2 (2014), 895–912.

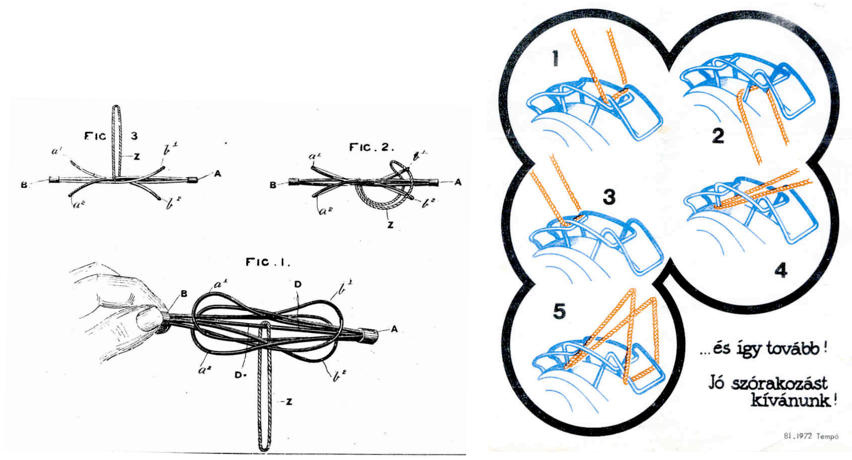


Figure 28. Left: From the patent for the Loony Loop. Right: Bogi instruction sheet.

Figure 29. Brain, bottom of box, and SpinOut, from box.

mechanical versions appeared in the 1980s: The Brain (trademarked by Mag-Nif in 1989); Figure 29, and Spin Out (patented by William Keister in 1972, produced by Binary Arts in 1986); Figure 29 (right).

A similar looking, but quite different, puzzle called Panex, invented by Toshio Akanuma, appeared in Japan in 1983. It looks like two 10-disc piles in two of three channels; see Figure 30 (but the middle channel is not clear). The frame has concealed notches so that a piece cannot move down further than its natural position. With this restriction, one doesn't worry about such trivial matters as putting large discs on smaller. (An essentially identical puzzle was patented in the US

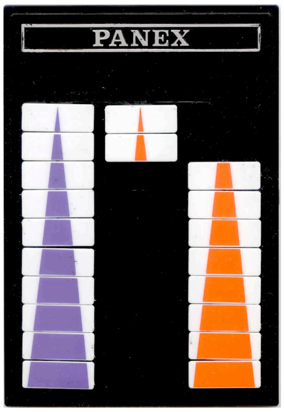


Figure 30. Panex.

by someone else in 1993.) Mark Manasse, Danny Sleator and Victor Wei, at Bell Labs (unpublished study, 1983) have shown that one can move one pile to the centre peg in 4875 moves and one can exchange the piles in a number of moves between 27564 and 31537. They found minimal solutions for up to six levels. At the Fifth Gathering for Gardner in 2002, Nick Baxter gave out a sheet which said that levels seven and eight had been solved, but with current computing power, the next two levels would take 10 and 1200 years. See http://www.baxterweb.com/puzzles/panex.

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