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About this book

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This book consists of 23 invited, original peer-reviewed papers in Combinatorial Game Theory (CGT) [5; 11; 46][1] — seven surveys and sixteen research papers. This is the fifth volume in the subseries Games Of No Chance (GONC) of the Mathematical Sciences Research Institute Publications. The name emphasizes these volumes' focus on play with no dice and no hidden cards, situating them in the landscape of game theory at large, where incomplete and/or imperfect information is common. Considering our class of games, *perfect play* can in theory be computed, and thus we include games such as CHESS, GO[2] and CHECKERS, but not YAHTZEE, BACKGAMMON and POKER.

Another characterizing feature is that combinatorial games are usually zero-sum, typically win-loss situations, although in some games draws are also possible. Players alternate in moving, so for any game description, we include a move flag of who starts. Game positions can be very sensitive to this move flag, and a common question is, given a combinatorial game, if I offer you to start, should you accept?

Often it is better to start, but not always. In the popular game of GO, the second player is rewarded a "komi" advantage of about 6.5 points before the game begins. In CHESS it is also regarded that White has a slight advantage. In neither of these games there is a mathematical proof, of this believed advantage, but since the games have been played for thousands of years, the belief seems well founded through overwhelming play-evidence.

[1]This book was initiated at the Combinatorial Game Theory Workshop, January 2011, at the Banff International Research Station (BIRS). As usual, this workshop attracted many researchers from Asia, Europe and North America, and it was organized by Richard Nowakowski (Dalhousie University), Elwyn Berlekamp (University of California, Berkeley), Aviezri Fraenkel (Weizmann Institute of Science), Martin Mueller (University of Alberta), and Tristan Cazenave (Paris-Dauphine University).

[2]On page 8, Carlos Santos reviews briefly DeepMind's AI advances of AlphaZero, a generalization of AlphaGo Zero, which recently beat the previously highest ranked CHESS program Stockfish, after just a few hours of training, alas using a massive computing power.

There are play-games which are also math-games.[3] The first player loses the game TWENTYONE. The rules are as follows: start with the number 21. The players alternate in subtracting 1 or 2 from the current number. If you start, then (in perfect play) the other player will "complement your move modulo 3", and win after 7 such rounds. Here, the game is specially *designed* to be a second player win.

We include three papers (13 on p. 313, 14 on p. 333 and 19 on p. 403) in the spirit of mechanism design in game theory; here, given a candidate set of P-positions[4], related to Beatty's classical theorem [2; 3], these contributions construct three classes of game rules with this set as the set of P-positions. The problem originates in the traditional combinatorial game of WYTHOFF NIM [53], which has a Beatty type solution on the modulus the Golden section. Generalizations of this game have begun to accumulate a lot of work, and we present the first comprehensive survey related to the heritage of Wilhelm Abraham Wythoff (paper 2, p. 35).

Singmaster famously proved [48] that almost no combinatorial game is a second player win. In this volume, we have a contribution by Singmaster (paper 7, p. 207), where he surveys the history of binary arithmetic in connection with puzzles, that is "one-player games" such as the CHINESE RINGS and the TOWER OF HANOI [27].

The game of HEX [25] is a classical game related to Brouwer's fixed-point theorem: two players compete in being the first to connect opposite sides of a hexagonal grid. A convention is often added, to compensate for the first player's advantage, namely, immediately after the first move the second player is given the opportunity to switch players. We have an amazing contribution of the theory of HEX in this book, by its current master, Ryan Hayward (paper 17, p. 387).

The game of CHOMP [9] has become famous for various reasons. Two players alternate to "chomp" pieces from a chocolate bar, by pointing at one piece and

[3]This is an informal distinction, but can be helpful to some extent: typical play-games are GO, CHESS, HEX, CHECKERS, KONANE, BLOKUS, FOUR IN A ROW, MANCALA, TIC TAC TOE, DOTS&BOXES, FOX&GEESE etc, while typical Math-games are DOMINEERING, HACKENBUSH, NIM, HEX, FOX&GEESE, WYTHOFF NIM, EUCLID, FIBONACCI NIM, etc. So, for example HEX and FOX&GEESE belong to both classes. One way of classification is to use the literature to claim membership of the latter class, while membership in the former class is due to being a popular social game. Other, more formal ways of classification may be suitable depending on purpose, but at least play-games should exclude games of values such as OMEGA; on the other hand many loopy/cyclic games are perfectly playable for human players. The distinction can be important when we aspire to build game rules, knowing beforehand "the solution"; see discussion on page 12 related to papers 13, 14 and 19 in this book.

[4]The class of second player win positions is usually called P-positions (*P*revious player wins), and they can be recursively computed for a game with a finite number of positions, starting with the terminal(s).

eating everything to the right and above, trying to avoid eating the lower left poisonous piece. The first player has a winning move, and the argument is as follows. The first player chomps the single right uppermost piece.[5] Now, any move the second player makes, could instead have been played also by the first player. Therefore the second player cannot have a winning strategy in this game. Of course, this argument does not give a clue of how to play. No general rigorous method is known, even for three row CHOMP, but to great surprise a method in physics, called renormalization, gives an estimate of where the first winning move must be [20]. This result, among others, were also published in a previous volume in this series (3.349 in the Index). Here, we are happy to report yet another finding where the renormalization approach gives precise estimates of solutions of a novel generalization of WYTHOFF NIM, called LINEAR NIMHOFF (paper 15, p. 343), indicating that in an infinite class of "linear" Wythoff extensions, all second player winning positions are distributed along thin lines.

The theory of combinatorial games was initiated by Charles Leonard Bouton in 1901 [8], when he discovered that elementary binary arithmetics solves the game of NIM, in a way that any finite number of heaps can be replaced by one single heap. For the rules of this game, see page 171 in Siegel's article (paper 6, p. 169). The next major development was in the 1930s, when Roland Percival Sprague [50] and Patrick Michael Grundy [23] independently discovered that any impartial 2-player game with the normal ending convention (last move wins) is equivalent to a one heap NIM position (and this development is also surveyed in Siegel's chapter). Again, observe how brilliant and surprising this is. Played on its own, of course, the one heap NIM game is a ridiculous thing — if the heap is nonempty, you win by removing all pebbles, and otherwise you already lost — but, by playing in a "disjunctive sum" with other NIM heaps, then this simple game encodes *any* other game in its class, and the class is huge (!), and moreover, the simple arithmetics of solving NIM then suffices to solve any such game. So, "there is something going on here" (attempting to read the minds of Sprague and Grundy). Although, this class of games is solved in theory, computationally, the games are often hard, and we include a famous yet unsolved problem in this book, presented by Grossman (paper 16, p. 373).

The next big discovery occurred in the 1950s when John Milnor [42] and Olof Hanner [24] developed a similar theory for a wide class of scoring-play games (highest score wins) without zugzwangs, that is games where it is never (!) bad to move first. Although "scoring games" are standard in game theory

[5] It has been conjectured that, in a natural generalization of CHOMP (SUBSET TAKEAWAY), "taking the largest element" is a winning move [22], but more recently counterexamples were found [10]. Note also that, given a quadratic chocolate bar of size larger than 1, you will win if you chomp off all except the lower row and left column.

at large, via concepts such as "utility", "revenue" etc., in CGT, the main idea usually concerns the "move ability"; when is it beneficial to have move options? In fact Milnor's games belong to a class of scoring games where either no player can move, or both players can move, so they remain closer to "economic games" for these two reasons.

In this volume, we proudly present Stewart's eye-opener on the full class of scoring games (paper 22, p. 447), where he pinpoints the difficulty of analyzing the full class; the problem boils down to a subset of games, where you want to start, but you do not have any move option (!). Those readers who study the misère play convention (last move loses) would acknowledge with a nod, that this situation usually induces severe complications. Many standard CGT (normal-play) tools fail.

These type of problems are discussed in three survey papers in this volume: papers 6 (p. 169) and 4 (p. 113) on impartial and partisan misere play developments respectively, and paper 3 (p. 89) on recent progress in scoring-play. In fact, a theory has recently been developed for those scoring games which exclude exactly Stewart's problematic games, the class of guaranteed scoring games [35; 34], and it is shown that the normal-play games are order embedded into this class. In this landscape, intersecting scoring- with normal-play, we find also the master pieces on DOTS&BOXES [6] and Mathematical GO [7].

The huge leap forward was in the 1970s-80s when Elwyn Berlekamp, John Horton Conway and Richard Kenneth Guy developed the normal-play theory to encompass so called partizan games, where players do not necessarily have the same move options [11; 5]. They adapted Milnor's definitions of disjunctive sum and game comparison [42], which was inspired by the apparent decomposition of GO positions into independent components towards the end of play.

Let G, H be normal-play games (without draws). Then $G \geq H$ if, for any normal-play game X, Left wins the game $H+X$ then Left wins the game $G+X$.

The intuition is as follows: let us imagine that you (playing Left) are in the middle of a complex game, a game which is decomposed in several (a finite number) of components — you are allowed to play only in one component at the time — and get an offer by a passerby to exchange one of the game components for another one. Let us say, your game is $G+X$, where X denotes a complex part of the game that you do not quite understand, and the passerby offers the game H in exchange for G, both much simpler games. Should you accept this offer?

One of the main theorems of normal-play CGT without draws [11] is that you can ignore the complicated X component, and simply play out the game $G-H=G+(-H)$, where the negative denotes that the players have swapped positions; then check whether you win this game when the opponent starts, which is the same as checking whether $G - H \geq 0$.[6]

In this book we have a unique contribution (paper 10, p. 271), by Carvalho and Santos, where the authors describe a ruleset, a modification of the traditional Hawaiian ruleset KONANE to "PORTUGUESE KONANE", which has a position in each equivalence class of *short* (acyclic games with finite ranks and out-degrees) normal-play games. One could think of this as a CGT analogy of a universal Turing machine: one ruleset encodes it all. This relates to computational complexity. Again, we can imagine CGT relatives to simple universal machines, such as Emil Post's classical Tag productions; many extremely simple rulesets, such as OFFICERS (paper 16, p. 373) has so far defied all solution attempts by human and computer. In fact, recent development in the field contributes three classes of Turing complete classes of combinatorial games [16; 38; 39]. In this volume (paper 9, p. 299), Burke and George show that a generalization of NIM on a graph is PSPACE-complete, a more common hardness measure for combinatorial games [26] (see also 3.3 in the Index).

Some games are cyclic (or loopy), i.e., they have infinite game trees, and such games can be hard to analyze, although FOX&GEESE is an example of a ruleset, where analysis has been fruitful [5].[7] Moreover, one full class of games is fully understood; for the class of loopy normal-play impartial games (on finitely many positions), a complete theory is known. The first solution was given by Smith [49], and then using a more constructive algorithmic approach, Fraenkel and Yesha [19] generalize the classical Sprague–Grundy theory by letting "infinities", enumerate the loopy game values.[8] In this volume (paper 21,

[6] Let us illustrate with an example: the game component is $G = *$, and the game offer is "up" is defined by $H = \uparrow = \{0 \mid *\}$, where the game (class) "$*$" denotes a NIM heap of size one, and where "0" denotes the equivalence class containing as simplest element the empty game, that is the game where no player can move. Suppose that you are playing Left. In this case, you should not exchange G for H. The reason for this is the following: play the game $G - H$, and ask the other player, Right, to start. The negative of "up" is "down", which is the game $-H = \downarrow = \{* \mid 0\}$. That is, the test is to let Right start the game $* + \{* \mid 0\} = \{\{* \mid 0\}, 0 \mid \{* \mid 0\}, *\}$. Right has a good move. Which one?

[7] The original 1982 version of Winning Ways included a lot of examples of loopy games, including subclasses such as stoppers and enders, and more complicated examples, such as BACH'S CAROUSEL. The usual rules of canonical forms still apply to stoppers. In the second edition of Winning Ways, the much-enlarged chapter on FOX&GEESE pretty much solved all initial positions of that game, and many others, thanks also to Siegel's popular program CGSuite [47], much of which he developed in the course of those studies. So loopy games have long played a prominent role in the core content of CGT.

[8] As a personal note by the editor, optimal play may be infinite, but a slightest mistake by your opponent may lead to your easy victory; throughout childhood I played hundreds of games of the traditional game of PICARIA (which was proved drawn in [37]) an extension of THREE MEN MORRIS, both cyclic generalizations of TIC-TAC-TOE, and those plays always concluded with a winner.

p. 439) Sarkar establishes an infinite class of drawn positions of the classical CGT ruleset PHUTBALL, so here, players are indifferent to an offer of choosing side, and playing first or second. See also 3.91 in the Index (and 3.125) for a brilliant introduction to this topic.

We have more pioneers in this book. Some rulesets encourage questions of the form "how many game positions are there?"; this holds true for the class of *placement games* (papers 9 on p. 259, 11 on p. 285 and 12 on p. 297), where relations with simplicial complexes and generating functions are skillfully exposed by Brown et al., Faridi, Huntemann and Nowakowski. This type of games are particularly appealing to "conjoin", which is demonstrated by Huggan and Nowakowski in paper 18 (p. 395). In paper 23 (p. 469), Weimerskirch presents a CGT framework which generalizes the normal and misère conventions and ingeniously includes the notion of disjunctive sum, which brings us back to the heat of the matter; Berlekamp gives a splendid performance in surveying the temperature of the field (paper 1, p. 21). How urgent is it to move in the game component X? He also shows how urgency, and temperature, can be precisely captured by playing the original game in conjunction with an idealized stack of coupons.

The measure of "importance to move" is also captured in the setting of bidding games (paper 20, p. 421), where each play consists in two phases, first make your bid, and if you win the bid you get to move, hence mixing in "auction play" a popular subject in algorithmic game theory to the setting of combinatorial games.

Since its start in the early 1990s, this series of books has captured much of the core of the CGT-development. To celebrate its 20th anniversary, and as suggested by Elwyn Berlekamp, we include an index of all published GONC papers, compiled by Silvio Levy.[9]

An elementary introduction to combinatorial games is contained in the first part of paper 6 (p. 169), by Aaron Siegel, before he plunges into the complexity of the misère quotients and more; see also his current state of the art reference [46], a monumental contribution to the field of combinatorial games.

The book is divided into two sections: Survey articles and Research articles. In the Survey section, we find:

(1) Temperatures of games and coupons (Berlekamp)

(2) Wythoff visions (Duchêne, Fraenkel, Gurvich, Ho, Kimberling, Larsson)

[9] The previous books are *Games of no chance*, volume **29** in the MSRI Publications series (1998), *More games of no chance*, **42** (2002), *Games of no chance 3*, **56** (2009), and *Games of no chance 4*, **66** (2015); all were edited by Richard J. Nowakowski, GONC 3 jointly with Michael H. Albert.

(3) Scoring games: the state of play (Larsson, Nowakowski, Santos)

(4) Restricted developments in partizan misère game theory (Milley, Renault)

(5) Unsolved problems in combinatorial games (Nowakowski)

(6) Misère games and misère quotients (Siegel)

(7) An historical tour of binary and tours (Singmaster)

The Research papers are:

(8) A note on polynomial profiles of placement games (Brown et al.)

(9) A PSPACE-complete Graph Nim (Burke, George)

(10) A nontrivial surjective map onto the short Conway group (Carvalho, Santos)

(11) Games and complexes I: Transformation via ideals (Faridi, Huntemann, Nowakowski)

(12) Games and complexes II: Weight games and Kruskal–Katona type bounds (Faridi, Huntemann, Nowakowski)

(13) Chromatic Nim finds a game for your solution (Fischer, Larsson)

(14) Take-away games on Beatty's theorem and the notion of k-invariance (Fraenkel, Larsson)

(15) Geometric analysis of a generalized Wythoff game (Friedman, Garrabrant, Landsberg, Larsson, Phipps-Morgan). Related to work in these volumes: Friedman, Landsberg (3.349)

(16) Searching for periodicity in Officers (Grossman)

(17) Good pass moves in no-draw HyperHex: two proverbs (Hayward). Related work in these volumes: Anshelevich (2.151); Hayward (3.151); Payne, Robeva (4.207); Henderson, Hayward (4.129)

(18) Conjoined games: Go-cut and Sno-Go (Huggan, Nowakowski)

(19) Impartial games whose rulesets produce given continued fractions, (Larsson, Weimerskirch)

(20) Endgames in Bidding Chess (Larsson, Wästlund). Related work (a.k.a. Richman games) in these volumes: Lazarus, Loeb, Propp, Ullman (1.427, 1.439) (the field initiator); Payne, Robeva (4.207)

(21) Scoring play combinatorial games (Stewart)

(22) Phutball draws (Sarkar). Related work in these volumes: Demaine, Demaine, Eppstein (2.351); Grossman Nowakowski (2.361); Siegel (3.91)

(23) Generalized misère play (Weimerskirch)

Before we move on, we would like to say a few words about the recent blooming development of AI and deep neural networks in playing combinatorial games. Thanks to Carlos Santos for contributing this discussion:

> DeepMind team published an arXiv-preprint (December 5th, 2017) about AlphaZero, a computer program developed to play GO, and generalized to play CHESS (and SHOGI). Within 24 hours, it achieved an outstanding level of play.
>
> AlphaZero was trained with no opening theory or endgame tables. Comparing with the previous Monte Carlo algorithms, AlphaZero used just 80 000 positions per second, whereas Stockfish used 70 million. Even so, it won against Stockfish: in 100 games AlphaZero scored 25 wins and 25 draws with White, while with Black it scored 3 wins and 47 draws. It didn't lose a game, with the final score 64:36. The 9th game of the match showed an amazing attacking player with profound positional play. The 10th game was a masterpiece with identical characteristics.
>
> Therefore, as humans know, sometimes less is more! It seems a historical moment, AlphaZero Chess presents a very good "human" CHESS style. But with the incredible power of precise calculations.
>
> Garry Kasparov said "It is a remarkable achievement, even if we should have expected it after AlphaGo."

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Overview of contents

Let us overview the contributions of this volume in their various contexts.

Surveys. We highlight some popular CGT topics.

Paper 1 (p. 21). *Temperatures of games and coupons* (Berlekamp)

For each game position, the normal-play theory defines a temperature, which is a measure of the importance of the next move [5; 46]. Here the author discusses many classical rulesets and divides them into "temperature classes" visualized by thermographic spectral lines. He comments on these games, starting with the simplest games at the lowest temperatures and continues on upwards into the hotter ones.

This is a survey, which includes some new results, using coupons to indicate "how hot a game is". In this way, normal-play games become economic-style games, whose outcome is determined by scores. Moreover, with the normal termination rule, it is hard to define a good strategy for the losing player, as all of his possible strategies will lose against an optimal opponent. In this new approach both players have well-defined optimal strategies.

Paper 2 (p. 35). *Wythoff visions* (Duchêne, Fraenkel, Gurvich, Ho, Kimberling, Larsson)

This is the first comprehensive survey of WYTHOFF NIM type games and related sequences. The game of WYTHOFF NIM (Wythoff 1907) was originally played on two heaps of stones, with rules: remove the same number of stones from both heaps or any number from exactly one of the heaps. It became famous because of its elegant solution via the Golden ratio and the floor function [53], and its popularity increased further via its more geometric interpretation CORNER THE LADY (played with a single queen of CHESS, Isaacs 1963), which inspired numerous variations such as "disjunctive sum play" (using several Queens of CHESS, "blocking maneuvers" and "linear extensions". Its richness never ceases to surprise; here we emphasize connections with various fields such as number theory, combinatorics on words and recently also cellular automata and physics.

Paper 3 (p. 89). *Scoring games: the state of play* (Larsson, Nowakowski, Santos)

Scoring-play combinatorial games started with Milnor and Hanner in the 1950s [42; 24]. Recent progress by Ettinger [15], Johnson [29], Larsson, Nowakowski, Santos [34; 35] and Stewart [51] (paper 21, p. 439) motivates a survey on the subject. There are similarities with classical settings in normal-play and misère-play, but the subject is richer than both these combined. This survey has a threefold purpose, first to survey the existing Combinatorial Game Theory in the area, such as disjunctive sum, game comparison, game reduction and game values. Secondly, and this is a novelty, important ideas, in relation with normal- and misère-play rulesets are introduced: e.g., when does a scoring-play ruleset have more interesting behavior than a normal-play ditto? The last topic in this survey is a discussion of many scoring combinatorial rulesets from Graph Theory, which have not yet been studied in the broader CGT context.

Paper 4 (p. 113). *Restricted developments in partizan misère game theory* (Milley, Renault)

Although initially designed only for impartial games, the restricted misère analysis works equally well for partizan games. The study of restricted partizan misère games began with the doctoral theses of Allen [1] and Ottaway [43].

This brilliant survey highlights the most significant results from recent research, including canonical forms of partizan misère games, the invertibility of games under restricted misère-play, and applications to specific partizan misère versions of Nim, Kayles, and Hackenbush [5].

Paper 5 (p. 125). *Unsolved problems in combinatorial games* (Nowakowski)

This survey is an updated version of Nowakowski's popular list of unsolved combinatorial games' problems.

Paper 6 (p. 169). *Misère games and misère quotients* (Siegel)

These comprehensive lecture notes concern impartial games and contain a complete course in the theory of misère quotients, illustrated through many examples. They are based on a short course offered at the Weizmann Institute of Science in 2006. In normal-play there are just six games with birthday smaller than six, whereas in misère-play, there are 4171780, and on day six there are more than $2^{4171779}$. Hence, the full theory of misère games modulo "=" is not so useful; we cannot hope for much structure if we require that games behave identically in any context. Through the theory of misère quotients, it suffices to understand how a ruleset position interacts with other positions in the same ruleset.

The notes begin with the basic definitions of combinatorial game theory and a proof of the Sprague–Grundy Theorem [50; 23], then proceeds via misère quotients to the full proof of the Guy–Smith–Plambeck Periodicity Theorem [44]. It concludes with a discussion of major open problems and directions for future research.

Paper 7 (p. 207). *An historical tour of binary and tours* (Singmaster)

This original survey by Singmaster, and edited by Larsson, dwells on topics in recreational mathematics in connection with binary representations and paths on graphs.

Highlights include the history of the CHINESE RINGS and its Gray code solution, which are elaborated in various directions, one of which leads to an appealing result on the classical TOWER OF HANOI puzzle [27]. Singmaster determines the "distance" of an arbitrary TOWER OF HANOI position to a terminal tower (see also [28]).

Workshop topics. Some contributions were presented at the workshop or were authored in direct response to workshop questions.

Placement games and simplicial complexes. New theory for *placement games* were discussed at the workshop, and resulted in four papers. Early work counted the number of positions in the game of GO [16], although this game is not strictly a placement game, because, by playing, pieces can be removed. In a placement game pieces cannot be moved or removed, only "placed".

Paper 8 (p. 243). *A note on polynomial profiles of placement games* (Brown et al.)

In this paper, placement games are recognized as a class of games. A game-specific polynomial encodes the number of distinct positions of games such as COL, SNORT [5] and NOGO (A variation of GO), provided they are played on a single path. The enumeration uses a bijection between the positions and the independent sets of an associated graph.

Paper 11 (p. 285). *Games and Complexes I: Transformation via Ideals* (Faridi, Huntemann, Nowakowski)

The authors show that for a given board and placement game, played on a graph, there are two associated simplicial complexes, defined via ideals on square-free monomials. The first simplicial complex is in terms of the legal positions, the second in terms of the illegal positions.

Paper 12 (p. 297). *Games and Complexes II: Weight games and Kruskal–Katona type bounds* (Faridi, Huntemann, Nowakowski)

This paper continues the work on simplicial complexes in relation with placement games. In the subclass of *strong* placement games, every move sequence between legal positions is legal. Moreover, for the class of *weight games*, game pieces may cover several neighboring nodes. By combining these classes, they find upper bounds on the number of positions with i pieces, or equivalently the number of faces with i vertices.

Paper 18 (p. 395). *Conjoined games: Go-cut and Sno-Go* (Huggan, Nowakowski)

This is a study of a recently introduced operation on rulesets, which generalizes math-games such as BUILDING NIM [13] and traditional play-games such as

NINE MEN MORRIS (which is a draw on a standard game board; see 1.101 in Index) and PICARIA [37]. In a pair of *conjoined games*, the first ruleset is played to a terminal position, then play continues under the second ruleset. They find the outcome classes for two such games, GO-CUT and SNO-GO, played on a strip.

Games and number theory. Fraenkel posed an intruiging problem at the workshop: "describe nice/short rulesets for games with so-called complementary Beatty sequences [2; 3] as sets of P-positions". The problem is the opposite of the main field of research in this area, which is, given some ruleset, to search for its set of P-positions.

This question resulted in three contributions. All but one requires a partial disclosure of the P-positions in the game rules. The one that does not depend on this studies a sub-class of Beatty sequences, described by periodic continued fractions.

A relaxation of Fraenkel's problem, to find so-called *invariant* rulesets [14], was recently resolved, by using a novel $\star$-operator [36], but without satisfying the proviso of "short".

We make a yet very informal distinction between rule sets for "math games" versus "play games"; rules (and/or solutions) in the former class appeals in particular to mathematicians, whereas rules in the latter class are more generally playable. These type of problems have also been discussed in (the introduction of) a recent PhD thesis [31]. Note the similarity with the field of mechanism design (or reverse game theory) in economics and game theory.

Paper 13 (p. 313). *Chromatic Nim finds a game for your solution* (Fischer, Larsson)

This contribution produces truly playable rules: "the rules can be understood by a five-year-old". This is done by a bi-coloring of the tokens in the stacks, thus mimicking one of the Beatty-type patterns on each heap. The rules are just classic NIM, with a proviso on the color of the top tokens in the current heaps.

Inspired by the resolution of the original problem, the authors continue by developing a general theory for this class of 2-heap games, and they study a "minimal sequence" for this theory, namely the famous Prouhet–Thue–Morse sequence.

Paper 14 (p. 333). *Take-away games on Beatty's theorem and the notion of k-invariance* (Fraenkel, Larsson)

This paper studies various short math-game rules, with the proviso that for the full game rules one must keep track of the number of tokens in the heaps, and moreover one of the Beatty sequences is revealed in giving the game rules.

The notion of k-invariance is introduced, and the games in this paper are 2-invariant. The succinctness of the game rules, together with their *invariance number*, seems to be related with the complexity of the proposed set of P-positions, and the paper ends with an intriguing problem interconnecting number-theory and computer science through game theory.

Paper 19 (p. 403). *Impartial games whose rulesets produce given continued fractions* (Larsson, Weimerskirch)

In this contribution, the moduli of the complementary Beatty sequences are given by the continued fractions $(1; k, 1, k, 1, \ldots)$ and $(k+1; k, 1, k, 1, \ldots)$, respectively.

The authors describe (short math-game) rules that satisfy two criteria: they are given by a closed formula and/or a simple recurrence, and they are invariant, that is, the available moves do not depend on the position played from (for all options with nonnegative coordinates).

The solution involves Sturmian word and morphism constructions of the Beatty Sequences [41]. Relating back to the problem in the previous paper, in this case, the smallest invariance number is possible with low complexity of game rules.

Classical GONC subjects. These four contributions (and more) are follow-ups to earlier GONC papers; indexed at the end of this issue.

Paper 17 (p. 387). *Good pass moves in no-draw HyperHex: two proverbs* (Hayward)

Hayward defines the concept of a "good move" and notes that in HEX, as in GO, it is not always true that your opponent's good move is your good move. The author studies a generalization of classical games: HYPERHEX is the hypergraph generalization of variants of HEX, where each player has a list of win-sets, and wins by coloring all cells of any of her win-sets. He finds a condition for this game where each move is good for both players, and he reminds us that "it is never too late to play a good move".

Previous HEX papers in the GONC series: Anshelevich (2.151); Hayward (3.151); Payne, Robeva (4.207); Henderson, Hayward (4.129).

Paper 15 (p. 343). *Geometric analysis of a generalized Wythoff game* (Friedman, Garrabrant, Landsberg, Larsson, Phipps-Morgan)

Using methods from physics, namely *renormalization* [20], the authors study a class of combinatorial games, LINEAR NIMHOFF, for which a probabilistic geometry explains the global behavior of observed outcome patterns. Let us highlight two novelties: (1) By axiomatizing the "renormalization properties", they present a rigorous method, where the question of existence is left for a future study. (2) Via a related *reorganization* property, they demonstrate precisely when game rules can be omitted, because they do not contribute to the global geometry.

Moreover, "symmetric rules" sometimes cause an underlying geometry to transform into quasi log-periodic fluctuations, centered in the proposed probabilistic geometry solution. As a special case, the method provides a solution to a class of games which has defied all previous analysis, namely the class GENERALIZED DIAGONAL WYTHOFF NIM [32; 33].

The GONC-series contributed a field pioneer on the subject of renormalization in combinatorial games: Friedman, Landsberg (3.349).

Paper 20 (p. 421). *Endgames in Bidding Chess* (Larsson, Wästlund)

A bidding combinatorial game is a combinatorial game, where instead of alternating play, the players bid for the opportunity to move. Richman showed in the 1980s that, in a finite setting for impartial games, the bidding games are equivalent to "random turn games" (1.427 and 1.439 in Index) and [40].

Develin and Payne claim on page 3 in their comprehensive paper on discrete bidding games [12] that "...the basic results and arguments of Richman game theory go through unchanged for partisan games...". Here, the authors develop the theory for partisan Bidding games, and show that this claim is not true in general. It is demonstrated that in the special case of 3-piece endgames in BIDDING CHESS [4], the claim holds. This is an area which recently has attracted research from the larger games' community, and where zero-sum has been generalized to general-sum [30].

The earlier GONC-papers about Bidding games (a.k.a. Richman games) are: Lazarus, Loeb, Propp, Ullman (1.427, 1.439) (the field initiater); Payne, Robeva (4.207).

Paper 22 (p. 447). *Phutball draws* (Sarkar)

The game PHILOSOPHER'S PHUTBALL was introduced by Conway et al. and appeared in Winning Ways [5]. It is usually played on a GO-board with one black stone (the ball) and the remaining stones white (the chaps). A move consists in either jumping a line of chaps with the ball and removing them, or placing a new chap on the board. The goals are the top and bottom edge of the board, respectively.

This game appeared twice in *More Games of no chance*; first: it is NP-complete to decide if one has a "win in one" (Demaine et al.), and secondly, the one-dimensional restriction was analyzed for a restricted class (Grossman et al.). In *Games of no chance 3* there is a survey paper on cyclic games by Siegel. He claims that, in one-dimensional PHUTBALL, it is not even known if both players have a drawing strategy. In this paper, the author finds an infinite class of drawn 2-dimensional positions.

Previous GONC-papers on PHUTBALL: Demaine, Demaine, Eppstein (2.351); Grossman Nowakowski (2.361); Siegel (3.91)

Conceptualizers. We have some aspiring trailblazers.

Paper 10 (p. 271). *A nontrivial surjective map onto the short Conway group* (Santos, Carvalho)

This is a unique contribution to classical combinatorial game theory. Every impartial game is equivalent to a unique NIM-heap, but no partizan game with an analogous property was known before this volume.

The authors demonstrate universality of a specific ruleset, namely a modification of the classical Hawaiian game of KONANE (see also 3.287 in the Index), and thereby generalizing [45]. Each short loop-free game value in normal-play theory can be described by a position in this game.

Paper 22 (p. 447). *Scoring play combinatorial games* (Stewart)

Stewart introduces a general class of scoring play games (generalizing Milnor's nonnegative incentive games [42] and Ettinger's dicot games [15]), and he points towards the difficulty of having a full class of scoring games. Namely, the troublesome games are those games where you benefit by not being able to move. Indeed his observation already inspired development of the theory of guaranteed scoring games [34; 35], where exactly those type of games are excluded; see also the related scoring play survey in this proceedings.

Stewart studies also the special case of "impartial" (or symmetric) Scoring games. He defines a variation of the 'Sprague–Grundy theory' for take-and-break scoring play, and studies analogies of subtraction- and octal games in this setting. He finishes off with several accompanying conjectures.

Paper 23 (p. 469). *Generalized misère-play* (Weimerskirch)

Weimerskirch develops a framework by which to view classical impartial games as an infinite array of game boards, or lattice points. Instead of thinking of normal-play and misère-play as differing in their winning condition, they are here viewed as differing in which set of positions are "in the field of play". This approach leads to a novel class of generalizations, and moreover, a separate notion of "disjunctive sum" becomes obsolete, because it is inherent in the lattice point definition of a position.

Computational aspects. Hardness problems lie at the heart of combinatorial game theory.

Paper 16 (p. 373). *Searching for periodicity in Officers* (Grossman)

OFFICERS [5] is a take-and-break game in which a move consists of removing a bean from a heap and leaving the remaining beans from that heap in exactly one or two nonempty heaps. It is an open question as to whether or not the Sprague–Grundy values of this game are eventually periodic; answering this question in the positive for take-and-break games generally requires computing enough values to explicitly find the period, but appearances of large "rare values" has continued to perplex the games community.

The presented method, which involves two novel parallelization strategies, is general and can be applied to other take-and-break games. The trick is to exploit the "rare values". If an oracle told us they are finitely many, then since the remaining values are bounded in size, one could conclude that OFFICERS were periodic. Without an oracle, Grossman shows how to exploit the rare value phenomenon to accelerate the computation, and after computation of more than 140 trillion values, no periodicity has been found.

Paper 9 (p. 259). *A PSPACE-complete Graph Nim* (Burke, George)

The game of NEIGHBORING NIM is played on graphs; each node contains a heap of pebbles and a move consists of removing some pebbles from some node then moving along a neighboring edge. This game is a generalization of a wide class of games including GEOGRAPHY [18] and NIM. The authors use methods from GEOGRAPHY to prove that the general game is PSPACE-hard and that a restricted variant is PSPACE-complete.

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