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Temperatures of games and coupons

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This paper gives an overview of many popular combinatorial games and the temperatures which occur in them. It also includes an explanation of Coupons, a temperature-related construction which has proved very useful in the study of relatively complicated combinatorial games such as Go and Amazons.

Overview

In its broadest sense, *combinatorial game theory* (CGT) is the study of two-person, perfect information games of no chance. For each position in such a game, the theory defines a temperature, which is a measure of the importance of the next move. CGT differs from economic game theory, which emphasizes multiplayer games including elements of chance and imperfect information. Most economic games focus on maximizing some payoff or score; CGT was originally more concerned with getting the last move, but it now also applies to games whose outcomes are determined by scores.

CGT is a branch of mathematics. It seeks to find and understand strategies which can provably succeed against any opposition. This differs from the primary goal of human or computer competitors, who are more focused on making fewer serious mistakes than their opponents. CGT seeks to understand *every* position, including composed problems. It assigns no special importance to any official "opening" position, nor to who gets the first move. Each position is treated as its own game, and both possibilities for who moves next are given appropriate consideration. Most CGT results employ the "divide and conquer" methodology:

(1) partition the board into disjoint regions;

(2) analyze each region, condensing it into an appropriate data structure;

(3) analyze the entire board position as the (disjunctive) sum of these disjoint regions.

The results are so interesting that many combinatorial game theorists now also play and analyze hybrid games, which are sums of positions in different

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games. Such a hybrid sum is called a gallimaufry.

The most successful application of CGT to Anglo-American checkers has been to composed problems (e.g., [Berlekamp 2002]). Elkies [1996] has successfully applied CGT to composed chess problems, and even to at least one position which occurred in a world championship chess match. But in most historical games of chess and checkers, every position that occurs is already as well understood by players who know no CGT as by those who do. But in every one of the other games considered in this paper, most, and sometimes even all, well-played games pass through a sequence of endgame positions about which CGT provides significant extra insights to those who have learned it. In some cases, it provides a complete solution. Details of the histories and rule variants of checkers and many other ancient and modern combinatorial games may be readily found from numerous sources on the web.

Introductions to CGT may be found in *Winning Ways* (WW) [Berlekamp, Conway and Guy 2001–2004] and in [Albert, Nowakowski and Wolfe 2007]. Siegel [2013] provides the major mathematical results and their proofs. His Appendix C is an excellent historical summary of how this subject has evolved from its early roots in ancient board games and in recreational mathematics.

Temperatures

In Winning Ways, the temperature of a game was viewed as a specific number, determined as the base of its thermograph's mast. But in cases where the lower portion of the mast coincides with one or both of its walls, it is now considered more convenient to allow the Left-temperature and the Right-temperature to be viewed as intervals of numbers, whose lower endpoints coincide at the value originally called "the" temperature. It is a measure of the importance of the next move. It can be computed by thermography, a graphical method described in WW and extended in [Berlekamp 1996].

Table 1 lists several combinatorial games and the known temperatures of their positions, listed in approximately descending order of their hottest known positions with finite temperature. More information about these games is summarized in Table 2. The reader is challenged to find improvements and/or corrections to these tables!

I'll now comment on these games, starting with the simplest games at the lowest temperatures and continuing on upwards into the hotter ones.

Temperature -1

Only integers have temperature -1. The "cutcake" family of games at this temperature has solutions which provide challenging examples for beginners.

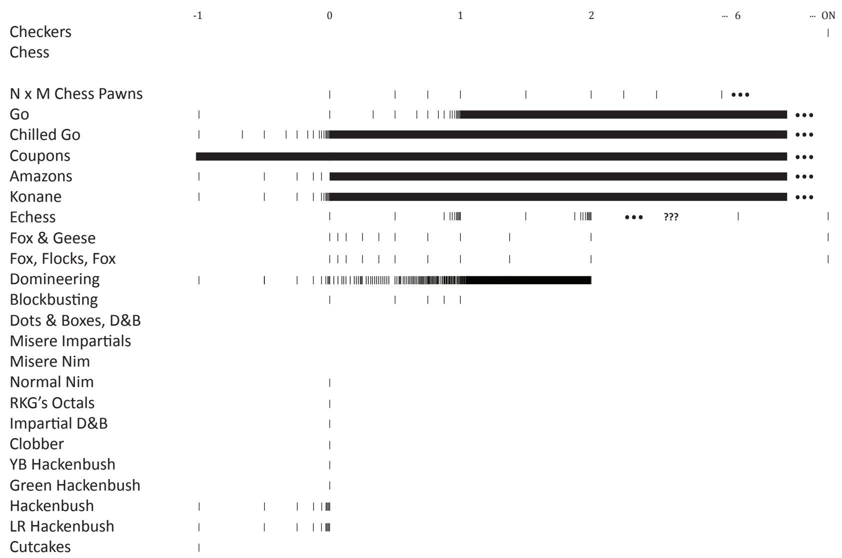

Table 1. Known temperatures of several combinatorial games.

Temperatures −1 to 0

The values of all such normal finite games are numbers. Blue-Red Hackenbush, (alias LR Hackenbush) is the best outstanding example. A basic theorem states that if G is a game for which there are one or more numbers greater than all of G's Left followers, G^L, and less than all of its Right followers, G^R, then G is the *simplest* such number. Among integers, the simplest is the one of least magnitude, and among other numbers, the simplest is the dyadic rational, $J/2^K$, with smallest nonnegative denominator, K. Several structural theorems in WW provide polynomial-time algorithms for sums of strings, trees, spiders, and several other classes of LR Hackenbush positions. But when applied to "Redwood beds", they yield a proof of NP-hardness, as indicated in Table 2.

Temperature 0

After subtracting out its numerical mean, every game of temperature 0 becomes a nonzero infinitesimal. So every game of temperature 0 is number-ish, where "ish" can be viewed as an abbreviation of "infinitesimally shifted". A game of temperature 0 is confused with at most one number, which is its mean.

The most common nonzero infinitesimal, by far, is the game STAR $= \{0 \mid 0\}$, denoted by $*$. It is confused with 0. The most common positive infinitesimal is UP $= \{0 \mid *\}$, denoted by $\uparrow$. Up turns out to be the first of many orders of infinitesimals, several of which appear in the complete, explicit solution of

Combinatorial Games	Guestimated # of Players	Origin	Winner	Loopy ?	★	NP hard ?	Most relevant theory
Checkers	10^8	~3000 BCE	N	L		n/a	c
Chess	10^8	~1810		L		n/a	c
N x M Chess Pawns	10^2	1996	N	-	*8191	?	a c
Go (Total)	10^8				*		
Chinese Weiqi		~2000 BCE	S	?	*	NP	c d e+ f′ g
Taiwan [Ing] Goe		~2000 BCE	S	?	*	NP	c d e+ f′ g
Korean Baduk		~ 800 CE	S	L	*	NP	c d e+ f′ g
Japanese Go		~ 800 CE	S	L	*	NP	c d e+ f′ g
American Go		~1930s	S	L	*	NP	c d e+ f′ g
Mathematical Go	10^4	1989	N	L	*	NP	w c d e+ f′ g
Chilled Go	10^3	1989	N	L	*3	NP	d e f g
Coupons	10^2	1997	S	-	0	P	g
Amazons	10^4	1988	N	-	*3	NP	c d e g
Konane	10^5	Medieval	N	-	★	NP	c d e h
Echess	10^2	~2000 CE			0		w c e+
Fox & Geese, F&G[1]	10^5	Medieval	N	-	?	P?	
Fox, Flocks, Fox	10^2	2002	N	L	?	?	c
Domineering	10^4	~1980	N	-	*3	?	c d e+
Blockbusting	10^3	1984	N	-	*	P	w e+
Dots & Boxes, D&B	10^6	1889	S	-	-	NP	c
Misere Impartials	10^6	<1600	M	-	-	?	c′ i
Misere Nim	10^6	<1600	M	-	-	P	a
Normal Nim	10^6	<1600	N	-	★	P	a
RKG's Octals	10^5	~1950	N	-	?	?	ab c
Impartial D&B	10^4	1967	N	-	?	?	ab c
Clobber	10^3	2001	N	-	?	?	c f
YB Hackenbush	10^2	1993	N	-	*	P	c f
Green Hackenbush	10^4	1971	N	-	★	P	a c
Hackenbush	10^4	1971	N	-	★	NP	c d f
LR Hackenbush	10^4	1970	N	-	0	NP	c
Cutcakes	10^3	1970	N	-	0	P	c

[1] alias Fox & Hounds

N = Normal rule, player unable to move loses
M = Misere rule, player unable to move wins
S = Scored

Entries in the column headed "★" show *K for the largest K known to exist within that game. ★ = Remote star

w	Warming inverts chilling
a	Bouton
b	Grundy-Guy
c	Canonical
c′	Misere canonical
d	Thermography
e	Heating
f	Atomic weights
f′	Liberties as combinatorial games
g	Coupons and orthodoxy
h	Universality
i	Finite Semi-groups

Table 2. Additional information of games listed in Table 1.

sums of strings of Yellow-Brown Hackenbush. A class of much-smaller positive infinitesimals are called *tinies*. Their negatives are called *minies*, although sometimes the word "tinies" may refer to both.

All finite impartial games are infinitesimal; each of them is equivalent to a green Hackenbush string. The value of the string of length N is called $*$ if $N = 1$; otherwise it is called $*N$. There is an important impartial infinitesimal called *remote star*, ★, which is an easy first-player win when played with any sum of finite stars. All members of a common class of infinitesimals, including all that occur in Hackenbush, can be approximated (to within at most 1 or 2), by an integer multiple of $\uparrow$s. This integer is called its atomic weight. Using a definition which includes ★ in a basic way, the atomic weight becomes a homomorphism from infinitesimals to all games. The importance of atomic weights is exemplified in a popular game called *Clobber*, all of whose positions are infinitesimals but whose atomic weights span a very wide variety of games.

Although the outcome of conventional (i.e., partizan) Dots and Boxes is determined by scores, many of its positions depend on the outcome of impartial Dots and Boxes, in which the loser is the player who is unable to complete his turn. Both variants of Dots and Boxes have an unconventional rule: you must continue making moves until you complete your turn with a move which completes no box(es). This gives rise to a peculiar value, called *loony*, denoted by ☽. It has the property that the next player to move can either play the loony to zero and complete his turn, or play the loony to zero and continue to move elsewhere. ☽ is an idempotent, ☽ + ☽ = ☽. It is a win for the first player: OUTCOME(☽) = first player, unlike OUTCOME(0) = second player. But when added to any nonzero game, either impartial or partizan, ☽ is negligible. It has even less effect than any tiny, in the sense that OUTCOME(G+☽) = OUTCOME(G) unless $G =$ 0. So ☽ is an infinitesimal, and like all infinitesimals, its temperature must be 0.

Temperatures up thru 1

Cooling is a homomorphism that reduces temperature while preserving the mean. Cooling by any positive amount transforms all infinitesimals to 0. In general, it is therefore a many-to-one homomorphism.

Intrigued by $2 \times N$ and $3 \times N$ Domineering, Berlekamp [1988] invented a simplified related game called Blockbusting, which is played on $1 \times N$ strips, each of whose ends is either L or R. I discovered that all but one of Blockbusting's positions become a number when *chilled* (i.e., cooled by one degree), whence the recursions to solve chilled Blockbusting are much simpler than the computations to solve Blockbusting. Almost all values of chilled Blockbusting are all dyadic rationals, and the sequences of them are arithmeticoperiodic.

Whereas in typical games cooling and chilling are many-to-one homomorphisms, Blockbusting satisfies two atypical conditions: All scores in Blockbusting are integers and $*$ is the only infinitesimal. Hence, chilling is only a two-to-one homomorphism. The parity turns out to be resolvable, and so chilling is reversible by an appropriately defined operator called *warming*. This yielded explicit tractable expressions for all Blockbusting positions.

Temperatures up thru 2

Domineering. Further refinements of heating operators and their application to Blockbusting values led to precise solutions of infinitely many $2 \times N$ and $3 \times N$

Domineering games, and to many others within "ish". These analyses remain far from complete. Kim [1996] included constructions and analyses of several new sequences and a summary of all classes of positions whose values were then known explicitly. More temperatures were found by Shankar and Sridharan [2005]. Drummond-Cole [2004] discovered a Domineering position which has temperature 2, the hottest value yet found.

Drummond-Cole [2005] composed a position of value $*2$, which could not be realized by legal moves starting from any empty rectangular board. Another position of value $*2$, which was realizable, was found by Uiterwijk and Barton [2015]. This explains Domineering's $*3$ in Table 2.

Fox and Geese. Although there are loopy positions in Fox and Geese, they are all stoppers with tractable canonical forms. The simplest two of these are $\text{OVER} = \{ 0 \mid \text{OVER} \}$ and $\text{OFF} = \{ \ \mid \text{OFF} \}$. Every position in which the fox has escaped is easily seen to have value OFF, so the outcome of any Fox and Geese position on any $N \times 8$ board is reduced to deciding whether the value of the position is finite (a win for the Geese), or OFF (a win for the Fox), or infinitely HOT as in $\{ x \mid \text{OFF} \}$, (a win for the next player to move). OVER and OFF are idempotents. The temperature of $\{ x \mid \text{OFF} \}$ is ON.

With reasonable play, Uiterwijk and Barton [2015] showed that by retreating when appropriate, the Fox can easily avoid getting trapped anywhere except the bottom edge of the board, or possibly in the side square of the double-corner adjacent to the bottom of the board. So modifying the rules to prohibit the Fox from moving into a simple trap along any other side or top edge has no significant effect on the game. It is this modified variation of Fox and Geese whose known temperatures are shown in Table 1. If one also allows composed positions in which the Fox is about to be trapped along the side edge on an $N \times 8$ board, then there is such a position which has $G^L = 2N$; $G^R < 2$; and temperature exceeding $N - 1$.

Fox, Flocks, Fox. This game is the sum of two games, in one of which a traditional flock of four black geese seeks to trap a white fox, and another in which four white geese seek to trap a black fox on the other side of the board. It can be conveniently played on a single 8×8 board, viewed as two 4×8 boards. When the two starting positions are symmetric, their values are putative negatives of each other. Second player can survive indefinitely by responding to each move with its opposite, so the question becomes whether first player can force a draw. He can iff a loopy position (i.e., OVER, or OFF) appears anywhere in the canonical form. The game of Fox, Flocks, Fox provides interest and relevance in the canonical values of Fox and Geese, many of which are given in Chapter 20 of 2nd edition of WW for four geese versus a fox on an $N \times 8$ board. The temperatures of these positions are shown in Table 1, but that list might be incomplete.

Temperatures slightly higher

Entrepreneurial chess (Echess) by Berlekamp and Low [2017] is naturally played on an infinite board, comprising one quarter of the entire plane. Its hottest known finite temperature is $5\frac{5}{8}$. Many Echess positions have relatively tractable canonical values. When cooled or quenched (i.e., cooled by 2), many become a number plus an infinitesimal of integral atomic weight. When played alone, the outcome depends only on the stopping positions. Each of the finite stopping positions is an integer plus an additional OVER. As in Blockbusting and Go, there is another warming operator which inverts chilling.

Echess becomes more interesting when played as a component of a gallimaufry. The set of Echess' thermographic spectral lines shown in Table 1 is conjectured to be incomplete.

Temperatures yet higher

Coupons. As we look at earlier positions of a well-played endgame of Amazons or Go, we often find positions which are both hotter and more complicated.

Coupons were initially developed to facilitate more quantitative discussions with expert human Go players. It turned out that they are also helpful in studying other games, including Amazons.

When played in isolation, the game of Coupons is degenerate because there is never more than one legal move, which is to take the top coupon. The game becomes much more interesting when played as a summand added to another game, such as Amazons or Go. However, the analyses of Coupon Amazons and Coupon Go both depend on the following analysis of an ideal stack of Coupons played in isolation.

This game is played with a stack of coupons. Each coupon has a value printed on its face. The coupons in the stack are in monotonic nonascending order. The *ideal stack* consists of two large substacks. The bottom substack is a large number of coupons of the same terminal temperature, T_0. The top substack, in ascending order, contains coupons of these values: $T_0 + \frac{1}{2}\delta$, $T_0 + \frac{3}{2}\delta$, $T_0 + \frac{5}{2}\delta, \ldots, T_{\text{top}}$. Every consecutive pair of coupons above T_0 has values which differ by the same amount, δ. At any intermediate stage of play, when the top coupon has value C, the stack is said to have an *ambient temperature* of $(C + \frac{1}{2}\delta)$, unless the top coupon has value T_0, in which case the ambient temperature is T_0. In all other cases, the ambient temperature is the average of the previous coupon and the next coupon. We also define a *current komi*, whose value is half the current ambient temperature.

Consider a game of Coupons played on a consecutive subset of the ideal stack described above, with the ambient temperature running from T_2 down to T_1. Without komis, a simple calculation shows that the net sum of all those consecutive coupons will be $\pm\frac{1}{2}T_2 \pm \frac{1}{2}T_1$, where the signs depend on who moves first and who moves last, respectively. To make the game fair, we need the komis. The interval's initial komi of magnitude $\frac{1}{2}T_2$ is assigned to the opponent of the player who makes the first move, at an ambient temperature of T_2; its terminal komi, of magnitude $\frac{1}{2}T_1$, is assigned to the opponent of the player who makes the final move in this interval at an ambient temperature of T_1. When the game is over, your score is the sum of all coupons you have taken, including komis. When this ideal interval coupon game is played in isolation, the net final score will be precisely zero.

When Coupons are played with any chosen (possibly composed) starting position, the ideal initial temperature should be large enough that both players will take a few coupons before either chooses to play on the board. The terminal temperature should be $T_0 = -1$, and the number of coupons of this temperature should exceed the number of empty squares on the Amazons position. When the terminal temperature is reached, at every turn the player will prefer to fill a point of his territory on the board rather than take the -1 point coupon. So when the game eventually ends, all scores on the board will have been converted into the coupons. The winner is the player with the higher score. The terminal komi obviates any advantage or disadvantage of getting the last move. So a tie is a possible outcome. This has effectively converted a combinatorial game with normal termination rule into an economic-style game whose outcome is determined by scores. One advantage of this viewpoint is that *both* players now have well-defined optimal strategies. With the normal termination rule, it is hard to define a good strategy for the losing player, as all of his possible strategies will lose against an optimal opponent.

Expediting play with thicker stacks. To simplify the analysis when Coupons are added to another game, it is convenient to let δ to be a small nonnegative number. If δ is small but positive, the number of coupon moves will be so large that it is convenient to expedite the game by the following procedure, which has no effect on the eventual score.

Whenever the players take three coupons on consecutive turns, the game is interrupted. (The reason that we do not interrupt after only two coupons is that in some ko positions in Go, one player may use coupons as ko threats). The opponent of the player who took the third of these three coupons is awarded the current komi. If the ambient temperature is T_0, the game is terminated. If not, each player is required to submit a sealed bid $\geq T_0$. The bid must be the temperature of a coupon which, if play continued, it would be his turn to take. The winning bidder and his bid are announced. Larger coupons are removed from the stack. The opponent of the winning bidder is awarded the new komi. Unless the new temperature is T_0, the winning bidder is required to make a move on the board, and play resumes. If instead the winning bid is T_0, then the winning bidder may either take a coupon or play on the board, and play resumes until three consecutive coupons of value T_0 are taken, at which point the terminal komi is awarded and the game ends.

Infinitely thick stacks. Using these conventions, it is feasible to play with $\delta = 0$. There is no physical stack of coupons. Instead, only the current ambient temperature is relevant. Initially it is so large that the first three moves take coupons, followed by a komi, an auction, another komi, and the first move on the board. The players may then either play on the board or take a coupon at this temperature. After three consecutive coupons are taken at the current temperature, a komi is awarded; a bidding auction yields a new lower temperature, at which another komi is awarded and play resumes. If the auction ends in a tie, you may let your opponent break it arbitrarily.

Encores. The first play at a negative ambient temperature begins a phase of the game called the *encore*. As explained in Berlekamp and Wolfe [1994], the encore is often lengthy, tedious, and dull. The easiest way for players to avoid it is to agree on a forecast of how it would turn out if played. This is usually straightforward when the temperature is 0. However, there are rare examples in which even very good players may not agree. Some Amazonian territories are defective. Other Amazonian territories may have a value that might not be obvious even to very good players, such as the value $\frac{1}{16}$ in [Snatzke 2002]. In Coupon Go, even though the score on the board at the beginning of an encore is an integer, the appendix of the 1989 Japanese rules contain several interesting examples in which the value of that score has been debated. The mathematically simple procedures stated in this paper resolve all such disputes by continued play of the encore. In the very rare cases in which different dialects of Go yield different scores, the results of the encore with infinitely thick coupons tend to be more consistent with Chinese or American scoring than with Japanese scoring.

Orthodoxy. A theorem states that if $\delta = 0$, the optimum final score is the mean value of the board's starting position. Each player has a strategy which ensures an outcome at least that good for him. Moves which are consistent with any such strategy are called *orthodox* moves. The orthodox viewpoint yields much simpler game graphs than the canonical viewpoint. It also sometimes enables refinement of the decomposition of the board into "independent" regions. Even when playing a Go endgame in a traditional way, without coupons, orthodox accounting in Berlekamp [1996] facilitates a prediction of the final net score and an itemization of how much of this score is due to each region of the board and how much is dependent on who gets the next move. Refinements facilitate locally computable quantitative estimates of the values of kos. For further discussion of orthodoxy see this link: https://math.berkeley.edu/~berlek/pubs/videos.html.

In Amazons, Berlekamp [2000] composed a hard problem featuring four opposing pairs of Amazons, each pair in a region of size 11×2 or smaller; Snatzke [2002] built a database big enough to analyze each of them. Several had canonical forms with many thousands of positions, yet their thermographs were very simple, with temperatures up to about 5. In Go, a team of three expert combinatorial gamesmen and two Go players of the highest rank (9p) spent several months analyzing a position they had encountered 66 moves before the end of a full game. It was published by Spight [2002]. When they had played it, they had estimated the temperature as slightly under 4, but analysis showed it was actually about 5. The determination of the temperature of another region entailed the compilation and study of a computer database of over 20000 positions. The answer was 3.42.

Amazons

Amazons is a conventional loop-free combinatorial game with the normal ending condition. Hence, it has no positions of infinite temperature, although on large boards it contains positions with arbitrarily large temperatures. It has numbers and interesting infinitesimals. It includes positions which have very complicated canonical values but simple orthodox values. Its largest known negative temperature is $-\frac{1}{16}$, found independently by Snatzke [2002] and Tegos [2002].

Tegos [2002] also found positions on a 4×4 board with positive temperatures whose denominators were 256. Song and Mueller [2015] provide more results and references, with a primary focus on who can win from certain conventional starting positions on rectangular boards of various sizes. The definitive list of positive Amazonian temperatures will evidently require more resolution than the printing constraints of this journal can provide in Table 1.

Go

As explained in the Appendices of Berlekamp and Wolfe [1994], there are many dialects of the rules of Go. The mathematical study of this game is complicated by the common occurrence of positions called “kos”. More information about that can be found in [Berlekamp and Kim 1996] and in [Spight 2003].

Unlike chess, where White gets the first move, in Go it is Black who gets the first move. In chess, it is now widely believed that despite White’s advantage, Black has a reasonable chance of getting a draw. But in Go, modern experts believe that if uncompensated, Black’s first-move advantage would be decisive.

Hence, in most modern professional tournaments White is given a special compensation of 6.5 points, which is added to his score. This is called the “komi”. If we presume that the temperature of the empty 19×19 Go board is about 13, then this “komi” plays approximately the same role in conventional Go as what we call the “initial komi” in Coupon Go. Berlekamp [1996] explains how the absence of other komis and coupons in conventional Go corresponds to comparable terms in an “orthodox accounting” which, in principle, itemizes the score of a well-played Go endgame in terms of each move and each region of the board.

Chess pawns

This is a degenerate form of chess, in which all pieces are pawns, and the “normal” objective is to get the last move.

$N \times 1$ chess pawns is a degenerate form of chess pawns, in which there is only one file. Its positions provide all of the temperatures shown in the relevant row of Table 1. In Table 2, the construction of $\ast 8191$ is described by Elkies [2002].

Chess

Noam Elkies [1996; 2002] has shown that some real chess positions, including a nontrivial one that occurred in a world championship match, simplify to sums of positions in the simpler game I’m now calling “chess pawns”. Carlos Santos [2015] has composed more real chess positions which can be solved by CGT.

Due to the very unusual termination rules of conventional chess, including such notions as “stalemate”, it isn’t clear how temperature could be meaningfully defined.

Anglo-American checkers

Although relatively few native English speakers realize it, there are many variations of checkers now popular in different countries of the world. In most countries in continental Europe, checkers has “flying kings”, who can jump opposing checkers at some distance away on the same diagonal. Board sizes of 10×10 rather than 8×8 are also common. Some people even regard Konane as the Hawaiian variation of the checkers family.

Since Anglo-American checkers positions only rarely, if ever, decompose into sums of disjoint regions, there has been little, if any, study of temperatures of positions in this game, so I’ve left this row of Table 1 blank.

Nevertheless, the temperature theory of CGT has been very successfully applied to at least one composed gallimaufry problem including a checkers position (i.e., “four games for Gardner” in [Berlekamp 2002]). Surprisingly, the results of CGT are so robust that the solution of that gallimaufry is independent of what initially might appear to be important details of the rules:

(1) What is the goal of a game whose components include such diverse components as Go and chess?

(2) What is the scope of the compulsory capture rule in checkers? Does it compel the opponent to take the capture immediately, or does it only prevent him from making any other move on the checker board.

(3) Can a move elsewhere, perhaps in chess, be used as a ko threat in Go?

Open problems

(1) Debug and extend the entries in the existing rows of Table 1.

(2) Insert more rows into Table 1. Obvious candidates include more restricted versions of some games already listed there. In particular, most of the Konane positions constructed by Santos and Nuno-Silva [2008] have checkers of both colors on both colors of squares, although in all positions that can arise from the ancients' official starting position, black checkers can only occupy black squares and white checkers can only occupy white squares. So composed Konane problems can be partitioned into two sets: unrestricted Portugese Konane and restricted ancient Hawaiian Konane.

(3) Several families of Domineering positions are known, each containing an infinite number of different temperatures. Compose a sum of them which maximizes the difference between the orthodox result and the result when played optimally.

(4) Make a similar study of which atomic weights occur in which infinitesimal games. Hopefully, this would include corridors in Go, many of which are infinitesimal but not "all-small".

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