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Wythoff visions

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Six authors tell their stories from their encounters with the famous combinatorial game WYTHOFF NIM and its sequences, including a short survey on exactly covering systems. The volume of the mathematical study of this game is 59% of that of the most ubiquitous game CHESS (MathSciNet). The former originated in 1907, the latter in antiquity. Thus the mathematical study of WYTHOFF NIM may surpass that of CHESS!

1. A modification of the game of NIM

The game of NIM only preceded Wythoff's modification by a few years. By the famous theory of Sprague and Grundy some decades later, NIM drew a lot of attention. WYTHOFF NIM (here also called Wythoff's game), on the other hand, only became regularly revisited towards the end of the 20th century, but its winning strategy is related to Fibonacci's old discovery of the evolution of a rabbit population. The subject has been receiving more attention in recent decades thanks in part to new studies of WYTHOFF NIM and its variants by Fraenkel, and investigations into related sequences and arrays by Kimberling. In this paper we provide surveys of six different aspects of this subject by six of its current masters. Let us recall Wythoff's original definition of the game, given in item 1 in his paper [158]:

> *The game is played by two persons. Two piles of counters are placed on the table, the number of each pile being arbitrary. The players play alternately and either take from one of the piles an arbitrary number of counters or from both piles an equal number. The player who takes up the last counter or counters, wins.*

Wythoff proceeds by designating the *safe* positions (P-positions in current jargon) of his game, first by noting that the heaps are unordered, which implies

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n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n	0	1	3	4	6	8	9	11	12	14	16	17	19	21	22
B_n	0	2	5	7	10	13	15	18	20	23	26	28	31	34	36

Table 1. The first few terms of the A and B sequences.

that (x, y) is safe if and only if (y, x) is, where x and y denote the respective number of counters in each pile. Then he proceeds to the nowadays celebrated *minimal exclusive algorithm* (mex), but without giving it a name. Let U be a finite subset of the nonnegative integers. Then the minimal excludant of U, $\operatorname{mex} U$, is the smallest nonnegative integer not in U.

Theorem 1 (recursive characterization of WYTHOFF NIM's P-positions). *Let* $\{(A_n, B_n), (B_n, A_n) : n \geq 0\}$ *be the set of P-positions of* WYTHOFF NIM. *Then, for all* $n \geq 0$,

$$A_n = \operatorname{mex}\{A_i, B_i : 0 \leq i < n\}, \tag{1}$$

$$B_n = A_n + n. \tag{2}$$

We display the first few terms of the sequences in Table 1.

This result implies an exponential-time winning strategy in the input size $\log(xy)$ of an arbitrary input position (x, y). Wythoff's game has become famous because of the algebraic characterization of the P-positions (item 6 of [158]), the solution via the floor function and the golden ratio, which implies a polynomial-time winning strategy.

Theorem 2 (algebraic characterization of WYTHOFF NIM's P-positions). *A combination of pile sizes of* WYTHOFF NIM *is a P-position if and only if it is of the form* $\lfloor \varphi n \rfloor$, $\lfloor \varphi^2 n \rfloor$, *for some nonnegative integer* n, *and where* $\varphi = \frac{1}{2}(1+\sqrt{5})$ *is the golden ratio.*

It also became famous because of the difficulty to compute the nonzero Sprague–Grundy values [69; 147] (see Section 6 for a definition and some recent progress), and the problem to find a generalization to several heaps that preserves properties of WYTHOFF NIM (see Section 4.3).

We will often encounter the more general concept of a *Beatty sequence* [7]. Let α denote a positive irrational. Then $(\lfloor \alpha n \rfloor)$ is a Beatty sequence, where n ranges over the positive integers. Two or more sets of positive integers are *complementary* if each positive integer occurs in precisely one of them. Such systems are also known under the name of *exactly covering systems*, or *splitting systems*. It is a well-known result that the sets $\{\lfloor \alpha n \rfloor\}$ and $\{\lfloor \beta n \rfloor\}$, where n ranges over the positive integers, are complementary if and only if α, β are positive irrationals satisfying $\alpha^{-1} + \beta^{-1} = 1$. Such a pair of sequences $(\lfloor \alpha n \rfloor)$, $(\lfloor \beta n \rfloor)$ is often called a pair of *complementary Beatty sequences*, although the result was discovered by Rayleigh in the book *The theory of sound* [138] (without giving a proof) and independently proved by Hyslop, Ostrowski and Aitken [8]. WYTHOFF NIM's P-positions give a special case of this, where α is the golden ratio.

In a survey paper [24] concerning the golden ratio, phyllotaxis and WYTHOFF NIM in 1953, Coxeter sketches a simple proof of Theorem 2, recalling the proof of Hyslop and Ostrowski [8], using also Theorem 1. He omits elaborating on the formula for B_n (a similar shortcoming appears in Wythoff's original proof). Namely, the mex-property for A_n holds by the (graph theory) kernel property of the P-positions of an impartial game. For the relation $B_n = A_n + n$, however, an inductive argument of a fill-rule property of diagonal parents of P-positions is also required. It is a geometric argument, and we display the idea in Figure 1, noting that if (x, y) is a P-position, then for example $(x+t, y+t)$ is an N-position for all $t > 0$. This *fill-rule property* is further generalized in a renormalization approach of GENERALIZED DIAGONAL WYTHOFF NIM and LINEAR NIMHOFF; see Section 8.

WYTHOFF NIM has been considered in the context of phyllotaxis more recently in Chapter 17 in the book "Symmetry in Plants" [82] and here the *maximal Fibonacci representation* is used to represent Wythoff's sequences (Adamson's Wythoff Wheel [82, Chapter 17]), while we more often encounter the *minimal Fibonacci representation* (often called the Zeckendorff [159] representation, although it was discovered by Ostrowski [134] and Lekkerkerker [121]) together with left and right shifts, e.g., [140; 38].

In the book *Theory of graphs*, by C. Berge [22], R. P. Isaac mentions the game of WYTHOFF NIM in Example 1 in Section 6, but played with a queen of chess; this variation of the game is often called "Corner the Lady" [60]. Without reference, the winning strategy is mentioned, and for the first time, a nice illustration of the fill rule of the diagonal moves is presented. The picture also gives the initial Grundy values of Wythoff's game. The section concerns "Nim-type games" and illustrates "the kernel of a graph" idea for positions of Grundy value 0.

In 1959, Connell [20] restricted Wythoff's game by requiring to take a multiple of $b \geq 1$ from a single pile. For the P-positions, he obtained b pairs of sequences, each pair consisting of two complementary nonhomogenous Beatty sequences [36].

In 1968, Holladay [79] extended Wythoff's game to a k-WYTHOFF NIM, by the extension of the diagonal rule:

> *take from both piles, but do not take more than k more counters from one pile than from the other.*

This game gives a simple but elegant generalization of both the minimal exclusive description in Theorem 1, and also the algebraic description for the

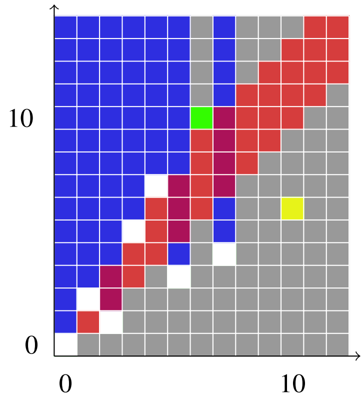

Figure 1. The P-position (6, 10) (green) of WYTHOFF NIM has been computed via a fill rule algorithm, using the P-positions closer to the origin (light). We also indicated the symmetric P-position $(10, 6)$ (yellow). We colored vertical (blue) and diagonal (red) N-positions that have these smaller P-positions as options. View the old P-positions as sources of light, and the N-positions as colored light-beams: the green and yellow cells will next become light sources and invoke new colored beams, giving birth to new green and yellow cells, and so on. Note that the diagonal red beam is tightly packed, and this is part of the induction hypothesis, whereas the vertical beams leave a-periodic gaps, related to the famous rabbit-birth algorithm of Fibonacci: one baby rabbit if and only if we find two *upper* green cells (P-positions) within the same gap (so the picture illustrates an immature rabbit). We omitted the horizontal beams in the picture because they are not required in the algorithm. Neither are the diagonal beams below the line $y = x$ needed. The fill-rule idea gives a nice conjecture for a generalization of WYTHOFF NIM (see also Figure 7).

P-positions in Theorem 2. In fact, he defines four variations to these game rules with exactly the same set of P-positions. The k-Wythoff game was revisited by Fraenkel [38], where computational aspects and connections to continued fractions are emphasized. Holladay also studies related games where at most k counters may be removed from both piles, or any number from just one pile, but these variations do not invoke the fill-rule-property, and therefore neither do they generalize Wythoff's characterizations of the P-positions.

In 1973, Fraenkel and Borosh [49] generalized Wythoff's game in a way that includes both Connell's and Holladay's games, preserving the complementary (nonhomogeneous) Beatty sequences property, and in 2009 Larsson [106; 111]

13	8	5	3	2	1	n	8	5	3	2	1
					1	1					1
				1	0	**2**				1	0
			1	0	0	3				1	1
			1	0	1	4			1	0	1
		1	0	0	0	**5**			1	1	0
		1	0	0	1	6			1	1	1
		1	0	1	0	**7**		1	0	1	0
	1	0	0	0	0	8		1	0	1	1
	1	0	0	0	1	9		1	1	0	1
	1	0	0	1	0	**10**		1	1	1	0
	1	0	1	0	0	11		1	1	1	1
	1	0	1	0	1	12	1	0	1	0	1
1	0	0	0	0	0	**13**	1	0	1	1	0

Table 2. The minimal (no two consecutive 1s) and maximal (no two consecutive 0s) Fibonacci representation of the first few integers, respectively. The bold numbers are the ones in the B sequence. These representations satisfy a number of interesting number theoretical properties. Note that the numbers in the A sequence have even number of rightmost "0"s in the minimal representation. In the maximal representation they are the ones ending in a "1". The so-called left-shift property holds for both representations: the number $B(n)$ is obtained by shifting the digits of $A(n)$ one step to the left and putting a "0" as the least significant bit.

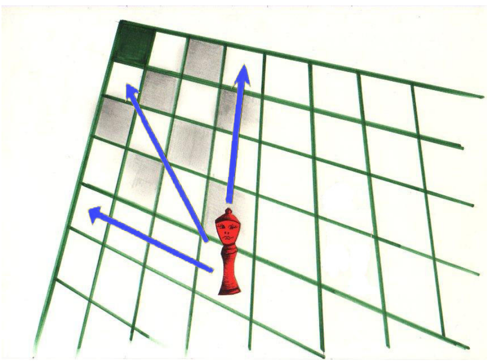

Figure 2. WYTHOFF NIM is often played with a single queen of chess on a semi-infinite chess board. By moving, the queen must get closer to the single corner, labeled $(0, 0)$. Martin Gardner coined the other classical name for this game, "Corner the Lady", and attributed this variation to Rufus P. Isaacs.

found yet three such games, with distinct complementary Beatty sequences. See Sections 4.1 and 8.1, respectively.

As we already noted, Wythoff's game is closely connected to complementary and to disjoint integer, rational and irrational Beatty sequences, a concept which generalizes arithmetic progressions. Such sequences are considered in many papers, with or without references to Wythoff's game, and often concerning exactly covering systems. In Section 2, we review some of this work, which has partly been inspired by Wythoff's sequences but also stem from diverse origins. Starting with E. Duchêne's vision in Section 3, the author's contributions are presented in alphabetical order[1].

2. Exactly covering sequences with some game applications

Obvious exactly covering systems, i.e., partitions of the set of positive integers, are arithmetic sequences, such as $2n-1$, $2n$, $n \geq 1$; or $4n-3$, $4n-1$, $4n$, $n \geq 1$. In these two examples, the two largest *moduli* (2 and 4 respectively) are the same. This is a general property of exactly covering arithmetic sequences: If all the moduli α_i are integers with $m \geq 2$ and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m$, then $\alpha_{m-1} = \alpha_m$. A proof using complex numbers and roots of unity was given by Mirsky, Newman, Davenport and Rado — see Erdös [33]. Since every math paper — even a survey paper! — should contain at least one proof, we present here a "proof from the Book" of their result.

Let $\{na_i + b_i : 1 \leq i \leq m;\, n = 1, 2 \ldots\}$ be an exactly covering system of $m \geq 2$ arithmetic sequences, where $a_1 \leq \cdots \leq a_m$. Consider the generating function $z^{b_i}/(1 - z^{a_i}) = \sum_{n \geq 1} z^{na_i + b_i}$. The fact that the system is exactly covering is expressed by the identity $\sum_{i=1}^{m} z^{b_i}/(1 - z^{a_i}) = z/(1-z)$. Let ξ be a primitive a_m-th root of unity and let $z \to \xi$. If $a_{m-1} < a_m$, then the only unbounded term in the identity is $z^{b_m}/(1 - z^{a_m})$, a contradiction. Hence $a_{m-1} = a_m$. (See [102] for another application of this proof method.).

A first elementary proof of this result was given independently by Berger et al. [14] and by Simpson [141]. *Beatty sequences* are normally associated with irrational *moduli* α, β. Recent studies deal with rational moduli α, β. Clearly if $a/b \neq g/h$ are rational, then the sequences $\{\lfloor na/b \rfloor\}$ and $\{\lfloor ng/h \rfloor\}$ cannot be complementary, since $kbg \times a/b = kha \times g/h = kag$ for all $k \geq 1$. Also the former sequence is missing the integers $ka - 1$ and the latter $kg - 1$, so both are missing the integers $kag - 1$ for all $k \geq 1$. However, complementarity can be maintained for the *nonhomogeneous* case: In [36; 133], necessary and sufficient conditions on α, γ, β, δ are given so that the sequences $\{\lfloor n\alpha + \gamma \rfloor\}$ and

[1] U. Larsson wrote the introduction and edited the paper, assisted by E. Duchêne, A. S. Fraenkel and N. B. Ho.

$\{\lfloor n\beta+\delta\rfloor\}$ are complementary — for both irrational moduli and rational moduli. We are not aware of any previous work in this direction, except that in Bang [4] necessary and sufficient conditions are given for $\{\lfloor n\alpha\rfloor\} \supseteq \{\lfloor n\beta\rfloor\}$ to hold, both for the case α, β irrational and the case α, β rational. Results of this sort also appear in Niven [131], for the homogeneous case only. In Skolem [143; 144], the homogeneous and nonhomogeneous cases are studied, but only for α and β irrational. Incidentally, Skolem set out from the point of view of Steiner systems [148] and discovered Wythoff's sequences, but without making the connection to Wythoff Nim. For related work, see also [66; 12; 13].

Uspensky [155] demonstrated, by using a well-known theorem of Kronecker, that if we have $k>1$ homogeneous Beatty sequences with real moduli $a, b, \ldots$ that partition the positive integers, then $k=2$, $1/a+1/b=1$ and a, b are irrational. Graham [64] later demonstrated that $k=2$ by elementary means in a one page proof.

These investigations spawned the following interesting conjecture [37] (see also Erdös and Graham [34]): If the system $\bigcup_{i=1}^{m}\{\lfloor n\alpha_i+\gamma_i\rfloor\}$, $n=1,2,\ldots$ splits the positive integers with $m\geq 3$ and $\alpha_1<\alpha_2<\cdots<\alpha_m$, then

$$\alpha_i=(2^m-1)/2^{m-i}, \quad i=1,\ldots,m. \tag{3}$$

Graham [65] showed that if all the m moduli are irrational and $m\geq 3$, then two moduli are equal. Thus distinct integer moduli or irrational moduli cannot exist for $m\geq 2$ or $m\geq 3$ respectively in a splitting system.

The conjecture was proved for $m=3$ by Morikawa [126], $m=4$ by Altman et al. [3], for all $3\leq m\leq 6$ by Tijdeman [154] and for $m=7$ by Barát and Varjú [5] and was generalized by Graham and O'Bryant [67]. Other partial results were given by Morikawa [127] and Simpson [142]. Many others have contributed partial results; see Tijdeman [153] for a detailed history. The conjecture has some applications in job scheduling and related industrial engineering areas (just-in-time manufacturing); see e.g., Altman et al. [3], Brauner and Jost [17]; also in [123; 156]. However, the conjecture itself has not been settled.

So, this is a problem that has been solved for the integers, has been solved for the irrationals, and is wide open for the rationals!

The conjecture provides a challenge to find game rules using the sequences as candidate sets of P-positions [48]. Thus, for example, for the rat (rat — rational) game, the P-positions are $\{(\lfloor 7n/4\rfloor, \lfloor 7n/2-1\rfloor, 7n-3),\ n=1,2,\ldots\}\cup\{0\}$. For the related mouse game on two piles, the use for an invariant analogue, a mouse trap [112], became apparent, which brings us to a modern trend in CGT. A typical interest in CGT is, given a finite rule set describing a game, find its P-positions, or also, when possible, its Sprague–Grundy function. A modern trend is to reverse this process: given a subsequence R of nonnegative vectors, is there a game whose set of P-positions is precisely R? Suppose we gave a family of games for which the moves and the outcomes have the same description, for example t-tuples of nonnegative integers (for WYTHOFF NIM $t = 2$). Any such game for which some move cannot be made from all game-positions (sometimes because it would be a move connecting two P-positions) is a *variant* game (e.g., the rat and mouse games). Duchêne and Rigo [31] conjectured that if R is the set of numbers produced by a pair of complementary homogeneous Beatty sequences (with irrational moduli), then there is an *invariant* game whose set of P-positions is R, together with $(0, 0)$. Larsson et al. proved a generalization thereof [116]. Informally, a game is invariant if every move can be done from every position, provided only that the result is a game position. Much earlier Golumb [63] defined the notion of a *vector subtraction game*, which is an instance of the family of invariant games, including WYTHOFF NIM and many other impartial heap games. The *move-size dynamic* games FIBONACCI NIM [157; 117; 118] and IMITATION NIM [105], also have winning strategies related to the P-sequences of WYTHOFF NIM, although they are noninvariant in this sense. From the games' perspective it leads us to a general territory, with many open problems: when do exactly covering sequences or variations hereof provide efficient procedures for the outcomes of nice/short combinatorial games? This type of question is addressed also in three research papers in this book [114; 115; 119], in response to a question of Fraenkel at the BIRS workshop in CGT 2011.

Before we move on, one should note that Stolarsky has contributed some interesting papers related to Beatty sequences; for example, one with Porta and Fraenkel [56], where many curious identities involving φ (= golden section) are proved. For example, the reals $\{n\varphi\}$ are closed under ordinary multiplication, where $\{x\}$ is the fractional part of x. In fact, $\{m\varphi\}\{n\varphi\} = \{k\varphi\}$, where $k = mn - m\lfloor\varphi\rfloor - n\lfloor m\varphi\rfloor$. Another more recent contribution of Stolarsky and Kimberling [100] concerns interesting kinds of convergence. A sequence (x_n) *converges deviously* to L if, in addition to converging to L, it is true that for every real B, there exists $\ell \neq L$ such that $x_n = \ell$ for more than B numbers n. For example, let

$$g(n) = \frac{n}{\varphi\lfloor n/\varphi\rfloor},$$

where $\varphi = \frac{1}{2}(1 + \sqrt{5})$. The sequence $(\lfloor n/\varphi\rfloor)$ is an example of a *slow Beatty sequence*, and $(g(n))$ converges deviously to 1. Next, suppose that sequences w_1 and w_2 partition the positive integers. Suppose further that (a_n) is a sequence such that $a_{w_1(n)} \to L_1$ and $a_{w_2(n)} \to L_2$, so that (a_n) converges if and only if $L_1 = L_2$. Otherwise, (a_n) *diverges partitionally* on w. Define

$$h(n) = n(g(n+1) - g(n)),$$

and for irrational $t > 1$, let $w_1(n) = \lfloor nt \rfloor$ and $w_2(n) = \lfloor nt/(t-1) \rfloor$, these being complementary Beatty sequences. Then $(h(n))$ is partitionally divergent, with $h(w_1(n)) \to 1-t$ and $h(w_2(n)) \to 1$.[2]

3. Wythoff Nim seen as a vector subtraction game, and solved with infinite words

Written by Eric Duchêne

In this section, I have chosen to consider Wythoff Nim under two "visions":

- Wythoff Nim can be considered as an instance of the more general *vector subtraction games* introduced by Golomb in [63]. A large set of variants of Wythoff Nim found in the literature can be seen as instances of Golomb's game. In Section 3.1, I will give a couple of personal results obtained for some particular cases of this game.
- My second vision is about the link between Wythoff Nim and Fibonacci, and more precisely the Fibonacci word. Variants of Wythoff Nim based on other words, such as the so-called *Tribonacci word* will be described.

3.1. Wythoff Nim *as a vector subtraction game.* In [63], Golomb introduced the notion of *t-vector subtraction games*. Given t piles of counters, a *position* of such a game is a t-tuple of nonnegative integers, corresponding to the number of counters in each pile. A *move* is also a t-tuple of nonnegative integers corresponding to the number of counters that are removed from each pile. Let $p = (p_1, \ldots, p_t)$ be a position and $m = (m_1, \ldots, m_t)$ be a nonzero move. The move m can be applied to the position p provided that $m \le p$, i.e., for all i, $m_i \le p_i$. The position resulting from the application of m is the t-tuple $p - m$. Given a set $\mathcal{M}$ of allowed moves and a starting position p, two players alternately apply a move from $\mathcal{M}$. The first player unable to move loses the game. Clearly, Wythoff Nim is an instance of the vector subtraction game with

$$\mathcal{M}_{WYT} = \{(0, i), (i, 0), (i, i) : i > 0\}.$$

In the literature, several games can be seen as instances of the vector subtraction game. In particular for $t = 2$, some of them have a set $\mathcal{M}$ corresponding to a proper subset (restriction) or superset (extension) of Wythoff Nim. Some of these games will be detailed in the following section.

Remark 3. Note that in [31], such games are also called take-away *invariant* games, since the allowed moves do not depend on the initial position (i.e., if $m \in \mathcal{M}$, playing $p - m$ is allowed for any p provided $m \le p$). If invariant

[2] A.S. Fraenkel and U. Larsson wrote this section, and the last paragraph was composed by A.S. Fraenkel and C. Kimberling.

take-away games and vector subtraction games are equivalent, the notion of invariance is devoted to be expanded in a more general context than the one of take-away games.

2-vector subtraction games. Some instances of the 2-vector subtraction game will be considered further in the current paper. The first one is Connell's game [20] (see Section 4.1), defined as

$$\mathcal{M}_{Con}(b) = \{(0, i), (i, 0) : i = kb, k > 0\} \cup \{(i, i) : \ i > 0\}.$$

In other words, Connell's game (of parameter b) is a restriction of WYTHOFF NIM where the Nim moves must be multiples of b. Connell proved that the P-positions of his game can be seen as a set of b pairs of homogeneous Beatty sequences $(A_{i,n}, B_{i,n})$ for $i = 0, \dots, b-1$, such that for all $n \geq 0$,

$$A_{i,n} = \Big\lfloor \Big(n + \frac{i}{b}\Big)\Big(1 + \frac{1}{\alpha})\Big\rfloor, \quad B_{i,n} = \Big\lfloor \Big(n + \frac{i}{b}\Big)\Big(1 + \alpha)\Big\rfloor,$$

where $\alpha = \frac{1}{2}(b + \sqrt{b^2 + 4})$.

Another example is the case of t-WYTHOFF NIM defined by Fraenkel [38], where

$$\mathcal{M}_{GW}(t) = \{(0, i), (i, 0) : i > 0\} \cup \{(i, j) : |i - j| < t, \ i, j > 0\}.$$

For this game, the P-positions can be characterized with an algebraic formula close to the one of WYTHOFF NIM (this formula will be given in Section 4.1).

The games NIM(a, b) [73] (see Section 5), MAHARAJA NIM [120] (see Section 8.4) are other invariant WYTHOFF NIM variations. One can also mention the recent game WYT(F) [58], where f is a given function $\mathbb{N} \to \mathbb{N}$, and defined as follows:

$$\mathcal{M}_{WYT}(f) = \{(0, i), (i, 0) : i > 0\} \cup \{(i, j) : 1 \leq i \leq j < f(k)\}.$$

When the number of moves is finite, one can mention the game where the allowed vectors correspond to the moves of a knight in chess. In that case, the Grundy function was proved to be periodic [6]. The same kind of result is also true for the king and its powers [28] (in other words, this is WYTHOFF NIM where one can remove at most k counters per heap, for a given k).

In [28], Duchêne and Gravier have considered a restriction of WYTHOFF NIM, namely the $[a, b]$-vector game (a, b being two positive integers), which can be expressed as a particular family of 2-vector subtraction games:

$$\mathcal{M}(a, b) = \{(0, i), (i, 0), (ia, ib) : i > 0\}.$$

If $a \neq b$, it is proved that the P-positions of this game are exactly the set $\{(i, i) : i \geq 0\}$. The situation is more tricky when $a = b$. For $a = b = 1$, the game is equivalent to WYTHOFF NIM. In [28], an acceptable exponential algorithm is given to compute the P-positions of the [2, 2]-game. Yet, it seems to us that a closed formula should be available, since it is conjectured that the P-positions follow the progression of $\lfloor \frac{1}{4}(n(3+\sqrt{17})) \rfloor$. Note that the $[a, b]$-game can be naturally extended to n heaps, including the most natural extension of WYTHOFF NIM:

> *Given n piles of counters, both players alternately take either from one of the piles an arbitrary number of counters or from all piles an equal number. The player who takes up the last counter or counters, wins.*

When n is odd, it was shown in [28] that this game has the same set of P-positions as NIM. When n is even, the resolution remains open (except for $n = 2$, i.e., WYTHOFF NIM).

Given a subset K of $\mathbb{N}$, another natural restriction of WYTHOFF NIM, namely WYT$_K$, is the following instance of the 2-vector subtraction game:

$$\mathcal{M}_{WYT}(K) = \{(0, i), (i, 0) : i > 0\} \cup \{(k, k) : k \in K\}.$$

The game WYT$_K$ with $K = \mathbb{N}$ is WYTHOFF NIM. In [26; 53], this game has been investigated for $|K| = 1$. In such a case, the P-positions are known. A full characterization of the $\mathcal{G}$-function is even proved for $K = \{2^k\}$ for some $k \geq 0$, and also for every subset of the powers of 2 including 1. In a certain manner, this characterization shows that these WYTHOFF NIM restrictions are "closer" to NIM than WYTHOFF NIM, since their Grundy functions behave like the one of NIM (i.e., a latin square with a strong regularity).

In [31], another set of 2-vector subtraction games is considered:

$$\mathcal{M}_{DR}(k) = \mathcal{M}_{WYT} \setminus \{(2i, 2i) \mid 0 < i < k\} \cup \{(2k+1, 2k+2), (2k+2, 2k+1)\}.$$

In other terms, these games can be described as follows:

> *Given a positive integer k, either take a positive number from a single pile, or* (i, i) *from both as in* WYTHOFF NIM*, or* $2k+1,\ k > 0$ *from one and* $2k+2$ *from the other, except that the* WYTHOFF NIM *moves of taking* $(2i, 2i)$, $i < k$ *from both are disallowed.*

For this set of games, it was proved [31] that the P-positions can be expressed as a pair of complementary Beatty sequences, as it is the case for WYTHOFF NIM. More precisely:

Theorem 4. *The P-positions of the game* $\mathcal{M}_{DR}(k)$ *are of the form* $(\lfloor n\alpha_k \rfloor, \lfloor n\beta_k \rfloor)_{n \geq 0}$*, where* α_k *is the quadratic irrational number having* $(1; \overline{1, k})$ *as continued fraction expansion, and* $1/\alpha_k + 1/\beta_k = 1$.

Remark 5. For $t > 2$, there are few instances of the t-vector subtraction that are considered in the literature. Some of them were mentioned in [28].

3.2. WYTHOFF NIM *as the "Fibonacci game".* In [29], Duchêne and Rigo introduced a new characterization of the P-positions of WYTHOFF NIM, which deals with the Fibonacci word (see e.g. [9]). Given a two-letter alphabet $\{a, b\}$, take the morphism $\phi : \{a, b\} \to \{a, b\}^*$ defined as follows:

$$\phi(a) = ab, \quad \phi(b) = a.$$

By iterating this morphism from a, we get the famous *Fibonacci word* $w_F = (w_n)_{n \geq 1} = \lim_{n \to +\infty} \phi^n(a)$,

$$w_F = abaababaabaababaababaabaababa \ldots$$

In this word, we will use the convention that the first letter has index 1. For $X = A, B$ (resp. $x = a, b$), we define the sets

$$X = \{X_1 < X_2 < \cdots\} = \{n \in \mathbb{N} \mid w_n = x\}.$$

Roughly speaking, the indices of the letters a (resp. b) in w_F correspond to the sequence (A_n) (resp. (B_n)). In addition, we set $A_0 = B_0 = 0$. According to this definition, the P-positions of WYTHOFF NIM exactly correspond to the sequence (A_n, B_n).

Remark 6. Note that a similar characterization has been obtained [30] for the P-positions of k-WYTHOFF NIM using the morphism

$$\phi'(a) = a^k b, \quad \phi'(b) = a.$$

Since the P-positions of WYTHOFF NIM are correlated to the Fibonacci word, a natural question arose: does there exist a 3-heap game whose P-positions can be coded by the so-called *Tribonacci word* w_T, defined as the unique fixed-point of the morphism $\psi : \{a, b, c\} \to \{a, b, c\}^*$, starting from a:

$$\psi(a) = ab, \quad \psi(b) = ac, \quad \psi(c) = a.$$

Hence w_T starts with

$$w_T = abacabaabacababacabaabacabacabaabacababaca \ldots$$

The first values of the sequence (A_n, B_n, C_n) derived from the Tribonacci word are given in Table 3. In [29], a 3-heap game is built, whose P-positions exactly correspond to the sequence (A_n, B_n, C_n) (with all their permutations adjoined). This game has been called the TRIBONACCI GAME:

Given 3 piles of counters, the rules are the following:

- *Any positive number of tokens from up to two piles can be removed.*

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A_n	0	1	3	5	7	8	10	12	14	16	18	20	21	23	25	27
B_n	0	2	6	9	13	15	19	22	26	30	33	37	39	43	46	50
C_n	0	4	11	17	24	28	35	41	48	55	61	68	72	79	85	92

Table 3. First values of the sequences $(A_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$.

- *Let* α, β, γ *be three positive integers such that*

  $$2\max\{\alpha, \beta, \gamma\} \leq \alpha + \beta + \gamma.$$

  Then one can remove α (*resp.* β, γ) *from the first* (*resp. second, third*) *pile.*
- *Let* $\beta > 2\alpha > 0$. *From position* (a, b, c) *one can remove the same number* α *of counters from any two piles and* β *counters from the unchosen one with the following condition. If* a' (*resp.* b', c') *denotes the number of counters in the pile which contained a* (*resp. b, c*) *tokens before the move, then the configuration*

  $$a' < c' < b'$$

  is not allowed.

In [29; 62; 30; 31], several take-away games were deeply investigated with the use of such words. In many cases, deciding whether a given position is N or P is proved to be polynomial thanks to a numeration system derived from the underlying morphism.

Remark 7. Note that the TRIBONACCI GAME is not invariant (i.e., it is not an instance of the 3-vector subtraction game). In [32], the authors provide an algorithm which decides whether invariant rules could have been proposed to fit this set of P-positions.

4. Some of the ramified depths and wisdoms of WYTHOFF NIM

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This section is a partial survey — *partial* in two senses: It is tailored to our own partial taste (not impartial), and it contains only a small part of the appetizing WYTHOFF NIM curiosities (not comprehensive). In contrast — and in a third sense of partiality — it contains only one study, in Section 4.2, of partial games, more often termed partizan games. Otherwise only impartial games are surveyed: occasionally we refer to properties of all games, not just WYTHOFF NIM. Then all *impartial* games are meant.

Notation 8. The set of all P-positions of a game is denoted $\mathcal{P}$; the set of all its N-positions is $\mathcal{N}$.

There are extensions and restrictions of WYTHOFF NIM.

4.1. ***Three 2-pile extensions.***

(i) A Nim move restriction. Connell [20] restricted Wythoff's game by requiring to take a multiple of $b \geq 1$ from a single pile. For the P-positions, he obtained b pairs of sequences, each pair consisting of two complementary nonhomogenous Beatty sequences [36].

(ii) Nim-move restriction and diagonal move extension/restriction. In [49] we analyzed the following generalization of Connell's game, dubbed bt-WYTHOFF NIM NIM: for fixed positive integer parameters, b and t, remove a positive multiple of b tokens from a *single* pile, or $k > 0$ from one and $\ell > 0$ from the other, provided that $|k - \ell| < bt$, $k - \ell \equiv 0 \pmod b$.

(iii) A diagonal move extension, i.e., t-WYTHOFF NIM:

> *The moves are of two types: remove any positive number from a* single *pile* (NIM *move*)*, or $k > 0$ from one and $\ell > 0$ from the other, provided that $|k - \ell| < t$, where t is a fixed positive integer parameter* (diagonal *move*).

Notice that in (ii), taking $b = 1$ gives t-WYTHOFF NIM and taking $t = 1$ gives Connell's game; in (iii), taking $t = 1$ is classical WYTHOFF NIM, where the *same* amount has to be taken from both piles.

In [38] three strategies are presented for computing the P-positions of t-WYTHOFF NIM:

- Recursive. $\mathcal{P} = \bigcup_{i=0}^{\infty}\{(A_i, B_i)\}$, where $A_n = \operatorname{mex}\{A_i, B_i : 0 \leq i < n\}$, $B_n = A_n + tn$, $n \geq 0$.
- Algebraic. $\mathcal{P} = \bigcup_{n=0}^{\infty}\{(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)\}$, where $\alpha^{-1} + (\alpha + t)^{-1} = 1$, $\beta = \alpha + t$, so $\alpha = \frac{1}{2}(2 - t + \sqrt{t^2 + 4})$, $\beta = \frac{1}{2}(2 + t + \sqrt{t^2 + 4})$.
- Arithmetic. In the exotic numeration system whose basis elements are the numerators of the convergents of the simple continued fraction expansion $\alpha = [1, t, t, t, \ldots]$, the numbers A_i end in an even number of 0s; B_i is the "left shift" of A_i, that is, it is the representation A_i with a 0 adjoined at the end of the representation. For further details see [38].

The input size of any t-WYTHOFF NIM-position (x, y) is $\log(xy)$. The computation for deciding whether $(x, y) \in \mathcal{P}$ is exponential for the first of the three strategies, but polynomial for the last two.

4.2. ***Misère t*-WYTHOFF NIM.** In [39], t-WYTHOFF NIM in misère play was studied. As for normal play, recursive, algebraic and arithmetic strategies were given. Let S_1 and S_2 denote the P-positions for normal and misère play respectively. Curiously, for $t = 1$, $S_1 = S_2$ except for the first two P-positions, where $(A_0, B_0) = (2, 2)$, $(A_1, B_1) = (0, 1)$ and $S_1 \cap S_2 = \varnothing$ for all $t > 1$. Specifically,

- Recursive. For $t = 1$, $(A_0, B_0) = (2, 2)$, $A_n = \text{mex}\{A_i, B_i : 0 \le i < n\}$, $B_n = A_n + n$, $n \ge 1$. For $t > 1$, $A_n = \text{mex}\{A_i, B_i : 0 \le i < n\}$, $B_n = A_n + n + 1$, $n \ge 0$.
- Algebraic. For $t = 1$, $(A_0, B_0) = (2, 2)$, $(A_1, B_1) = (0, 1)$, $A_n = \lfloor \frac{1}{2} n(1 + \sqrt{5}) \rfloor$, $B_n = \lfloor \frac{1}{2} n(3 + \sqrt{5}) \rfloor$, $n \ge 2$. For $t > 1$, $A_n = \lfloor n\alpha + \gamma \rfloor$, $B_n = \lfloor n\beta + \delta \rfloor$, $n \ge 2$, where $\alpha = \frac{1}{2}(2 - t + \sqrt{t^2 + 4})$, $\beta = \alpha + t$, $\gamma = \alpha^{-1}$, $\delta = \gamma + 1$.
- Arithmetic. For $t = 1$, $(A_0, B_0) = (2, 2)$, $(A_1, B_1) = (0, 1)$; for $n \ge 2$, A_n, B_n are the same as for normal play. For $t > 1$, The characterization is too long to state here — see [39].

Also a subset of binary trees, dubbed *cedar trees*, was constructed and used for conducting generalized searches and consolidating the three strategies of WYTHOFF NIM in both normal and misère play.

4.3. ***Multipile* WYTHOFF NIM.** When our interest in combinatorial games first arose, we noticed that WYTHOFF NIM seems to be rather more difficult than NIM in at least two aspects, though both are acyclic two-player games with perfect information and no chance moves:

(i) Computation of the Sprague–Grundy function g.

(ii) Generalization to more than two piles. No generalization seemed to preserve the properties $B_n = A_n + n$ for the P-positions of some two piles, and the role of $\varphi := (1 + \sqrt{5})/2$ (golden ratio) in the strategy.

Study of the 1-values of g and related aspects was done in [15]. Some light was shed on the approximate distance from the nonzero g-values to the 0s by Nivasch [130]. In Dress, Flammenkamp, Pink's work [25], the additive periodicity of the Sprague–Grundy function of WYTHOFF NIM was established: the g-function of each row minus its saltus is periodic. A much simpler proof was given independently by Landman [103] (see also Section 6.1). All of these studies attest to the difficulty of computing the g-function of WYTHOFF NIM.

We asked the experts for an explanation of this discrepancy. We were told that it is due to the nondisjunctive move of taking from both piles simultaneously. We tested this claim by replacing the diagonal move by taking k from one, ℓ from the other for any $k \ne \ell$. To our surprise we saw that the experts were wrong: the P-position strategy remained precisely as that of NIM (though not necessarily the nonzero g-values). This was true for both $k \neq \ell$ fixed, say $(k, \ell) = (4, 7)$, or selecting any $k \neq \ell$ at each move.

Many authors attempted to generalize WYTHOFF NIM to multipile WYTHOFF NIM by taking the same number from couples or triples or all the piles, and many similar variations. None of those preserved the properties (ii) above.

It turns out that taking the same number of tokens from both piles in WYTHOFF NIM is a red herring! Rather, taking k from one and ℓ from the other pile such that $k \oplus \ell = 0$ (so $k = \ell$), where $\oplus$ denotes Nim-addition, is the key for understanding the nature of WYTHOFF NIM: The couples (k, k) are the P-positions of 2-pile NIM. Adjoining them as moves necessarily destroys the P-positions of NIM, since $\mathcal{P}$ of any game is an independent set. The independence follows from the fundamental properties of any acyclic game:

$$u \in \mathcal{P} \Longleftrightarrow F(u) \subseteq \mathcal{N}, \quad u \in \mathcal{N} \Longleftrightarrow F(u) \cap \mathcal{P} \neq \varnothing,$$

where $F(u)$ denotes the set of direct followers of position u in the game. More precise information is given in [16] and [52, §5.1].

These observations led us to the conclusion that the proper set of diagonal moves for N-pile WYTHOFF NIM is the set of P-positions of N-pile NIM (only moves that leave nonnegative pile sizes). This, in turn, led to our conjectures stated below. Its budding is in [40, §6]. See Nowakowski and Guy [75]; see also [41].

Define an N-pile WYTHOFF NIM game as follows:

Given $N \geq 2$ piles of finitely many tokens, whose sizes are $p_1, \ldots, p_N$. The moves are to take any positive number of tokens from a single pile or to take $(a_1, \ldots, a_N) \in \mathbb{Z}^N_{\geq 0}$ from all the piles — a_i from the i-th pile — subject to the conditions:

(1) $a_i > 0$ *for some* i,

(2) $a_i \leq p_i$ *for all* i,

(3) $a_1 \oplus \cdots \oplus a_N = 0$.

The player making the last move wins and the opponent loses.

Notice that WYTHOFF NIM is the case $N = 2$.

For $N \geq 3$, denote the P-positions for N-pile WYTHOFF NIM by

$$(A^1, \ldots, A^{N-2}, A_n^{N-1}, A_n^N), \quad A^{N-2} \leq A_n^{N-1} \leq A_n^N$$

and $A_n^{N-1} < A_{n+1}^{N-1}$ for all $n \geq 0$. The notation is intended to imply that $A^1, \ldots, A^{N-2}$ are fixed.

Conjecture 9. *There exists an integer m_1, depending only on $A^1, \ldots, A^{N-2}$, such that $A_n^{N-1} = \operatorname{mex}(\{A_i^{N-1}, A_i^N : 0 \leq i < n\} \cup T)$, $A_n^N = A_n^{N-1} + n$ for all $n \geq m_1$ where T is a (small) set of integers depending only on $A^1, \ldots, A^{N-2}$.*

Conjecture 10. *There exist integers m_2, a such that $A_n^{N-1} = \lfloor n\varphi \rfloor + a + \epsilon_n$ and $A_n^N = A_n^{N-1} + n$, $-1 \le \epsilon_n \le 1$ for all $n \ge m_2$.*

Both conjectures were proved by Sun and Zeilberger [152] for the special case $N = 3$ and $1 \le A^1 \le 10$. In [52] it was shown, inter alia, that Conjecture 9 implies Conjecture 10. This was also proved, inter alia, by Sun [151, Corollary 4.6]. (In his abstract — not in the paper itself — it is stated erroneously that the two conjectures are proven to be equivalent.) See also Coxeter's work [24], and [54; 59; 40].

4.4. ***Bridges between* NIM *and* WYTHOFF NIM.** Motivated by the difficulty to compute the Sprague–Grundy function g for WYTHOFF NIM and the ease to do the same for NIM, we attempted to bridge these two games with in-between games. In NIMHOFF (hybrid of NIM and WYTHOFF NIM) [53], the diagonal move is restricted in various ways. A closed formula for the Sprague–Grundy function g is given for most of these games. A generalized Nim-sum is given to guarantee the polynomiality of g for these games. A second bridge between the two games is established in [26], which continues the above first bridge. The diagonal moves are restricted by taking k from both piles only if k belongs to a predetermined given set K. The P-positions are computed; it is determined under what conditions on K is $(a_j, a_j + j) \in \mathcal{P}$; the g-function is computed, and the regularity properties of g are studied.

4.5. ***The game of* END-WYTHOFF.** Motivated by the game END-NIM of Albert and Nowakowski [1], we studied END-WYTHOFF in normal play [57]:

> *A position in* END-WYTHOFF *is a vector of finitely many piles of finitely many tokens. Two players alternate in taking a positive number of tokens from either end-pile ("burning-the-candle-at-both-ends"), or taking the same positive number of tokens from both ends.*

A recursive characterization of the P-positions (a_i, K, b_i) is presented. For special cases of the vector K of middle-piles, the recursive characterization can be improved. It is also shown that $b_i - a_i = i$ for sufficiently large i (which holds for all i in WYTHOFF NIM). Further, it is shown that if K is a P-position, then (a, K, b) is a P-position if and only if (a, b) is a P-position of WYTHOFF NIM. Finally, a polynomial algorithm is given for computing the P-positions (a_i, K, b_i).

4.6. ***Extensions, restrictions of* WYTHOFF NIM *preserving its P-positions.*** In the paper [27], we show that no strict subset of rules of WYTHOFF NIM is the ruleset of a game having the same set of P-positions as WYTHOFF NIM [27]. On the other hand, we characterize all moves that can be adjoined while preserving the set of P-positions of WYTHOFF NIM. Testing if a move belongs to such an extended set of rules is shown to be doable in polynomial time.

Many arguments rely on the infinite Fibonacci word, automatic sequences and the corresponding numeration system. With these tools, we provide new two-dimensional morphisms generating an infinite picture encoding P-positions of WYTHOFF NIM and moves that can be adjoined.

4.7. *Rat games.* The general rat game considered here is played on $m \geq 2$ piles. The k-th component of its P-positions has the form

$$\left\lfloor \frac{2^m - 1}{2^{m-k}} n \right\rfloor - 2^{k-1} + 1, \quad k = 1, \ldots, m; \quad n = 1, 2, \ldots \tag{4}$$

The m terms $\lfloor (2^m - 1)/2^{m-k} \rfloor$, $k = 1, \ldots, m$ are called the *moduli* of the system (4).

Rat games (rat — rational) are studied in [48]. For $m = 3$, the rat game is played on 3 piles of tokens. Positions are denoted (x, y, z) with $0 \leq x \leq y \leq z$, and moves $(x, y, z) \to (u, v, w)$, where also $0 \leq u \leq v \leq w$. The following is the essence of the game rules:

(I) Take any positive number of tokens from up to 2 piles.

(II) Take $\ell > 0$ from the x pile, $k > 0$ from the y pile, and an arbitrary positive number from the z pile, subject to the constraint $|k - \ell| < a$, where

$$a = \begin{cases} 1 & \text{if } y - x \not\equiv 0 \pmod 7, \\ 2 & \text{if } y - x \equiv 0 \pmod 7. \end{cases}$$

(III) Take $\ell > 0$ from the x pile, $k > 0$ from the z pile, and an arbitrary positive number from the y pile, subject to the constraint $|k - \ell| < b$, where $b = 3$ if $w = u$; otherwise,

$$b = \begin{cases} 5 & \text{if } w - u \not\equiv 4 \pmod 7, \\ 6 & \text{if } w - u \equiv 4 \pmod 7. \end{cases}$$

Also for $m = 2$ (the "mouse" game), game rules were given there. But for $m \geq 4$, we didn't find "nice" game rules.

I have shown [37] that for every $m \geq 2$, the m sequences of the form (4) split the positive integers into m nonintersecting complementary sets. I further conjectured that this splitting is unique: it is the only system that splits the positive integers with *distinct* moduli [41]; see also Erdös and Lin [66]; and Erdös and Graham [34, p. 19]. The motivation for this study is thus 4-fold:

(i) to try a games approach, which might help to settle the conjecture;

(ii) demonstrate existence of a take-away game whose P-positions depend on *rational* numbers;

(iii) find another analyzable non-NIM take-away game played on more than 2 piles; and

(iv) present another challenge of finding "nice" game rules, given the game's P-positions. (Such a challenge is implicit in Duchêne and Rigo [31] and Larsson et al. [116])

Since "nice" game rules based on a given set of P-positions were mentioned, I feel that a tentative definition thereof should be given here, though this is a survey: The game rules are *nice* if they depend on at most a finite number of the P-positions or range values of functions thereof. Notice that according to this definition, the game rules given above for $m = 3$ are nice. (Nice game rules may, perhaps, be called *invariant* game rules.)

4.8. RATWYT. This is another game played with rational numbers (rat — rational, Wyt — Wythoff). Given a rational number p/q in lowest terms, a *step* is defined by

$$\frac{p}{q} \to \frac{p-q}{q},$$

if $p/q \geq 1$, otherwise

$$\frac{p}{q} \to \frac{p}{q-p}.$$

RATWYT [47] *is played on a pair of reduced rational numbers* $(p_1/q_1, p_2/q_2)$. *A move consists of either doing any positive number of steps to precisely one of the rationals, or doing the same number of steps to both. The first player unable to play* (*because both numerators are* 0) *loses.*

A winning strategy using the Calkin Wilf tree [18] is given.

4.9. *Games played by Boole and Galois.* In [42] we proved the following:

Theorem 11. *Let* $S = \bigcup_{i \geq 0}(a_i, b_i)$, *where for all* $n \geq 0$,

$$a_n = \text{mex}\{a_i, b_i : 0 \leq i < n\},$$

$b_0 = 0$, *and for all* $n > 0$,

$$b_n = f(a_{n-1}, b_{n-1}, a_n) + b_{n-1} + a_n - a_{n-1}.$$

If f *is positive*, *monotone and semiadditive* (*defined in* [42, §4]), *then* S *is the set of* P-*positions of a general* 2-*pile subtraction game with* **constraint function** f, *and the sequences* $A = \{a_i\}_{i \geq 0}$, $B = \{b_i\}_{i \geq 0}$ *have the following properties*:

(i) *they partition the positive integers*;

(ii) $b_{n+1} - b_n \geq 2$ *for all* $n \geq 0$;

(iii) $a_{n+1} - a_n \in \{1, 2\}$ *for all* $n \geq 0$.

The case $f = t$ is t-WYTHOFF NIM considered above.

In [42] we illustrated Theorem 11 with a collection of games based on sequences A and B, including known ones such as Prouhet–Thue–Morse, Hofstadter sequence, and mainly on new sequences. In [44] we gave an assortment of games based on constraint functions over Boolean variables or GF(2) (Galois).

4.10. WYTHOFF-*like games.* We introduced a class of variants of WYTHOFF NIM whose diagonal move is constrained by a function f [58]:

> *Three types of functions f are considered: f a constant, f strictly increasing and superadditive, $f(k) = \sum_{i=0}^{s} a_i k^i$ a polynomial of degree $s > 1$ with nonnegative integer coefficients and $a_0 > 0$.*

A function from the nonnegative integers to the nonnegative integers is *superadditive* if it satisfies $f(k) \geq k$ and $f(k+\ell) \geq f(k) + f(\ell)$ for all $k, \ell \geq 0$. The P-positions are pairs (A_n, B_n), $n \geq 0$, where A_n is computed by the mex function, and B_n is a function of A_n.

4.11. *Harnessing the unwieldy* mex *function.* A pair of integer sequences that splits the positive integers is often — especially in the context of combinatorial game theory such as WYTHOFF NIM-like games — defined recursively by $a_n = \text{mex}\{a_i, b_i : 0 \leq i < n\}$, $b_n = a_n + c_n$ $(n \geq 0)$. A typical problem is this: given integers $0 \leq x \leq y$, decide whether $x = a_n$, $y = b_n$. For general functions c_n, the best known algorithm for this decision problem is exponential in the input size $|\Omega(\log x + \log y)|$.

In [55] we produced a polynomial-time algorithm for solving this problem for the case of approximately linear functions c_n. We call the sequence $\bigcup_{i\geq 0} c_i$ *approximately linear* if there exist real constants α, u_1, u_2 such that $u_1 \leq c_n - n\alpha \leq u_2$ for all $n \geq 0$.

This result solves constructively and efficiently the complexity question of a number of previously analyzed take-away games of various authors.

4.12. *Translations of* WYTHOFF NIM*'s P-positions.* In [51], the translation phenomenon of the P-positions of WYTHOFF NIM was studied. The question was whether there exists a variant of WYTHOFF NIM whose P-positions, except for a finite number, are translations of those of WYTHOFF NIM, forming the set

$$S \cup \{(\lfloor \varphi n \rfloor + k, \lfloor \varphi^2 n \rfloor + k) : n \geq n_0\},$$

where $k \neq 0$, $n_0 \geq 0$ and S is a finite set of pairs of integers. Two variants of WYTHOFF NIM that answer the question for all positive integers k were established:

> *Given $k \geq 1$, in the variant called $\mathcal{W}_k$, each move is either removing a number of tokens from a single pile or removing an equal number of*

tokens from both piles, provided that none of the resulting piles has size less than k*: the move from* (a, b) *to* $(a-i, b-i)$ *with* $\min(a-i, b-i) < k$ *is not allowed.*

The P-positions of $\mathcal{W}_k$ form the set

$$\{(i, i) : 0 \le i < k\} \cup \{(\lfloor \varphi n \rfloor + k, \lfloor \varphi^2 n \rfloor + k) : n \ge 0\}.$$

A variant of $\mathcal{W}_k$ that also exhibits the translation phenomenon is then introduced. Let $0 \le j \le k$.

In the variant $\mathcal{W}_{j,k}$*, each move from a position* (a, b) *with* $a \le b$ *is either removing a number of tokens from a single pile or removing an equal number* $i > 0$ *of tokens from both piles provided that the resulting position* $(a-i, b-i)$ *satisfies both* $a - i \ge j$ *and* $b - i \ge k$*.*

Notice that the two games $\mathcal{W}_{k,k}$ and $\mathcal{W}_k$ are identical. The set $\mathcal{P}$ of $\mathcal{W}_{j,k}$ is identical to that of $\mathcal{W}_k$ for all $j \le k$. It is important to note that the set $\mathcal{P}$ of $\mathcal{W}_{j,k}$ depends only on k.

The other variant of WYTHOFF NIM, called $\mathcal{T}_k$, is as follows:

From a position (a, b) *with* $a \le b$*, one can either*

(i) *remove a positive number of tokens from a single pile, or*

(ii) *remove an equal positive number, say* s*, of tokens from both piles provided that* $a - s > 0$ *and*

$$\left| \left\lfloor \frac{b-s}{a-s} \right\rfloor - \left\lfloor \frac{b}{a} \right\rfloor \right| \le k.$$

Note that the diagonal move (ii) is a restriction of the diagonal move of WYTHOFF NIM. In this move, the condition $a - s > 0$ guarantees that the ratio $(b-s)/(a-s)$ is defined. Thus, when making a diagonal move in $\mathcal{T}_k$, one must ensure that the difference between the ratios of the bigger entry over the smaller entry before and after the move must not exceed k.

Consider the special case $k = \infty$. The game $\mathcal{T}_\infty$ is the variant of WYTHOFF NIM in which the only restriction is that the diagonal move cannot make any pile empty.

The following general question has been proposed: Does there exist a variant of WYTHOFF NIM whose P-positions, except possibly a finite number, are $(A_n + k, B_n + l)$ for some fixed integers $k \ne l$ [51]?

4.13. WYTHOFF NIM *and* EUCLID. The game EUCLID, like WYTHOFF NIM, is played on two piles of tokens, though we usually play it, equivalently, on a pair of positive integers. Unlike WYTHOFF NIM, the integers remain positive throughout; a move consists of decreasing the larger number by any positive multiple of the smaller, as long as the result remains positive. The player first unable to move loses; see Lengyel [122].

Two exotic characterizations of the Sprague–Grundy function (g-function) values of EUCLID's game, in terms of the winning strategy of t-WYTHOFF NIM, are given in [43]. A novel polynomial-time algorithm for computing the g-function for EUCLID is given in Nivasch [129].

5. The game NIM(a, b); recursive solution, asymptotic, and polynomial algorithm based on the Perron–Frobenius theory

Written by Vladimir Gurvich

For any positive integer a and b, a game NIM(a, b) was introduced in [73] as follows:

> *Two piles contain x and y matches. Two players alternate turns. By one move, it is allowed to take x' and y' matches from these two piles such that*
>
> $$\begin{gathered} 0 \le x' \le x, \quad 0 \le y' \le y, \quad 0 < x' + y', \\ \textit{and either } |x' - y'| < a \quad \textit{or} \quad \min(x', y') < b. \end{gathered} \tag{5}$$

In other words, a player can take "approximately equal" (differing by at most $a-1$) numbers of matches from both piles or any number of matches from one pile but at most $b-1$ from the other. This game, NIM(a, b), extends further the game NIM(a) = NIM$(a, 1)$ considered by Fraenkel [38; 39], which, in its turn, is a generalization of the classic game NIM$(1, 1)$ introduced by Wythoff [158]; see also [24].

A position of NIM(a, b) is a nonnegative integer pair (x, y). Due to obvious symmetry, positions (x, y) and (y, x) are equivalent. By default, we will assume that $x \le y$.

Obviously, $(0, 0)$ is a unique terminal position. By definition, the player entering this position is the winner in the *normal* version of the game and (s)he is the loser in its *misére* version.

The normal version of NIM(a, b) was solved in [73]. It was shown that the P-positions (x_n, y_n) are characterized by the recursion

$$x_n = \text{mex}_b(\{x_i, y_i \mid 0 \le i < n\}), \quad y_n = x_n + an; \quad n \ge 0, \tag{6}$$

where $x_n \le y_n$ and the function mex_b is defined as follows:

Given a finite nonempty subset $S \subset \mathbb{Z}_+$ of m nonnegative integers, let us order S and extend it by $s_{m+1} = \infty$ and by $s_0 = -b$ to get the sequence $s_0 < s_1 < \cdots < s_m < s_{m+1}$. Obviously there is a unique minimum i such that $s_{i+1} - s_i > b$. By definition, let us set $\text{mex}_b(S) = s_i + b$; in particular, $\text{mex}_b(\varnothing) = 0$.

n	0	1	2	3	4	5	6	7	8	9
x_n	0	1	3	4	6	8	9	11	12	14
y_n	0	2	5	7	10	13	15	18	20	23

n	0	1	2	3	4	5	6	7	8	9
x_n	0	1	2	4	5	7	8	9	11	12
y_n	0	3	6	10	13	17	20	23	27	30

Table 4. P-positions of NIM(a, b) for $(a = b = 1)$, and $(a = 2, b = 1)$, respectively.

It is easily seen that mex_b is well-defined and for $b = 1$ it is exactly the classic minimum excludant mex, which assigns to S the (unique) minimum nonnegative integer missing in S. Thus, $\text{mex}_1 = \text{mex}$ and (6) turns into the recursive solution of NIM$(a, 1)$ given by Fraenkel [38; 39].

The first ten P-positions (with $x \leq y$) of the games NIM(1, 1) and NIM(2, 1) are given in Table 4.

Furthermore, Fraenkel solved the recursion for NIM$(a, 1)$ and got the following explicit formula for (x_n, y_n): Let $\alpha_a = \frac{1}{2}(2 - a + \sqrt{a^2 + 4})$ be the (unique) positive root of the quadratic equation $\hat{z}^2 + (a - 2)\hat{z} - a = 0$, or equivalently, $1/\hat{z} + 1/(\hat{z} + a) = 1$. In particular, $\alpha_1 = \frac{1}{2}(1 + \sqrt{5})$ is the *golden section* and $\alpha_2 = \sqrt{2}$. Then, it follows that for all $n \in \mathbb{Z}_+$ we have

$$x_n = \lfloor \alpha_a n \rfloor \quad \text{and} \quad y_n = x_n + an \equiv \lfloor n(\alpha_a + a) \rfloor. \tag{7}$$

This recursion implies the asymptotic

$$\lim_{n\to\infty} x_n(a)/n = \alpha_a \quad \text{and} \quad \lim_{n\to\infty} y_n(a)/n = \alpha_a + a.$$

As it was mentioned in [38], the explicit formula (7) solves the game in linear time, in contrast to recursion (6), which provides only an exponential algorithm. Yet, it looks too difficult to solve (6) explicitly when $b > 1$, because of the following bounds from [73]:

$$b \leq x_{n+1} - x_n \leq 2b \quad \text{and} \quad b + a \leq y_{n+1} - y_n \leq 2b + a. \tag{8}$$

For $b = 1$ the difference $x_{n+1} - x_n$ is either 1 or 2, and thus $\alpha_a n$ is a good approximation of x_n. When $b > 1$, it seems harder to find a similar estimate, since the bound of (8) for $x_{n+1} - x_n$ is looser. Although no closed form expressions for x_n and y_n are known in case $b > 1$ (yet), in [11], these values were computed (and thus NIM(a, b) solved) by a polynomial time algorithm based on the Perron–Frobenius theory.

n	0	1	2	3	4	5	6	7	8	9
x_n	0	2	5	9	11	14	17	21	25	27
y_n	0	3	7	12	15	19	23	28	33	36

n	0	1	2	3	4	5	6	7	8	9
x_n	0	3	8	11	15	20	26	29	33	36
y_n	0	5	12	17	23	30	38	43	49	54

Table 5. P-positions of NIM(a, b) for $(a=1, b=2)$ and $(a=2, b=3)$, respectively.

The first ten P-positions (with $x \le y$) are given in Table 5 for $(a = 1, b = 2)$ and $(a = 2, b = 3)$.

The linear asymptotic still holds, not only for $b = 1$ but for $b > 1$ as well. In [73] it was conjectured that the limits $\ell(a, b) = \lim_{n\to\infty} x_n(a, b)/n$ exist for all positive integers a, b and are irrational algebraic numbers. This conjecture was proven in [11]; moreover, the following explicit formula for the limiting values was obtained: The limit $\ell(a, b)$ exists for all positive integers a, b and, when they are coprime ($\gcd(a, b) = 1$), it is given by the fraction $\ell(a, b) = a/(r - 1)$, where $r > 1$ is a unique positive real root of the polynomial

$$P(z) = z^{b+1} - z - 1 - \sum_{i=1}^{a-1} z^{\lceil ib/a \rceil}, \tag{9}$$

which is the characteristic polynomial of a nonnegative integer $(b+1) \times (b+1)$ matrix M that depends only on a and b. For any (coprime) a and b there exist unique integers α and β such that $\alpha \ge 0,\ 0 < \beta \le b$ and $a = \alpha b + \beta$. For example, if $b = 1$ then $\alpha = a - 1$ and $\beta = 1$. The entries $M_{i,j}$ of M for $i, j \in \{0, 1, \dots, b\}$ are defined by

$$M_{i,j} = \begin{cases} \alpha & \text{if } i = 0 \text{ and } 0 \le j \le b - \beta, \\ \alpha + 1 & \text{if } i = 0 \text{ and } b - \beta < j \le b, \\ 1 & \text{if } i > 0 \text{ and } (j + a - i \mod b) = 0, \\ 0 & \text{if } i > 0 \text{ and } (j + a - i \mod b) \ne 0. \end{cases} \tag{10}$$

Note that, for all j, $M_{0,j} = \lfloor (a+j-1)/b \rfloor$. By the Perron–Frobenius theorem, we have $|r'| < r$ for any other root r' of $P(z)$.

The case $\gcd(a, b) > 1$ is easily reduced to the case $\gcd(a, b) = 1$ considered above, since, as it was shown in [73], $x_n(a, b)$ (and, hence $y_n(a, b)$ and $\ell(a, b)$)

$i \downarrow j \rightarrow$	0	1	$\cdots$	$b-\beta-1$	$b-\beta$	$b-\beta+1$	$\cdots$	$b-1$	b
0	α	α	$\cdots$	α	α	$\alpha+1$	$\cdots$	$\alpha+1$	$\alpha+1$
1	0	0	$\cdots$	0	0	1	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\beta-1$	0	0	$\cdots$	0	0	0	$\cdots$	1	0
β	1	0	$\cdots$	0	0	0	$\cdots$	0	1
$\beta+1$	0	1	$\cdots$	0	0	0	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$b-1$	0	0	$\cdots$	1	0	0	$\cdots$	0	0
b	0	0	$\cdots$	0	1	0	$\cdots$	0	0

Table 6. The matrix M.

as well) are uniform functions of a and b; that is,

$$x_n(ka, kb) = kx_n(a, b), \quad y_n(ka, kb) = ky_n(a, b), \\ \text{and} \quad \ell(ka, kb) = k\ell(a, b). \tag{11}$$

The main results of [11] were derived with help of the Perron–Frobenius theorem and the Collatz–Wielandt formula for the nonnegative matrices; see [124, Chapter 8]. Alternatively, these results can be derived from the Cauchy–Ostrovsky theorem; see [136, Theorems 1.1.3 and 1.1.4] and verify that our polynomial $P(z)$ satisfies all conditions of the latter.

For the joint consideration of the normal and misère versions of an impartial game we refer the reader to the books [6] and [19, Chapter 12]. This approach was applied to NIM$(a, 1)$ in [39] and to NIM(a, b) in [72; 73; 74].

However, the results differ in the cases $a = 1$ and $a > 1$. In the case $a = 1$ (for any $b \geq 1$) the set of P-positions P^N and P^M (of the normal and misère versions, respectively) "almost coincide". More precisely, their symmetric difference consists of only six positions:

$$P^N \setminus P^M = \{(0,0), (b, b+1), (b+1, b)\}, \\ \text{while} \quad P^M \setminus P^N = \{(0,1), (1,0), (b+1, b+1)\}.$$

This result was obtained in [39] for $b = 1$ and extended for any positive integer b in [73; 74].

Also in [72; 74] the following three properties were shown for $a = 1$:

(i) from any position of $P^M \setminus P^N$ there is a move to $P^N \setminus P^M$;

(ii) from any nonterminal position of $P^N \setminus P^M$, that is, from $(b, b+1)$ or $(b+1, b)$, there is a move to $P^M \setminus P^N$;

(iii) from any position $(x, y) \notin P^N \cup P^M$, either each set P^N and P^M can be reached in one move, or none of them can.

In the case $a > 1$ (for any $b \geq 1$), the kernel of the misère version is defined by the recursion

$$\begin{aligned} \tilde{x}_n &= \text{mex}_b(\{\tilde{x}_i, \tilde{y}_i \mid 0 \leq i < n\}), \\ \tilde{y}_n &= \tilde{x}_n + an + 1; \quad n \in \mathbb{Z}_+. \end{aligned} \tag{12}$$

This formula was proven in [39] for $b = 1$ and extended to any positive integer b in [73]. Let us notice that formulas (6) and (12) differ just slightly. Comparing them we immediately conclude that for any integer $a > 1$ and $b \geq 1$ the sets of P-positions of the normal and misère versions are disjoint, in contrast to the case $a = 1$; see [72] for more details and, in particular, for the cases $a = 0$ or $b = 0$. According to terminology of [70; 72; 74], NIM(a, b) is a *strongly miserable* game when $a > 1$ and it is *miserable* (but not strongly) when $a = 1$.

Some conclusions and open problems. Two main recursions (6) and (12) are deterministic, yet their solutions (the kernels, or equivalently, the P-positions of the normal and misère versions of NIM(a, b)) behave in a "pseudochaotic way" when $b > 1$. For which other combinatorial games do their kernels demonstrate such behavior? It seems that the four-parametric game NIM$(a, b; p, q)$, introduced in [71], is a good candidate. This game is a generalization of NIM(a, b) and Larsson's NIM(a, p) from [76; 111]; see also [105; 106]. Yet, the class in question might be much larger.

Both recursions (6) and (12) can be solved by a polynomial algorithm based on the Perron–Frobenius theorem. Which other recursions can be solved in such a way?

For $b = 1$ the solutions of both recursions are given by closed formulas, while for $b > 1$ this is hardly possible.

Cases $a = 1$ and $a > 1$ also differ substantially. In the first case, the symmetric difference $P^N \,\Delta\, P^M$ consists of only six positions, while in the second case these two sets are disjoint, $P^N \cap P^M = \varnothing$. In [72; 74] such two types of games are named miserable and strongly miserable, respectively, and simple characterizations for both classes are obtained.

Do recursions (6) and (12) or similar ones have other applications, perhaps beyond game theory?

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6. Sprague–Grundy values and preserving P-positions

Written by Nhan Bao Ho

Recall the definition of Sprague–Grundy values [69; 147] for impartial normal-play games: the value of a position x is the smallest nonnegative integer not in the set of values of the positions that can be reached from x by one move. Thus, the recurrence starts with the value zero of each terminal (final) position. Moreover, a position is a P-position if and only if it has value zero.

Section 6.1 was composed by N. B. Ho and U. Larsson, and Section 6.2 by N. B. Ho.

6.1. ***Additive periodicity of* WYTHOFF NIM*'s Sprague–Grundy function.*** We give an overview of Landman's FSM-based proof of arithmetic periodicity of Sprague–Grundy values of WYTHOFF NIM [103]. An FSM (or finite state machine) is a restricted Turing machine which has a finite memory, and moreover it can only move in one direction, and it cannot print. Its simple features implies that its states must eventually enter a cycle, at least if there is no structured input (i.e., the new state depends only on the previous state).

We are concerned with the Sprague–Grundy function G of WYTHOFF NIM along a fixed y-coordinate, indicated with the horizontal dashed red line in Figure 3, and we want to show arithmetic periodicity. It suffices to show that an FSM, with no input, can compute a function $H(x, y) = G(x, y) - x + 2y$. As we remarked in the previous paragraph, then H must be periodic and hence G must be arithmetic periodic. One can prove that G satisfies the inequalities $x - 2y \leq G(x, y) \leq x + y$, and hence H is bounded, as a function of x, $0 \leq H(x, y) \leq 3y$. As an FSM has finite memory, this bound is necessary. Moreover, Landman shows that it suffices for establishing periodicity. For each $0 \leq y_i \leq y$, it suffices to register the H-values in the sets L'_i (bounded left), D_i (down) and S_i (slant), as in Figure 3.

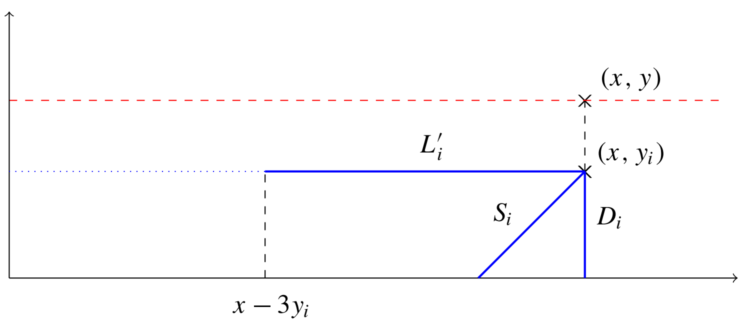

Figure 3. An FSM-analysis of Grundy values of WYTHOFF NIM.

These can be stored in three bit strings of length $3y_i + 1$, where a "1" at index j indicates membership of H-value j. Then the NOR operator finds the smallest index for which there is no "1" in either string, and $G(x, y_i)$ can be computed via $H(x, y_i)$ for each i. Of course, the machine is never concerned with the actual Grundy values, and even if modified with a printing ability, it could not print out G-values because they will become arbitrarily large and hence no finite memory could represent them.

To update the bit strings to binary code $H(x+1, y_i)$, the machine shifts the entries and drops the left most entry. Hence an FSM can simulate the H-function along any given y-coordinate, and so H must be ultimately periodic. The number of bit strings is $O(y)$ and each bit string is of length $O(y)$. Hence the machine requires at most $O(2^{y^2})$ states. Thus the Sprague–Grundy function is ultimately arithmetic periodic along any given y-coordinate (or x-coordinate, by symmetry of WYTHOFF NIM).

Note that a key ingredient for Landman's method is the establishment of the bounds for $G(x, y)$. Using similar bounds, one can apply this technique for other 2-pile NIM-like games. One such example was analyzed in [81].

6.2. *Two variants that preserve P-positions of* WYTHOFF NIM. Duchêne et al. [27] characterize modifications of WYTHOFF NIM that preserves its P-positions (also see Section 4.6). In the context of that work, modifications of WYTHOFF NIM are invariant in the sense that if the move $(a, b) \to (a-i, b-j)$ is allowed then the move $(a, b) \to (a-j, b-i)$ is also allowed, provided that $a \geq j$ and $b \geq i$.

In [77], the author studies two noninvariant modifications, one extension and one restriction, of WYTHOFF NIM preserving its P-positions.

> *In the restriction called* $\mathcal{R}$-WYTHOFF*, the constraint is that removing tokens from the smaller pile is not allowed.*

In other words, from an $\mathcal{R}$-WYTHOFF position (a, b) with $a \leq b$, one can move either $(a, b) \to (a, b-i)$ or $(a, b) \to (a-i, b-i)$. The author also proves that there is no restriction of $\mathcal{R}$-WYTHOFF that preserves $\mathcal{R}$-WYTHOFF's P-positions.

> *In the extension called* $\mathcal{E}$-WYTHOFF*, along with original moves of* WYTHOFF NIM*, there exists an extra move of the form* $(a, b) \to (a-k, b-l)$ *in which* $a \leq b$ *and* $l < k$.

The author also establishes positions whose Sprague–Grundy values are 1 of both games as follows:

$\{(2,2), (4,6), (\lfloor \phi n \rfloor - 1, \lfloor \phi n \rfloor + n - 1) \mid n \geq 1, n \neq 2\}$ for $\mathcal{R}$-WYTHOFF;

$\{(\lfloor \phi n \rfloor - 1, \lfloor \phi n \rfloor + n - 1) \mid n \geq 1\}$ for $\mathcal{E}$-WYTHOFF.

These positions are remarkably close to the P-positions of WYTHOFF NIM, being determined by a translation except for three initial positions in the first case.

The author also proves the following property of Sprague–Grundy functions for both games: for any nonnegative integer a and Sprague–Grundy value g, there exists b such that $\mathcal{G}(a, b) = g$. Actually, this property is equivalent to the following feature of the sequence of positions whose Sprague–Grundy values are g. Let $((a_n, b_n))_{n\geq 0}$ be the sequence of positions whose Sprague–Grundy values are g, in which $a_i < a_j$ if $i < j$. Then the set $\{a_n, b_n \mid n \geq 0\}$ contains every nonnegative integer.

Another feature of Sprague–Grundy values of both $\mathcal{R}$-WYTHOFF and $\mathcal{E}$-WYTHOFF NIM is the additive periodicity of the sequence $(\mathcal{G}(a, n))_{n\geq 0}$ in the sense that there exist n_0 and $p > 0$ such that $\mathcal{G}(a, n + p) = \mathcal{G}(a, n) + p$ for all $n \geq n_0$. This type of periodicity of WYTHOFF NIM is discussed in Section 6.1.

7. The Wythoff array and associated arrays and sequences

Written by Clark Kimberling

In response to an invitation, the author surveys the Wythoff array and its many associates, concentrating on his own contributions and those of others.

7.1. ***The Wythoff array.*** During the first year of existence of the Fibonacci Association, one of the founders published a short article [2] in the first volume of *The Fibonacci Quarterly*. There, Brother Alfred Brousseau discusses an ordering of the set of *all* Fibonacci sequences of positive integers. He concludes with these words: “The above approach in representing Fibonacci sequences and ordering them is all by way of suggestion. There are doubtless other ways of achieving the same objective. It would be very helpful if additional proposals were aired before a final standard is adopted.”

A second ordering of the positive Fibonacci sequences appears in Kenneth Stolarsky’s one-page article [149], in which he introduces an ordering in a form now known as a *Stolarsky array*. Three years later, David Morrison, then a student at Harvard University, published another ordering [128]. If — to borrow Brother Alfred’s words — there is a “final standard”, this must be it. In Morrison’s ordering, the Fibonacci sequences appear as rows of the array shown in Table 7.

Morrison named this array after Wythoff because the rows consist of Wythoff pairs — these being the winning pairs for Wythoff’s game. For example, in row 1, the Wythoff pairs are (1, 2), (3, 5), (8, 13), ...; in row 2, they are (4, 7), (11, 18), (29, 47), ...; and so on. The Wythoff pairs are given by $(\lfloor n\alpha \rfloor, \lfloor n\alpha^2 \rfloor)$, where $\alpha = \frac{1}{2}(1 + \sqrt{5})$, the golden ratio. Properties of the Wythoff array W include the following:

(1) Every row is a Fibonacci sequence; i.e., the recurrence $x_n = x_{n-1} + x_{n-2}$ holds.

(2) The rows extend indefinitely to the left by " precursion" ($x_{n-2} = x_n - x_{n-1}$), resulting in an array that contains every Fibonacci sequence of integers [83].

The array W is an interspersion and a dispersion. The first of these means, briefly, that every row is interspersed by every other row, and the second means that the first column can be used to disperse its complement using certain iterated compositions. More precise definitions [84] follow:

An array $A = A(i, j)$ of positive integers is an *interspersion* if

(I1) every positive integer occurs exactly once in A;

(I2) every row of A is an increasing sequence;

(I3) every column of A is an increasing sequence;

(I4) (u_i) and (v_j) are distinct rows of A, and if i and h are indices for which $u_i < v_h < u_{i+1}$, then $u_{i+1} < v_{h+1} < u_{i+2}$.

To define *dispersion,* suppose that s is an increasing sequence of positive integers, that the complement t of s is infinite, and that $t(1) = 1$. The dispersion of s is the array whose n-th row is

$$t(n), s(t(n)), s(s(t(n))), s(s(s(t(n)))), \ldots$$

The Wythoff array is the dispersion of the complement of its first column; indeed, the odd-numbered columns form the lower Wythoff sequence, and the rest of W forms the upper Wythoff sequence. The main theorem on interspersions and dispersions is that they are equivalent [84].

Another property of W is observed when, for each n, we write the number of the row of W that contains n, resulting in this sequence:

$$f = (1, 1, 1, 2, 1, 3, 2, 1, 4, 3, 2, 5, 1, 6, 4, 3, 7, 2, 8, 5, 1, \ldots).$$

1	2	3	5	8	13	21	34	55	89	144	⋯
4	7	11	18	29	47	76	123	199	322	521	⋯
6	10	16	26	42	68	110	178	288	466	754	⋯
9	15	24	39	63	102	165	267	432	699	1131	⋯
12	20	32	52	84	136	220	356	576	932	1508	⋯
14	23	37	60	97	157	254	411	665	1076	1741	⋯
17	28	45	73	118	191	309	500	809	1309	2118	⋯
19	31	50	81	131	212	343	555	898	1453	2351	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Table 7. The Wythoff array.

1	2	5	13	34	89	233	610	1597	⋯
3	7	18	47	123	322	843	2207	5778	⋯
4	10	26	68	178	466	1220	3194	8362	⋯
6	15	39	102	267	699	1830	4791	12543	⋯
8	20	52	136	356	932	2440	6388	16724	⋯
9	23	60	157	411	1076	2817	7375	19308	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Table 8. The Wythoff difference array.

Deleting the first occurrence of each positive integer leaves the same sequence f. Because this upper-trim operation can be repeated indefinitely and always returns f, this sequence, and any other that arises similarly from an interspersion, is called a *fractal sequence* [89]. The lower-trim of f, obtained by deleting all the 0s from $f - 1$, is also a fractal sequence.

There are several modifications of the Wythoff array W that appear in the Online Encyclopedia of Integer Sequences (OEIS) [132]. Aside from W itself, which is indexed as $A033513$ in the OEIS, the left-justified Wythoff array, $A165357$, formed from W by left extending each row to a pair (a, b) such that $a > b$, has the following property: every (a, b) satisfying $a > b \geq 0$ occurs exactly once, and every (c, d) satisfying $0 \leq c \leq d$ occurs exactly once.

A second array obtained from W is the Wythoff difference array D ([93] and $A080164$), formed by differences between Wythoff pairs in W; see Table 8.

Properties of D include the following:

(1) The difference between adjacent terms in every column is a Fibonacci number.

(2) Every term of column 1 of W is in column 1 of D.

(3) Every term in a row of D, except the first, is in the corresponding row of W.

(4) D is an interspersion.

(5) D is the dispersion of the upper Wythoff sequence $A001950$, whereas W is the dispersion of the lower Wythoff sequence $A000201$.

7.2. ***Zeckendorf arrays and the Wythoff array.*** Every positive integer n is a sum of Fibonacci numbers, no two of which are consecutive. This unique sum is known as the Zeckendorf representation of n. The *Zeckendorf array* $Z = Z(i, j)$ is defined [87] as follows: column j of Z is the increasing sequence of all n whose Zeckendorf representation the least term is F_{j+1}. For example, row 1 of Z is given by

$$z(1, 1) = 1 = F_2, \quad z(1, 2) = 2 = F_3, \ldots, z(1, j) = F_{j+1}, \ldots$$

When working with Zeckendorf representations, it is often helpful to refer to a shift function s defined from a Zeckendorf representation as follows:

$$n = \sum_{i=1}^{\infty} c_i F_{i+1} \Longrightarrow s(n) = \sum_{i=1}^{\infty} c_i F_{i+2}.$$

Using s, it is proved [87] that the Zeckendorf array is identical to the Wythoff array. Related developments, including higher-order Zeckendorf arrays and Zeckendorf/Wythoff trees, are found in Lang's work [104]; Bicknell-Johnson's paper [10]; a paper on Fibonacci Phyllotaxis [146] by Spears, Bicknell-Johnson, and Yan; Cooper's work [23]; and Ericksen and Anderson's paper [35]. Other related arrays are discussed in Hegarty and Larsson [76] and Kimberling [85; 86; 88; 97].

7.3. ***Lower and upper Wythoff sequences.*** The winning solutions of Wythoff's game are the previously mentioned pairs $(\lfloor n\alpha \rfloor, \lfloor n\alpha^2 \rfloor)$. Separating the components gives the lower Wythoff sequence, $L = (\lfloor n\alpha \rfloor) = A000201$ and the upper Wythoff sequence $U = (\lfloor n\alpha^2 \rfloor) = A001950$. Clearly, the terms of L fill the odd numbered columns of the Wythoff array, and those of U, the even numbered columns.

It is easy to write out terms of L and U without reference to an irrational number or Wythoff's game. Consider the rows in Table 9.

The first step is to write row 1 and then to place 1 below the 1 in row 1. Add the two numbers to get $U(1) = 2$. Thereafter, take each $L(n)$ to be the least positive integer not yet in rows 2 and 3, and take $U(n) = n + L(n)$.

A related procedure, from a comment by Roland Schroeder at $A000201$, produces L from a Mancala-type game as follows: n stacks of chips are aligned and numbered from left to right as #1, #2, #3, etc., with stack #n consisting initially of n chips. One step in the game consists of transferring from the leftmost stack all of its chips so that the stacks to the right each gain 1 chip until one of two things happens: either there are no more chips, or otherwise, the leftover chips are used to create new stacks, one chip per stack, lined up to the right of the stacks already present. The game continues until there are n stacks, no two of which have the same number of chips. The number of steps for the whole game is $L(n)$.

A variant of the Schroeder–Mancala game is described in a proposal by Ron Knott, with a solution by Sam Northshield [101]: "As an infinite Mancala game,

n	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$L(n)$	1	3	4	6	8	9	11	12	14	16	17	19	$\cdots$
$U(n)$	2	5	7	10	13	15	18	20	23	26	28	31	$\cdots$

Table 9. Values of n, L, and U.

suppose a line of pots contains pebbles, 1 in the first, 2 in the second, and n in the n-th, without end. The pebbles are taken from the leftmost nonempty pot and added, one per pot, to the pots to the right. Prove that the number of pebbles in pot n as it is emptied is $\lfloor n\varphi \rfloor$, where φ is the golden ratio $\frac{1}{2}(1+\sqrt{5})$."

Another method for generating both L and U is by sending light rays through a certain 2-dimensional array of half-silvered mirrors (the type used in the Michelson–Morley experiment) and seeing which integer points on the coordinate axes become illuminated. See Porta and Stolarsky [135].

Yet another method for generating both L and U is to decree that $L(1)=1$, that $a(n+1)=a(n)+2$ if $a(n)$ is already determined, and that $a(n+1)=a(n)+1$ otherwise. This procedure can be generalized (e.g., [94], $A184117$) to produce Beatty sequences other than L and U. See also [96] and [98].

Quite a different approach to L and U is to arrange in increasing order all the numbers j/α and k/α^2 (or equivalently, $j\alpha$ and k), so that the list begins with

$$\frac{1}{\alpha^2}, \frac{1}{\alpha}, \frac{2}{\alpha^2}, \frac{3}{\alpha^2}, \frac{2}{\alpha}, \frac{4}{\alpha^2}, \frac{3}{\alpha}, \frac{5}{\alpha^2}, \frac{6}{\alpha^2}, \frac{4}{\alpha}, \frac{7}{\alpha^2}. \frac{8}{\alpha^2}, \frac{5}{\alpha}, \frac{9}{\alpha^2}, \frac{5}{\alpha}.$$

Here, for every n, the position of n/α is $\lfloor n\alpha^2 \rfloor$, and that of n/α^2 is $\lfloor n\alpha \rfloor$. The same method works for many pairs of irrational numbers: if $r>1$ and $1/r+1/s=1$, then the positions of n/r and n/s in the joint ranking of all j/r and k/s are $\lfloor ns \rfloor$ and $\lfloor nr \rfloor$, respectively. This may be the shortest route for introducing pairs of Beatty sequences and proving that they partition the positive integers. The method extends to more than two sequences; e.g., Paul Hanna's three-way splitting of $\mathbb{N}$ using a zero γ of $\gamma^3=\gamma^2+\gamma+1$, at $A184820$–$A184822$. See also $A187950$.

It would be of interest to see more of algebraic irrationalities of degree greater than 2 playing a role in this subject. See also Section 3.2, [27], and [137]. But it is the case that the original Wythoff pairs have in fact a (somewhat hidden) quartic algebraic structure. See [150] and [139].

The lower and upper Wythoff sequences both occur in connection with both the greedy and lazy Fibonacci representations of positive integers. We have already discussed the "greedy" case, since the greedy algorithm simply finds the Zeckendorf representation, for which the numbers in L are those whose representation ends with an even number of 0s, and, of course, those in U that end with an odd number of 0s.

Another way to find Zeckendorf representations starts with the sequence $(n)_{\text{base }2}$: simply delete every term that contains "11", so that the remaining terms comprise $(n)_{\text{Zeckendorf}}$. Analogously, deleting every term that contains "00" leaves $(n)_{\text{lazy}}$. It can be shown that

$$(\#\text{ terms in }(n)_{\text{Zeckendorf}}) \leq (\#\text{ terms in }(n)_{\text{lazy}}),$$

as in $A095792$; indeed, the Zeckendorf representation is often called the minimal Fibonacci representation, and the lazy, the maximal Fibonacci representation.

An interesting way to present lazy Fibonacci representations is as a graph consisting of two components, L^* and U^*, each being a binary tree. The tree L^* is rooted in 1, of which the children are $1+2$ and $1+3$. The children of $1+2$ are $1+2+3$ and $1+2+5$, and the children of $1+3$ are $1+3+5$ and $1+3+8$; in general, the children of each

$$m = F_{i_1} + F_{i_2} + \cdots + F_{i_k}, \quad \text{where} \quad i_1 < i_2 < \cdots < i_k,$$

are $m + F_{i_k+1}$ and $m + F_{i_k+2}$. The same rule of generation applies starting with the root 2 of U^*. The numbers in L^* and U^*, taken in order as generated, form the sequences $A255773$ and $A255774$, which are permutations of L and U, respectively; see also $A095903$.

Among the zero-one sequences known as the infinite Fibonacci word is

$$A003849 = (0, 1, 0, 0, 1, 0, 1, \ldots),$$

definable as the fixed point of the morphism $0 \to 01$, $1 \to 0$, starting with 0. The lower Wythoff sequence L tells the positions of 0 in $A003849$, and U the positions of 1. Let S denote the infinite Fibonacci word $A003849$, and let $S(n) = (s(1), s(2), \ldots, s(n))$ be the initial segment of S that has length n. Then $S(n)$ occurs infinitely many times in S. We ask where each appearance starts and answer as follows: for $n = 1$, the segment consists solely of 0, and starts at positions given by L; for $n = 2$ and $n = 3$, the segment starts at positions given by the Wythoff AA numbers $A003622$; for $n = 4, 5, 6$, the segment starts at the Wythoff AAA numbers $A134859$; for the next F_5 numbers ($n = 7, 8, 9, 10, 11$), the segments start at positions given by the Wythoff AAAA numbers $A151915$, and so on. See $A246354$.

Other appearances of U and L are as solutions to complementary equations, as introduced in [91]; that is, equations that can be put into the form $f(a, b) = 0$, where a and b are complementary sequences of positive integers. Four examples, for which the initial condition is $a(1) = 1$ and the unique solution is $a = L$ (or equilvalently, $b = U$) are shown here:

(1) $a(a(n)) = b(n) - 1$,

(2) $a(b(n)) = a(n) + b(n)$,

(3) $b(a(n)) = a(n) + b(n) - 1$,

(4) $b(b(n)) = a(n) + 2b(n)$.

These four equations are used as lemmas for developing more elaborate complementary equations in which the columns of the Wythoff array play a central role

[50; 91; 92; 29; 46; 45; 95]. In particular, Fraenkel introduces the game of *flora* in [45].

7.4. ***Wythoff-related trees.*** Let T_1 be the tree generated by these rules: the root is 1, and for each node x, the children are $\lfloor nx \rfloor$ and $\lfloor nx^2 \rfloor$. The first four generations of T_1 are given by

$$\{1\}, \quad \{2\}, \quad \{3, 5\}, \quad \{4, 7, 8, 13\}, \quad \{6, 10, 11, 18, 12, 20, 21, 34\}.$$

In T_1, every Wythoff pair except $(1, 2)$ occurs as a pair of children. Taken in order of appearance in T_1, the numbers comprise $A074049$.

Next, let T_2 be the tree, essentially $A052499$, generated [90] as follows: $1 \in T_2$, and if $x \in T_2$, then $2x \in T_2$ and $4x - 1 \in T_2$. When all the terms of T_2 are arranged in increasing order and the initial 1 is removed, the remaining even numbers are in positions $1, 3, 4, 6, 8, \ldots$; i.e., the lower Wythoff sequence L and the odd numbers are in positions given by U.

The next tree T_3 contains every integer exactly once. Here, $0 \in T_3$, and if $x \in T_3$, then $2x \in T_3$ and $1 - x \in T_3$, and duplicates are deleted as they occur. As in $A232723$, the numbers in order of generation are

$$0, 1, 2, 4, -1, 8, -3, -2, 16, -7, -6, -4, 3, 32, \ldots$$

The even integers occupy the positions given by L, and the odds by U.

Another tree T_4 gives an ordering of the positive rational numbers. To generate T_4, start with 1, and if $x \in T_4$, then $x + 1 \in T_4$ and $1/x \in T_4$, and duplicates are deleted as they occur. The first few fractions in T_4 are

$$\tfrac{1}{1}, \tfrac{2}{1}, \tfrac{3}{1}, \tfrac{1}{2}, \tfrac{4}{1}, \tfrac{1}{3}, \tfrac{3}{2}, \tfrac{5}{1}, \tfrac{1}{4}, \tfrac{4}{3}, \tfrac{5}{2}, \tfrac{2}{3}, \tfrac{6}{1}.$$

Here, the positions of the positive integers comprise row 1 of the Wythoff array W and the positions of the numbers $n + \frac{1}{2}$ comprise row 2. In general, the positions of denominators ($A226080$) congruent to r mod n, where $0 < r < n$ and $\gcd(n, r) = 1$, comprise a row of W.

In T_4, taken as a sequence, the fractions ≤ 1 occupy positions given by $U - 1$, and those > 1 by $L - 1$. Other trees, including trees consisting of all the rational numbers, all the Gaussian integers, and all the Gaussian rational numbers, are introduced in [99]. See $A226080$ for an overview of such trees.

8. Goldilocks principle in combinatorial games

Written by Urban Larsson

WYTHOFF NIM interacts well with number theory, computer science, physics, biology and more. Its generosity lets us ask questions about rearranging game rules, or we may begin by looking at Wythoff-type sequences or recurrences, and search for games with those sequences as P-positions. We show that new theories get revealed by altering old patterns, keeping some properties and shifting others — usually not too many at a time, just enough to be able to recognize some new features, and not losing sight of known ones. The game truly inspires a multitude of experiments, founded in its connection with the golden mean.[3]

8.1. IMITATION NIM *and* BLOCKING WYTHOFF NIM. In this section we describe various dynamic restrictions of 2-heap NIM and WYTHOFF NIM respectively. In the first variation, the previous move(s) gives the restriction whereas in the subsequent examples the restriction is a temporary blocking of options imposed by the other player (but independently of the previous moves).

The game of IMITATION NIM [105] is a move-size dynamic restriction of the classical game of NIM on two piles:

> *Suppose that the previous player removed x tokens from the smaller heap* (*any heap if they have equal size*). *Then the next player may not remove x tokens from the larger heap. In case of a starting position* (*with no move dynamic restriction*), *any nim-type move is allowed.*

Notice that by this move restriction, the winning strategy of 2-pile NIM is altered. For example, the player who moves from the position $(1, 1)$ will lose in NIM, but win in IMITATION NIM (independent of previous move). It turns out that, regarded as starting positions, the P-positions correspond to those of WYTHOFF NIM. To begin to see this, note that the position $(1, 2)$ does not have a winning move (independent of move-size dynamics). Namely the possible options are the N-positions $(0, 2; 1)$, $(1, 0)$, $(1, 1)$. Here, the move-size dynamic notation $(x, y; z)$, with $x \le y - z$, means that the nim move $(x, y - z)$ is not allowed. Note for example that $(0, 1; 1)$ is a terminal P-position, but $(0, 1)$ is an N-position. Hence, the P-equivalence with WYTHOFF NIM only regards the starting positions of IMITATION NIM.

The game generalizes nicely. Suppose that $k - 1$ consecutive imitations from one and the same player are allowed, but not the k-th one. For example, with $k = 2$ and $0 < x \le y$, suppose that the three most recent moves were $(x, y) \to (x - z, y) \to (x - z, y - z) \to (x - z - w, y - z)$, alternating between the two players. Then precisely the move to $(x - z - w, y - z - w)$ is prohibited.

[3] According to Wikipedia there are many interpretations of Goldilocks principle, of which I pick just two: “In ancient Greek philosophy, especially that of Aristotle, the golden mean or golden middle way or Goldilocks Theory is the desirable middle between two extremes, one of excess and the other of deficiency” and “In cognitive science and developmental psychology, the Goldilocks principle refers to an infant’s [my remark: or scientist’s?] preference to attend to events which are neither too simple nor too complex according to their current representation of the world.” My tentative interpretation: if we depart too far from the rules of Wythoff Nim, we see either unordered chaos or else trivial regularity, but in its neighborhood there is a rich and thriving environment.

The P-positions of this k-IMITATION NIM correspond to those of a variation of WYTHOFF NIM with a certain *k-blocking* on the diagonal options [76].

The author has studied three *blocking* variations of WYTHOFF NIM. The first game has the same set of P-positions as those of k-IMITATION NIM.

> *Let k be a positive integer. The first blocking variation is as* WYTHOFF NIM*, with one exception: the previous player may, before the current player moves, block off $k-1$ of the diagonal type options and declare them forbidden. When a player has moved, any blocking maneuver is forgotten.*

Thus the parameter $k = 1$ gives WYTHOFF NIM. The P-positions of an m-Wythoff type generalization of this game approximate closely pairs of complementary homogeneous Betty sequences of the form

$$\left(\left\lfloor n\frac{\sqrt{m^2+4k^2}+2k-m}{2k}\right\rfloor, \left\lfloor n\frac{\sqrt{m^2+4k^2}+2k+m}{2k}\right\rfloor\right),$$

for positive integers n. However, there is no Beatty-type solution to this game for $k > 1$ [105; 55, Appendix].

Combinatorial games with a blocking maneuver, or so-called Muller Twist, were proposed via the game Quarto in "Mensa Best Mind Games Award" in 1993. Later the idea appeared in the literature [80; 145; 61].

Since a blocking maneuver on the diagonal type options gives rise to interesting sequences of integers [76], we set out to find other natural blocking maneuvers of WYTHOFF NIM. In another study [111], blocking is instead exclusive to the NIM-type options (and three P-equivalent games are defined). The P-positions of these games can be described *exactly* via k complementary pairs of (nonhomogenous) Beatty sequences for all blocking parameters, generalizing simultaneously the P-positions of Connell's and Holladay's games (in the following formulas, put $m = 1$, and $k = 1$ respectively). For positive integers x, let

$$\phi(x) = \frac{2-x+\sqrt{x^2+4}}{2}.$$

The Beatty sequences are $a = (a_n)$ and $b = (b_n)$, for nonnegative integers n, where

$$a_n = a_n^{m,k} = \left\lfloor \frac{n\phi(mk)}{k} \right\rfloor \quad \text{and} \quad b_n = b_n^{m,k} = \left\lfloor \frac{n(\phi(mk)+mk)}{k} \right\rfloor.$$

For a third variation [107], blocking is allowed on any option of WYTHOFF NIM:

> *Let k be a positive integer. The game of k-*BLOCKING WYTHOFF NIM *is played as* WYTHOFF NIM*, except that, before the current player*

moves, the previous player may block off at most $k - 1$ *options. After a move, any blocking maneuver is forgotten.*

In the case of an unrestricted blocking maneuver, an exact formula for the P-positions is known for the case where at most one option may be blocked. For this game, the *upper* P-positions have *split* into two sequences of P-positions, one with slope ϕ, similar to the Beatty type formula for WYTHOFF NIM, and the other with slope 2. A position (x, y) is *upper* if $y \geq x$. Somewhat surprisingly, there is a closed formula expression for the P-positions of the game with at most two blocked options, but then the game becomes harder.

Consider a game with $k = 5$, where the queen is now at (3, 3); yellow in Figure 4. It is player A's turn, and player B is blocking the four positions

$$\{(0, 0), (1, 1), (0, 3), (3, 0)\}$$

(dark brown and light olive). This leaves A with the options

$$\{(3, 1), (3, 2), (2, 2), (2, 3), (1, 3)\}$$

(each is black or blue). Regardless of which of these A chooses, B will then have at least five winning moves to choose from (ones marked yellow, or light, medium, or dark olive). These are winning moves because it is possible when moving there to block all possible moves of the other player and thereby immediately win. Therefore player B will win.

In general this game is hard to analyze, but nevertheless quite interesting due to an elegant self-organization for large blocking parameters, and due to the fact that a generalization of the outcomes can be simulated by a cellular automaton [21]; see Figure 5. We cite, from the abstract:

> "As k becomes large, parts of the pattern of winning positions converge to recurring chaotic patterns that are independent of k. The patterns for large k display a surprising amount of self-organization at many scales."

This self-organization is illustrated in Figure 6. The cellular automaton (CA for short) computes the "palace numbers"$-k$ (the palace number of a given position counts the number of P-positions among the options of Wythoff's queen). It updates diamond shaped cells in parallel, and time is running southeast; see Figure 6.

The main result connecting the CA to the combinatorial game [21] is this: the k-BLOCKING WYTHOFF NIM position (x, y) is a P-position if and only if the CA gives a negative value at that position when the CA is started from an initial condition defined by

$$CA(x, y) = \begin{cases} k & x < 0 \text{ and } y < 0, \\ 0 & x < 0 \text{ and } y \geq 0, \\ 0 & x \geq 0 \text{ and } y < 0. \end{cases}$$

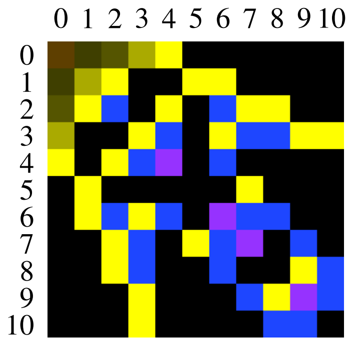

Figure 4. 5-BLOCKING WYTHOFF NIM: (0, 0) is the upper left position on this 11×11 chessboard, and the queen is allowed to move north, west, or northwest. The *palace number* is the number of *palaces* (P-positions) visible, as shown in this picture, and the *surplus number* is the palace number minus k (in this case $k = 5$): $-5 =$ brown, $-4 =$ dark olive, $-3 =$ olive, $-2 =$ light olive, $-1 =$ yellow, $0 =$ black, $1 =$ blue, $2 =$ indigo, and in general, yellowish-olive colors are winning moves (the queen wants to move to her palaces and eat olives) and bluish colors are losing moves.

The behavior of our game/CA generalizes Wythoff Nim in various ways, but mostly proofs are hard to catch. For example: in Figure 6 (the leftmost picture), the uppermost white colored "threads" of P-positions have horizontal thickness "a small Fibonacci number", and, for each such thread, the number of P-positions to the left remains constant.

8.2. ***A* GENERALIZED DIAGONAL WYTHOFF NIM *and splitting beams of P-positions.*** The P-positions of NIM lie on the single *beam* of slope 1, whereas those of WYTHOFF NIM lie on the beams of slopes ϕ and ϕ^{-1}. Therefore, going from NIM to WYTHOFF NIM has *split* the single *P-beam* in NIM into two new P-beams for WYTHOFF NIM of distinct slopes. Let p, q be positive integers. If we adjoin to the game of WYTHOFF NIM new moves of the form (pt, qt) and (qt, pt), for all positive integers t, will the upper P-positions of the new game, denoted (p, q)-GDWN, split once again into two new distinct slopes?

The first paper on a GENERALIZED DIAGONAL WYTHOFF NIM (GDWN) [108] proves that the ratio of the coordinates of the upper P-positions of this game do not have a unique accumulation point if $(p, q) = (1, 2)$ (see also the rightmost picture in Figure 10). Via experimental results it was also conjectured that the upper P-positions of (p, q)-GDWN "split" if and only if (p, q) is either a Wythoff pair or a dual Wythoff pair, that is of the form $(p, q) = (\lfloor \phi n \rfloor, \lfloor n\phi^2 \rfloor)$ or $(\lceil \phi n \rceil, \lceil n\phi^2 \rceil)$, for n a positive integer.

The conjecture was subsequently proved [110] for (1, 2)-GDWN. Two discoveries made this possible. We sketch the idea: Suppose that the sequences (a_i) and (b_i) satisfy a certain property W (roughly $\{(a_i, b_i)\}$ represents the upper P-positions of some WYTHOFF NIM *extension*, where "extension" means that

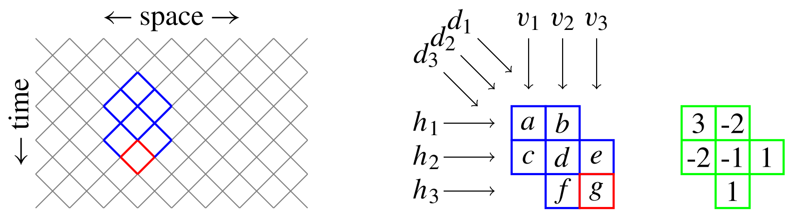

Figure 5. The updates of BLOCKING THE QUEEN's cellular automaton (for references to colors, see an online version of the survey). The red g-cell's value is computed according to the formula $g = a - b - c + e + f + p$, where a, b, c, e, f are the values of previous states, and p represents the total contribution of the *palace compensation terms*. The green squares (to the right) correspond to the blue cells (in the middle) and show the palace compensation terms. If a blue cell contains a palace (a negative value), then the corresponding palace compensation term is "added"; for example, if $a = -1$, $b = -2$, $c = 0$, $d = 2$, $e = -1$, $f = 0$, then $p = 3 - 2 + 1 = 2$ which gives $g = -1 + 2 - 1 + p = 2$. The initial condition of our cellular automaton is given in text.

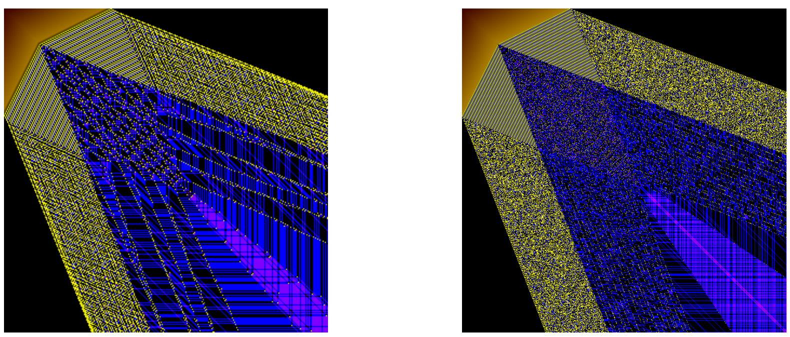

Figure 6. Self-organized regions in 100-BLOCKING WYTHOFF NIM (left, 300×300 region) and 1000-BLOCKING WYTHOFF NIM (right, 3000×3000 region). Here, the upper left corner is the (0, 0)-position.

Figure 7. The fill-rule properties of the rules $(0, t)$, (t, t) and $(3t, 5t)$ respectively, computed for the game (3,5)-GDWN. Apart from the striking dynamics of the fill rules, initial fluctuations give rise to "quasi-log-periodicity" visible in the two left most pictures, with a scale factor ≈ 1.48. For a further discussion and some conjectures, see "Geometric Analysis of a Generalized Wythoff Game" in these proceedings.

new "moves" may be adjoined but no moves have been removed). Then

$$\frac{\#\{i > 0 \mid a_i < n\}}{n} \geq \phi^{-1} - o(1) \tag{13}$$

and

$$\frac{\#\{i > 0 \mid b_i < n\}}{n} \leq \phi^{-2} + o(1),$$

where n tends to infinity. The bound (13) was used to prove that there is a positive lower asymptotic density of x-coordinates of P-positions above the line $y = 2x$, and it was demonstrated that this implies that the upper P-positions $\{(a_n, b_n)\}$ of $(1, 2)$-GDWN split.

Moreover, the conjecture is that there are precisely two accumulation points for the upper P-beams, namely to the ratio of coordinates 1.477 . . . and 2.247 . . . respectively; see Figure 10, the rightmost picture. The conjecture has been further strengthened in the *Linear Nimhoff* project (geometric analysis of a generalized Wythoff game) in this book. This leads us to dwell a bit upon the idea of a "fill-rule property" of a class of (Wythoff) Nim extensions. Central to the hypothesis built in that paper is a renormalization idea from physics. Figure 7 shows some of $(3, 5)$-GDWN's behavior: N-positions obtained by given fill-rules in the respective regions are colored black. The P-positions are not visible, but we can see their impact on the horizontal, $(1, 1)$-diagonal and $(3, 5)$-diagonal N-positions respectively. Each game rule independently and completely fills one designated region with N-positions (the regions are obtained by renormalization equations).

8.3. MAHARAJA NIM *and a dictionary process.* MAHARAJA NIM [120] is an *extension* of WYTHOFF NIM, and a restriction of $(1, 2)$-GDWN, where

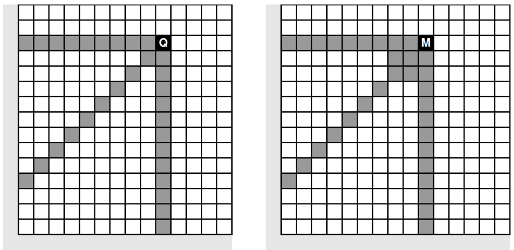

Figure 8. The rules of WYTHOFF NIM, left, and MAHARAJA NIM, right.

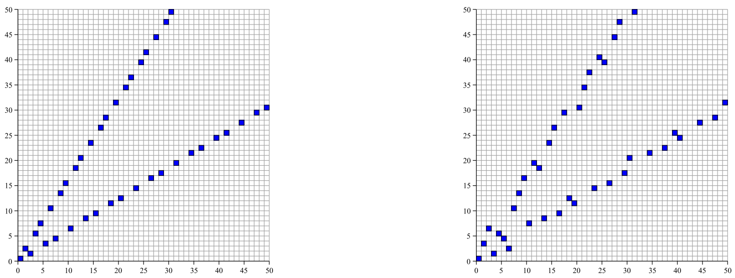

Figure 9. The initial P-positions of WYTHOFF NIM, left, and MAHARAJA NIM, right.

the queen and knight of chess are combined in one and the same piece, the Maharaja (*no coordinate increases by moving*). *See also Figure 8.*

It is clear that the P-positions of WYTHOFF NIM will be altered for this game. Namely, the "smallest" nonzero P-positions of WYTHOFF NIM are $(1, 2)$ and $(2, 1)$, corresponding precisely to the new move options introduced for MAHARAJA NIM.

Figure 9 illustrates that there is indeed a lot of reordering of P-positions in going from WYTHOFF NIM to MAHARAJA NIM. Nevertheless, the P-positions remain within a bounded distance to the half-lines of slopes ϕ^{-1} and ϕ respectively. This is established by relating the upper P-positions to a certain dictionary process on binary words (a process that is proved Turing complete in [120]). The dictionary is constructed by generalizing the "fill-rule property" of WYTHOFF NIM. A binary sequence indicates whether the x-th P-position (x, y_x) is below or above the main diagonal. After a few initial bits, whenever each difference in an interval $0 < y_x - x < n$ is obtained for some $x > 0$, then any new word is written to the dictionary, and the read-head resets. Iteratively, each new word is translated by computing the upper P-positions and using the fill rule property (and the new upper P-positions are copied symmetrically to the lower P-positions).

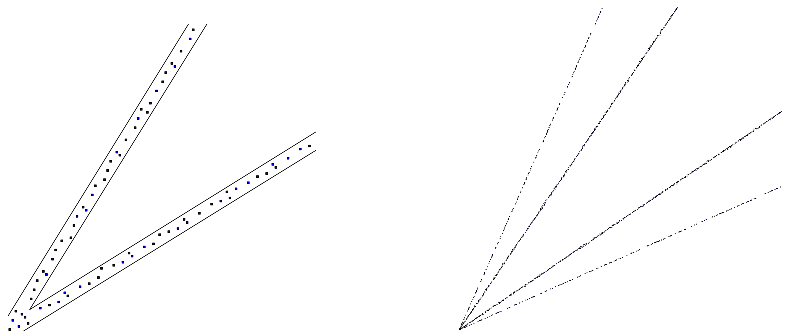

Figure 10. Initial P-positions for MAHARAJA NIM and (1, 2)-GDWN respectively. The leftmost picture shows that MAHARAJA NIM's upper P-positions fits within a narrow stripe of slope ϕ. On the other hand, the P-positions of GDWN to the right will eventually depart from any such stripe, no matter how wide we make it; a renormalization approach (from this book) suggests that the slopes of the upper pair of P-beams of (1, 2)-GDWN satisfy a pair of fourth degree equations, converging to 1.47779977 ... and 2.24772558 ... respectively.

In this way, it is proved that a short Dictionary exists (less than 15 words), and by some lemmata an $O(1)$ bound is obtained as shown in Figure 10, left.

The proof depends on a relaxation of an already very nice result [55] (they require (y_n) increasing), as follows. Suppose (x_n) and (y_n) are complementary sequences of positive integers with (x_n) increasing. Suppose further that there is a positive real constant δ such that, for all n, $y_n - x_n = \delta n + O(1)$. Then there are constants $1 < \alpha < 2 < \beta$ such that, for all n, $x_n - \alpha n = O(1)$ and $y_n - \beta n = O(1)$. See [77] for a variation of MAHARAJA NIM with a surprising connection to Lucas representations of positive integers.

8.4. ***A game creating operator and Wythoff Nim.*** The game of WYTHOFF NIM motivated the definition of a certain *game creating* operator[4], also dubbed the $\star$-operator [116] of impartial vector subtraction games (see Section 3 for a discussion of this general class and its relation to a bigger class of "invariant games"). Let G be a vector subtraction game in any finite dimension. Then $G^\star = \mathcal{P}(G) \setminus \{\mathbf{0}\}$ is another vector subtraction game. If $G = G^{\star\star}$, then we say that G is reflexive. It is easy to see that WYTHOFF NIM is not reflexive. However, WYTHOFF NIM's set of nonzero P-positions constitutes a reflexive vector subtraction game; in Figure 12, the game (WYTHOFF NIM*)*, which is P-equivalent to WYTHOFF NIM [116], is displayed.

[4]Thanks to Silvia Heubach for proposing the new name.

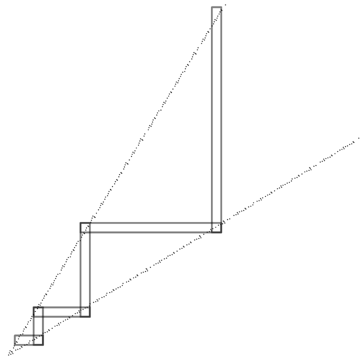

Figure 11. An open question: is it possible to decide in polynomial time whether a given position is in P for MAHARAJA NIM? A "telescope" with focus $O(1)$ and reflectors along the lines ϕn and n/ϕ attempts to determine the outcome (P or N) of some position (x, y) at the top of the picture. The method is successful for a similar game called (2, 3)-MAHARAJA NIM [113]. (It gives the correct value for all extensions of WYTHOFF NIM with a finite nonterminating converging dictionary). The focus is kept sufficiently wide (a constant) to provide correct translations in each step. The number of steps is linear in $\log(xy)$.

More generally, in [31] it was conjectured that, given a pair of complementary Beatty sequences (a_i) and (b_i) (as described in the Section 1), there is an invariant subtraction game for which the P-positions constitute precisely all the pairs (a_i, b_i) and (b_i, a_i), together with the terminal position $(0, 0)$. This was subsequently proved in more generality [116].

8.5. ***Wythoff partizan subtraction.*** One can play a one heap partizan subtraction game using the Wythoff sequences as infinite subtraction sets [125], and this generalizes to the class of *complementary subtraction games* (take your favorite sequence of positive integers as Left's subtraction set and let Right subtract any other positive integer). Here Left subtracts numbers from the lower Wythoff sequence and Right removes numbers from the upper sequence.

This is an example where game outcomes are almost trivial; it is easy to see that each heap is either a Left- or Next-player win (Left wins from heaps in the lower sequence and the Next player wins from heaps in the upper sequence). Playing this game as a disjunctive sum of several heaps leads to a yet rare encounter of Wythoff's sequences with Conway's famous theory for partizan normal-play games. Moreover, one benefits greatly by using a novel theory for game approximation: namely each heap size is either a number or a *reduced canonical form* [68] (equivalence classes ignoring infinitesimals) switch [125].

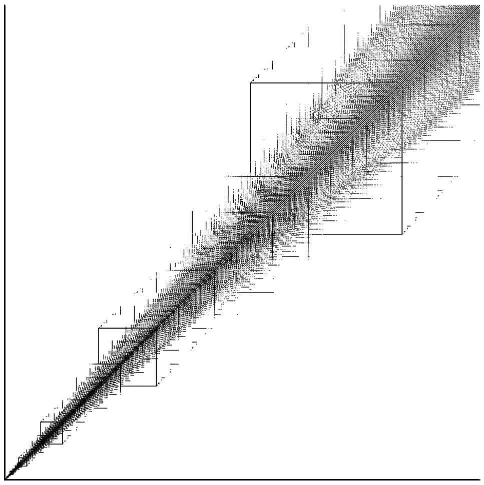

Figure 12. The initial P-positions of the game (WYTHOFF NIM)* (coordinates less than 5000), or equivalently $(0, 0)$ (the lower left corner) together with the moves of the game (WYTHOFF NIM)$^{\star\star} \neq$ WYTHOFF NIM (!). The overall pattern remains a mystery, although one can prove that its chaotic/self-organized part is contained between half lines from the origin of slopes ϕ^{-1} and ϕ. Moreover, a characterization of infinitely many log-periodic positions has been obtained [109]. (WYTHOFF NIM)** is not easily viewed as a "play game", although it is P-equivalent to WYTHOFF NIM. However, the former game has a very nice property, which is absent in WYTHOFF NIM, namely that it is reflexive; that is, (WYTHOFF NIM)**=(WYTHOFF NIM)^{2k*} for all $k \geq 1$. Thus, "simplest" rules do not always give the "nicest" game properties.

The result is accomplished by numerous intertwinings of the classical Fibonacci words and sequences.

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