8 Spectral algebraic geometry

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8.1 Introduction

This chapter is a very modest introduction to some of the ideas of spectral algebraic geometry, following the approach due to Lurie. The goal is to introduce a few of the basic ideas and definitions, with the goal of understanding Lurie's characterization of highly structured elliptic cohomology theories.

A motivating example: elliptic cohomology theories

Generalized cohomology theories are functors which take values in some abelian category. Traditionally, we consider ones which take values in *abelian groups*, but we can work more generally. For instance, take cohomology theories which take values in *sheaves of graded abelian groups (or rings)* on some given topological space, or in *sheaves of graded $\mathcal{O}_S$-modules (or rings)* on S, where S is a scheme, or possibly a more general kind of geometric object, such as a *Deligne–Mumford stack*, and $\mathcal{O}_S$ is its *structure sheaf.*

Given a scheme (or Deligne–Mumford stack) S, it is easy to construct an example of a cohomology theory taking values in graded $\mathcal{O}_S$-algebras; for instance, using ordinary cohomology, we can form

$$\mathcal{F}^*(X) := \big(U \mapsto H^*(X, \mathcal{O}_S(U))\big),$$

which is a presheaf of graded $\mathcal{O}_S$-algebras on S, which in turn can be sheafified into a sheaf of graded $\mathcal{O}_S$-modules on S.

A more interesting example is given by *elliptic cohomology theories.* These consist of

1. an elliptic curve $\pi\colon C \to S$ (which is in particular an algebraic group with an identity section $e\colon S \to C$),
2. a multiplicative generalized cohomology theory $\mathcal{F}^*$ taking values in sheaves graded commutative $\mathcal{O}_S$-algebras, which is *even and weakly 2-periodic* in the sense that $\mathcal{F}^{\mathrm{odd}}(\text{point}) \approx 0$ while $\mathcal{F}^0(\text{point}) \approx \mathcal{O}_S$ and $\mathcal{F}^2(\text{point})$ is an invertible $\mathcal{O}_S$-module, together with

3. a choice of isomorphism

$$\alpha\colon \operatorname{Spf}\mathcal{F}^0(\mathbb{CP}^\infty) \xrightarrow{\sim} C_e^\wedge$$

of *formal groups*, where the right-hand side denotes the formal completion of the elliptic curve $\pi\colon C \to S$ at the identity section.

This is easiest to think about when S is affine, i.e., $S = \operatorname{Spec} A$ for some ring A. Then the above data corresponds exactly to what is known as an ***elliptic spectrum*** [4]: a weakly 2-periodic spectrum E with $\pi_0 E = A$, together an isomorphism of formal groups $\operatorname{Spf} E^0\mathbb{CP}^\infty \approx C_e^\wedge$, where C is an elliptic curve defined over the ring A. Many such elliptic spectra exist, including some which are structured commutative ring spectra.

For a more general elliptic cohomology theory defined over some base scheme (or stack) S, one may ask that it be "represented" by a ***sheaf of (commutative ring) spectra*** on S, which I'll call $\mathcal{O}_S^{\mathrm{top}}$. E.g., for an open subset U of the scheme S, and a finite CW-complex X, we would have

$$\mathcal{F}^q(X)(U) \approx \pi_0 \operatorname{Map}_{\mathrm{Spectra}}(\Sigma^{-q}\Sigma^\infty X, \mathcal{O}_S^{\mathrm{top}}(U))$$

where $\mathcal{O}_S^{\mathrm{top}}(U) \in \mathrm{Spectra}$ are the sections of $\mathcal{O}_S^{\mathrm{top}}(U)$ over U.

Goerss, Hopkins, and Miller showed that such an object exists, where $S = \mathcal{M}_{\mathrm{Ell}}$ is the moduli stack of (smooth) elliptic curves, and $C \to S$ is the universal elliptic curve. This can be viewed as giving a "universal" example of an elliptic cohomology theory. As a consequence you can take global sections of $\mathcal{O}_S^{\mathrm{top}}$ over the entire moduli stack S, obtaining a ring spectrum called TMF, the **topological modular forms**. (There is also an extension of this theory to the "compactification" of $\mathcal{M}_{\mathrm{Ell}}$, the moduli stack of ***generalized*** elliptic curves; I will not discuss this version of the theory here.)

From the point of view of spectral algebraic geometry, the pair $(\mathcal{M}_{\mathrm{Ell}}, \mathcal{O}^{\mathrm{top}})$ is an example of a ***nonconnective spectral Deligne–Mumford stack***, i.e., an object in ***spectral algebraic geometry***.

Lurie proves a further result, which precisely characterizes the nonconnective spectral Deligne–Mumford stack $S = (\mathcal{M}_{\mathrm{Ell}}, \mathcal{O}^{\mathrm{top}})$. Namely, it is the classifying object for a suitable type of "derived elliptic curve", called an ***oriented elliptic curve***. More precisely, for each nonconnective spectral Deligne–Mumford stack X there is an equivalence of ∞-groupoids

$$\operatorname{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(X, S) \approx \{\text{oriented elliptic curves over } X\},$$

natural in X; here $\mathrm{SpDM}^{\mathrm{nc}}$ denotes the ∞-category of nonconnective spectral Deligne–Mumford stacks. In particular, there is a "universal" oriented elliptic curve $C \to S$.

Organization of this chapter

We describe some of the basic concepts of spectral algebraic geometry. This chapter is written for algebraic topologists, with the example of elliptic cohomology as a prime motivation. This chapter will only give an overview of some of the ideas. I'll give precise definitions and complete proofs when I can (rarely); more often, I will try to give an idea of a definition and/or proof, sometimes by appealing to an explicit example, or to a "classical" analogue.

I will not try to describe applications to geometry or representation theory. The reader should look at Lurie's introduction to [170], as well as Toën's survey [290], to get a better idea of motivations from classical geometry.

We will follow Lurie's approach. This was originally presented in the book *Higher Topos Theory* [169], together with the sequence of "DAG" preprints [163]. Some of the DAG preprints have been incorporated in/superseded by the book *Higher Algebra* [168], while others have been absorbed by the book-in-progress *Spectral Algebraic Geometry* [170]. I try to use notation consistent with [170], and give references to it when possible (references are to the February 2018 version). Note that [170] is still under construction and its numbering and organization is likely to change. Lurie's approach to elliptic cohomology is sketched in [162], and described in detail in [166] and [167].

Derived algebraic geometry had its origins in problems in algebraic geometry, and was first pursued by geometers. We note in particular the work of Toën and Vezzosi, which develops a theory broadly similar to Lurie's; the aforementioned survey [290] is a good introduction.

Notation and terminology

I'll use the "naive" language of ∞-categories pretty freely. When I say "category" I really mean "∞-category", unless "1-category" or "ordinary category" is explicitly indicated. An "isomorphism" in an ∞-category is the same thing as an "equivalence"; I use the two terms interchangeably. Sometimes I will say that a construction is "essentially unique", which means it is defined up to contractible choice.

I write Cat_∞ and $\widehat{\mathrm{Cat}}_\infty$ for the ∞-categories of small and locally small ∞-categories respectively. I write $\mathcal{S}$ for the ∞-category of small ∞-groupoids. "Sets" are implicitly identified with the full subcategory of "0-truncated ∞-groupoids": thus, $\mathrm{Set} \approx \tau_{\leq 0}\mathcal{S} \subseteq \mathcal{S}$. I write $\mathrm{Map}_{\mathcal{C}}(X, Y)$ for the space (= ∞-groupoid) of maps between two objects in an ∞-category $\mathcal{C}$. I use the notations $\mathcal{C}_{X/}$ and $\mathcal{C}_{/X}$ for the slice categories under and over an object X of $\mathcal{C}$.

I will consistently notate adjoint pairs of functors in the following way. In

$$L\colon \mathcal{C} \rightleftarrows \mathcal{D} : R \qquad \text{or} \qquad R\colon \mathcal{D} \leftrightarrows \mathcal{C} : L,$$

the arrow corresponding to the left adjoint is always *above* that for the right adjoint.

I use the notation $C \rightarrowtail D$ for a *fully faithful* functor, and $C \twoheadrightarrow D$ for a *localization* functor, i.e., the universal example of formally inverting a class of arrows in C. Note that any adjoint (left or right) of a fully faithful functor is a localization, and any adjoint (left or right) of a localization functor is fully faithful.

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8.2 The notion of an ∞-topos

A scheme is a particular kind of ***ringed space***, i.e., a topological space equipped with a sheaf of rings. Spectral algebraic geometry replaces "rings" with an ∞-categorical generalization, namely ***commutative ring spectra***, which (following Lurie) we will here call $\mathbb{E}_\infty$*-rings*. Similarly, spectral algebraic geometry replaces "topological space" with its ∞-categorical generalization, which is called an ∞*-topos*.

The key observation motivating ∞-topoi is that a topological space X is determined[1] by the ∞-category of ***sheaves*** of ∞-groupoids on X. I will try to justify this in the next few sections.

The notion of ∞-topos is itself a generalization of a more classical notion, that of a ***1-topos*** (or ***Grothendieck topos***), which can be thought of as the 1-categorical generalization of topological space. I will not have much to say about these, instead passing directly to the ∞-case (but see (8.2) below). However, the theory of ∞-topoi does parallel the classical case in many respects; a good introduction to 1-topoi is [173].

There is a great deal to say about ∞-topoi, so I'll try to say as little as possible. Note that to merely understand the basic definitions of spectral algebraic geometry, only a small part of the theory is necessary: much as, to understand the definition of a scheme, you need enough topology to understand the "Zariski spectrum" of a ring, without any need to inhale large quantities of esoteric results in point-set topology.

We refer to a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ as a **presheaf** of ∞-groupoids on $\mathcal{C}$, and write

$$\mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

for the ∞-category of presheaves.

We first describe two examples of ∞-topoi arising from "classical" constructions.

The ∞-topos of a topological space

Let X be a topological space, with $\mathrm{Open}_X =$ its poset of open subsets. A **sheaf** of ∞-groupoids on X is a functor $F\colon \mathrm{Open}_X^{\mathrm{op}} \to \mathcal{S}$ such that, for every open cover $\{U_i \to U\}_{i\in I}$ of an element U of Open_X, the evident map

$$F(U) \xrightarrow{\sim} \lim_\Delta\Big[[n] \mapsto \prod_{i_0,\dots,i_n\in I} F(U_{i_0}\cap\cdots\cap U_{i_n})\Big] \tag{8.2.1}$$

is an equivalence; the target is the limit of functor $\Delta \to \mathcal{S}$, i.e., of a cosimplicial space. We let $\mathrm{Shv}(X) \subseteq \mathrm{PSh}(\mathrm{Open}_X)$ denote the full subcategory of sheaves. It turns out that this embedding admits a left adjoint $a\colon \mathrm{PSh}(\mathrm{Open}_X) \to \mathrm{Shv}(X)$ which is **left exact**, i.e., a preserves finite limits.

The ∞-topos of sheaves on the étale site of a scheme

Let X be a scheme, and let $\mathrm{Ét}_X =$ a full subcategory of the category of schemes over X spanned by a suitable collection of étale morphisms $U \to X$, (e.g., morphisms which

[1] This is not exactly true; see (8.5) below.

factor as $U \xrightarrow{f} V \rightarrowtail X$ where f is a finitely presented étale map to an open affine subset of X). An **étale cover** is a collection of étale maps $\{U_i \to U\}_{i\in I}$ in Ét_X which are jointly surjective on Zariski spectra. We get full subcategory $\text{Shv}(X^{\text{ét}}) \subseteq \text{PSh}(\text{Ét}_X)$ of **étale sheaves** on X, whose objects are functors $F\colon \text{Ét}_X^{\text{op}} \to \mathcal{S}$ such that the evident map

$$F(U) \xrightarrow{\sim} \lim_\Delta \Big[[n] \mapsto \prod_{i_0,\dots,i_n\in I} F(U_{i_0} \times_X \cdots \times_X U_{i_n}) \Big]$$

is an equivalence for every étale cover. (This makes sense because Ét_X is an essentially small category which is closed under finite limits.) As in (8.2), the embedding $\text{Shv}(X^{\text{ét}}) \subseteq \text{PSh}(\text{Ét}_X)$ admits a left exact left adjoint.

Definition of ∞-topos

An ∞**-topos** is an ∞-category $\mathcal{X}$ such that

1. there exists a small ∞-category $\mathcal{C}$, and
2. an adjoint pair
$$i\colon \mathcal{X} \rightleftarrows \text{PSh}(\mathcal{C}) : a$$
where the right adjoint i is fully faithful (whence a is a localization), and such that
3. i is **accessible**, i.e., there exists a regular cardinal λ such that i preserves all λ-filtered colimits, and
4. a is left exact.

Remark 8.2.1 (Presentable ∞-categories). An $\mathcal{X}$ for which there exists data (1)–(3) is called a **presentable** ∞**-category** [169, 5.5]. This class includes many familiar examples such as: small ∞-groupoids, chain complexes of modules, spectra, $\mathbb{E}_\infty$-ring spectra, functors from a small ∞-category to a presentable ∞-category, etc. (Note: [169, 5.5.0.1] defines this a little differently, but it is equivalent to what I just said by [169, 5.5.1.1].)

All presentable ∞-categories are complete and cocomplete. The "presentation" $(\mathcal{C}, i, a)$ of $\mathcal{X}$ leads to an explicit recipe for computing limits and colimits in $\mathcal{X}$: apply i to your diagram in $\mathcal{X}$ to get a diagram in $\text{PSh}(\mathcal{C})$, take limits or colimits there, and apply a to get the desired answer. (Since i is a fully faithful right adjoint, the last step of applying a is not even needed when computing limits.)

Remark 8.2.2 (Adjoint functors between presentable ∞-categories). It turns out that a very strong form of an "adjoint functor theorem" applies to presentable ∞-categories [169, 5.5.2.9].

1. If $\mathcal{A}$ is presentable, then a functor $F\colon \mathcal{A} \to \mathcal{B}$ admits a right adjoint if and only if it preserves small colimits.
2. If $\mathcal{A}$ and $\mathcal{B}$ are presentable, then a functor $F\colon \mathcal{A} \to \mathcal{B}$ admits a left adjoint if and only if it preserves small limits and is accessible.

In particular, if $\mathcal{A}$ is presentable, then a functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{S}$ to ∞-groupoids is ***representable*** if and only if it preserves limits, and $\mathcal{A} \to \mathcal{S}$ is ***corepresentable*** if and only if it preserves limits and is accessible.

Remark 8.2.3. The presentation $(\mathcal{C}, i, a)$ is not part of the structure of an ∞-topos (or presentable ∞-category): it merely needs to exist, and it is not in any sense unique.

Any presheaf category $\mathrm{PSh}(\mathcal{C})$ is an ∞-topos, and in particular $\mathcal{S}$ is one.

Both the examples (8.2) and (8.2) given above are ∞-topoi. They are special cases of sheaves on a ***Grothendieck topology*** on an ∞-category $\mathcal{C}$; see (8.5) below and [169, 6.1, 6.2].

Relation to the classical notion of topos

Recall that an object U of any ∞-category $\mathcal{X}$ is 0-**truncated** if $\mathrm{Map}_{\mathcal{X}}(-, U)$ takes values in $\tau_{\leq 0}\mathcal{S} \subseteq \mathcal{S}$, i.e., in "sets". For an ∞-topos $\mathcal{X}$, its full subcategory $\mathcal{X}^{\heartsuit} \subseteq \mathcal{X}$ of 0-truncated objects is called the **underlying 1-topos** of $\mathcal{X}$. This $\mathcal{X}^{\heartsuit}$ is equivalent to a 1-category, and is a "classical" topos in the sense of Grothendieck; in fact all Grothendieck topoi arise from ∞-topoi in this way.

For instance, if X is a topological space then $\mathrm{Shv}(X)^{\heartsuit}$ is the 1-category of sheaves of *sets* on X.

Example 8.2.4. As we'll see (8.4), the slice category $\mathcal{S}_{/X}$ is an ∞-topos for any $X \in \mathcal{S}$, and it is easy to verify that $(\mathcal{S}_{/X})^{\heartsuit} \approx \mathrm{Fun}(\Pi_1 X, \mathrm{Set})$. Thus $(\mathcal{S}_{/X})^{\heartsuit}$ only depends on the fundamental groupoid of X, while $\mathcal{S}_{/X}$ itself depends on the homotopy type of X. Thus, non-equivalent ∞-topoi can share the same underlying 1-topos.

8.3 Sheaves on an ∞-topos

There is an obvious notion of sheaves on a topological space which take values in an arbitrary complete ∞-category $\mathcal{A}$. These are functors $F \colon \mathrm{Open}_X^{\mathrm{op}} \to \mathcal{A}$ which satisfy the "sheaf condition", i.e., that the map in (8.2.1) is an equivalence for every open cover. We can reformulate this definition so that it depends only on the ∞-category $\mathcal{X} = \mathrm{Shv}(X)$, rather than on the category of open sets in X. This leads to a definition of $\mathcal{A}$-valued sheaf which makes sense in an arbitrary ∞-topos.

Sheaves valued in an ∞-category

For a general ∞-topos, an $\mathcal{A}$-**valued sheaf** on $\mathcal{X}$ is a limit preserving functor $F \colon \mathcal{X}^{\mathrm{op}} \to \mathcal{A}$. These objects form a full subcategory $\mathrm{Shv}_{\mathcal{A}}(\mathcal{X}) \subseteq \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{A})$.

Example 8.3.1 ($\mathcal{A}$-valued sheaves on a presheaf ∞-topos). If $\mathcal{X} = \mathrm{PSh}(\mathcal{C})$, then $\mathrm{Shv}_{\mathcal{A}}(\mathcal{X})$ is equivalent to the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ of "$\mathcal{A}$-valued presheaves" on $\mathcal{C}$. This is because the Yoneda embedding $\rho \colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ is the "free colimit completion" of $\mathcal{C}$ [169, 5.1.5]: for any cocomplete $\mathcal{B}$, restriction along ρ gives an equivalence

$$\mathrm{Fun}(\mathrm{PSh}(\mathcal{C}), \mathcal{B}) \supseteq \mathrm{Fun}^{\text{colim pres.}}(\mathrm{PSh}(\mathcal{C}), \mathcal{B}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{B})$$

between the full subcategory of ***colimit preserving*** functors $\mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$ and all functors $\mathcal{C} \to \mathcal{B}$; the inverse of this equivalence is defined by left Kan extension along ρ. Taking $\mathcal{B} = \mathcal{A}^{\mathrm{op}}$ we obtain the equivalence

$$\mathrm{Fun}(\mathrm{PSh}(\mathcal{C})^{\mathrm{op}}, \mathcal{A}) \supseteq \mathrm{Fun}^{\text{lim pres.}}(\mathrm{PSh}(\mathcal{C})^{\mathrm{op}}, \mathcal{A}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}).$$

Example 8.3.2 ($\mathcal{A}$-valued sheaves on a space, revisited). For $\mathcal{X} = \mathrm{Shv}(X)$ the two definitions coincide: limit preserving functors $F' \colon \mathcal{X}^{\mathrm{op}} \to \mathcal{A}$ correspond to functors $F \colon \mathrm{Open}_X^{\mathrm{op}} \to \mathcal{A}$ satisfying the sheaf condition.

To see this, recall the adjoint pair $i \colon \mathrm{Shv}(X) \rightleftarrows \mathrm{PSh}(\mathrm{Open}_X) : a$. For each open cover $\mathcal{U} = \{U_i \to U\}_{i \in I}$ in X, the functor a carries the evident map

$$s_{\mathcal{U}} \colon \mathrm{colim}_{\Delta^{\mathrm{op}}}\Big[[n] \mapsto \coprod_{i_0,\dots,i_n \in I} \rho_{U_{i_0} \cap \cdots \cap U_{i_n}}\Big] \to \rho_U$$

in $\mathrm{PSh}(\mathrm{Open}_X)$ to an isomorphism in $\mathrm{Shv}(X)$, where $\rho_U := \mathrm{Map}_{\mathrm{Open}_X}(-, U)$ denotes the representable functor. (Proof: applying $\mathrm{Map}_{\mathrm{PSh}(\mathrm{Open}_X)}(-, F)$ to this exactly recovers the map (8.2.1) exhibiting the sheaf condition for a presheaf F, and if F' is a sheaf we have $\mathrm{Map}_{\mathrm{PSh}(\mathrm{Open}_X)}(-, iF') = \mathrm{Map}_{\mathrm{Shv}(X)}(a(-), F')$.)

More is true: the functor a is the ***initial example*** of a colimit preserving functor which takes all such maps $s_{\mathcal{U}}$ to isomorphisms. (In the terminology of [169, 5.5.4] $\mathrm{Shv}(X)$ is the ***localization*** of $\mathrm{PSh}(\mathrm{Open}_X)$ with respect to the ***strongly saturated class generated*** by $\{s_{\mathcal{U}}\}$, and universality is [169, 5.5.4.20].)

Thus, objects $F \in \mathrm{Shv}_{\mathcal{A}}(\mathcal{X})$ coincide with limit preserving $F' \colon \mathrm{PSh}(\mathrm{Open}_X)^{\mathrm{op}} \to \mathcal{A}$ such that $F'(s_{\mathcal{U}})$ is an equivalence for every open cover $\mathcal{U}$, which coincide with functors $F \colon \mathrm{Open}_X^{\mathrm{op}} \to \mathcal{A}$ satisfying the sheaf condition.

Example 8.3.3 (Sheaves of ∞-groupoids). Every limit preserving functor $\mathcal{X}^{\mathrm{op}} \to \mathcal{S}$ is representable by an object of $\mathcal{X}$ (8.2.2). Therefore, the Yoneda embedding restricts to an equivalence $\mathcal{X} \xrightarrow{\sim} \mathrm{Shv}_{\mathcal{S}}(\mathcal{X}) \subseteq \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{S})$: the underlying ∞-category of the ∞-topos $\mathcal{X}$ is also the category of sheaves of ∞-groupoids on $\mathcal{X}$.

Example 8.3.4 (Sheaves of sets). We have that $\mathrm{Shv}_{\mathrm{Set}}(\mathcal{X}) \approx \mathcal{X}^{\heartsuit}$.

Remark 8.3.5 (Sheaves of ∞-groupoids as "generalized open sets"). The above displays the first instance of a philosophy you encounter a lot of in this theory. For an ∞-topos $\mathcal{X}$, objects $U \in \mathcal{X}$ can be thought of ***either*** as "sheaves of ∞-groupoids" on $\mathcal{X}$ via $\mathcal{X} \approx \mathrm{Shv}_{\mathcal{S}}(\mathcal{X})$, ***or*** as "generalized open sets of $\mathcal{X}$", in the sense that it makes sense to evaluate any sheaf $F \in \mathrm{Shv}_{\mathcal{A}}(\mathcal{X})$ at any object U.

Given an $\mathcal{A}$-valued sheaf $F \colon \mathcal{X}^{\mathrm{op}} \to \mathcal{A}$ on $\mathcal{X}$, its **global sections** are defined to be

$$\Gamma(\mathcal{X}, F) := F(1_{\mathcal{X}}).$$

8.4 Slices of ∞-topoi

We give a quick tour through some basic general constructions and properties involving ∞-topoi. First, we look at slices of ∞-topoi, which give more examples of ∞-topoi.

Slices of ∞-topoi are ∞-topos

Given an object U in an ∞-category $\mathcal{X}$, we get a slice ∞-category $\mathcal{X}_{/U}$.

Proposition 8.4.1 ([169, 6.3.5.1]). *Every slice $\mathcal{X}_{/U}$ of an ∞-topos $\mathcal{X}$ is an ∞-topos.*

Proof. Choose a presentation $(\mathcal{C}, i, a)$ of $\mathcal{X}$ with fully faithful $i\colon \mathcal{X} \rightarrowtail \mathrm{PSh}(\mathcal{C})$, which induces a fully faithful $i'\colon \mathcal{X}_{/U} \rightarrowtail \mathrm{PSh}(\mathcal{C})_{/iU}$, which furthermore admits a left adjoint a' induced by a (since $U \to aiU$ is an equivalence). The functor a' is seen to be accessible and left exact since a is.

Note that $\mathrm{PSh}(\mathcal{C})_{/iU}$ is itself equivalent to presheaves on $\mathcal{C}/iU := \mathcal{C} \times_{\mathrm{PSh}(\mathcal{C})} \mathrm{PSh}(\mathcal{C})_{/iU}$, which is itself a equivalent to small ∞-category. We therefore obtain a presentation for $\mathcal{X}_{/U}$ as a full subcategory of $\mathrm{PSh}(\mathcal{C}/iU)$. □

Example 8.4.2. Let X be a topological space. The Yoneda functor $\mathrm{Open}_X \to \mathrm{Shv}(X)$ factors through the full subcategory $\mathrm{Shv}(X)$. Thus for any open set U of X, we have the representable sheaf $\rho_U \in \mathrm{Shv}(X)$, which we simply denote U by abuse of notation. It is straightforward to show that $\mathrm{Shv}(X)_{/U} \approx \mathrm{Shv}(U)$: the slice category over the sheaf U is exactly sheaves on the topological space U.

Remark 8.4.3 (Relativized notions). Any morphism $f\colon V \to U$ in an ∞-topos $\mathcal{X}$ is also an object in an ∞-topos (namely $\mathcal{X}_{/U}$). Thus any general concept defined for objects in an ∞-topos can be “relativized” to a concept defined on morphisms (assuming the definition is preserved by equivalence of ∞-topoi). Conversely, any concept defined for morphisms in an arbitrary ∞-topos can be specialized to objects, by applying it to projection maps $U \to 1$.

Colimits are universal in ∞-topoi

Given a morphism $f\colon U \to V$ in an ∞-topos $\mathcal{X}$, we get an induced **pullback functor** $f^*\colon \mathcal{X}_{/V} \to \mathcal{X}_{/U}$, which on objects sends $V' \to V$ to $V' \times_V U \to U$.

Proposition 8.4.4. *Colimits are “universal” in ∞-topoi; i.e., $f^*\colon \mathcal{X}_{/V} \to \mathcal{X}_{/U}$ preserves small colimits.*

Proof. The statement of the proposition only involves colimits and *finite* limits in $\mathcal{X}$. Thus via a choice of presentation $(\mathcal{C}, i, a)$ for $\mathcal{X}$ we can reduce to the case of $\mathcal{X} = \mathrm{PSh}(\mathcal{C})$. As colimits and limits of presheaves are computed “objectwise”, we can reduce to the case of infinity groupoids $\mathcal{X} = \mathcal{S}$. In this case the statement is “well-known” [169, 6.1.3.14]. □

∞-topoi have internal homs

A consequence of universality of colimits is that $U \times (-)\colon \mathcal{X} \to \mathcal{X}$ is colimit preserving, and therefore (8.2.2) has a right adjoint which we may denote $[U, -]\colon \mathcal{X} \to \mathcal{X}$. This is an internal function object, so any ∞-topos is *cartesian closed*, and so is *locally cartesian closed* (i.e., every slice is cartesian closed).

∞-topoi have descent

Given any ∞-category $\mathcal{X}$, let $\mathrm{Cart}(\mathcal{X}) \subseteq \mathrm{Fun}(\{0 \to 1\}, \mathcal{X})$ denote the (non-full) subcategory of the arrow category of $\mathcal{X}$, consisting of all the objects, and morphisms $f \to g$ which are pullback squares in $\mathcal{X}$. This is a subcategory because pullback squares paste together.

We say that $\mathcal{X}$ has **descent** if $\mathrm{Cart}(\mathcal{X})$ has small colimits, and if the inclusion functor $\mathrm{Cart}(\mathcal{X}) \to \mathrm{Fun}(\{0 \to 1\}, \mathcal{X})$ preserves small colimits.

Proposition 8.4.5 (Descent [169, 6.1.3]). *Every ∞-topos has descent.*

Let's spell out the consequences of this. Suppose given a functor $F\colon \mathcal{C} \to \mathcal{X}$ from a small ∞-category to an ∞-topos. We obtain a family of slice categories $\mathcal{X}_{/F(c)}$, which is a contravariant functor of $\mathcal{C}$ via the functors $F(\alpha)^*\colon \mathcal{X}_{/F(c')} \to \mathcal{X}_{/F(c)}$ for $\alpha\colon c \to c'$ in $\mathcal{C}$. This functor $\mathcal{C}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ extends to a cone $(\mathcal{C}^{\triangleright})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$, where the value at the cone point is the slice category $\mathcal{X}_{/\overline{F}}$ over the colimit $\overline{F} = \mathrm{colim}_{c\in\mathcal{C}} F(c)$ of F.[2]

We can also form the limit $\lim_{c\in\mathcal{C}^{\mathrm{op}}} \mathcal{X}_{/F(c)}$ in $\widehat{\mathrm{Cat}}_\infty$. An object of this limit amounts to: a functor $A\colon \mathcal{C} \to \mathcal{X}$ and a natural transformation $f\colon A \to F$ such that for each $\alpha\colon c \to c'$ in $\mathcal{C}$ the square

$$\begin{array}{ccc} A(c') & \xrightarrow{A(\alpha)} & A(c) \\ \downarrow & & \downarrow \\ F(c') & \xrightarrow[F(\alpha)]{} & F(c) \end{array}$$

is a pullback in $\mathcal{X}$. Descent implies the following.

Proposition 8.4.6. *The functor*

$$\mathcal{X}_{/\overline{F}} \to \lim_{c\in\mathcal{C}^{\mathrm{op}}} \mathcal{X}_{/F(c)}$$

sending $\overline{A} \to \overline{F}$ to $\big(c \mapsto (\overline{A} \times_{\overline{F}} F(c) \to F(c))\big)$ is an equivalence. The inverse equivalence is a functor which sends $(A \to F) \in \lim_{\mathcal{C}^{\mathrm{op}}} \mathcal{X}_{/F(c)}$ to the object of $\mathcal{X}_{/\overline{F}}$ represented by the evident map

$$\mathrm{colim}_{\mathcal{C}} A \to \mathrm{colim}_{\mathcal{C}} F.$$

Thus, descent in an ∞-topos has a very beautiful interpretation in terms of the definition of "sheaves on $\mathcal{X}$" as functors: the functor $\mathcal{X}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ which sends $U \mapsto \mathcal{X}_{/U}$ is limit preserving, and so is a *sheaf* on $\mathcal{X}$ valued in locally small ∞-categories.

Example 8.4.7. Let X be a topological space. Recall that (after identifying an open set U with its representable sheaf on X), we have that $\mathrm{Shv}(X)_{/U} \approx \mathrm{Shv}(U)$. If U and

[2] This is not a complete description of a functor $(\mathcal{C}^{\triangleright})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$, as there is also "higher coherence" data to keep track of. A correct description is implemented using the theory of *Cartesian fibrations* [169, 2.4]. I am not going to try to be precise about such matters here.

V are open sets of X, then $U \cup V$ is the pushout of $U \leftarrow U \cap V \rightarrow V$ as sheaves. Given this, descent says that there is an equivalence

$$\mathrm{Shv}(U \cup V) \xrightarrow{\sim} \lim\,[\,\mathrm{Shv}(U) \rightarrow \mathrm{Shv}(U \cap V) \leftarrow \mathrm{Shv}(V)].$$

That is, the category of sheaves of ∞-groupoids on $U \cup V$ is equivalent to a category of "descent data" involving sheaves on U, V, and $U \cap V$.

This particular example works "the same way" in the classical topos $\mathrm{Shv}(X)^{\heartsuit}$ of sheaves of sets on X: the category of sheaves of sets on $U \cup V$ can be reconstructed from appropriate descent data, i.e., as an ∞-categorical pullback of a diagram of categories of sheaves of sets on U, V, and $U \cap V$. However, 1-categorical descent in this form fails for general pushout diagrams in $\mathrm{Shv}(X)^{\heartsuit}$. This is one way in which the theory of ∞-topoi shows advantages over the classical theory.

8.5 Truncation and connectivity in ∞-topoi

n-Truncation and n-connectivity in ∞-categories

An ∞-groupoid X is n-**truncated** if

$$\pi_k(X, x_0) \approx \{*\} \qquad \text{for all } k > n \text{ and all } x_0 \in X.$$

In particular, 0-truncated ∞-groupoids are equivalent to sets (discrete spaces), while (-1)-truncated ∞-groupoids are equivalent to either the empty set $\varnothing$ or the terminal object. By fiat, (-2)-truncated ∞-groupoids are those which are equivalent to the terminal object.

An object $X \in \mathcal{A}$ in a general ∞-category is n-**truncated** if $\mathrm{Map}_{\mathcal{A}}(A, X)$ is an n-truncated ∞-groupoid for all objects A in $\mathcal{A}$. We relativize to the notion of n-**truncated morphism**: i.e., an $f\colon X \rightarrow Y$ which is n-truncated as an object of the slice $\mathcal{A}_{/Y}$. I write $\tau_{\leq n}\mathcal{A} \subseteq \mathcal{A}$ for the full subcategory of n-truncated objects.

In many ∞-categories (including all presentable ∞-categories and thus all ∞-topoi), there is an n-**truncation functor** which associates to each object X the initial example $X \rightarrow \tau_{\leq n}X$ of a map to an n-truncated object. When this exists, the essential image of the n-truncation functor $\tau_{\leq n}\colon \mathcal{A} \rightarrow \mathcal{A}$ is $\tau_{\leq n}\mathcal{A}$, and we have an adjoint pair $\tau_{\leq n}\mathcal{A} \rightleftarrows \mathcal{A}$.

Relativized, we obtain for a morphism $f\colon X \rightarrow Y$ in $\mathcal{A}$ an n-truncation factorization

$$X \xrightarrow{g} \tau_{\leq n}(f) \xrightarrow{h} Y,$$

so that h is the initial example of an n-truncated map over Y which factors f.

Following Lurie, we say that an object U in an ∞-category is n-**connective** if $\tau_{\leq n-1}U \approx 1$. Likewise an n-**connective morphism** $f\colon X \rightarrow Y$ in $\mathcal{A}$ is one which is an n-connective object of $\mathcal{A}_{/Y}$.

Remark 8.5.1. In $\mathcal{S}$, an n-connective object is the same as what is usually called an $(n{-}1)$-connected space (so 1-connective means connected). However, an n-connective map is the same as what is classically called an n-connected map of spaces.

The n-truncation factorization is in fact a factorization into "$(n+1)$-connective followed by n-truncated".

Proposition 8.5.2. *If $X \xrightarrow{g} \tau_{\le n}(f) \xrightarrow{h} Y$ is the n-truncation factorization of $f\colon X \to Y$ in $\mathcal{A}$, then g is an $(n+1)$-connective map in $\mathcal{A}$. (Assuming all the relevant truncations exist in $\mathcal{A}$.)*

Proof. By replacing $\mathcal{A}$ with $\mathcal{A}_{/Y}$, we can assume $Y \approx 1$. Thus we need to show that $g\colon X \to \tau_{\le n}X$, the "absolute" n-truncation of the object X, is also the "relative" n-truncation of the map g, i.e., that in the n-truncation factorization

$$X \xrightarrow{g'} \tau_{\le n}(g) \xrightarrow{g''} \tau_{\le n}X$$

of the object g of $\mathcal{A}_{/\tau_{\le n}X}$, the map g'' is an equivalence.

Both $\tau_{\le n}X \to 1$ and g'' are n-truncated maps of $\mathcal{A}$, from which it is straightforward to show that $\tau_{\le n}(g)$ is an n-truncated object of $\mathcal{A}$. Thus, the universal property for $g\colon X \to \tau_{\le n}X$ gives $s\colon \tau_{\le n}X \to \tau_{\le n}(g)$ such that $sg = g'$ and $g''s = \mathrm{id}_{\tau_{\le n}X}$. The universal property for $g'\colon X \to \tau_{\le n}(g)$ then implies that $sg'' = \mathrm{id}_{\tau_{\le n}(g)}$. □

Remark 8.5.3. n-truncation of objects in an ∞-topos preserves finite products, as can be seen by choosing a presentation and reducing to the case of $\mathcal{S}$ [169, 6.5.1.2].

Čech nerves and effective epimorphisms

For ∞-topoi, the case of truncation when $n = -1$ is especially important. An (-1)-truncated map in an ∞-category is the same thing as a **monomorphism**, i.e., a map $i\colon A \to B$ such that all the fibers of all induced maps $\mathrm{Map}(C,A) \to \mathrm{Map}(C,B)$ are either empty or contractible. Equivalently, i is a monomorphism if and only if the diagonal map $A \to A \times_B A$ is an equivalence (if the pullback exists), if and only if either projection $A \times_B A \to A$ is an equivalence.

In an ∞-topos, an **effective epimorphism** is defined to be a 0-connective morphism. The (-1)-truncation factorization in an ∞-topos (also called **epi/mono factorization**) can be computed using *Čech nerves*.

Given a morphism $f\colon U \to V$ in an ∞-topos $\mathcal{X}$, its **Čech nerve** is an augmented simplicial object $\check{C}(f)\colon \Delta_+^{\mathrm{op}} \to \mathcal{X}$ of the form

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; U \times_V U \times_V U \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; U \times_V U \;\substack{\longrightarrow\\\longrightarrow}\; U \xrightarrow{f} V$$

Proposition 8.5.4. *Given a map $f\colon U \to V$ in an ∞-topos, the factorization*

$$U \xrightarrow{p} \mathrm{colim}_{\Delta^{\mathrm{op}}}\, \check{C}(f) \xrightarrow{i} V$$

defined by taking the colimit of the underlying simplicial object of the Čech nerve is equivalent to the factorization of f into an effective epimorphism p followed by a monomorphism i.

Proof. Without loss of generality assume $V \approx 1$ (since the slice $\mathcal{X}_{/V}$ is an ∞-topos). Write $\overline{U} = \operatorname{colim}_{\Delta^{\mathrm{op}}} \check{C}(F) = \operatorname{colim}[[n] \mapsto U^{n+1}]$. Because colimits are universal in an ∞-topos (8.4.4), we have that $\overline{U} \times U^{k+1} \approx \operatorname{colim}[[n] \mapsto U^{n+1} \times U^{k+1}]$. For any $k \geq 0$ the augmented simplicial object $[n] \mapsto U^{n+1} \times U^{k+1}$ admits a contracting homotopy, so $\overline{U} \times U^{k+1} \xrightarrow{\sim} U^{k+1}$. Universality of colimits again gives $\overline{U} \times \overline{U} \xrightarrow{\sim} \overline{U}$, whence $\overline{U} \to 1$ is monomorphism, i.e., $\overline{U}$ is a (-1)-truncated object

To show that $p\colon U \to \overline{U}$ is the universal (-1)-truncation is easy: for any $f\colon U \to Z$ to a (-1)-truncated object, we have

$$\operatorname{Map}_{\mathcal{X}_{U/}}(p, f) \approx \lim_{\Delta} [[n] \mapsto \operatorname{Map}_{\mathcal{X}_{U/}}(U \to U^{n+1}, f)],$$

which is easy to evaluate since all the mapping spaces must be contractible if non-empty, since Z is (-1)-truncated.

□

Warning 8.5.5. In an ∞-topos the class of ***effective epimorphisms*** contains, but is ***not*** equal to the class of ***epimorphisms***. This is very unlike the classical case: in a 1-topos the two classes coincide.

Remark 8.5.6 (Covers). A set $\{U_i\}$ of objects in an ∞-topos $\mathcal{X}$ is called a **cover** of $\mathcal{X}$ if $\coprod U_i \to 1$ is an effective epimorphism in $\mathcal{X}$. We also speak of a cover of an object V in $\mathcal{X}$, which is a set $\{U_i \to V\}$ of maps in $\mathcal{X}$ such that $\coprod U_i \to V$ is an effective epi.

If X is a topological space, then a set $\{U_i\} \subseteq \mathrm{Open}_X$ of open sets of X is a open cover of X if and only if the corresponding set $\{U_i\} \subseteq \mathrm{Shv}(X)$ of sheaves on X is a cover in the above sense.

Sometimes we see the following condition on a collection $\{U_i\}$ of objects in $\mathcal{X}$: that it generates $\mathcal{X}$ under small colimits. This condition implies that there exists a subset of $\{U_i\}$ which covers $\mathcal{X}$.

Example 8.5.7 (Effective epis in ∞-groupoids). A map in $\mathcal{S}$ is an effective epimorphism if and only if it induces a surjection on sets of path components. The epi/mono factorization of a map $f\colon U \to V$ in $\mathcal{S}$ is through $\overline{U} \subseteq V$, the disjoint union of path components of V which are in the image of f.

Homotopy groups

Given a ***pointed*** object $(U, u_0\colon 1 \to U)$ in an ∞-topos $\mathcal{X}$, there is an object $(U, u_0)^K$ in $\mathcal{X}$ for every pointed space $K \in \mathcal{S}_*$, which represents the functor

$$\operatorname{Map}_{\mathcal{S}_*}(K, \operatorname{Map}_{\mathcal{X}}(-, U))\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{S}$$

(which is clearly limit preserving, so by (8.2.2) defines a $\mathcal{S}$-valued sheaf on $\mathcal{X}$). We let

$$\pi_n(U, u_0) := \tau_{\leq 0}((U, u_0)^{S^n}) \in \mathcal{X}^{\heartsuit},$$

the ***n*th homotopy sheaf** of (U, u_0). This is in general a sheaf of based sets on $\mathcal{X}$, a sheaf of groups for $n \geq 1$, and a sheaf of abelian groups for $n \geq 2$.

Remark 8.5.8. An object U in an ∞-topos can easily fail to have "enough" global sections, or even any global sections. Thus it is often necessary to use a more sophisticated formulation of homotopy sheaves of U allowing for arbitrary "local" choices of basepoint. These are objects $\pi_n U \in (\mathcal{X}_{/U})^{\heartsuit}$, defined as the homotopy sheaves (as defined above) of $(\mathrm{proj}_2\colon U \times U \to U, \Delta\colon U \to U \times U)$ in $\mathcal{X}_{/U}$, the projection map "pointed" by the diagonal map. See [169, 6.5.1].

For instance, with this more sophisticated definition, an object U is n-connective if and only if $\pi_k U \approx 1$ for all $k < n$ [169, 6.5.1.12].

Example 8.5.9 (Eilenberg–Mac Lane objects and sheaf cohomology). An **Eilenberg–Mac Lane object** of dimension n is a *pointed* object (K, k_0) in $\mathcal{X}$ such that K is both n-truncated and n-connective. One can show [169, 7.2.2.12] that taking $(K, k_0) \mapsto \pi_n(K, k_0)$ gives a correspondence between Eilenberg–Mac Lane objects of dimension n and: abelian group objects in $\mathcal{X}^{\heartsuit}$ (if $n \geq 2$), group objects in $\mathcal{X}^{\heartsuit}$ (if $n = 1$), and pointed objects in $\mathcal{X}^{\heartsuit}$ (if $n = 0$).

Thus, given a sheaf A of (classical) abelian groups on $\mathcal{X}$, we can define the cohomology group

$$H^n(\mathcal{X}; A) := \pi_0 \operatorname{Map}_{\mathcal{X}}(1, K(A, n))$$

of the ∞-topos $\mathcal{X}$.

∞-connectedness and hypercompletion

An object or morphism is **∞-connected** if it is n-connective for all n. It turns out that the obvious analogue of the "Whitehead theorem" can fail in an ∞-topos: ∞-connected maps need not be equivalences.

We say that an object U in $\mathcal{X}$ is **hypercomplete** if $\operatorname{Map}(V', U) \to \operatorname{Map}(V, U)$ is an equivalence for any ∞-connected map $V \to V'$.

Example 8.5.10. All n-truncated objects are hypercomplete, for any n. Any limit of hypercomplete objects is hypercomplete.

We write $\mathcal{X}^{\mathrm{hyp}} \subseteq \mathcal{X}$ for the full subcategory of hypercomplete objects of $\mathcal{X}$. It turns out that the inclusion is accessible, and admits a left adjoint which is itself left exact. So $\mathcal{X}^{\mathrm{hyp}}$ is an ∞-topos in its own right [169, 6.5.2].

We say that $\mathcal{X}$ is itself **hypercomplete** if all ∞-connected maps are equivalences, i.e., if $\mathcal{X}^{\mathrm{hyp}} = \mathcal{X}$.

Example 8.5.11. Any presheaf ∞-category is hypercomplete, including $\mathcal{S}$ itself.

Truncation towers

Given an object U in $\mathcal{X}$, we may consider the tower

$$U \to \cdots \to \tau_{\leq n} U \to \tau_{\leq n-1} U \to \cdots \to \tau_{\leq -1} U \to *$$

of truncations of U. There is a limit $U_\infty := \lim \tau_{\le n} U$, together with a tautological map $U \to U_\infty$. It is generally not the case that $U \to U_\infty$ is an equivalence. For instance, U_∞ is necessarily hypercomplete, whereas U may not be. Furthermore, even if U is hypercomplete, $U \to U_\infty$ can fail to be an equivalence.

There are various general conditions which ensure that $U \xrightarrow{\sim} U_\infty$ for all objects U in $\mathcal{X}$ (and in fact ensure a stronger fact, called convergence of Postnikov towers). For instance, this is the case when $\mathcal{X}$ is *locally of homotopy dimension* $\le n$ for some n [169, 7.2.1.12]. (Say $\mathcal{X}$ is of **homotopy dimension** $\le n$ if every n-connective object $U \in \mathcal{X}$ admits a global section $1 \to U$. We say $\mathcal{X}$ is **locally of homotopy dimension** $\le n$ if there exists a set $\{U_i\}$ of objects which generate $\mathcal{X}$ under colimits and such that each $\mathcal{X}_{/U_i}$ is of homotopy dimension $\le n$.)

Constructing ∞-topoi

We defined an ∞-topos $\mathcal{X}$ to be an ∞-category which admits a ***presentation*** $(\mathcal{C}, i, a)$. It is natural to ask: given a small ∞-category $\mathcal{C}$, can we classify the presentations of ∞-topoi which use it?

Given any left exact accessible localization $\mathcal{X} \subseteq \mathrm{PSh}(\mathcal{C})$, let $\mathcal{T}$ denote the collection of morphisms j in $\mathrm{PSh}(\mathcal{C})$ which

1. are monomorphisms of the form $S \rightarrowtail \rho_C$ for some object C of $\mathcal{C}$, and
2. are such that $a(j)$ is an isomorphism in $\mathcal{X}$.

The class of maps $\mathcal{T}$ is an example of a ***Grothendieck topology*** on $\mathcal{C}$. When $\mathcal{C}$ is a 1-category this precisely recovers the classical notion of a Grothendieck topology on a 1-category.

It can be shown [169, 6.4.1.5] that if $F \in \mathrm{PSh}(\mathcal{C})$ is n-truncated for some $n < \infty$, then $F \in \mathcal{X}$ if and only if $F(j)$ is an isomorphism for all $j \in \mathcal{T}$. That is, the ***n-truncated objects*** in left exact accessible localizations of $\mathrm{PSh}(\mathcal{C})$ are entirely determined by $\mathcal{T}$.

Conversely, given a Grothendieck topology $\mathcal{T}$ on $\mathcal{C}$, the full subcategory $\mathrm{Shv}(\mathcal{C}, \mathcal{T}) := \{F \mid F(j) \text{ iso for all } j \in \mathcal{T}\} \subseteq \mathrm{PSh}(C)$ is an example of an ∞-topos. This includes the examples (8.2) and (8.2).

A general left exact localization of $\mathrm{PSh}(\mathcal{C})$ can be obtained by (i) choosing a Grothendieck topology $\mathcal{T}$ on $\mathcal{C}$, and then (ii) possibly localizing $\mathrm{Shv}(\mathcal{C}, \mathcal{T})$ further with respect to a suitable class of ∞-connected maps [169, 6.5.2.20].

Remark 8.5.12 (1-localic reflection). Given any classical topos, i.e., a 1-topos $\mathcal{X}_1$, we can upgrade it to an ∞-topos denoted $\mathrm{Shv}_{\mathcal{S}}(\mathcal{X}_1)$; this is called its **1-localic reflection**. In general this can be difficult to describe. In the case that $\mathcal{X}_1 \approx \mathrm{Shv}_{\mathrm{Set}}(\mathcal{C}, \mathcal{T})$ is an identification of $\mathcal{X}_1$ as a category of sheaves of sets on a 1-category $\mathcal{C}$ equipped with a Grothendieck topology $\mathcal{T}$, and if $\mathcal{C}$ ***has finite limits***, then $\mathrm{Shv}_{\mathcal{S}}(\mathcal{X}_1) := \mathrm{Shv}(\mathcal{C}, \mathcal{T})$ is the 1-localic reflection of $\mathcal{X}_1$ [169, 6.4.5, esp. 6.4.5.6].

For instance, we constructed $\mathrm{Shv}(X)$ and $\mathrm{Shv}(X^{\text{ét}})$, sheaves on a topological space or on the étale site of a scheme, in exactly this way, so they are 1-localic.

As can be seen from (8.2.4), an ∞-topos $\mathcal{X}$ is not generally equivalent to the 1-localic reflection of $\mathrm{Shv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit})$ of its underlying 1-topos $\mathcal{X}^{\heartsuit}$.

Warning: $\mathrm{Shv}_{\mathcal{S}}(\mathcal{X}_1)$ is not the same as the construction of (8.3): it is *not* equivalent to limit preserving functors $\mathcal{X}_1^{\mathrm{op}} \to \mathcal{S}$.

Remark 8.5.13 (Simplicial presheaves). Given a small 1-category $\mathcal{C}$ with a Grothendieck topology $\mathcal{T}$, Jardine [135] produced a model category structure on the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{sSet})$ of presheaves of simplicial sets. The ∞-category associated to that model category is equivalent to what we have called $\mathrm{Shv}(\mathcal{C}, \mathcal{T})^{\mathrm{hyp}}$ [169, 6.5.2].

8.6 Morphisms of ∞-topoi

To justify the claim that ∞-topoi are the ∞-categorical generalization of topological spaces, we need an appropriate notion of morphism between ∞-topoi that generalizes the notion of continuous map. This is called a *geometric morphism.* In fact, I won't consider any other kind of morphism between ∞-topoi here.

Geometric morphisms

A **geometric morphism** (or just **morphism**) of ∞-topoi $f: \mathcal{X} \to \mathcal{Y}$ is an adjoint pair of functors

$$f_*: \mathcal{X} \leftrightarrows \mathcal{Y} : f^*$$

such that the left adjoint f^* is left exact (i.e., preserves finite limits). The functor f_* is **direct image**, and f_* is **pullback** or **preimage**.

The collection of geometric morphisms from $\mathcal{X}$ to $\mathcal{Y}$, together with natural transformations between the *left* adjoints of the geometric morphisms, forms an ∞-category, sometimes denoted $\mathrm{Fun}^*(\mathcal{Y}, \mathcal{X})$. We note that this ∞-category is *not in general* equivalent to a small ∞-category, although it is in some cases; it is always an *accessible* ∞-category [169, 6.3.1.13]. We will mostly be concerned with the maximal ∞-groupoid inside this ∞-category, which we denote $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{Y})$, and regard as mapping spaces of $\infty\mathcal{T}\mathrm{op}$, the ∞-category of ∞-topoi.

Remark 8.6.1. Since ∞-topoi are presentable ∞-categories, to construct a geometric morphism $f: \mathcal{X} \to \mathcal{Y}$ it suffices to produce a functor $f^*: \mathcal{Y} \to \mathcal{X}$ which preserves colimits and finite limits; presentability then implies (8.2.2) that a right adjoint f_* exists. Typically, having a "presentation" for $\mathcal{Y}$ gives an explicit recipe for describing colimit preserving f^*, so constructing morphisms amounts to finding such functors which also preserve finite limits.

Example 8.6.2 (The terminal ∞-topos). The ∞-category $\mathcal{S}$ of infinity groupoids is the *terminal* ∞-topos, i.e., there is an essentially unique geometric morphism $\mathcal{X} \to \mathcal{S}$ from any ∞-topos. To see this, note that a colimit preserving $\pi^*: \mathcal{S} \to \mathcal{X}$ is precisely determined by its value on the terminal object $1_{\mathcal{S}}$ of $\mathcal{S}$, while to preserve finite limits it is necessary that π^* take $1_{\mathcal{S}}$ to the terminal object of $\mathcal{X}$. This is also sufficient, by the fact the colimits are universal in $\mathcal{X}$ (8.4.4). Thus $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{S}) \approx *$ for any $\mathcal{X}$.

Example 8.6.3. Every presentation of an ∞-topos $\mathcal{X} \subseteq \mathrm{PSh}(\mathcal{C})$ as in (8.2) corresponds to a geometric morphism $\mathcal{X} \to \mathrm{PSh}(\mathcal{C})$.

Example 8.6.4. Hypercompletion (8.5) gives a geometric morphism $\mathcal{X}^{\mathrm{hyp}} \to \mathcal{X}$.

Continuous maps vs. geometric morphisms

Let $\mathcal{X} = \mathrm{Shv}(X)$ for some topological space X, and let $\mathcal{Y}$ be any ∞-topos. We can describe $\mathrm{Fun}^*(\mathrm{Shv}(X), \mathcal{Y})$ as follows. It is equivalent to the full subcategory of

$$\mathrm{Fun}^{\text{colim pres.}}(\mathrm{PSh}(\mathrm{Open}_X), \mathcal{Y}) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{Open}_X, \mathcal{Y}),$$

spanned by those $\phi\colon \mathrm{Open}_X \to \mathcal{Y}$ such that

1. for each open cover $\{U_i \to U\}$, the map $\coprod_i \phi(U_i) \to \phi(U)$ is an effective epi in $\mathcal{Y}$,
2. $\phi(X) \approx *$, and
3. $\phi(U \cap V) \approx \phi(U) \times_{\phi(X)} \phi(V)$.

Condition (1) ensures that $\mathrm{PSh}(\mathrm{Open}_X) \to \mathcal{Y}$ factors through the localization

$$a\colon \mathrm{PSh}(\mathrm{Open}_X) \twoheadrightarrow \mathrm{Shv}(X),$$

while conditions (2) and (3) ensure that the resulting functor $f^*\colon \mathrm{Shv}(X) \to \mathcal{Y}$ preserves finite limits. (This is a special case of [169, 6.1.5.2].)

Note that since $U \cap U \approx U$, (2) and (3) imply that each $\phi(U) \to \phi(X) \approx *$, is a monomorphism, i.e., that each $\phi(U)$ is a (-1)-truncated object of $\mathcal{Y}$.

For instance, if $\mathcal{Y} = \mathrm{Shv}(Y)$ for some topological space Y, then $\tau_{\leq -1}\mathcal{Y} \approx \mathrm{Open}_Y$. Under this identification, morphisms of topoi $\mathcal{Y} \to \mathcal{X}$ correspond to functors $\mathrm{Open}_X \to \mathrm{Open}_Y$ which (1) take covers to covers, (2) take X to Y, and (3) preserve finite intersections.

Example 8.6.5. If X is a scheme, we have both $\mathrm{Shv}(X^{\mathrm{Zar}})$ (sheaves on the underlying Zariski space of X) and $\mathrm{Shv}(X^{\text{ét}})$ (sheaves in the étale topology (8.2)). There is an evident geometric morphism $\mathrm{Shv}(X^{\text{ét}}) \to \mathrm{Shv}(X^{\mathrm{Zar}})$ induced by $\mathrm{Open}_{X^{\mathrm{Zar}}} \to \mathrm{Shv}(X^{\text{ét}})$ sending an open set to the étale sheaf it represents.

A space X is **sober** if every irreducible closed subset is the closure of a unique point (e.g., Hausdorff spaces, or the Zariski space of a scheme). One can show that if X is sober, then

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathrm{Shv}(Y), \mathrm{Shv}(X)) \approx (\text{set of continuous maps } Y \to X).$$

This justifies the assertion that "∞-topos" is a generalization of the notion of a topological space.

Remark 8.6.6. The sobriety condition is necessary. For instance, if $Y = \{*\}$, then the $\phi\colon \mathrm{Open}_X \to \mathrm{Open}_Y \approx \{0 \to 1\}$ satisfying (1)–(3) are in bijective correspondence with irreducible closed $C \subseteq X$: we have

$$\big(\phi \leftrightarrow C\big) \iff \big(C = \textstyle\bigcap_{\phi(U)=0}(X \smallsetminus U)\big) \iff \big(\phi(U) = 0 \text{ iff } U \cap C = \varnothing\big).$$

That is, the underlying point set of X can be recovered from Open_X only if X is sober.

Locales

We see that it is not quite correct to say that ∞-topoi generalize topological spaces; rather, they generalize *locales*.

A **locale** is a poset $\mathcal{O}$ equipped with all the formal algebraic properties of the poset of open sets of a space: i.e., it is a complete lattice such that finite meets distribute over infinite joins. A map $f : \mathcal{O}' \to \mathcal{O}$ of locales is a function $f^* : \mathcal{O} \to \mathcal{O}'$ which preserves all joins and all finite meets. Any locale $\mathcal{O}$ has an ∞-category of sheaves $\mathrm{Shv}(\mathcal{O})$ (defined exactly as sheaves on a space), and $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathrm{Shv}(\mathcal{O}), \mathrm{Shv}(\mathcal{O}')) \approx \{\text{locale maps } \mathcal{O} \to \mathcal{O}'\}$.

Every topological space determines a locale, though not every locale comes from a space. From the point of view of sheaf theory, a space is indistinguishable from its locale. For spaces we care about (i.e., sober spaces), we can recover their point sets from their locale, and this is good enough for us.

Remark 8.6.7. From the point of view that "objects in an ∞-topos are generalized open sets" (8.3.5), the preimage functor $f^* : \mathcal{Y} \to \mathcal{X}$ of a geometric morphism is the operation of "preimage of generalized open sets".

Remark 8.6.8. Every ∞-topos $\mathcal{X}$ has an associated locale, whose lattice of "open sets" $\mathrm{Open}_{\mathcal{X}}$ consists precisely of the (-1)-truncated objects of $\mathcal{X}$.

Limits and colimits of ∞-topoi

The ∞-category of ∞-topoi itself (remarkably) has all small limits and colimits.

Colimits are easy to describe (modulo the technical issues involved in making precise statements; see [169, 6.3.2]): given $F : \mathcal{C} \to \infty\mathcal{T}\mathrm{op}$, consider the functor $F^* : \mathcal{C}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ which sends an arrow $\alpha : C \to C'$ to the left adjoint $F(\alpha)^* : F(C') \to F(C)$ of the geometric morphism. Then the underlying ∞-category of the colimit of F in ∞-topoi is just the limit of the diagram F^* of ∞-categories.

Limits are more difficult. As we have seen, the terminal object in $\infty\mathcal{T}\mathrm{op}$ is $\mathcal{S}$. *Filtered* limits are computed by a pointwise construction much like colimits [169, 6.3.3]. To get general limits we also need pullbacks; see [169, 6.3.4] for details.

Remark 8.6.9. The product of two ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$ has a nice description. It is equivalent to

$$\mathrm{Fun}^{\text{lim pres./lim pres.}}(\mathcal{X}^{\mathrm{op}} \times \mathcal{Y}^{\mathrm{op}}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{X}^{\mathrm{op}} \times \mathcal{Y}^{\mathrm{op}}, \mathcal{S}),$$

the full subcategory consisting of functors F which preserve limits separately in each variable, i.e., such that $F(\mathrm{colim}_i U_i, V) \xrightarrow{\sim} \lim_i F(U_i, V)$ and $F(U, \mathrm{colim}_j V_j) \xrightarrow{\sim} \lim_j F(U, V_j)$. This ∞-category is also equivalent to both of

$$\mathrm{Fun}^{\text{lim pres.}}(\mathcal{X}^{\mathrm{op}}, \mathcal{Y}) \approx \mathrm{Fun}^{\text{lim pres.}}(\mathcal{Y}^{\mathrm{op}}, \mathcal{X}),$$

by the adjoint functor theorem for presentable ∞-categories (8.2.2). That is,

$$\mathcal{X} \times^{\infty\mathcal{T}\mathrm{op}} \mathcal{Y} \approx \mathrm{Shv}_{\mathcal{Y}}(\mathcal{X}) \approx \mathrm{Shv}_{\mathcal{X}}(\mathcal{Y})$$

[168, 4.8.1.18]. This construction is a special case of the "tensor product" of presentable ∞-categories; see [168, 4.8].

Remark 8.6.10. Recall that in scheme theory, the underlying topological space of the pullback of schemes is *not* usually equivalent to the pullback of the underlying spaces of the schemes, as is already easily seen in the case of affine schemes. The analogous fact applies in the setting of derived geometry. Thus, we won't actually need to worry about general limits of ∞-topoi.

Sheaves and geometric morphisms

We are going to be interested in sheaves on ∞-topoi with values in things like *spectra* or $\mathbb{E}_\infty$*-ring spectra*. Thus we need to know how these behave under geometric morphisms.

For any complete ∞-category $\mathcal{A}$, any geometric morphism $f\colon \mathcal{X} \to \mathcal{Y}$ induces a **direct image** functor $f_*\colon \mathrm{Shv}_{\mathcal{A}}(\mathcal{X}) \to \mathrm{Shv}_{\mathcal{A}}(\mathcal{Y})$, which is defined by precomposition with f^*. That is, it sends a limit preserving $F\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{A}$ to the composite functor

$$\mathcal{Y}^{\mathrm{op}} \xrightarrow{(f^*)^{\mathrm{op}}} \mathcal{X}^{\mathrm{op}} \xrightarrow{F} \mathcal{A},$$

which is limit preserving because f^* is colimit preserving. The construction $F \mapsto f_*F$ is itself limit preserving, and thus, if $\mathcal{A}$ is presentable, admits a left adjoint f^*.

The left adjoint f^* is in general difficult to describe explicitly. However, in many of the cases we are interested in (e.g., spectra, $\mathbb{E}_\infty$-rings, topological abelian groups) $\mathcal{A}$ is a **compactly generated ∞-category** (see [169, 5.5.7]). This means[3] that there exists a small and finite cocomplete $\mathcal{A}_0$, and a left exact functor $\mathcal{A}_0 \to \mathcal{A}$ inducing an equivalence

$$A \mapsto \mathrm{Map}_{\mathcal{A}}(-,A)\colon \mathcal{A} \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}((\mathcal{A}_0)^{\mathrm{op}},\mathcal{S}) \subseteq \mathrm{Fun}((\mathcal{A}_0)^{\mathrm{op}},\mathcal{S}),$$

where "lex" indicates the full subcategory of left exact (= finite limit preserving) functors.

Example 8.6.11. For instance, if $\mathcal{A} = \mathrm{Sp}$ is the ∞-category of spectra, we can take $\mathcal{A}_0$ to be the full subcategory of "finite" spectra, i.e., those built from finitely many cells.

For such $\mathcal{A}$, we then have equivalences

$$\mathrm{Shv}_{\mathcal{A}}(\mathcal{X}) = \mathrm{Fun}^{\mathrm{lim.\ pres}}(\mathcal{X}^{\mathrm{op}},\mathcal{A}) \approx \mathrm{Fun}^{\mathrm{lim.\ pres}}(\mathcal{A}^{\mathrm{op}},\mathcal{X}) \approx \mathrm{Fun}^{\mathrm{lex}}((\mathcal{A}_0)^{\mathrm{op}},\mathcal{X}),$$

(where the middle equivalence sends a limit preserving functor $\mathcal{X}^{\mathrm{op}} \to \mathcal{A}$ to the right adjoint of its opposite, using (8.2.2)). It turns out that in this case a geometric morphism $f\colon \mathcal{X} \to \mathcal{Y}$ induces direct image and pullback functors $\mathrm{Shv}_{\mathcal{A}}(\mathcal{X}) \leftrightarrows \mathrm{Shv}_{\mathcal{A}}(\mathcal{Y})$ by

[3] To see this combine [169, 5.3.5.10] and [169, 5.5.1.9].

postcomposition with $f_*\colon \mathcal{X} \to \mathcal{Y}$ and $f^*\colon \mathcal{Y} \to \mathcal{X}$ respectively (defined because both of these are left exact). (See [163, V 1.1.8].)

Remark 8.6.12 (Descent for sheaves). An immediate consequence of this is *descent* for sheaves with values in compactly generated ∞-categories $\mathcal{A}$: if $\mathcal{X} \approx \operatorname{colim}_i \mathcal{X}_i$ in $\infty\mathcal{T}\mathrm{op}$, then $\mathrm{Shv}_{\mathcal{A}}(\mathcal{X}) \approx \lim_i \mathrm{Shv}_{\mathcal{A}}(\mathcal{X}_i)$, where the limit is taken over pullback functors. In particular, if $U \approx \operatorname{colim}_i U_i$ in $\mathcal{X}$, then $\mathrm{Shv}_{\mathcal{A}}(\mathcal{X}_{/U}) \approx \lim_i \mathrm{Shv}_{\mathcal{A}}(\mathcal{X}_{/U_i})$.

8.7 Étale morphisms

Any morphism $f\colon U \to V$ in $\mathcal{X}$ gives rise to a geometric morphism, denoted $f\colon \mathcal{X}_{/U} \to \mathcal{X}_{/V}$, where the left exact left adjoint f^* is defined by pullback along f. (We already met this functor in (8.4).) In particular, for any $U \in \mathcal{X}$ there is a geometric morphism $\pi\colon \mathcal{X}_{/U} \to \mathcal{X}$.

Maps to slices of ∞-topoi

Proposition 8.7.1. *Given $U \in \mathcal{X}$ and a geometric morphism $f\colon \mathcal{Y} \to \mathcal{X}$, there is an equivalence*

$$\left\{ \begin{array}{ccc} & & \mathcal{X}_{/U} \\ & \overset{s}{\nearrow} & \downarrow \pi \\ \mathcal{Y} & \xrightarrow[f]{} & \mathcal{X} \end{array} \right\} \xrightarrow{\sim} \left\{ 1 \dashrightarrow f^*U \right\}$$

between the ∞-category of "sections" of π over $\mathcal{Y}$, and the ∞-groupoid of global sections of f^*U on $\mathcal{Y}$. It is defined by sending s to $s^*(t)$, where $t\colon 1 \to \pi^*U$ is the map in $\mathcal{X}_{/U}$ represented by the diagonal map $\Delta\colon U \to U \times U$. (See [169, 6.3.5.5] for a more precise statement and proof.)

As a consequence, we see that $U \mapsto \mathcal{X}_{/U}$ describes a fully faithful functor $\mathcal{X} \rightarrowtail \infty\mathcal{T}\mathrm{op}_{/\mathcal{X}}$. Thus, objects of $\mathcal{X}$, which as we have seen (8.3.5) can be thought of as "generalized open sets" of $\mathcal{X}$, can also be identified with particular kinds of geometric morphisms to $\mathcal{X}$, and we lose no information by doing so.

Example 8.7.2 (Espace étalé). Given a sheaf of *sets* F on a topological space X, the **espace étalé** of F is a topological space X_F equipped with a map $\pi\colon X_F \to X$, defined so that $\mathrm{Open}_{X_F} = \coprod_{U \in \mathrm{Open}_X} F(U)$. It is not hard to show that $\mathrm{Shv}(X_F) \approx \mathrm{Shv}(X)_{/F}$, and that there is a bijection between maps $F \to F'$ in $\mathrm{Shv}_{\mathrm{Set}}(X)$, and maps $X_F \to X_{F'}$ of topological spaces which are compatible with the projection to X.

Any local homeomorphism $f\colon Y \to X$ of spaces is equivalent to the espace étalé of a sheaf of sets. Local homeomorphisms are also called *étale* maps of spaces, which motivates the terminology of the next section.

Étale morphisms of ∞-topoi

A geometric morphism is **étale** if it is equivalent to a morphism of the form $\pi\colon \mathcal{X}_{/U} \to \mathcal{X}$ for some ∞-topos $\mathcal{X}$ and object $U \in \mathcal{X}$. This class includes the geometric morphism $\mathcal{X}_{/U} \to \mathcal{X}_{/V}$ induced by a map $f\colon U \to V$ in $\mathcal{X}$, as f also represents object of the ∞-topos $\mathcal{X}_{/V}$.

Remark 8.7.3 (Pullbacks of étale morphisms). Pullbacks of étale morphisms of ∞-topoi are étale: (8.7.1) implies a pullback diagram

$$\begin{array}{ccc} \mathcal{Y}_{/f^*U} & \longrightarrow & \mathcal{X}_{/U} \\ \downarrow & & \downarrow \\ \mathcal{Y} & \xrightarrow[f]{} & \mathcal{X} \end{array}$$

in $\infty\mathcal{T}\mathrm{op}$.

Remark 8.7.4 (Characterization of étale morphisms). For any étale morphism $f\colon \mathcal{Y} \to \mathcal{X}$, the pullback functor f^* admits a left adjoint $f_!\colon \mathcal{Y} \to \mathcal{X}$. In the case of the projection $\pi\colon \mathcal{X}_{/U} \to \mathcal{X}$, this is the evident functor which on objects sends $V \to U$ to V.

The left adjoint $f_!$ associated to an étale morphism $f\colon \mathcal{Y} \to \mathcal{X}$ is conservative, and has the property that the evident map $f_!(f^*U \times_{f^*V} Z) \xrightarrow{\sim} U \times_V f_!Z$ is an equivalence for all $Z \in \mathcal{Y}$ and all $U \to V$ and $f_!Z \to V$ in $\mathcal{X}$. Furthermore, étale morphisms f are characterized by the existence of an $f_!$ with these properties [169, 6.3.5.11].

Remark 8.7.5 ("Restriction" of sheaves along étale maps). For an étale morphism $f\colon \mathcal{Y} \to \mathcal{X}$ and any ∞-category $\mathcal{A}$, the induced functor $f^*\colon \mathrm{Shv}_{\mathcal{A}}(\mathcal{X}) \to \mathrm{Shv}_{\mathcal{A}}(\mathcal{Y})$ on $\mathcal{A}$-valued sheaves admits a very simple description using $f_!$: it sends $F\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{A}$ to $F(f_!)^{\mathrm{op}}\colon \mathcal{Y}^{\mathrm{op}} \to \mathcal{A}$. When f is the projection $\mathcal{X}_{/U} \to \mathcal{X}$ this amounts to saying that $(f^*F)(V \to U) \approx F(V)$. It is easy to think of this as a "restriction" functor, so sometimes we will use the notation "$F|_U$" for f^*F in this case.

Colimits along étale maps of ∞-topoi

Let $\infty\mathcal{T}\mathrm{op}_{\text{ét}} \subseteq \infty\mathcal{T}\mathrm{op}$ denote the (non-full) subcategory consisting of étale morphisms between arbitrary ∞-topoi.

Proposition 8.7.6 ([169, 6.3.5.13]). *The ∞-category $\infty\mathcal{T}\mathrm{op}_{\text{ét}}$ has all small colimits, and the inclusion $\infty\mathcal{T}\mathrm{op}_{\text{ét}} \to \infty\mathcal{T}\mathrm{op}$ preserves small colimits.*

For instance, given an ∞-topos $\mathcal{X}$, the descent property (8.4.5), plus the fact that colimits in $\infty\mathcal{T}\mathrm{op}$ are computed as limits in $\widehat{\mathrm{Cat}}_\infty$ (8.5), implies that the functor

$$U \mapsto \mathcal{X}_{/U}\colon \mathcal{X} \to \infty\mathcal{T}\mathrm{op}$$

is itself colimit preserving. This functor clearly factors through the subcategory $\infty\mathcal{T}\mathrm{op}_{\text{ét}}$. In fact, every colimit in $\infty\mathcal{T}\mathrm{op}_{\text{ét}}$ is equivalent to one of this form.

Example 8.7.7. Any equivalence of ∞-topoi is étale. Thus, if $\mathcal{X}\colon \mathcal{G} \to \infty\mathcal{T}\mathrm{op}$ is a functor from a small ∞-groupoid $\mathcal{G}$, it factors through $\infty\mathcal{T}\mathrm{op}_{\text{ét}} \to \infty\mathcal{T}\mathrm{op}$, so its colimit is a "quotient ∞-topos" $\mathcal{X}/\!/\mathcal{G}$, with the property that $\mathcal{X}(c) \to \mathcal{X}/\!/\mathcal{G}$ is étale for all objects $c \in \mathcal{G}$.

For instance, let $\mathcal{X} = \mathrm{Shv}(X)$ be the ∞-topos of sheaves on a topological space X, and let G be a discrete group acting on X. Then $\mathcal{X}/\!/G$ is equivalent to an ∞-category of "G-equivariant sheaves on X", and the projection map $\pi\colon \mathcal{X} \to \mathcal{X}/\!/G$ is étale.

Remark 8.7.8. The proof of (8.7.6) is pretty technical, but ultimately it is a generalization of the following observation: given open immersions $U \leftarrow W \to V$ of topological spaces, the pushout X in spaces can be constructed so that a basis of open sets is described by the category $\mathrm{colim}[\mathrm{Open}_U \leftarrow \mathrm{Open}_W \to \mathrm{Open}_V]$.

8.8 Spectra and commutative ring spectra

Now that we have ∞-categorical versions of spaces, we can put sheaves of spectra or commutative ring spectra on them. In this section I collect some notation and observations about these; some familiarity with spectra and structured ring spectra on the part of the reader is assumed.

Spectra

We write Sp for the ∞-category of spectra. It is an example of a *stable ∞-category* [168, 1.1.1.9], and so is pointed, has suspension and loop functors which are inverse to each other, has fiber sequences and cofiber sequences which coincide, and so forth.

The ∞-category Sp has a symmetric monoidal structure with respect to "smash product", here denoted "$\otimes$", with unit object being the sphere spectrum $\mathbb{S}$. The monoidal structure is closed, so there are internal hom objects.

We write $\Omega^{\infty-n}\colon \mathrm{Sp} \to \mathcal{S}$ for the usual "forgetful" functors, and define homotopy groups of spectra by $\pi_n X = \pi_{n+k}\Omega^{\infty-k}X$ for $n \in \mathbb{Z}$, and any $k \geq -n$. We say that a spectrum X is n-**truncated** if $\Omega^{\infty-k}X \approx 1$, or equivalently if $\pi_k X \approx 0$ for $k < n$. We say a spectrum is n-**connective** if $\pi_k X \approx 0$ for $k > n$, and **connective** if 0-connective.

We write $\mathrm{Sp}_{\leq n}$ and $\mathrm{Sp}_{\geq n}$ respectively for the full subcategories in Sp of n-truncated and n-connective objects. The intersection

$$\mathrm{Sp}^{\heartsuit} = \mathrm{Sp}_{\geq 0} \cap \mathrm{Sp}_{\leq 0}$$

is equivalent to the ordinary category of abelian groups: every abelian group A corresponds to an Eilenberg–MacLane spectrum in $\mathrm{Sp}^{\heartsuit}$, which we also denote A by abuse of notation.

Warning 8.8.1. The notion of n-truncated spectrum described above is *not* the same as the general notion of n-truncation in an ∞-category that we described earlier (8.5): since every spectrum is a suspension of one, every n-truncated object in Sp (in the earlier sense) is equivalent to 0. The pair $(\mathrm{Sp}_{\leq 0}, \mathrm{Sp}_{\geq 0})$ is instead an example of a *t-structure* on Sp [168, 1.2.1].

Commutative ring spectra

By an $\mathbb{E}_\infty$**-ring**, we mean a commutative ring object with respect to the symmetric monoidal structure on the ∞-category of spectra. The ∞-category of commutative rings is denoted CAlg. (We are following the notation and terminology of [170] here. This notion of $\mathbb{E}_\infty$-ring is an ∞-categorical manifestation of the notion of structured commutative ring spectrum/commutative $\mathbb{S}$-algebra as defined in, e.g., [94].)

Given $A \in \mathrm{CAlg}$ we write $\mathrm{CAlg}_A = \mathrm{CAlg}_{A/}$ for the category of $\mathbb{E}_\infty$-rings under A, also called **commutative** A**-algebras**. The initial $\mathbb{E}_\infty$-algebra is the sphere spectrum $\mathbb{S}$, so $\mathrm{CAlg} = \mathrm{CAlg}_{\mathbb{S}}$.

There is a forgetful functor $\mathrm{CAlg} \to \mathrm{Sp}$ which is conservative. The homotopy groups of an $\mathbb{E}_\infty$-algebra are those of its underlying spectrum, and likewise we may speak of an $\mathbb{E}_\infty$-ring being n-truncated or n-connective by reference to its underlying spectrum. In particular we distinguish the full subcategory $\mathrm{CAlg}^{\mathrm{cn}}$ of **connective** $\mathbb{E}_\infty$-rings, i.e., those $A \in \mathrm{CAlg}$ such that $\pi_k A \approx 0$ for $k < 0$.

We further consider the full subcategory $\mathrm{CAlg}^{\heartsuit}$ of $\mathbb{E}_\infty$-algebras which are both 0-connective and 0-truncated. This is equivalent to the ordinary category of commutative rings, so we will identify an ordinary commutative ring with its corresponding Eilenberg–Mac Lane spectrum in $\mathrm{CAlg}^{\heartsuit}$.

We have adjoint pairs

$$\mathrm{CAlg}^{\heartsuit} \rightleftarrows \mathrm{CAlg}^{\mathrm{cn}} \rightleftarrows \mathrm{CAlg}$$

of fully faithful and localization functors relating these subcategories; the localization functors of these pairs are denoted $\tau_{\geq 0}\colon \mathrm{CAlg} \to \mathrm{CAlg}^{\mathrm{cn}}$ and $\tau_{\leq 0}\colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{CAlg}^{\heartsuit}$. Note that $\mathbb{S} \in \mathrm{CAlg}^{\mathrm{cn}}$ and that $\mathbb{S} \to \tau_{\leq 0}\mathbb{S} \approx \mathbb{Z}$.

Remark 8.8.2 (General truncation of $\mathbb{E}_\infty$-rings). The ∞-category CAlg of $\mathbb{E}_\infty$-rings, being a presentable ∞-category, has n-truncation functors $\tau_{\leq n}\colon \mathrm{CAlg} \to \mathrm{CAlg}$ for $n \geq -1$ (8.5). However, these are not generally compatible with the n-truncation functors on spectra defined in (8.8). For example, the periodic complex K-theory spectrum KU admits the structure of an $\mathbb{E}_\infty$-ring, but its nth truncation as an $\mathbb{E}_\infty$-ring is equivalent to 0 for all $n \geq -1$.

However, the n-truncation functors on CAlg restrict to functors on connective $\mathbb{E}_\infty$-rings $\tau_{\leq n}\colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{CAlg}^{\mathrm{cn}}$, which are in fact the n-truncation functors for $\mathrm{CAlg}^{\mathrm{cn}}$, and which are in fact compatible with n-truncation of the underlying spectra.

Modules

To each $\mathbb{E}_\infty$-ring A there is an associated ∞-category of (left) modules Mod_A, which is itself closed symmetric monoidal: we write $M \otimes_A N$ for the monoidal product and $\underline{\mathrm{Hom}}_A(M, N)$ for the internal hom. We have that $\mathrm{Mod}_{\mathbb{S}} \approx \mathrm{Sp}$, an equivalence of symmetric monoidal ∞-categories.

Example 8.8.3. If $A \in \mathrm{CAlg}^{\heartsuit}$ is an ordinary ring, then Mod_A is equivalent to the ∞-category obtained from chain complexes of A-modules and quasi-isomorphisms

[266]. Thus, the homotopy category of Mod_A is the *derived category* of the ring A. The tensor product on Mod_A corresponds to the *derived* tensor product of complexes.

Remark 8.8.4 ($\mathbb{Z}$-modules are abelian groups). We will write $\mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}} \subseteq \mathrm{Mod}_{\mathbb{Z}}$ for the full subcategory of (-1)-connected $\mathbb{Z}$-modules. The ∞-category $\mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}}$ is equivalent to those obtained from each of the following examples by inverting the evident weak equivalences: (-1)-connected chain complexes of abelian groups, simplicial abelian groups, topological abelian groups.

An object X in an ∞-category $\mathcal{A}$ is called an **abelian group object** if it represents a functor $\mathcal{A}^{\mathrm{op}} \to \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}}$.

Every commutative A-algebra has an underlying A-module. The coproduct of A-algebras coincides with tensor product of A-modules. For this reason, we typically denote coproduct in CAlg_A by $B \otimes_A C$.

The homotopy groups $\pi_* M$ of an A-module are automatically a graded $\pi_* A$-module. To get a feel for how these things behave, it is useful to be aware of two spectral sequences:

$$E_2 = \mathrm{Tor}_*^{\pi_* A}(\pi_* M, \pi_* N) \Longrightarrow \pi_*(M \otimes_A N),$$
$$E_2 = \mathrm{Ext}_{\pi_* A}^*(\pi_* M, \pi_* N) \Longrightarrow \pi_* \underline{\mathrm{Hom}}_A(M, N).$$

The Tor spectral sequence satisfies complete convergence, while the Ext spectral sequence satisfies conditional convergence [94, Ch. IV].

Flat modules and $\mathbb{E}_\infty$-rings

An A-module M is said to be **flat** if

1. $\pi_0 M$ is flat as a $\pi_0 A$-module, and
2. the evident maps $\pi_0 M \otimes_{\pi_0 A} \pi_n A \to \pi_n M$ are isomorphisms for all n.

Likewise, a map $A \to B$ of $\mathbb{E}_\infty$-rings is **flat** if B is flat as an A-module, In view of the tor spectral sequence, we see that if $A \to B$ is flat then $\pi_*(B \otimes_A N) \approx \pi_0 B \otimes_{\pi_0 A} \pi_* N$ for $N \in \mathrm{Mod}_A$.

Remark 8.8.5 (Flatness and connective covers). Let's pause to note the following. Consider the map $\tau_{\geq 0} A \to A$ from the connective cover to an $\mathbb{E}_\infty$-ring A. The base change functor $A \otimes_{\tau_{\geq 0} A} -\colon \mathrm{Mod}_{\tau_{\geq 0} A} \to \mathrm{Mod}_A$ restricts to an *equivalence*

$$\mathrm{Mod}_{\tau_{\geq 0} A}^{\flat} \xrightarrow{\sim} \mathrm{Mod}_A^{\flat}$$

of full subcategories of *flat* modules; the inverse equivalence sends an A-module N to its connective cover $\tau_{\geq 0} N$ viewed as a $\tau_{\geq 0} A$-module. Similarly, we obtain an equivalence

$$\mathrm{CAlg}_{\tau_{\geq 0} A}^{\flat} \xrightarrow{\sim} \mathrm{CAlg}_A^{\flat}$$

of full subcategories of algebras which are flat over the ground ring. Thus, any flat morphism of $\mathbb{E}_\infty$-rings is a base change of one between connective $\mathbb{E}_\infty$-rings. This phenomenon turns out to extend to nonconnective spectral Deligne–Mumford stacks (8.13.6).

Examples of $\mathbb{E}_\infty$-rings

Example 8.8.6 (Polynomial rings). Given any space K, we obtain a spectrum $\mathbb{S}[K] =$ the suspension spectrum of K_+. If K is equipped with the structure of an $\mathbb{E}_\infty$-space (i.e., space with an action by an $\mathbb{E}_\infty$-operad), then $\mathbb{S}[K]$ is equipped with a corresponding structure of $\mathbb{E}_\infty$-ring. A particular example of this is when K is a discrete commutative monoid.

For instance, we can form **polynomial rings**: $\mathbb{S}[x] := \mathbb{S}[\mathbb{Z}_{\geq 0}]$, and more generally $A[x_1,\ldots,x_n] := A \otimes \mathbb{S}[(\mathbb{Z}_{\geq 0})^n] \approx A \otimes \mathbb{S}[\mathbb{Z}_{\geq 0}]^{\otimes n}$. We have

$$\pi_*\Big(A[x_1,\ldots,x_n]\Big) \approx (\pi_* A)[x_1,\ldots,x_n].$$

Thus, $A[x_1,\ldots,x_n]$ is a flat A-algebra. In particular, if A is an ordinary ring, then $A[x_1,\ldots,x_n]$ is also an ordinary ring.

Example 8.8.7 (Free rings). Let $\mathbb{S}\{x\}$ denote the **free $\mathbb{E}_\infty$-ring on one generator**, which is characterized by the existence of isomorphisms

$$\mathrm{Map}_{\mathrm{CAlg}}(\mathbb{S}\{x\}, R) \xrightarrow{\sim} \Omega^\infty(R)$$

natural in $R \in \mathrm{CAlg}$. We have that $\mathbb{S}\{x\} \approx \mathbb{S}[\coprod_k B\Sigma_k]$.

We may similarly define $A\{x_1,\ldots,x_n\} := A \otimes \mathbb{S}\{x\}^{\otimes n}$, the free commutative A-algebra on n generators.

There is a canonical map $A\{x_1,\ldots,x_n\} \to A[x_1,\ldots,x_n]$ from the free ring to the polynomial ring. It is generally not an equivalence, but is an equivalence if $\mathbb{Q} \subseteq \pi_0 A$. When A is connective so is $A\{x_1,\ldots,x_n\}$, and then $\pi_0\Big(A\{x_1,\ldots,x_n\}\Big) \approx \pi_0 A[x_1,\ldots,x_n]$; however, no such isomorphism on π_0 holds for general non-connective $\mathbb{E}_\infty$-rings.

$\mathbb{E}_\infty$-rings of finite characteristic

We note the following curious fact, conjectured by May and proved by Hopkins; see [188]. It is a generalization of the Nishida nilpotence theorem, which is the special case $R = \mathbb{S}$.

Theorem 8.8.8. *For any $R \in \mathrm{CAlg}$, all elements in the kernel of the evident map $\pi_* R \to \pi_*(R \otimes \mathbb{Z})$ are nilpotent. In particular, $R \otimes \mathbb{Z} \approx 0$ implies $R \approx 0$.*

Many spectra which arise in chromatic homotopy theory have the property that $R \otimes \mathbb{Z} \xrightarrow{\sim} R \otimes \mathbb{Q}$; e.g., if $R \approx L_n^f R$ for some n at some prime p. Therefore, if $R \in \mathrm{CAlg}$ is such that $R_{(p)} \approx L_n^f R_{(p)} \not\approx 0$ for some prime p and some $n < \infty$, then $1 \in \pi_0 R$ has infinite order. So there are no non-trivial $\mathbb{E}_\infty$-rings of finite characteristic in chromatic homotopy.

A related result of Hopkins–Mahowald is: any $R \in \mathrm{CAlg}$ such that $p = 0 \in \pi_0 R$ admits the structure of a $\mathbb{Z}/p$-module [188, Theorem 4.18]. In particular, the underlying spectrum of an $\mathbb{E}_\infty$-ring of positive characteristic p is always a product of Eilenberg–MacLane spectra.

Other kinds of commutative rings

We note several other flavors of commutative ring which can be used in derived versions of algebraic geometry.

1. Given an ordinary ring R, there is a notion of chain-level $\mathbb{E}_\infty$-R-algebra, consisting of an unbounded chain complex of abelian groups equipped with the action of a chain-level $\mathbb{E}_\infty$-operad. The resulting ∞-category of chain level $\mathbb{E}_\infty$-R-algebras is equivalent to CAlg_R [236].
2. Over any ordinary ring R we may consider the category of differential graded commutative R-algebras. In general it is not possible to extract a useful ∞-category from this notion. However, it is possible when $R \supseteq \mathbb{Q}$, in which case the resulting ∞-category is equivalent to CAlg_R.
3. The category of ***simplicial commutative rings*** gives rise to an ∞-category CAlg^Δ. This ∞-category is related to $\mathrm{CAlg}_\mathbb{Z}$ but is quite distinct from it. In fact, there is a conservative "forgetful" functor
$$\mathrm{CAlg}^\Delta \to \mathrm{CAlg}_\mathbb{Z}^{\mathrm{cn}}$$
which is both limit and colimit preserving. This implies that simplicial commutative rings are intrinsically connective objects, and that pushouts in CAlg^Δ are computed as tensor products on underlying $\mathbb{Z}$-modules.
   However, the above functor is far from being an equivalence. For instance, the "free simplicial commutative ring on one generator" maps to $\mathbb{Z}[x] \in \mathrm{CAlg}_\mathbb{Z}^{\mathrm{cn}}$, rather than to $\mathbb{Z}\{x\}$. See [170, 25.1].

Spectrally ringed ∞-topoi

The categories Sp and CAlg are presentable ∞-categories (and in fact are compactly generated), so it is straightforward to consider sheaves on an ∞-topos valued in each of these. For any such sheaf $\mathcal{O}$ on $\mathcal{X}$ we have homotopy sheaves $\pi_k\mathcal{O}$ on $\mathcal{X}^\heartsuit$.

A **spectrally ringed ∞-topos** is a pair $X = (\mathcal{X}, \mathcal{O}_X)$ consisting of an ∞-topos $\mathcal{X}$ and a sheaf $\mathcal{O}_X \in \mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X})$ of $\mathbb{E}_\infty$-rings. These are objects of an ∞-category $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$, in which morphisms $X \to Y$ are pairs consisting of a geometric morphism $f\colon \mathcal{X} \to \mathcal{Y}$ together with a map $\phi\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{Y}$ (see [170, 1.4.1.3]).

8.9 The étale site of a commutative ring

Our objects of study will be spectrally ringed ∞-topoi which are "locally affine". There are two such notions of affine we can use here, corresponding in the classical case to the Zariski and étale topologies of a ring. We are going to focus on the étale case (which is in some sense strictly more general). Thus, in this section we describe the spectrally ringed ∞-topos $\mathrm{Spét}\,A$ associated to an $\mathbb{E}_\infty$-ring A. It is an "étale topology version" of an analogous construction of a spectrally ringed ∞-topos $\mathrm{Spec}\,A$, which generalizes the classical construction of affine schemes.

Warning: this notion of "étale" map of rings is not to be confused with that of étale maps of ∞-topoi (8.7), though the notions will be linked later on (8.13).

Étale maps of $\mathbb{E}_\infty$-rings

A map $R \to S$ of ordinary commutative rings is **étale** if:

1. S is finitely presented over R,
2. $R \to S$ is flat, and
3. the fold map $S \otimes_R S \to S$ is projection onto a factor (or equivalently, there exists idempotent $e \in S \otimes_R S$ inducing $(S \otimes_R S)[e^{-1}] \xrightarrow{\sim} S$).

Example 8.9.1. If K is a field, then $K \to R$ is étale if and only if $R \approx \prod_{i=1}^d F_i$, where each $K \to F_i$ is a finite separable field extension.

We say that a map $A \to B$ of $\mathbb{E}_\infty$-rings is **étale** if

1. the underlying map $\pi_0 A \to \pi_0 B$ of ordinary commutative rings is étale, and
2. $\pi_n A \otimes_{\pi_0 A} \pi_0 B \to \pi_n B$ is an isomorphism for all n (so that $A \to B$ is flat in the sense of (8.8)).

Remark 8.9.2. If $A \in \mathrm{CAlg}^\heartsuit$ is an ordinary commutative ring, then the two notions of étale coincide.

Theorem 8.9.3 (Goerss–Hopkins–Miller). *Let $A \in \mathrm{CAlg}$.*

1. *For every étale map $\psi\colon \pi_0 A \to B_0$ of ordinary rings, there exists an étale map $\phi\colon A \to B$ of $\mathbb{E}_\infty$-rings and an isomorphism $\pi_0 B \approx B_0$ with respect to which $\pi_0\phi\colon \pi_0 A \to \pi_0 B$ is identified with ψ.*
2. *Let $\phi\colon A \to B$ be an étale map of $\mathbb{E}_\infty$-rings. Then for every $C \in \mathrm{CAlg}_A$, the evident map*
$$\mathrm{Map}_{\mathrm{CAlg}_A}(B, C) \to \mathrm{Map}_{\mathrm{CAlg}^\heartsuit_{\pi_0 A}}(\pi_0 B, \pi_0 C)$$
is an equivalence.

See [168, 7.5.4] for a proof of a generalized formulation of this.

Remark 8.9.4. A consequence of this theorem is that $\mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$ is a *set* (i.e., 0-truncated) whenever $\phi\colon A \to B$ is étale. This consequence can be proved directly from the definition of étale morphism. In fact, when ϕ is étale, then the evident map $B \otimes_{(B \otimes_A B)} B \to B$ must be an equivalence (using that both $A \to B$ and $B \otimes_A B \to B$ are flat). Writing $X = \mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$, this equivalence implies that $X \to X \times_{(X \times X)} X$ is an equivalence, which says exactly that X is 0-truncated.

Remark 8.9.5. Given an étale morphism $\pi_0 A \to B_0$ of ordinary rings, it is not hard to show that the functor $\mathrm{CAlg}_A \to \mathrm{Set} \subseteq \mathcal{S}$ defined by $\mathrm{Map}_{\mathrm{CAlg}^{\heartsuit}_{\pi_0 A}}(B_0, \pi_0(-))$ preserves limits[4] and is accessible, so is corepresentable by a $B \in \mathrm{CAlg}_A$. The hard part of (8.9.3) is to show that $B_0 \to \pi_0 B$ is an isomorphism.

Remark 8.9.6. Statement (2) of the theorem is equivalent to: for every étale map $A \to B$ and $R \in \mathrm{CAlg}$, the square

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}}(B,R) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}}(\pi_0 B, \pi_0 R) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}}(A,R) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}}(\pi_0 A, \pi_0 R) \end{array}$$

is a pullback of ∞-groupoids.

Let $\mathrm{CAlg}^{\text{ét}}_A \subseteq \mathrm{CAlg}_A$ be the full subcategory of A-algebras whose objects are maps $A \to B$ which are étale. As we have seen, it is equivalent to a 1-category.

Remark 8.9.7. If $A \xrightarrow{f} B \xrightarrow{g} C$ are maps of $\mathbb{E}_\infty$-rings such that f and gf are étale, then g is also étale [168, 7.5.1.7]. Thus every morphism in $\mathrm{CAlg}^{\text{ét}}_A$ is itself étale.

Corollary 8.9.8. *For any $A \in \mathrm{CAlg}$, the functor $\mathrm{CAlg}^{\text{ét}}_A \to \mathrm{CAlg}^{\text{ét}}_{\pi_0 A}$ defined by taking π_0 is an equivalence of ∞-categories.*

Example 8.9.9 (Localization of $\mathbb{E}_\infty$-rings). Let $A \in \mathrm{CAlg}$, and suppose $f \in \pi_0 A$. Then $\pi_0 A \to (\pi_0 A)[f^{-1}]$ is an étale morphism of commutative rings. By (8.9.3), (i) there exists a map $A \to A[f^{-1}]$ of $\mathbb{E}_\infty$-rings such that (i) $\pi_*(A[f^{-1}]) \approx (\pi_* A)[f^{-1}]$, and (ii) for any $C \in \mathrm{CAlg}$, $\mathrm{Map}_{\mathrm{CAlg}}(A[f^{-1}], C) \to \mathrm{Map}_{\mathrm{CAlg}}(A, C)$ is the inclusion of those path components consisting of $\phi\colon A \to C$ which take f to a unit in $\pi_0 C$.

This special case predates the proof of the Goerss–Hopkins–Miller theorem for $\mathbb{E}_\infty$-rings. In fact, one can in a similar way invert any multiplicative subset $S \subseteq \pi_* A$ of the *graded* homotopy ring to obtain A_S with $\pi_*(A_S) \approx (\pi_* A)_S$.

Example 8.9.10 (Adjoining primitive roots of unity). Here is a hands-on construction of an étale morphism, due to [260]. Given any $\mathbb{E}_\infty$-ring A, prime p, and $k \geq 1$, consider the group ring $B' := A[\mathbb{Z}/p^k]$ (8.8.6), with $\pi_0 B' \approx (\pi_0 A)[t]/(t^{p^k} - 1)$. Let $f = \sum_{j=0}^{p-1} (1 - t^{j p^{k-1}})$ in $\pi_0 B'$, and note that $f^2 = pf$. Formally inverting f we obtain

$$B := B'[f^{-1}], \quad \text{with} \quad \pi_* B \approx (\pi_* A)[\tfrac{1}{p}, t]/(1 + t^{p^{k-1}} + \cdots + t^{(p-1)p^{k-1}}).$$

It turns out that $A \to B$ is an étale morphism, and $\pi_0 B$ is obtained from $\pi_0 A$ by (i) inverting p and (ii) adjoining a primitive p^kth root of unity.

Remark 8.9.11. In general, you can always construct étale maps of $\mathbb{E}_\infty$-rings using "generators and relations" (using free rings (8.8.7)), which in fact leads to an alternate proof of (8.9.3); see [170, B.1]. In particular, this shows that every étale map in CAlg is a base change of one between compact objects in CAlg ([170, B.1.3.3] with $R = \mathbb{S}$).

[4] Using the fact that étale maps of rings are also "formally étale".

(An object A in an ∞-category is **compact** if $\mathrm{Map}_{\mathcal{A}}(A,-)\colon \mathcal{A}\to\mathcal{S}$ preserves filtered colimits.)

The étale site of an $\mathbb{E}_\infty$-ring

Given $A\in\mathrm{CAlg}$, consider the category $\mathrm{CAlg}_A^{\text{ét}}$ of étale morphisms under A. A finite set $\{A\to A_i\}_{i=1}^d$ of maps in $\mathrm{CAlg}_A^{\text{ét}}$ is an **étale cover** if $\pi_0A\to\prod_{i=1}^d \pi_0A_i$ is *faithfully flat*.

We define $\mathrm{Shv}_A^{\text{ét}}\subseteq \mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}},\mathcal{S})$ to be the full subcategory of functors F such that

$$F(A)\to \lim_\Delta\Big[[n]\mapsto \prod_{i_0,\dots,i_n} F(A_{i_0}\otimes_A\cdots\otimes_A A_{i_n})\Big]$$

is an equivalence for every étale cover $\{A\to A_i\}_i$ in $\mathrm{CAlg}_R^{\text{ét}}$. This $\mathrm{Shv}_A^{\text{ét}}$ is an ∞-topos; in fact, it is equivalent to the ∞-topos $\mathrm{Shv}_{\pi_0A}^{\text{ét}}$ of étale sheaves on the ordinary commutative ring π_0A. I'll call its objects of **sheaves on the étale site** of A.

The étale spectrum of an $\mathbb{E}_\infty$-ring

Let $\mathcal{O}\colon \mathrm{CAlg}_A^{\text{ét}}\to\mathrm{CAlg}$ denote the forgetful functor.

Proposition 8.9.12. *The functor $\mathcal{O}$ is a sheaf of $\mathbb{E}_\infty$-rings on the étale site of A.*

We thus define the **étale spectrum** of $A\in\mathrm{CAlg}$ to be the spectrally ringed ∞-topos $\mathrm{Spét}\,A=(\mathrm{Shv}_A^{\text{ét}},\mathcal{O})$.

Proof of (8.9.12). We must show that for every finite étale cover $\{A\to A_i\}_{i=1}^d$ the evident map

$$A\to \lim_\Delta\Big[[n]\mapsto \prod_{i_0,\dots,i_n} A_{i_0}\otimes_A\cdots\otimes_A A_{i_n}\Big]$$

is an equivalence of $\mathbb{E}_\infty$-rings. This is a special case of a much more general statement, called *flat descent* for $\mathbb{E}_\infty$-rings; see [170, D.5] for the general theory.

In this case, the proof amounts to computing the spectral sequence computing the homotopy groups of the inverse limit, whose E_1-term takes the form

$$E_1^{s,t}=\pi_t(A_{i_0}\otimes_A\cdots\otimes_A A_{i_s})\approx \pi_tA\otimes_{\pi_0A}(\pi_0A_{i_0}\otimes_{\pi_0A}\cdots\otimes_{\pi_0A}\pi_0A_{i_s})$$

because étale morphisms are flat. The classical version of flat descent for ordinary rings implies that

$$E_2^{s,t}\approx H^s[\pi_tA\otimes_{\pi_0A}(\pi_0A_{i_0}\otimes_{\pi_0A}\cdots\otimes_{\pi_0A}\pi_0A_{i_s})]\approx\begin{cases}\pi_tA & \text{if } s=0,\\ 0 & \text{if } s>0,\end{cases}$$

so the spectral sequence collapses to a single line at E_2. The claim follows because the inverse limit spectral sequence has conditional convergence. □

Remark 8.9.13. We actually have that $\mathcal{O}$ is a ***hypercomplete*** sheaf of spectra $\mathrm{Shv}_A^{\text{ét}}$. In fact, the argument of the proof of (8.9.12) shows that for each $n \geq 0$ the presheaf $\tau_{\leq n}\mathcal{O}\colon A \mapsto \tau_{\leq n}A$ of spectra obtained by truncation is a sheaf on the étale site, whence $\mathcal{O} \approx \lim_n \tau_{\leq n}\mathcal{O}$; this relies on the fact that $\mathrm{CAlg}^{\text{ét}}_{\tau_{\leq n}A} \approx \mathrm{CAlg}^{\text{ét}}_{\pi_0 A}$ for all $n \geq 0$, so all these rings have the same étale site.

The Zariski site and spectrum of an $\mathbb{E}_\infty$-ring

In the above we can replace $\mathrm{CAlg}_A^{\text{ét}}$ with the full subcategory $\mathrm{CAlg}_A^{\mathrm{Zar}}$ spanned by objects equivalent to localizations $A \to A[f^{-1}]$. Then $\{A \to A[f_i^{-1}]\}_{i=1}^d$ is a **Zariski cover** if $\pi_0 A \to \prod_{i=1}^d \pi_0 A[f_i^{-1}]$ is faithfully flat; equivalently, if $(f_1, \dots, f_d)\pi_0 A = \pi_0 A$. We obtain an ∞-topos $\mathrm{Shv}_A^{\mathrm{Zar}} \subseteq \mathrm{Fun}(\mathrm{CAlg}_A^{\mathrm{Zar}}, \mathcal{S})$ of Zariski sheaves. We have $\mathrm{Shv}_A^{\mathrm{Zar}} \approx \mathrm{Shv}_{\pi_0 A}^{\mathrm{Zar}}$, and these are equivalent to the ∞-categories of sheaves on a topological space, namely the prime ideal spectrum of $\pi_0 A$ equipped with the Zariski topology.

We can likewise define the **Zariski spectrum** to be the spectrally ringed ∞-topos $\mathrm{Spec}\, A = (\mathrm{Shv}_A^{\mathrm{Zar}}, \mathcal{O})$, as the forgetful functor $\mathcal{O}\colon \mathrm{CAlg}_A^{\mathrm{Zar}} \to \mathrm{CAlg}$ is sheaf of $\mathbb{E}_\infty$-rings on the Zariski site.

Example 8.9.14 (Points in étale site vs. the Zariski site). To get a sense of the difference between the Zariski and étale sites, let's compare $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{S}, \mathrm{Shv}_A^{\mathrm{Zar}})$ with $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{S}, \mathrm{Shv}_A^{\text{ét}})$. (A map of ∞-topoi of the form $\mathcal{S} \to \mathcal{X}$ is called a **point** of $\mathcal{X}$.)

First, suppose $K \in \mathrm{CAlg}^\heartsuit$ is an ordinary field. Then $\mathrm{CAlg}_K^{\mathrm{Zar}} \approx 1$, so $\mathrm{Shv}_K^{\mathrm{Zar}} \approx \mathcal{S}$, so there is a unique map $\mathcal{S} \to \mathrm{Shv}_K^{\mathrm{Zar}}$ of ∞-topoi. On the other hand, any separable closure $K \to K^{\mathrm{sep}}$ induces a geometric morphism $f\colon \mathcal{S} \to \mathrm{Shv}_K^{\text{ét}}$, characterized by the property that $f^*U \approx \mathrm{Map}_{\mathrm{CAlg}_K}(R, K^{\mathrm{sep}})$ when $U \in \mathrm{Shv}_K^{\text{ét}}$ is the sheaf represented by a map $K \to R \in \mathrm{CAlg}_K^{\text{ét}}$. Therefore,

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{S}, \mathrm{Shv}_K^{\text{ét}}) \approx B\mathrm{Gal}(K),$$

the classifying space of the absolute Galois group of K viewed as an ∞-groupoid.

For general $A \in \mathrm{CAlg}$, the ∞-groupoid $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{S}, \mathrm{Shv}_A^{\mathrm{Zar}})$ is equivalent to the ***set*** $|\mathrm{Spec}\, A|$ of prime ideals in $\pi_0 A$ (i.e., the prime ideal spectrum as a discrete set), while $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{S}, \mathrm{Shv}_A^{\text{ét}})$ is equivalent to a ***1-groupoid*** whose objects are pairs $(\mathfrak{p}, \pi_0 A/\mathfrak{p} \to F)$ consisting of a prime ideal $\mathfrak{p} \subset \pi_0 A$ and a separable closure F of the residue field $\pi_0 A/\mathfrak{p}$.

8.10 Spectral Deligne–Mumford stacks

We can now define the main notion, that of a *spectral Deligne–Mumford stack.*

First note that given a spectrally ringed ∞-topos $X = (\mathcal{X}, \mathcal{O}_X)$ and an object $U \in \mathcal{X}$, we obtain a new spectrally ringed ∞-topos

$$X_U := (\mathcal{X}_{/U}, \mathcal{O}_X|_U)$$

where $\mathcal{O}_X|_U := \pi^*\mathcal{O}_X$ is the preimage of $\mathcal{O}_X$ along the projection $\pi\colon \mathcal{X}_{/U} \to \mathcal{X}$. Furthermore, this comes with an evident map $X_U \to X$ of spectrally ringed ∞-topoi.

Example 8.10.1. If $X = \operatorname{Spét} A = (\operatorname{Shv}^{\text{ét}}_A, \mathcal{O})$ and $U \in \operatorname{Shv}^{\text{ét}}_A \subseteq \operatorname{PSh}((\operatorname{CAlg}^{\text{ét}}_A)^{\text{op}})$ is the sheaf represented by an étale map $(A \to B) \in \operatorname{CAlg}^{\text{ét}}_A$, then $X_U \approx ((\operatorname{Shv}^{\text{ét}}_A)_{/U}, \mathcal{O}|_U) \approx (\operatorname{Shv}^{\text{ét}}_B, \mathcal{O}) = \operatorname{Spét} B$.

The definition of spectral Deligne–Mumford stacks

We say that a spectrally ringed ∞-topos $X = (\mathcal{X}, \mathcal{O}_X)$ is **affine** if it is isomorphic to $\operatorname{Spét} A$ for some $A \in \operatorname{CAlg}$. Likewise, we say that an object $U \in \mathcal{X}$ is affine if X_U (as defined above) is affine.

A **nonconnective spectral Deligne–Mumford (DM) stack** is a spectrally ringed ∞-topos $X = (\mathcal{X}, \mathcal{O}_X)$ for which there exists a set of objects $\{U_i\}$ in $\mathcal{X}$ such that

1. the set $\{U_i\}$ covers $\mathcal{X}$ (i.e., $\coprod U_i \to 1$ is effective epi in $\mathcal{X}$), and
2. each U_i is affine.

Remark 8.10.2. The structure sheaf of a nonconnective spectral DM stack is always hypercomplete, as a consequence of the fact that this is so in the affine case (8.9.13).

A **spectral Deligne–Mumford (DM) stack** is a nonconnective DM stack $(\mathcal{X}, \mathcal{O}_X)$ such that the sheaf $\mathcal{O}_X$ is connective; i.e., such that the homotopy sheaves $\pi_k\mathcal{O}_X \in \mathcal{X}^{\heartsuit}$ satisfy $\pi_k\mathcal{O}_X \approx 0$ for $k < 0$.

Remark 8.10.3. $\operatorname{Spét} A$ is always a nonconnective spectral DM stack, and is a spectral DM stack if and only if A is connective.

Remark 8.10.4. If $X = (\mathcal{X}, \mathcal{O}_X)$ is a nonconnective spectral DM stack and $U \in \mathcal{X}$, then X_U is also a nonconnective spectral DM stack. Furthermore, if X is a spectral DM stack, so is X_U.

This is a consequence of the following claim: for a nonconnective spectral DM stack X, the collection $\mathcal{A} = \{V_j\}$ of all affine objects in $\mathcal{X}$ generates $\mathcal{X}$ under colimits [170, 1.4.7.9]. In particular, this implies that for any U we can find a set of maps of the form $V_j \to U$ with all $V_j \in \mathcal{A}$ which is a cover of $X_{/U}$ (8.5.6).

Here's a proof that affines generate $\mathcal{X}$ under colimits. First note that if $X \approx \operatorname{Spét} A$ is itself affine, then $\mathcal{X} \approx \operatorname{Shv}^{\text{ét}}_A$ which is manifestly generated by affines (i.e., by the image of $(\operatorname{CAlg}^{\text{ét}}_A)^{\text{op}} \rightarrowtail \operatorname{Shv}^{\text{ét}}_A$ (8.10.1)). In the general case, if $\{U_i\}$ is an affine cover of $\mathcal{X}$, choose for each i a set $\{V_{i,j} \to U_i\}$ of affine objects of $\mathcal{X}_{/U_i}$ which generate $\mathcal{X}_{/U_i}$ under colimits. Then the collection $\{V_{i,j}\}$ in $\mathcal{X}$ is a collection of affines which generate $\mathcal{X}$ under colimits (since $(X_{U_i})_{V_{i,j}} \approx X_{V_{i,j}}$).

Spectral schemes

We can carry out an analogous definition using the Zariski topology. A special case of this is a **nonconnective spectral scheme**, which is a spectrally ringed ∞-topos $X = (\mathcal{X}, \mathcal{O}_X)$ such that

1. $\mathcal{X} \approx \mathrm{Shv}(X_{\mathrm{top}})$ for some topological space X_{top}, and
2. there exists an open cover $\{U_i\}$ of X_{top} such that $X_{U_i} \approx \operatorname{Spec} A_i$ for some $A_i \in$ CAlg.

It is a **spectral scheme** if also $\pi_k \mathcal{O}_X \approx 0$ for $k < 0$. (This is not the definition given as [170, 1.1.2.8], but is equivalent to it by [170, 1.1.6.3, 1.1.6.4].)

8.11 Morphisms of spectral DM stacks

We need to work rather harder to get the correct notion of morphism of spectral DM stacks. Our goal is produce a category $\mathrm{SpDM}^{\mathrm{nc}}$ of nonconnective spectral DM stacks which includes Spét R for any $R \in \mathrm{CAlg}$, with the property that

$$\mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(\mathrm{Spét}\, S, \mathrm{Spét}\, R) \approx \mathrm{Map}_{\mathrm{CAlg}}(R, S).$$

More generally, we would like to have

$$\mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(X, \mathrm{Spét}\, R) \approx \mathrm{Map}_{\mathrm{CAlg}}(R, \Gamma(\mathcal{X}, \mathcal{O}_X)),$$

for any object $X \in \mathrm{SpDM}^{\mathrm{nc}}$, where $\Gamma(\mathcal{X}, \mathcal{O}_X) \in \mathrm{CAlg}$ is the global sections of the structure sheaf $\mathcal{O}_X$.

Let's make this more precise. Given a map $(f, \psi)\colon X \to \mathrm{Spét}\, R$ of spectrally ringed ∞-topoi, we obtain a map $R \to \Gamma(\mathcal{X}, \mathcal{O}_X)$ of $\mathbb{E}_\infty$-rings, by evaluating the composite of

$$\mathcal{O} \to f_* f^* \mathcal{O} \xrightarrow{f_*(\psi)} f_* \mathcal{O}_X$$

at global sections over $\mathrm{Shv}_R^{\mathrm{ét}}$. Thus we get a map of ∞-groupoids

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}(X, \mathrm{Spét}\, R) \to \mathrm{Map}_{\mathrm{CAlg}}(R, \Gamma(\mathcal{X}, \mathcal{O}_X)). \tag{8.11.1}$$

This map is rarely an equivalence, even when X is affine. It turns out that we obtain an equivalence when we require X to be ***strictly Henselian***, and restrict to a full subgroupoid $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}(X, \mathrm{Spét}\, R) \subseteq \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}(X, \mathrm{Spét}\, R)$, consisting of ***local*** maps.

Solution sheaves

To carry out this definition, we need to think locally. Given a spectrally ringed ∞-topos $X = (\mathcal{X}, \mathcal{O}_X)$ and an $\mathbb{E}_\infty$-ring R, define the **solution sheaf** $\mathrm{Sol}_R(\mathcal{O}_X) \in \mathrm{Fun}^{\text{lim. pres.}}(\mathcal{X}^{\mathrm{op}}, \mathcal{S}) \approx \mathcal{X}$ by

$$\mathrm{Sol}_R(\mathcal{O}_X)(U) := \mathrm{Map}_{\mathrm{CAlg}}(R, \mathcal{O}_X(U)).$$

Note that $R \mapsto \mathrm{Sol}_R(\mathcal{O}_X)$ is itself a functor $\mathrm{CAlg}^{\mathrm{op}} \to \mathcal{X}$, and is limit preserving.

We obtain a map of sheaves of ∞-groupoids on $\mathcal{X}$,

$$\left[U \mapsto \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}}(X_U, \mathrm{Spét}\, R)\right] \to \mathrm{Sol}_R(\mathcal{O}_X),$$

which can be thought of as a "local" version of the map (8.11.1), since evaluating the above map at the terminal object $U = 1_{\mathcal{X}}$ of $\mathcal{X}$ recovers the map (8.11.1).

Strictly Henselian sheaves

A sheaf $\mathcal{O}$ of $\mathbb{E}_\infty$-rings is **strictly Henselian** if for every étale cover $\{R \to R_i\}$ in $\mathrm{CAlg}^{\text{ét}}$, the induced map

$$\coprod \mathrm{Sol}_{R_i}(\mathcal{O}) \to \mathrm{Sol}_R(\mathcal{O}) \tag{8.11.2}$$

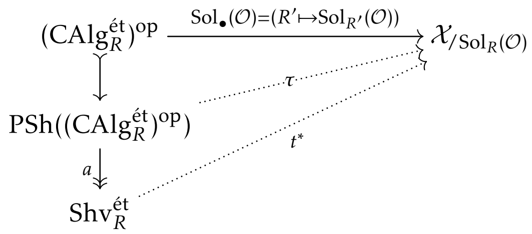
is an effective epi in $\mathcal{X}$. (This is not the definition of [170, 1.4.2.1], but is equivalent to it by [170, 1.4.3.9].)

Remark 8.11.1. The strictly Henselian condition on $\mathcal{O}$ gives rise to a map

$$\mathrm{Map}_{\mathrm{CAlg}}(R, \Gamma(\mathcal{X}, \mathcal{O})) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathrm{Shv}_R^{\text{ét}}),$$

i.e., from an $\mathbb{E}_\infty$-ring map $\overline{\alpha}\colon R \to \Gamma(\mathcal{X}, \mathcal{O})$ we can get a map $\mathcal{X} \to \mathrm{Shv}_R^{\text{ét}}$ of ∞-topoi.

To see how this works, note that in the diagram



there is an essentially unique colimit preserving functor τ extending $\mathrm{Sol}_\bullet(\mathcal{O})$. The strictly Henselian condition on $\mathcal{O}$ implies that τ factors through an essentially unique colimit preserving functor t^*. Because $\mathrm{Sol}_\bullet(\mathcal{O})$ preserves limits, t^* preserves finite limits. That is, t^* is the preimage of a geometric morphism $t\colon \mathrm{Shv}_R^{\text{ét}} \to \mathcal{X}_{\mathrm{Sol}_R(\mathcal{O})}$.

An $\mathbb{E}_\infty$-ring map $\alpha\colon R \to \Gamma(\mathcal{X}, \mathcal{O})$ corresponds to a section $1_{\mathcal{X}} \to \mathrm{Sol}_R(\mathcal{O})$, which induces an étale geometric morphism $\overline{\alpha}\colon \mathcal{X} \to \mathcal{X}_{/\mathrm{Sol}_R(\mathcal{O})}$. The composite $t \circ \overline{\alpha}$ is the desired map of ∞-topoi.

Remark 8.11.2. It can be shown [170, 1.4.3.8] that the map in (8.11.2) is a pullback of $\coprod \mathrm{Sol}_{\pi_0 R_i}(\pi_0\mathcal{O}) \to \mathrm{Sol}_{\pi_0 R}(\pi_0\mathcal{O})$, so $\mathcal{O}$ is strictly Henselian (or local) if and only if $\pi_0\mathcal{O}$ is so; this recovers the definition in [170, 1.4.2.1]. (The proof rather subtle: you need to use the fact that every étale map is a base change of an étale map between compact objects in CAlg (8.9.11), in order to reduce to the case of the pullback square (8.9.6) of mapping spaces. The issue here is that it is not the case that $f^*\mathrm{Sol}_R(\mathcal{O}) \to \mathrm{Sol}_R(f^*\mathcal{O})$ is an isomorphism in general, unless R is a compact object of CAlg.)

There is an analogous definition of **local** sheaf, in which étale covers are replaced with Zariski covers in the definition given above.

Example 8.11.3 (Local and strictly Henselian sheaves on a point). Let $\mathcal{X} = \mathcal{S}$, so $\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{S}) \approx \mathrm{CAlg}$, and for $\mathcal{O} \in \mathrm{CAlg}$ we have $\mathrm{Sol}_R(\mathcal{O}) \approx \mathrm{Map}(R, \mathcal{O}) \in \mathcal{S}$.

From the definitions and the universal property of localization maps $R \to R[f^{-1}]$ in CAlg, we see that $\mathcal{O}$ is local if and only if, for every pair $(R, \{f_1, \dots, f_d\} \subseteq \pi_0 R)$ consisting of $R \in \mathrm{CAlg}$ such that $(f_1, \dots, f_d)\pi_0 R = \pi_0 R$, every map $\alpha\colon R \to \mathcal{O}$ in CAlg is such that $\alpha(f_k)$ is an invertible element of $\pi_0\mathcal{O}$ for some $k \in \{1, \dots, d\}$.

It follows that $\mathcal{O}$ must be a local sheaf whenever $\pi_0\mathcal{O}$ is a local ring in the usual sense. The converse also holds: if $\mathcal{O}$ is a local sheaf, apply the condition with ($R = 0$, $\varnothing \subseteq \pi_0 R$) to see that $\pi_0\mathcal{O} \not\approx 0$, and with ($R = \mathbb{S}\{x,y\}[(x+y)^{-1}]$, $\{x,y\} \subseteq \pi_0 R$) to see that $\mathfrak{m} := \pi_0\mathcal{O} \smallsetminus (\pi_0\mathcal{O})^\times$ is an ideal.

A similar argument shows that $\mathcal{O} \in \mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{S})$ is strictly Henselian if and only if $\pi_0\mathcal{O}$ is a strictly Henselian ring in the classical sense, i.e., as defined in [281, Tag 04GE].

Spectral DM stacks are strictly Henselian

For an affine object $X = \mathrm{Sp\acute{e}t}\, A = (\mathrm{Shv}_A^{\mathrm{\acute{e}t}}, \mathcal{O})$, we see that $\mathrm{Sol}_R(\mathcal{O})(U) \approx \mathrm{Map}_{\mathrm{CAlg}}(R, B)$ when $U \in \mathrm{Shv}_A^{\mathrm{\acute{e}t}}$ is the object represented by the étale A-algebra $A \to B$. Using this it is straightforward to show that $\mathcal{O}$ is strictly Henselian.

Remark 8.11.4 (Spectral DM stacks are strictly Henselian). Observe that $\pi^* \mathrm{Sol}_R(\mathcal{O}) \approx \mathrm{Sol}_R(\pi^*\mathcal{O})$ when $\pi\colon \mathcal{X}_{/U} \to \mathcal{X}$ is the etale map of ∞-topoi associated to an object $U \in \mathcal{X}$. (Use (8.7.5).) Given this it is straightforward to prove that any nonconnective DM stack is strictly Henselian.

The category of strictly Henselian spectrally ringed ∞-topoi

We let $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{sHen}}$ denote the (non-full) subcategory of $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ whose *objects* are $X = (\mathcal{X}, \mathcal{O}_X)$ such that $\mathcal{O}_X$ is strictly Henselian, and whose *morphisms* $f\colon (\mathcal{X}, \mathcal{O}_X) \to (\mathcal{Y}, \mathcal{O}_Y)$ are such that

$$\begin{array}{ccc} f^* \mathrm{Sol}_{R'}(\mathcal{O}_Y) & \longrightarrow & \mathrm{Sol}_{R'}(\mathcal{O}_X) \\ \downarrow & & \downarrow \\ f^* \mathrm{Sol}_{R}(\mathcal{O}_Y) & \longrightarrow & \mathrm{Sol}_{R}(\mathcal{O}_X) \end{array}$$

is a pullback in $\mathcal{X}$ for every étale map $R \to R'$ in CAlg. Such morphisms are called **local**.

Remark 8.11.5. This is different than the definition given as [170, 1.4.2.1], but is equivalent by [170, 1.4.3.9].

Remark 8.11.6. If $X = (\mathcal{X}, \mathcal{O}_X)$ is a nonconnective spectral DM stack and $U \in \mathcal{X}$, then the evident map $X_U \to X$ of spectrally ringed ∞-topoi is local.

We can now state our goal.

Theorem 8.11.7. *For any strictly Henselian spectrally ringed ∞-topos $X = (\mathcal{X}, \mathcal{O}_X)$ and $\mathbb{E}_\infty$-ring R, the evident map*

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{sHen}}}(X, \mathrm{Sp\acute{e}t}\, R) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{CAlg}}(R, \Gamma(\mathcal{X}, \mathcal{O}_X))$$

is an equivalence.

Sketch proof. This is [170, 1.4.2.4]. Here is a brief sketch.

Geometric morphisms $f\colon \mathrm{Shv}_R^{\text{ét}} \to \mathcal{X}$ correspond (by restriction to representable sheaves) exactly to left-exact functors $\chi\colon (\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}} \to \mathcal{X}$ which send étale covers to effective epis. Given such an f, maps $\phi\colon \mathcal{O} \to f_*\mathcal{O}_X$ of sheaves of $\mathbb{E}_\infty$-rings on $\mathrm{Shv}_R^{\text{ét}}$ correspond to natural transformations $\phi'\colon \chi \to \mathrm{Sol}_\bullet(\mathcal{O})$ of functors; to see this, use the evident equivalence $\mathrm{Map}_{\mathcal{X}}(\chi(R'), \mathrm{Sol}_{R'}(\mathcal{O})) \approx \mathrm{Map}_{\mathrm{CAlg}}(R', \mathcal{O}_X(\chi(R')))$ for $R' \in \mathrm{CAlg}_R^{\text{ét}}$, and that $f_*\mathcal{O}_X|(\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}} \approx \mathcal{O}_X \circ \chi$ as functors $(\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}} \to \mathrm{CAlg}$.

One shows that if ϕ is local, then ϕ' is Cartesian, i.e., ϕ' takes morphisms in $(\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}$ to pullback squares of sheaves. But since $(\mathrm{CAlg}_R^{\text{ét}})$ has R as a terminal object, we discover that pairs (χ, ϕ') with ϕ' Cartesian correspond exactly to maps $1_{\mathcal{X}} = \chi(R) \to \mathrm{Sol}_R(\mathcal{O}_{\mathcal{X}})$, i.e., to maps $R \to \Gamma(\mathcal{X}, \mathcal{O}_X)$ of $\mathbb{E}_\infty$-rings. In particular, we learn that $\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{sHen}}}(X, \mathrm{Sp\acute{e}t}\, R) \to \mathrm{Map}_{\mathrm{CAlg}}(R, \Gamma(\mathcal{X}, \mathcal{O}_X))$ is a monomorphism.

Finally, given a map $\alpha\colon R \to \Gamma(\mathcal{X}, \mathcal{O}_X)$, there is an explicit procedure to construct a morphism $X \to \mathrm{Sp\acute{e}t}\, R$ in $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{sHen}}$ which projects to α; the underlying map $\mathcal{X} \to \mathrm{Shv}_R^{\text{ét}}$ of ∞-topoi is produced by the procedure of (8.11.1). □

The category of locally spectrally ringed ∞-topoi

We can play the same game with "local" replacing "strictly Henselian" as the condition on objects, resulting in a full subcategory $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{loc}}$ of $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{sHen}}$ and a version of (8.11.7) with Spét replaced with Spec [170, 1.1.5].

8.12 The category of spectral DM stacks

We have achieved our goal. We have full subcategories

$$\mathrm{SpDM} \subseteq \mathrm{SpDM}^{\mathrm{nc}} \subseteq \infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{sHen}}$$

of spectral DM stacks and nonconnective DM stacks respectively, inside the ∞-category of strictly Henselian spectrally ringed ∞-topoi and local maps, which is itself a non-full subcategory of the category $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ of spectrally ringed ∞-topoi. By (8.11.7) we see that there are adjoint pairs

$$\mathrm{Sp\acute{e}t}\colon \mathrm{CAlg}^{\mathrm{op}} \rightleftarrows \mathrm{SpDM}^{\mathrm{nc}} : \Gamma \quad \text{and} \quad \mathrm{Sp\acute{e}t}\colon (\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}} \rightleftarrows \mathrm{SpDM} : \Gamma.$$

Remark 8.12.1. There are analogous full subcategories of *spectral schemes* and *nonconnective spectral schemes* in $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{loc}}$.

Finite limits of DM stacks

The categories SpDM and $\mathrm{SpDM}^{\mathrm{nc}}$ have finite limits, and finite limits are preserved by the functors $\mathrm{Sp\acute{e}t}\colon (\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}} \to \mathrm{SpDM}$ and $\mathrm{Sp\acute{e}t}\colon \mathrm{CAlg}^{\mathrm{op}} \to \mathrm{SpDM}^{\mathrm{nc}}$. In particular, for a diagram $B \leftarrow A \to B'$ of rings, we have

$$\mathrm{Sp\acute{e}t}(B \otimes_A B') \approx \mathrm{Sp\acute{e}t}\, B \times_{\mathrm{Sp\acute{e}t}\, A} \mathrm{Sp\acute{e}t}\, B',$$

as an immediate consequence of (8.11.7). (See [170, 1.4.11.1], [163, V 2.3.21].)

Connective covers and truncation of DM stacks

The adjoint pairs

$$\mathrm{CAlg}^{\heartsuit} \rightleftarrows \mathrm{CAlg}^{\mathrm{cn}} \rightleftarrows \mathrm{CAlg}$$

relating classical, connective, and arbitrary $\mathbb{E}_\infty$-rings are paralleled by adjoint pairs

$$\mathrm{SpDM}^{\leq 0} \rightleftarrows \mathrm{SpDM} \rightleftarrows \mathrm{SpDM}^{\mathrm{nc}}$$

where $\mathrm{SpDM}^{\leq 0}$ is the ∞-category of 0-**truncated spectral DM stacks**, consisting of $X = (\mathcal{X}, \mathcal{O}_X)$ such that $\pi_q \mathcal{O}_X \approx 0$ for $q \neq 0$. The localization functors are obtained respectively by 0-truncating or taking connective cover of the structure sheaf [170, 1.4.5–6].

Classical objects as spectral DM stacks

We would like to connect this spectral geometry to some more “classical” (i.e., 1-categorical) kind of algebraic geometry.

Note that objects of $\mathrm{SpDM}^{\leq 0}$ are ∞-topoi $\mathcal{X}$ equipped with structure sheaves $\mathcal{O}_X$ of *classical* rings. However, the ∞-topos $\mathcal{X}$ is not necessarily a “classical” one, i.e., is not necessarily equivalent to the 1-localic ∞-topos $\mathrm{Shv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit})$ (8.5.12). So 0-truncated spectral DM stacks are not necessarily classical objects.

The classical analogue of spectral Deligne–Mumford stack is a **Deligne–Mumford stack**, which is a pair $X_0 = (\mathcal{X}, \mathcal{O}_{X_0})$ consisting of a 1-topos $\mathcal{X}$ with a sheaf $\mathcal{O}_{X_0}$ of ordinary commutative rings on it, which is “locally” affine, i.e., there exists a set $\{U_i\}$ of objects in $\mathcal{X}$ such that (i) $\coprod U_i \to 1$ is effective epi in $\mathcal{X}$ and (ii) $(\mathcal{X}_{/U_i}, \mathcal{O}|_{U_i}) \approx ((\mathrm{Shv}^{\text{ét}}_{A_i})^{\heartsuit}, \mathcal{O})$ for some ordinary ring A_i.

Given a nonconnective spectral DM stack $X = (\mathcal{X}, \mathcal{O}_X)$, we can form $X_{\mathrm{DM}} := (\mathcal{X}^{\heartsuit}, \pi_0 \mathcal{O}_X)$, which is in fact a classical Deligne–Mumford stack, called the **underlying DM stack** of X.

Conversely, given a classical DM stack $X_0 = (\mathcal{X}, \mathcal{O})$, we can upgrade it to a 0-truncated spectral DM stack

$$X_{\mathrm{SpDM}} = (\mathrm{Shv}_{\mathcal{S}}(\mathcal{X}), \mathcal{O}')$$

where $\mathrm{Shv}_{\mathcal{S}}(\mathcal{X})$ is the 1-localic reflection of $\mathcal{X}$ (8.5.12), and $\mathcal{O}'$ is the sheaf of connective $\mathbb{E}_\infty$-rings represented by the composite functor

$$\mathrm{Shv}_{\mathcal{S}}(\mathcal{X})^{\mathrm{op}} \xrightarrow{(\tau_{\leq 0})^{\mathrm{op}}} \mathcal{X}^{\mathrm{op}} \xrightarrow{\mathcal{O}} \mathrm{CAlg}^{\heartsuit} \rightarrowtail \mathrm{CAlg}^{\mathrm{cn}}.$$

It turns out that this construction describes a fully faithful embedding

$$(\text{classical DM stacks}) \rightarrowtail \mathrm{SpDM}^{\leq 0}.$$

See [170, 1.4.8] for more on the relation between DM stacks and spectral DM stacks.

Example 8.12.2. Here is a simple example which exhibits some of these phenomena. Let $K \in \mathrm{CAlg}^{\heartsuit}$ be an ordinary separably closed field, so that $\mathrm{Shv}^{\text{ét}}_K \approx \mathcal{S}$. Then

$\operatorname{Spét} K \approx (\mathcal{S}, K)$, where $K \in \mathrm{CAlg} \approx \mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{S})$, is an example of a 0-truncated spectral DM stack, whose ∞-topos is equivalent to sheaves on the 1-point space. It corresponds to the classical DM stack associated to K.

For any ∞-groupoid $U \in \mathcal{S}$ we can form $(\operatorname{Spét} K)_U = (\mathcal{S}_{/U}, \pi^* K)$, i.e., ∞-groupoids over U equipped with the constant sheaf associated to K. Then $(\operatorname{Spét} K)_U$ is also a 0-truncated spectral DM stack. If U is not a 1-truncated space, then $\mathcal{S}_{/U}$ is not 1-localic, and $(\operatorname{Spét} K)_U$ does not arise as a classical DM stack in this case.

In short, spectral DM stacks expand DM stacks in *two* ways: spectral DM stacks are allowed to have underlying ∞-topoi which are not classical, i.e., not 1-localic, and spectral DM stacks are also allowed to have structure sheaves which are not classical, i.e., not merely sheaves of ordinary rings.

Relation to schemes and spectral schemes

There are analogous statements for spectral schemes [170, 1.1]. Thus, a morphism of nonconnective spectral schemes is just a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of spectrally ringed ∞-topoi which is local; we get full subcategories $\mathrm{SpSch} \subseteq \mathrm{SpSch}^{\mathrm{nc}} \subseteq \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$; we have fully faithful $\mathrm{Spec}\colon (\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}} \to \mathrm{SpSch}$ and $\mathrm{Spec}\colon \mathrm{CAlg}^{\mathrm{op}} \to \mathrm{SpSch}^{\mathrm{nc}}$; and we have fully faithful embeddings

$$\mathrm{Sch} \xrightarrow{\sim} \mathrm{SpSch}^{\le 0} \rightarrowtail \mathrm{SpSch},$$

where $\mathrm{Sch}^{\le 0}$ is the full subcategory of 0-truncated spectral schemes. In this case we have an equivalence $\mathrm{Sch} \approx \mathrm{SpSch}^{\le 0}$, since underlying topos of a spectral scheme is already assumed to be a space.

What is the relation between spectral schemes and spectral DM stacks? Note that although both spectral schemes and spectral DM stacks are both types of spectrally ringed ∞-topoi, there is very little overlap between the two classes. What is true [170, 1.6.6] is that there exist fully faithful functors

$$\mathrm{SpSch} \rightarrowtail \mathrm{SpDM} \quad \text{and} \quad \mathrm{SpSch}^{\mathrm{nc}} \rightarrowtail \mathrm{SpDM}^{\mathrm{nc}}$$

which promote spectral schemes to spectral DM stacks. Objects in the essentially image of these functors are called **schematic**, and this property is easy to characterize: $X = (\mathcal{X}, \mathcal{O}_X)$ is schematic if and only if there exists a set $\{U_i\}$ of (-1)*-truncated* objects of $\mathcal{X}$ which are affine and which cover $\mathcal{X}$ [170, 1.6.7.3].

8.13 Étale and flat morphisms of spectral DM stacks

Étale morphisms in spectral geometry

A map $(\mathcal{X}, \mathcal{O}_X) \to (\mathcal{Y}, \mathcal{O}_Y)$ of spectrally ringed ∞-topoi is called **étale** if

1. the underlying map $f\colon \mathcal{X} \to \mathcal{Y}$ of ∞-topoi is étale (8.7), and
2. the map $f^*\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism in $\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X})$.

For instance, for any $X = (\mathcal{X}, \mathcal{O}_X)$ and $U \in \mathcal{X}$, the projection map $X_U \to X$ is étale in this sense, where $X_U = (\mathcal{X}_{/U}, \mathcal{O}_X|_U)$. In fact, any etale morphism of spectrally ringed ∞-topoi is equivalent to one of this form.

If $f: X \to Y$ is an étale map of spectrally ringed ∞-topoi and $Y \in \mathrm{SpDM}^{\mathrm{nc}}$, then also $X \in \mathrm{SpDM}^{\mathrm{nc}}$ (8.10.4), and in fact f is a morphism of $\mathrm{SpDM}^{\mathrm{nc}}$ (8.11.6).

This terminology turns out to be compatible with that of "étale map of $\mathbb{E}_\infty$-rings".

Proposition 8.13.1 ([170, 1.4.10.2]). *A map $A \to B$ of $\mathbb{E}_\infty$-rings is étale if and only if the corresponding map* $\mathrm{Sp\acute{e}t}\, B \to \mathrm{Sp\acute{e}t}\, A$ *is étale.*

We have the following for "lifting" maps over étale morphisms.

Proposition 8.13.2. *Given nonconnective spectral DM stacks $X = (\mathcal{X}, \mathcal{O}_X)$ and $Y = (\mathcal{Y}, \mathcal{O}_Y)$, a map $f: Y \to X$ of nonconnective spectral DM stacks, and an object $U \in \mathcal{X}$, there is an equivalence*

$$\left\{ \begin{array}{ccc} & & X_U \\ & \overset{s}{\nearrow} & \downarrow \pi \\ Y & \xrightarrow[f]{} & X \end{array} \right\} \xrightarrow{\sim} \left\{ 1 \dashrightarrow f^* U \right\}$$

between the ∞-groupoid of "sections" of π over Y in $\mathrm{SpDM}^{\mathrm{nc}}$*, and the ∞-groupoid of global sections of f^*U on $\mathcal{Y}$.*

Proof sketch. A map $s: Y \to X_U$ consists of a geometric morphism $s: \mathcal{Y} \to \mathcal{X}_{/U}$ together with a local map $\widetilde{s}: s^*\mathcal{O}_{X_U} \to \mathcal{O}_Y$ of sheaves of $\mathbb{E}_\infty$-rings. We already know (8.7.1) that geometric morphisms s which lift f correspond exactly to global sections of f^*U. We then have that $s^*\mathcal{O}_{X_U} = s^*\pi^*\mathcal{O}_X \approx f^*\mathcal{O}_X$, so there is an evident map $s^*\mathcal{O}_{X_U} \to \mathcal{O}_Y$, namely the one equivalent to the map $\widetilde{f}: f^*\mathcal{O}_X \to \mathcal{O}_Y$ which is part of the description of $f: Y \to X$. This is in fact the unique map making the diagram commute in $\mathrm{SpDM}^{\mathrm{nc}}$. (See [170, 21.4.6].) ☐

Corollary 8.13.3. *For any $f: Y \to X$ in* $\mathrm{SpDM}^{\mathrm{nc}}$ *and $U \in \mathcal{X}$ the square*

$$\begin{array}{ccc} Y_{f^*U} & \longrightarrow & X_U \\ \downarrow & & \downarrow \\ Y & \xrightarrow[f]{} & X \end{array}$$

is a pullback in $\mathrm{SpDM}^{\mathrm{nc}}$*. It is a pullback in* SpDM *if $X, Y \in$* SpDM.

Colimits along étale maps of spectral DM stacks

It turns out that we can "glue" spectral DM stacks along étale maps, much as one can construct new schemes by gluing together ones along open immersions.

Let $\mathrm{SpDM}_{\mathrm{\acute{e}t}} \subseteq \mathrm{SpDM}$ and $\mathrm{SpDM}^{\mathrm{nc}}_{\mathrm{\acute{e}t}} \subseteq \mathrm{SpDM}^{\mathrm{nc}}$ be the (non-full) subcategories containing just the étale maps.

Proposition 8.13.4. *The categories* $\mathrm{SpDM}_{\mathrm{\acute{e}t}}$ *and* $\mathrm{SpDM}^{\mathrm{nc}}_{\mathrm{\acute{e}t}}$ *have all small colimits, and the inclusions* $\mathrm{SpDM}_{\mathrm{\acute{e}t}} \to \mathrm{SpDM}$ *and* $\mathrm{SpDM}^{\mathrm{nc}}_{\mathrm{\acute{e}t}} \to \mathrm{SpDM}^{\mathrm{nc}}$ *preserves colimits.*

Proof. Here is a brief sketch; I'll describe the nonconnective case. (See [170, 21.4.4] or [163, V 2.3.5] for more details.)

Suppose $\big(c \mapsto X_c = (\mathcal{X}_c, \mathcal{O}_{X_c})\big) \colon \mathcal{C} \to \mathrm{SpDM}^{\mathrm{nc}}_{\text{ét}}$ is a functor from a small ∞-category. We know (8.7.6) that we can form the colimit $\mathcal{X} := \operatorname{colim}^{\infty\mathcal{T}\mathrm{op}}_{c\in\mathcal{C}} \mathcal{X}_c$ of ∞-topoi, and that each $\mathcal{X}_c \to \mathcal{X}$ is étale. In fact, there exists a functor $U \colon \mathcal{C} \to \mathcal{X}$ so that $(c \mapsto \mathcal{X}_c)$ is equivalent to $(c \mapsto \mathcal{X}_{/U_c})$ as functors $\mathcal{C} \to \infty\mathcal{T}\mathrm{op}_{/\mathcal{X}}$.

We also know (8.6.12) that we have descent for sheaves of $\mathbb{E}_\infty$-rings. That is, $\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X}) \approx \lim_{c\in\mathcal{C}} \mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X}_c)$, so there exists $\mathcal{O}_X \in \mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X})$ together with a compatible family of equivalences $\pi_c^* \mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_{X_c}$. In particular, we obtain a cone $\mathcal{C}^{\triangleright} \to \infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$, which in fact lands in the non-full subcategory consisting of étale maps. This cone is a colimit cone, presenting $X = (\mathcal{X}, \mathcal{O}_X)$ as the colimit of the diagram in spectrally ringed ∞-topoi.

To show that X is a nonconnective spectral DM stack, we need a set $\{V_j\}$ of objects in X such that each X_{V_j} is affine, and $\coprod V_j \to 1$ is an effective epi in $\mathcal{X}$. This is straightforward: there are sets $\{V_{c,i} \to U_c\}$ of maps for each object $c \in \mathcal{C}$ such that each $X_{V_{c,i}}$ is affine and $\coprod_i V_{c,i} \to U_c$ is effective epi in $\mathcal{X}_{/U_c}$, so just take the union $\bigcup_c \{V_{c,i}\}$.

Finally, show that the maps $X_{U_i} \to X$ of the cone are ***local***, so that the cone factors through $\mathcal{C}^{\triangleright} \to \mathrm{SpDM}^{\mathrm{nc}}$; this amounts to the fact that being "local" is itself a local condition in the domain.

□

Spectral DM stacks are colimits of affine objects

We obtain the following interesting consequence: every nonconnective spectral DM stack $X = (\mathcal{X}, \mathcal{O}_X)$ is a colimit of a small diagram of affines. That is,

$$X \approx \operatorname{colim}^{\mathrm{SpDM}^{\mathrm{nc}}}_{c\in\mathcal{C}} X_{U_c}$$

where $c \mapsto U_c \colon \mathcal{C} \to \mathcal{X}$ is a functor such that $\operatorname{colim}_{c\in\mathcal{C}} U_c \approx 1$ and each U_c is affine (which exists by (8.10.4)), and so each $X_{U_c} \approx \mathrm{Sp\acute{e}t}\, A_c$ for some $\mathbb{E}_\infty$-ring A_c. Analogous remarks apply to spectral DM stacks, which have the form

$$X \approx \operatorname{colim}^{\mathrm{SpDM}}_{c\in\mathcal{C}} X_{U_c}$$

with each $X_{U_c} \approx \mathrm{Sp\acute{e}t}\, A_c$ for some connective $\mathbb{E}_\infty$-ring A_c.

Flat morphisms in spectral geometry

A map $f \colon Y \to X$ of nonconnective spectral DM stacks is **flat** if for every commutative square

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}\, B & \longrightarrow & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Sp\acute{e}t}\, A & \longrightarrow & X \end{array}$$

in $\mathrm{SpDM}^{\mathrm{nc}}$ such that the horizontal maps are étale, the map g is induced by a flat morphism $A \to B$ of $\mathbb{E}_\infty$-rings [170, 2.8.2].

It is immediate that the base change of any flat morphism is flat. Also, if $Y \to X$ is flat and X is a spectral DM stack, then Y is a spectral DM stack.

Remark 8.13.5. A map $\mathrm{Spét}\, B \to \mathrm{Spét}\, A$ of nonconnective spectral DM stacks is flat in the above sense if and only if $A \to B$ is a flat morphism of $\mathbb{E}_\infty$-rings.

Given $A \in \mathrm{CAlg}$, let $\mathrm{SpDM}^{\mathrm{nc}}_A = (\mathrm{SpDM}^{\mathrm{nc}})_{/\mathrm{Spét}\, A}$, and let $\mathrm{SpDM}^{\flat}_A \subseteq \mathrm{SpDM}^{\mathrm{nc}}_A$ denote the full subcategory spanned by objects which are flat morphisms $X \to \mathrm{Spét}\, A$. It turns out that although the functor $\mathrm{SpDM}^{\mathrm{nc}}_{\tau_{\geq 0} A} \to \mathrm{SpDM}^{\mathrm{nc}}_A$ induced by base change is not an equivalence, it induces an equivalence on full subcategories of flat objects flat objects.

Proposition 8.13.6. *Base change induces an equivalence of* ∞*-categories*

$$\mathrm{SpDM}^{\flat}_{\tau_{\geq 0} A} \xrightarrow{\sim} \mathrm{SpDM}^{\flat}_A.$$

Proof. See [170, 2.8.2]. The inverse equivalence sends $X \to \mathrm{Spét}\, A$ to $\tau_{\geq 0} X \to \mathrm{Spét}(\tau_{\geq 0} A)$; compare (8.8.5). □

8.14 Affine space and projective space

Let's think about two basic examples: affine n-space and projective n-space. It turns out that these come in two distinct versions, depending on whether we use polynomial rings (8.8.6) or free rings (8.8.7).

Affine spaces

Given a connective $\mathbb{E}_\infty$-ring $R \in \mathrm{CAlg}^{\mathrm{cn}}$, define affine n-space over R to be the affine spectral DM stack

$$\mathbf{A}^n_R := \mathrm{Spét}\, R[x_1, \dots, x_n]$$

on a polynomial ring (8.8.6) over R. When $R \in \mathrm{CAlg}^{\heartsuit}$ is an ordinary ring, this is the "usual" affine n-space. In general, $\mathbf{A}^n_R \approx \mathbf{A}^n_{\mathbb{S}} \times_{\mathrm{Spét}\, \mathbb{S}} \mathrm{Spét}\, R$.

What are the "points" of $\mathbf{A}^n_{\mathbb{S}}$? If B is an *ordinary* ring, then

$$\mathbf{A}^n_{\mathbb{S}}(B) = \mathrm{Map}_{\mathrm{SpDM}_{/\mathrm{Spét}\, R}}(\mathrm{Spét}\, B, \mathbf{A}^n_{\mathbb{S}}) \approx \mathrm{Map}_{\mathrm{CAlg}}(\mathbb{S}[x_1, \dots, x_n], B) \approx B^n.$$

However, if B is not an ordinary ring, then things can be very different. For instance, the image of the evident map

$$\mathbf{A}^n_{\mathbb{S}}(\mathbb{S}) \to \mathbf{A}^n_{\mathbb{S}}(\mathbb{Z}) \approx \mathbb{Z}^n$$

consists exactly of the ordered n-tuples $(a_1, \dots, a_n) \in \mathbb{Z}^n$ such that each $a_i \in \{0, 1\}$.[5]

[5] Here's a quick proof. We need to understand the image of $\mathrm{Map}_{\mathrm{CAlg}}(\mathbb{S}[x], \mathbb{S}) \to \mathrm{Map}_{\mathrm{CAlg}^{\heartsuit}}(\mathbb{S}[x], \mathbb{Z}) \approx \mathbb{Z}$ induced by evaluation at $x \in \pi_0 \mathbb{S}[x]$. It is straightforward to construct maps realizing 0 or 1. To show these are the only possibilities, argue as

From this, we see that $\mathbf{A}^n_{\mathbb{S}}$ is ***not*** a group object with respect to addition; i.e., there is no map $\mathbb{S}[x] \to \mathbb{S}[x] \otimes_{\mathbb{S}} \mathbb{S}[x]$ of $\mathbb{E}_\infty$-rings which on π_0 sends $x \mapsto x \otimes 1 + 1 \otimes x$.

It is however true that $\mathbf{A}^1_{\mathbb{S}}$ is a monoid object under multiplication (the coproduct on $\mathbb{S}[x]$ is obtained by applying suspension spectrum to the diagonal map on $\mathbb{Z}_{\geq 0}$). Likewise,

$$\mathbf{G}_m := \text{Spét}\,\mathbb{S}[x, x^{-1}]$$

is an abelian group object in spectral DM stacks.

There is another affine n-space, which I'll call the ***smooth affine space***, namely

$$\mathbf{A}^n_{\text{sm}} := \text{Spét}\,\mathbb{S}\{x_1, \dots, x_n\},$$

defined using a free ring (8.8.7) instead of a polynomial ring. The points of this are easier to explain:

$$\mathbf{A}^n_{\text{sm}}(B) = \text{Map}_{\text{SpDM}}(\text{Spét}\,B, \mathbf{A}^n_{\text{sm}}) \approx \text{Map}_{\text{CAlg}}(\mathbb{S}\{x_1, \dots, x_n\}, B) \approx (\Omega^\infty B)^n.$$

The evident map $\mathbf{A}^n_{\mathbb{S}} \to \mathbf{A}^n_{\text{sm}}$, though not an equivalence, becomes an equivalence after base-change to any $R \in \text{CAlg}_{\mathbb{Q}}$.

Projective spaces

Given $R \in \text{CAlg}^{\text{cn}}_R$ we define projective n-space as follows [170, 5.4.1]. Let $[n] = \{0, 1, \dots, n\}$, and let $P^\circ([n])$ denote the poset of ***non-empty*** subsets. For each $I \in P^\circ([n])$ let

$$M_I := \{\, (m_0, \dots, m_n) \in \mathbb{Z}^{n+1} \mid m_0 + \cdots + m_n = 0,\ m_i \geq 0 \text{ if } i \in I \,\}.$$

We obtain a functor $P^\circ([n])^{\text{op}} \to \text{SpDM}_{\text{ét}}$ by

$$I \mapsto \text{Spét}(R[M_I]).$$

Define $\mathbf{P}^n_R := \text{colim}_{I \in P^0([n])^{\text{op}}} \text{Spét}(R[M_I])$, which exists by (8.13.4).

Example 8.14.1. $\mathbf{P}^1_R$ is the colimit of

$$\text{Spét}(R[x]) \leftarrow \text{Spét}(R[x, x^{-1}]) \to \text{Spét}(R[x^{-1}]).$$

This construction is compatible with base change, and for ordinary rings R recovers the "usual" projective n-space. You can use the same idea to construct spectral versions of toric varieties.

As for affine n-space, it is difficult to understand the functor that $\mathbf{P}^n_R$ represents when R is not an ordinary ring. On the other hand, one can import some of the classical apparatus associated to projective spaces. For instance, there are quasicoherent sheaves

follows. Given $f : \mathbb{S}[x] \to \mathbb{S}$, tensor with complex K-theory KU and take p-completions. The π_0 of p-complete commutative KU-algebras carries a natural "Adams operation" ψ^p, which is a ring endomorphism such that $\psi^p(a) \equiv a^p \mod p$, and on $\pi_0(KU[x])^\wedge_p$ acts via $\psi^p(f(x)) = f(x^p)$. Using this we can show that $a \in \mathbb{Z}$ is in the image if and only if $a^p = a$ for all primes p.
The same kind of argument shows that if $R = \mathbb{S}[\frac{1}{n}, \zeta_n]$ where ζ_n is a primitive nth root of unity as in (8.9.10), then the image of $\text{Map}_{\text{CAlg}}(\mathbb{S}[x], R) \to \pi_0 R = \mathbb{Z}[\frac{1}{n}, \zeta_n]$ is $\{0\} \cup \{\zeta_n^k \mid 0 \leq k < n\}$.

$\mathcal{O}(m)$ over $\mathbf{P}^n_R$ for any $R \in \mathrm{CAlg}^{\mathrm{cn}}$, constructed exactly as their classical counterparts, and $\Gamma(\mathbf{P}^n_R, \mathcal{O}(m))$ has the expected value [170, 5.4.2.6].

There is another projective n-space, the **smooth projective space $\mathbf{P}^n_{\mathrm{sm}}$**, defined to be the spectral DM stack representing a functor $R \mapsto \{$"lines in R^{n+1}"$\}$; see [170, 19.2.6].

8.15 Functor of points

We have defined an ∞-category $\mathrm{SpDM}^{\mathrm{nc}}$ of nonconnective spectral DM stacks. However, we have not yet shown that it is a *locally small* ∞-category: the definition of morphism involves morphisms of underlying ∞-topoi, and $\infty\mathcal{T}\mathrm{op}$ is not locally small. However, it is true that $\mathrm{SpDM}^{\mathrm{nc}}$ is locally small.

Proposition 8.15.1. *For any $X, Y \in \mathrm{SpDM}^{\mathrm{nc}}$, the space $\mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(Y, X)$ is essentially small, i.e., equivalent to a small ∞-groupoid.*

Note that when X is affine, (8.11.7) already implies that $\mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(Y, X)$ is essentially small: $\mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(Y, \mathrm{Sp\acute{e}t}\, B) \approx \mathrm{Map}_{\mathrm{CAlg}}(B, \Gamma(\mathcal{Y}, \mathcal{O}_Y))$.

Given this proposition, we can define the **functor of points** of a nonconnective spectral DM stack:

$$h^{\mathrm{nc}}_X \colon \mathrm{CAlg} \to \mathcal{S} \quad \text{by} \quad h^{\mathrm{nc}}_X(A) := \mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(\mathrm{Sp\acute{e}t}\, A, X).$$

For a spectral DM stack, we consider the restriction of h^{nc}_X to connective $\mathbb{E}_\infty$-rings:

$$h_X \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \quad \text{by} \quad h_X(A) := \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, A, X).$$

Note that if $B \in \mathrm{CAlg}$, then $h^{\mathrm{nc}}_{\mathrm{Sp\acute{e}t}\, B} \approx \mathrm{Map}_{\mathrm{CAlg}}(B, -)$ by the Yoneda lemma, and similarly in the connective case.

Proposition 8.15.2 ([170, 1.6.4.3]). *The functors*

$$X \mapsto h^{\mathrm{nc}}_X \colon \mathrm{SpDM}^{\mathrm{nc}} \rightarrowtail \mathrm{Fun}(\mathrm{CAlg}, \mathcal{S}) \quad \textit{and} \quad X \mapsto h_X \colon \mathrm{SpDM} \rightarrowtail \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$$

are fully faithful.

I'll sketch proofs of these below (giving arguments only in the nonconnective case).

Sheaves of maps into a spectral DM stack

To prove (8.15.1) that $\mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(Y, X)$ is essentially small, we can immediately reduce to the case that Y is affine, since every nonconnective spectral DM stack is a colimit of a small diagram of affines (8.8). So assume $Y = \mathrm{Sp\acute{e}t}\, A$ for some $A \in \mathrm{CAlg}$.

Given a nonconnective spectral DM stack X, consider the functor

$$H^A_X \colon \mathrm{CAlg}^{\mathrm{\acute{e}t}}_A \to \widehat{\mathcal{S}} \quad \text{defined by} \quad H^A_X(A') := \mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(\mathrm{Sp\acute{e}t}\, A', X).$$

This functor is in fact an object of the full subcategory

$$\widehat{\mathrm{Shv}^{\mathrm{\acute{e}t}}_A} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{\acute{e}t}}_A, \widehat{\mathcal{S}})$$

of sheaves on the étale site of A taking values in the category $\widehat{\mathcal{S}}$ of "large" ∞-groupoids; this is because for an étale cover $\{R \to R_i\}$ in $\mathrm{CAlg}_A^{\text{ét}}$, the evident map

$$\operatorname{colim}^{\mathrm{SpDM}^{\mathrm{nc}}}_{\Delta^{\mathrm{op}}}\Big([n] \mapsto \coprod \operatorname{Spét} R_{i_0} \times_{\operatorname{Spét} R} \cdots \times_{\operatorname{Spét} R} \operatorname{Spét} R_{i_n}\Big) \xrightarrow{\sim} \operatorname{Spét} R$$

is an equivalence by (8.13.4), which exactly provides the sheaf condition for H_X^A.

Note: the ∞-category $\widehat{\mathrm{Shv}_A^{\text{ét}}}$, although not locally small, behaves in many respects like an ∞-topos. For instance, it has descent for small diagrams, and in particular small colimits are universal in $\widehat{\mathrm{Shv}_A^{\text{ét}}}$. Furthermore, the inclusion $\mathrm{Shv}_A^{\text{ét}} \subseteq \widehat{\mathrm{Shv}_A^{\text{ét}}}$ preserves small colimits.

The key fact we need is the following.

Proposition 8.15.3. *The functor*

$$X \mapsto H_X^A : \mathrm{SpDM}^{\mathrm{nc}}_{\text{ét}} \to \widehat{\mathrm{Shv}_A^{\text{ét}}}$$

preserves small colimits.

Recall (8.8) that $X \approx \operatorname{colim}_{c\in\mathcal{C}} V_c$ for some functor $V : \mathcal{C} \to \mathrm{SpDM}^{\mathrm{nc}}_{\text{ét}}$ from a small ∞-category. Writing $V_c = \operatorname{Spét} B_c$, the proposition gives us the "formula"

$$\mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(\operatorname{Spét} A, X) \approx (aF)(A),$$

where aF is the sheafification of the presheaf $F : \mathrm{CAlg}_A^{\text{ét}} \to \widehat{\mathcal{S}}$ defined by

$$F(A') = \operatorname{colim}_c H^A_{V_c}(A') \approx \operatorname{colim}_c \mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(\operatorname{Spét} A', V_c) \approx \operatorname{colim}_c \mathrm{Map}_{\mathrm{CAlg}}(B_c, A'),$$

where the colimit is taken in $\widehat{\mathcal{S}}$. Since $\mathcal{C}$ and each $\mathrm{Map}_{\mathrm{CAlg}}(B_c, A')$ are small, we see that the value $F(A)$ is a small ∞-groupoid, as desired.

Sketch proof of (8.15.3). Let $X = \operatorname{colim}^{\mathrm{SpDM}^{\mathrm{nc}}_{\text{ét}}}_{c\in\mathcal{C}} V_c$ with $V : \mathcal{C} \to \mathrm{SpDM}^{\mathrm{nc}}_{\text{ét}}$. If $\mathcal{X}$ is the underlying ∞-topos of X, then we can factor this functor through a functor $U : \mathcal{C} \to \mathcal{X}$, so that $V_c = X_{U_c}$ and $\operatorname{colim}^{\mathcal{X}}_{c\in\mathcal{C}} U_c \approx 1$.

To show that

$$\operatorname{colim}^{\widehat{\mathrm{Shv}_A^{\text{ét}}}}_{c\in\mathcal{C}} H^A_{X_{U_c}} \xrightarrow{\sim} H_X^A,$$

it suffices to show that for any small sheaf $V \in \mathrm{Shv}_A^{\text{ét}}$ and any map $f : V \to H_X^A$ in $\widehat{\mathrm{Shv}_A^{\text{ét}}}$, the map

$$\operatorname{colim}^{\widehat{\mathrm{Shv}_A^{\text{ét}}}}_{c\in\mathcal{C}} (H^A_{X_{U_c}} \times_{H_X^A} V) \to V$$

induced by base change along f is an equivalence. (This is using descent in $\widehat{\mathrm{Shv}_A^{\text{ét}}}$, and the fact that any small sheaf is a small colimit of representables $\mathrm{Map}_{\mathrm{CAlg}_A^{\text{ét}}}(B, -)$, which are themselves small sheaves.)

Note that for small sheaves $V \in \mathrm{Shv}_A^{\text{ét}}$ there is a natural equivalence

$$\mathrm{Hom}_{\mathrm{SpDM}^{\mathrm{nc}}}((\operatorname{Spét} A)_V, X) \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathrm{Shv}_A^{\text{ét}}}}(V, H_X^A),$$

This is because $V \mapsto (\text{Spét}\, A)_v$ is colimit preserving (8.13.4) and the map is certainly an equivalence when V is representable. So let $g\colon (\text{Spét}\, A)_V \to X$ be the map corresponding to $f\colon V \to H_X^A$, and use (8.13.3) to obtain for any $U \in \mathcal{X}$ a pullback square

$$\begin{array}{ccc} (\text{Spét}\, A)_{g^*U} & \longrightarrow & X_U \\ \downarrow & & \downarrow \\ (\text{Spét}\, A)_V & \xrightarrow[g]{} & X \end{array}$$

in SpDM^{nc}, which on applying the functor $Y \mapsto H_Y^A$ gives a pullback square

$$\begin{array}{ccc} g^*U & \longrightarrow & H_{X_U}^A \\ \downarrow & & \downarrow \\ V & \xrightarrow[f]{} & H_X^A \end{array}$$

in $\widehat{\text{Shv}_A^{\text{ét}}}$. Because $g^*\colon \mathcal{X} \to (\text{Shv}_A^{\text{ét}})_{/V}$ is colimit preserving, we see that we get an equivalence $\text{colim}_{c\in\mathcal{C}}^{(\text{Shv}_A^{\text{ét}})_{/V}}\, g^*(U_c) \xrightarrow{\sim} 1_{(\text{Shv}_A^{\text{ét}})_{/V}}$, and the claim follows. □

Example 8.15.4 (Geometric points). Let K be a (classical) separable field, so that $\text{Shv}_K^{\text{ét}} \approx \mathcal{S}$. If $X = \text{colim}_{c\in\mathcal{C}}\, \text{Spét}\, B_c$ is a colimit of affines along étale morphisms, then our "formula" reduces to

$$\text{Map}_{\text{SpDM}^{\text{nc}}}(\text{Spét}\, K, X) \approx \text{colim}_{c\in\mathcal{C}}\, \text{Map}_{\text{CAlg}}(B_c, K).$$

Functor of points

Here is an idea of a proof of (8.15.2) (in the nonconnective case; the connective case is similar); see [163, V 2.4] which proves a more general statement in the framework of "geometries", or [170, 8.1.5] which proves a generalization to formal geometry. We want to show that

$$\text{Map}_{\text{SpDM}^{\text{nc}}}(Y, X) \to \text{Map}_{\text{Fun}(\text{CAlg},\mathcal{S})}(h_Y^{\text{nc}}, h_X^{\text{nc}})$$

is an equivalence for all $X, Y \in \text{SpDM}^{\text{nc}}$. Since nonconnective spectral DM stacks are colimits of small diagrams of affines along étale maps (8.13.4), we reduce to the case of affine $Y = \text{Spét}\, B$. Furthermore, if $X = \text{Spét}\, A$ is also affine, then $\text{Map}_{\text{SpDM}^{\text{nc}}}(Y, X) \approx \text{Map}_{\text{CAlg}}(A, B)$ by (8.11.7), and since $h_Y^{\text{nc}} \approx \text{Map}_{\text{CAlg}}(B, -)$ we see that the map is an equivalence by Yoneda.

Note that the composite functor $\text{CAlg}_A^{\text{ét}} \to \text{CAlg} \xrightarrow{h_X^{\text{nc}}} \mathcal{S}$ is precisely the functor H_X^A of the previous section. Thus h_X^{nc} lives in the full subcategory

$$\text{Shv}^{\text{ét}} \subseteq \text{Fun}(\text{CAlg}, \mathcal{S})$$

spanned by F such that $F|_{\text{CAlg}_A^{\text{ét}}}$ is an étale sheaf for all $A \in \text{CAlg}$.

It turns out that $\text{Shv}^{\text{ét}}$ is equivalent to the ∞-category of sections of a Cartesian fibration $\mathcal{D} \to \mathrm{CAlg}$, whose fiber over $A \in \mathrm{CAlg}$ is equivalent to $\mathrm{Shv}^{\text{ét}}_A$. Thus, by a standard argument, we see that (8.15.3) implies that

$$X \mapsto h^{\mathrm{nc}}_X : \mathrm{SpDM}^{\mathrm{nc}}_{\text{ét}} \to \mathrm{Shv}^{\text{ét}}$$

preserves colimits. The result then follows using X is also a colimit of a small diagram of affines along étale maps.

8.16 Formal spectral geometry

Let's briefly describe the generalization of these ideas to the spectral analogue of *formal* geometry.

Adic $\mathbb{E}_\infty$-rings

An **adic $\mathbb{E}_\infty$-ring** is a connective $\mathbb{E}_\infty$-ring A equipped with a topology on $\pi_0 A$ which is equal to the I-adic topology for some finitely generated ideal $I \subseteq \pi_0 A$. A map of adic $\mathbb{E}_\infty$-rings is a map $f : A \to B$ of $\mathbb{E}_\infty$-rings which induces a continuous map on π_0. Any finitely generated ideal I generating the topology of $\pi_0 A$ is called an **ideal of definition** for the topology; note that the ideal of definition is not itself part of the data of an adic $\mathbb{E}_\infty$-ring, only the topology it generates.

Remark 8.16.1. The **vanishing locus** of an adic $\mathbb{E}_\infty$-ring A is the set $X_A \subseteq |\mathrm{Spec}\, A|$ of prime ideals which are open neighborhoods of 0 in $\pi_0 A$; equivalently, primes which contain some (hence any) ideal of definition $I \subseteq \pi_0 A$. A map $\phi : A \to B$ of $\mathbb{E}_\infty$-rings is an adic map if and only if it sends X_B into X_A; equivalently, if $\phi(I^n) \subseteq J$ for some n where I and J are ideals of definition for A and B respectively [170, 8.1.1.3–4].

In particular, the topology on $\pi_0 A$ of an adic $\mathbb{E}_\infty$-ring A is entirely determined by the vanishing locus.

Completion at finitely generated ideals

Let $A \in \mathrm{CAlg}$ be an $\mathbb{E}_\infty$-ring (not necessarily connective). For every finitely generated ideal $I \subseteq \pi_0 A$ there is a notion of I**-complete** A-module. An A-algebra is called I-complete if its underlying module is so. There are adjoint pairs

$$M \mapsto M^\wedge_I : \mathrm{Mod}_A \rightleftarrows \mathrm{Mod}^{\mathrm{Cpt}(I)}_A, \qquad B \mapsto B^\wedge_I : \mathrm{CAlg}_A \rightleftarrows \mathrm{CAlg}^{\mathrm{Cpt}(I)}_A$$

whose right adjoint is the fully faithful inclusion of the category of I-complete objects, and whose left adjoint, called I**-completion**, is left exact. Furthermore, the notion of I-completeness and its associated completion functors depend only on the radical of I; hence, all ideals of definition of an adic $\mathbb{E}_\infty$-ring provide equivalent completion functors. See [170, 7.3] for more details.

Remark 8.16.2. Here is an explicit formula for I-completion on the level of modules. Given $a \in \pi_0 A$ let $\Sigma^{-1}(A/a^\infty) \in \mathrm{Mod}_A$ denote the homotopy fiber of the evident map $A \to A[a^{-1}]$. Then

$$M_I^\wedge \approx \underline{\mathrm{Hom}}_A(\Sigma^{-1}(A/a_1^\infty) \otimes_A \cdots \otimes_A \Sigma^{-1}(A/a_r^\infty), M)$$

where $(a_1, \ldots, a_r)$ is any finite sequence which generates the ideal I. The unit $M \to M_I^\wedge$ of the adjunction is induced by restriction along the evident map $\Sigma^{-1}(A/a_1^\infty) \otimes_A \cdots \otimes_A \Sigma^{-1}(A/a_r^\infty) \to A \otimes_A \cdots \otimes_A A \approx A$.

Example 8.16.3. If the vanishing ideal is $0 \subseteq \pi_0 A$, so that $\pi_0 A$ is equipped with the discrete topology, then every A-module is I-complete.

Example 8.16.4. If the vanishing ideal is $I = \pi_0 A$, so that $\pi_0 A$ is equipped with the trivial topology, then only the trivial A-module is I-complete.

Example 8.16.5. For a prime $p \in \mathbb{Z} = \pi_0 \mathbb{S}$, an $\mathbb{S}$-module is (p)-complete in the above sense if and only if it is a p-complete spectrum in the conventional sense, and (p)-completion coincides with the usual p-completion of spectra.

Example 8.16.6 (Completion and $K(n)$-localization). Suppose A is an $\mathbb{E}_\infty$-ring which p-local for some prime p, and is weakly 2-periodic and complex orientable (see (8.17) below). The complex orientation gives rise to a sequence of ideals $I_n = (p, u_1, \ldots, u_{n-1}) \subseteq \pi_0 A$; the ideal I_n is called the *nth Landweber ideal.* It turns out that the underlying spectrum of A is $K(n)$-local if and only if (i) A is I_n-complete and (ii) $I_{n+1}(\pi_0 A) = \pi_0 A$ [167, 4.5.2].

The formal spectrum of an adic $\mathbb{E}_\infty$-ring

Recall the ∞-topos $\mathrm{Shv}_A^{\text{ét}}$ of sheaves on the étale site of an $\mathbb{E}_\infty$-ring A. Given an adic $\mathbb{E}_\infty$-ring A, say that $F \in \mathrm{Shv}_A^{\text{ét}}$ is an **adic sheaf** if $F(A \to B) \approx *$ for étale morphisms $A \to B$ such that the image of $|\mathrm{Spec}\, \pi_0 B| \to |\mathrm{Spec}\, \pi_0 A|$ is disjoint from the vanishing locus X_A; i.e., if $I(\pi_0 B) = \pi_0 B$ for some (hence any) ideal of definition $I \subseteq \pi_0 B$. We thus obtain a full subcategory $\mathrm{Shv}_A^{\text{ad}} \subseteq \mathrm{Shv}_A^{\text{ét}}$ of adic sheaves, which in fact is an ∞-topos, and this inclusion is the right-adjoint of a geometric morphism $\mathrm{Shv}_A^{\text{ad}} \to \mathrm{Shv}_A^{\text{ét}}$.

Remark 8.16.7. That $\mathrm{Shv}_A^{\text{ad}}$ is an ∞-topos follows from the observation that $\mathrm{Shv}_A^{\text{ad}} \approx \mathrm{Shv}_{\pi_0 A/I}^{\text{ét}}$, where I is an ideal of definition for A. See [170, 3.1.4].

We can now define the **formal spectrum** of an adic $\mathbb{E}_\infty$-ring A to be the spectrally ringed ∞-topos $\mathrm{Spf}\, A := (\mathrm{Shv}_A^{\text{ad}}, \mathcal{O}_{\mathrm{Spf}\, A})$, where $\mathcal{O}_{\mathrm{Spf}\, A}$ is the composite functor

$$\mathrm{CAlg}_A^{\text{ét}} \rightarrowtail \mathrm{CAlg} \xrightarrow{(-)_I^\wedge} \mathrm{CAlg}.$$

Note that $\mathcal{O}_{\mathrm{Spf}\, A}$ is an adic sheaf because $B_I^\wedge \approx 0$ if $I(\pi_0 B) = \pi_0 B$ and because I-completion is limit preserving. It can be shown that $\mathrm{Spf}\, A$ is strictly Henselian and its structure sheaf is connective [170, 8.1.1.13].

Formal spectral DM stacks

A **formal spectral Deligne–Mumford stack** is a spectrally ringed ∞-topos $X = (\mathcal{X}, \mathcal{O}_X)$ which admits a cover $\{U_i\} \subseteq \mathcal{X}$ such that each $X_{U_i} = (\mathcal{X}_{/U_i}, \mathcal{O}_X|_{U_i})$ is equivalent to $\mathrm{Spf}\, A_i$ for some adic $\mathbb{E}_\infty$-ring A_i. There is a full subcategory

$$\mathrm{fSpDM} \subseteq \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$$

of formal spectral Deligne–Mumford stacks and local maps between them.

Example 8.16.8 (Spectral DM stacks are formal spectral DM stacks). If $A \in \mathrm{CAlg}^{\mathrm{ad}}$ is an adic $\mathbb{E}_\infty$-ring equipped with the discrete topology, then $\mathrm{Spf}\, A \approx \mathrm{Sp\acute{e}t}\, A$. In particular, any spectral DM stack is automatically a formal spectral DM stack, and $\mathrm{SpDM} \rightarrowtail \mathrm{fSpDM}$.

Example 8.16.9 (Formal functor of points). There is a fully faithful embedding $\mathrm{fSpDM} \rightarrowtail \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ defined by sending X to its functor of points $h_X(R) = \mathrm{Map}_{\mathrm{fSpDM}}(\mathrm{Sp\acute{e}t}\, R, X)$ on affine (not adic) spectral DM stacks [170, 8.1.5].

Furthermore, there is an explicit description of the functor of points of $\mathrm{Spf}\, A$:

$$h_{\mathrm{Spf}\, A}(R) = \mathrm{Map}_{\mathrm{fSpDM}}(\mathrm{Sp\acute{e}t}\, R, \mathrm{Spf}\, A) \approx \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}}(A, R) \subseteq \mathrm{Map}_{\mathrm{CAlg}}(A, R).$$

Here R is regarded as an adic $\mathbb{E}_\infty$-ring equipped with the discrete topology, so that $\phi\colon A \to R$ is a map of adic $\mathbb{E}_\infty$-rings if and only if $\phi(I^n) = 0$ for some n and ideal of definition $I \subseteq \pi_0 A$ [170, 8.1.5].

Remark 8.16.10. The formal spectrum functor $\mathrm{Spf}\colon (\mathrm{CAlg}^{\mathrm{ad}})^{\mathrm{op}} \to \mathrm{fSpDM}$ is not fully faithful, or even conservative. However, we have the following. Say that $B \in \mathrm{CAlg}^{\mathrm{ad}}$ is **complete** if $B \xrightarrow{\sim} B^\wedge_I$ for some (and hence any) ideal of definition $I \subseteq \pi_0 B$. For complete adic $\mathbb{E}_\infty$-rings B the evident map $\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}}(B, R) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{fSpDM}}(\mathrm{Spf}\, R, \mathrm{Spf}\, B)$ is always an equivalence [170, 8.1.5.4]. From this and the formal functor of points we see that the full subcategory of formal spectral DM stacks which are equivalent to $\mathrm{Spf}\, A$ for some adic ring A is equivalent to opposite of the full subcategory of complete objects in $\mathrm{CAlg}^{\mathrm{ad}}$.

Formal completion

Given a spectral DM stack X, one may form the **formal completion** $X^\wedge_K$ of X with respect to a "cocompact closed subset $K \subseteq |X|$", which is a formal spectral DM stack equipped with a map $X^\wedge_K \to X$. We refer to [170, 8.1.6] for details, but note that in the case $X = \mathrm{Sp\acute{e}t}\, A$ for $A \in \mathrm{CAlg}^{\mathrm{cn}}$ we have that $|X|$ is precisely the prime ideal spectrum $|\mathrm{Spec}\, A|$, while $X^\wedge_K = \mathrm{Spf}\, A$, where A is given the evident adic structure.

8.17 Formal groups in spectral geometry

Fix a connective $\mathbb{E}_\infty$-ring R. An *n-dimensional formal group* over R is, roughly speaking, a formal spectral DM stack $\widehat{G}$ over $\mathrm{Sp\acute{e}t}\, R$ which (i) is an abelian group object in formal spectral DM stacks, and (ii) as a formal spectral DM stack is equivalent to $\mathrm{Spf}(A)$ where A is an adic $\mathbb{E}_\infty$-ring which "looks like a ring of power series in n variables over R".

Smooth coalgebras

To make this precise, we need the notion of a **smooth commutative coalgebra**. Any symmetric monoidal ∞-category admits a notion of **commutative coalgebra** objects [166, 3.1]. If C is a commutative coalgebra object in Mod_R, then its R-linear dual $C^\vee := \underline{\mathrm{Hom}}_R(C, R)$ comes with the structure of a adic commutative R-algebra [167, 1.3.2].

We say that a commutative R-coalgebra C is **smooth** if (i) C is flat as an R-module and if (ii) there is an isomorphism of $\pi_0 R$-coalgebras

$$\pi_0 C \approx \bigoplus_{k\geq 0} \Gamma^k_{\pi_0 R}(M),$$

where the right-hand side is the divided polynomial coalgebra on some finitely generated projective $\pi_0 R$-module M; the rank of M (if defined) is also called the dimension of C [167, 1.2]. There is an associated ∞-category $\mathrm{cCAlg}^{\mathrm{sm}}_R$ of smooth commutative R-coalgebras.

Remark 8.17.1. The R-linear dual $C^\vee$ of C as above satisfies

$$\pi_0 C^\vee \approx \prod_{k\geq 0} \mathrm{Sym}^k_{\pi_0 R}(M^\vee), \qquad M^\vee = \mathrm{Hom}_R(M, R).$$

In particular, if M is free of rank n then $\pi_* C^\vee \approx \pi_* R[[t_1, \dots, t_n]]$ [167, 1.3.8].

For a *connective* $\mathbb{E}_\infty$-ring R, a functor $\mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ represented by $\mathrm{Spf}(C^\vee)$ for some smooth commutative R-coalgebra is called a **formal hyperplane** over R [167, 1.5.3]. It is said to be n-dimensional if C is n-dimensional in the sense above. (The "hyperplane" terminology arises because our $\mathrm{Spf}(C^\vee)$ does not come equipped with a "base-point", i.e., there is no distinguished R-algebra map $C^\vee \to R$, despite the fact that $\pi_0 C^\vee$ is equipped with an adic topology.)

Formal groups

An n-**dimensional formal group** over a *connective* $\mathbb{E}_\infty$-ring R is a functor

$$\widehat{G}\colon \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}}$$

such that the composite

$$\mathrm{CAlg}^{\mathrm{cn}}_R \xrightarrow{\widehat{G}} \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}} \xrightarrow{\Omega^\infty} \mathcal{S}$$

is represented by $\mathrm{Spf}(C^\vee)$ for some smooth commutative R-coalgebra C of dimension n.

Remark 8.17.2. The definition of formal group I have given here is different than, but equivalent to, the one given in [167, 1.6]; see [167, 1.6.7]. In particular, the basic definitions given there are expressed more directly in terms of commutative coalgebras.

In particular, the functor $\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ represented by $\mathrm{Spf}(C^\vee)$, where C is a commutative R-coalgebra, is equivalent to the **cospectrum** of C. The cospectrum is a functor sending an $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$ to a suitable space of "grouplike elements" in $R' \otimes_R C$ [167, 1.51].

Remark 8.17.3. What about the nonconnective case? Although smooth commutative coalgebras may be defined over any $\mathbb{E}_\infty$-ring, formal hyperplanes and formal groups have only been defined (following Lurie) over *connective* $\mathbb{E}_\infty$-rings.

This is awkward but it's okay! For instance, because smooth commutative R-coalgebras are ***flat*** over R, taking 0-connective covers gives an equivalence

$$\tau_{\geq 0} \colon \mathrm{cCAlg}_R^{\mathrm{sm}} \xrightarrow{\sim} \mathrm{cCAlg}_{\tau_{\geq 0}R}^{\mathrm{sm}}$$

between the ∞-categories of smooth commutative coalgebras over R and over its connective cover $\tau_{\geq 0}R$ [167, 1.2.8]; compare (8.13.6).

So you can extend the notions of formal hyperplane and formal group to nonconnective ground rings, so that a formal hyperplane or formal group over R is ***defined*** to be one over $\tau_{\geq 0}R$. In particular, for any $\mathbb{E}_\infty$-ring R you get an ∞-category $\mathrm{FGroup}(R)$ of formal groups over R, which ***by definition*** satisfies $\mathrm{FGroup}(R) = \mathrm{FGroup}(\tau_{\geq 0}R)$.

The Quillen formal group in spectral geometry

A complex oriented cohomology theory R gives rise to a 1-dimensional formal group over π_*R, whose function ring is $R^*\mathbb{CP}^\infty$. When the theory is represented by an $\mathbb{E}_\infty$-ring which is suitably periodic, then we can upgrade this formal group to an object in spectral geometry.

Given $R \in \mathrm{CAlg}$ and $X \in \mathcal{S}$, write $C_*(X; R) := R \otimes_{\mathbb{S}} \Sigma_+^\infty X \in \mathrm{Mod}_R$ for the "R-module of R-chains on X". This object is in fact a commutative R-coalgebra, via the diagonal map on X [168, 2.4.3.10].

An $\mathbb{E}_\infty$-ring R is **weakly 2-periodic** if $\pi_2 R \otimes_{\pi_0 R} \pi_n R \to \pi_{n+2} R$ is an isomorphism for all $n \in \mathbb{Z}$. If R is both weakly 2-periodic and complex orientable, then one can show that $C_*(\mathbb{CP}^\infty; R)$ is a ***smooth*** commutative R-coalgebra. Furthermore, it is a commutative group object in cCAlg_R (via the abelian group structure on $\mathbb{CP}^\infty$), and hence it gives rise to a 1-dimensional formal group $\widehat{G}_R^Q$, called the **Quillen formal group** of R [167, 4.1.3].

Remark 8.17.4. In view of what I said about connectivity in relation to formal groups (8.17.3), the formal spectral DM stack associated to the Quillen formal group of R is $\mathrm{Spf}((\tau_{\geq 0} C_*(\mathbb{CP}^\infty; R))^\vee)$. Note that $\tau_{\geq 0} C_*(\mathbb{CP}^\infty; R)$ is not at all the same as $C_*(\mathbb{CP}^\infty; \tau_{\geq 0} R)$, and that the latter does not give rise to a formal group in the sense defined above.

Remark 8.17.5. Let R be an $\mathbb{E}_\infty$-ring which is weakly 2-periodic and complex orientable, with Quillen formal group $\widehat{G}_R^Q$. Then every commutative R-algebra $R \to R'$

is *also* weakly 2-periodic and complex orientable, and so also has a Quillen formal group, and in fact $\widehat{G}^{Q}_{R'} \approx \widehat{G}^{Q}_{R} \times_{\mathrm{Sp\acute{e}t}\,\tau_{\geq 0}R} \mathrm{Sp\acute{e}t}\,\tau_{\geq 0}R'$.

Preorientations and orientations

Let R be an $\mathbb{E}_\infty$-ring, not necessarily assumed to be connective, and $\widehat{G} \in \mathrm{FGroup}(R)$ a 1-dimensional formal group over it. We ask the question: What additional data do we need to identify $\widehat{G}$ with the Quillen formal group over R? Note that I don't want to presuppose that the Quillen formal group actually exists in this case, i.e., I don't assume that R is weakly 2-periodic or complex orientable.

A **preorientation** of a 1-dimensional formal group $\widehat{G}$ over a (possibly nonconnective) $\mathbb{E}_\infty$-ring R is a map

$$e\colon S^2 \to \widehat{G}(\tau_{\geq 0}R)$$

of based spaces, where the base point goes to the identity of the group structure. We write $\mathrm{Pre}(\widehat{G}) = \mathrm{Map}_{\mathcal{S}_*}(S^2, \widehat{G}(\tau_{\geq 0}R))$ for the space of preorientations.

Proposition 8.17.6. *Suppose R is weakly 2-periodic and complex orientable. Then there is an equivalence*

$$\mathrm{Pre}(\widehat{G}) \approx \mathrm{Map}_{\mathrm{FGroup}(R)}(\widehat{G}^{Q}_{R}, \widehat{G})$$

between the space of preorientations and the space of maps from the Quillen formal group.

Proof. See [167, 4.3]. This is basically a formal consequence of the observation that the free abelian group on the based space S^2 is equivalent to $\mathbb{CP}^\infty$. □

Note that $\mathrm{Pre}(\widehat{G})$ is defined even when R does not admit a Quillen formal group. We will now describe a condition on a *preorientation* $e \in \mathrm{Pre}(\widehat{G})$ which implies simultaneously (i) that R is weakly 2-periodic and complex orientable, and (ii) that the map $\widehat{G}^{Q}_{R} \to \widehat{G}$ induced by e is an isomorphism in $\mathrm{FGroup}(R)$.

Given $\widehat{G} \in \mathrm{FGroup}(R)$, let $\mathcal{O}_{\widehat{G}}$ denote its ring of functions, so that $\widehat{G} \approx \mathrm{Spf}(\mathcal{O}_{\widehat{G}})$. Note that by our definitions (8.17.3) the ring $\mathcal{O}_{\widehat{G}}$ is a connective $\tau_{\geq 0}R$-algebra, even if R is not connective.

The **dualizing line** of a 1-dimensional formal group $\widehat{G}$ is an R-module defined by

$$\omega_{\widehat{G}} := R \otimes_{\mathcal{O}_{\widehat{G}}} \mathcal{O}_{\widehat{G}}(-\eta), \quad \text{where} \quad \mathcal{O}_{\widehat{G}}(-\eta) := \text{fiber of } (\mathcal{O}_{\widehat{G}} \xrightarrow{\eta} \tau_{\geq 0}R \to R),$$

where $\eta \in \widehat{G}(\tau_{\geq 0}R)$ is the identity element of the group structure. The R-module $\omega_{\widehat{G}}$ is in fact an R-module which is locally free of rank 1, and its construction is functorial with respect to isomorphisms of 1-dimensional formal groups [167, 4.1 and 4.2].

Example 8.17.7. Let R be weakly 2-periodic and complex orientable, and $\widehat{G}^{Q}_{R}$ its Quillen formal group. Then there is a canonical equivalence of R-modules

$$\omega_{\widehat{G}^{Q}_{R}} \approx \Sigma^{-2}R.$$

This object is also canonically identified with $C^*_{\mathrm{red}}(\mathbb{CP}^1; R)$, the function spectrum representing the reduced R-cohomology of $\mathbb{CP}^1 \approx S^2$ as a $C^*(S^2; R)$-module.

For a 1-dimensional formal group $\widehat{G}$ over an $\mathbb{E}_\infty$-ring R, any preorientation $e \in \mathrm{Pre}(\widehat{G})$ determines a map

$$\beta_e \colon \omega_{\widehat{G}} \to \Sigma^{-2} R$$

of R-modules, called the **Bott map** associated to e. This map is constructed in [167, 4.2–3].

Remark 8.17.8. Here is one way to describe the construction of the Bott map [167, 4.2.10].

For any suspension $X = \Sigma Y$ of a based space, the object $C^*_{\mathrm{red}}(X;R)$ is equivalent as a $C^*(X;R)$-module to the restriction of an R-module along the augmentation $\pi\colon C^*(X;R) \to R$ corresponding to the basepoint of X. ("The cup product is trivial on a suspension.") For instance if $X = S^2 = \Sigma S^1$ we have $C^*_{\mathrm{red}}(X;R) \approx \pi^*(\Sigma^{-2}R)$.

A preorientation $e\colon S^2 \to \widehat{G}(\tau_{\geq 0}R)$ corresponds exactly to a map of $\mathbb{E}_\infty$-rings, $\tilde{e}\colon \mathcal{O}_{\widehat{G}} \to C^*(S^2;\tau_{\geq 0}R)$, compatible with augmentations to $\tau_{\geq 0}R$, and in turn induces a map

$$\mathcal{O}_{\widehat{G}}(-\eta) \to C^*_{\mathrm{red}}(S^2;R) \approx \pi^*(\Sigma^{-2}R)$$

of $\mathcal{O}_{\widehat{G}}$-modules, which by the previous paragraph is adjoint to a map $\omega_{\widehat{G}} = R \otimes_{\mathcal{O}_{\widehat{G}}} \mathcal{O}_{\widehat{G}}(-\eta) \to \Sigma^{-2}R$ of R-modules, which is the Bott map of e.

An **orientation** of $\widehat{G}$ is a preorientation e whose Bott map $\beta_e\colon \omega_{\widehat{G}} \to \Sigma^{-1}R$ is an equivalence. We write $\mathrm{OrDat}(\widehat{G}) \subseteq \mathrm{Pre}(\widehat{G})$ for the full subgroupoid consisting of orientations.

Now we can state the criterion for a preoriented 1-dimensional formal group to be isomorphic to the Quillen formal group.

Proposition 8.17.9. *A preorientation $e \in \mathrm{Pre}(\widehat{G})$ of a formal group $\widehat{G}$ over an $\mathbb{E}_\infty$-ring R is an orientation if and only if (i) R is weakly 2-periodic and complex orientable, and (ii) the map $\widehat{G}^{Q}_{R} \to \widehat{G}$ of formal groups corresponding to e is an isomorphism.*

Proof. See [167, 4.3.23]. That R is weakly 2-periodic and complex orientable given the existence of an orientation is immediate from the fact that $\omega_{\widehat{G}}$ is locally free of rank 1, and also equivalent to $\Sigma^{-2}R$. □

8.18 Quasicoherent sheaves

Recall that we have defined a sheaf of $\mathbb{E}_\infty$-rings $\mathcal{O} \in \mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X})$ on an ∞-topos to be a limit preserving functor $\mathcal{X}^{\mathrm{op}} \to \mathrm{CAlg}$ (8.3). There is an alternate description: $\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X})$ is equivalent to the ∞-category of *commutative monoid objects* in the symmetric monoidal ∞-category $(\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}), \otimes)$ of sheaves of spectra, using a symmetric monoidal structure inherited from the usual one on spectra [163, VII 1.15].

This leads to notions of *sheaves of $\mathcal{O}$-modules* on a spectrally ringed ∞-topos, and eventually to *quasicoherent sheaves* on a nonconnective spectral DM stack.

Sheaves of modules

To each spectrally ringed ∞-topos $X = (\mathcal{X}, \mathcal{O})$, there is an associated ∞-category $\mathrm{Mod}_{\mathcal{O}}$ of **sheaves of $\mathcal{O}$-modules** on $\mathcal{X}$, whose objects are sheaves of spectra which are modules over $\mathcal{O}$. (A precise description of this category requires the theory of ∞-operads; see [168, 3.3].)

The ∞-category $\mathrm{Mod}_{\mathcal{O}}$ is presentable (so is complete and cocomplete), stable, and symmetric monoidal, and the monoidal structure $\otimes_{\mathcal{O}}$ preserves colimits and finite limits in each variable [170, 2.1].

Example 8.18.1. Given an $\mathbb{E}_\infty$-ring A, any A-module $M \in \mathrm{Mod}_A$ can be promoted to a sheaf $\mathcal{M} \in \mathrm{Mod}_{\mathcal{O}}$ of $\mathcal{O}$-modules on $\mathrm{Spét}\, A = (\mathrm{Shv}_A^{\text{ét}}, \mathcal{O})$, so that the underlying sheaf of spectra of $\mathcal{M}$ is

$$(A \to B) \mapsto B \otimes_A M : \mathrm{CAlg}_A^{\text{ét}} \to \mathrm{Sp}.$$

The resulting tuple $(\mathrm{Shv}_A^{\text{ét}}, \mathcal{O}, \mathcal{M})$ of ∞-topos, sheaf of rings, and sheaf of modules, is denoted $\mathrm{Spét}(A, M)$; see [170, 2.2.1] for details.

Quasicoherent sheaves

Now let $X = (\mathcal{X}, \mathcal{O}_X)$ be a nonconnective spectral DM stack. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$ is **quasicoherent** if there exists a set $\{U_i\}$ of objects in $\mathcal{X}$ which cover it (i.e., such that $\coprod_i U_i \to 1$ is effective epi), and there exist pairs (A_i, M_i), $A_i \in \mathrm{CAlg}$, $M_i \in \mathrm{Mod}_{A_i}$, and equivalences

$$(\mathcal{X}_{/U_i}, \mathcal{O}_X|_{U_i}, \mathcal{F}|_{U_i}) \approx \mathrm{Spét}(A_i, M_i)$$

of data consisting of (strictly Henselian spectrally ringed ∞-topos and sheaf of modules), where $\mathrm{Spét}(A_i, M_i)$ is as in (8.18.1).

The ∞-category

$$\mathrm{QCoh}(X) \subseteq \mathrm{Mod}_{\mathcal{O}_X}$$

of quasicoherent sheaves on X is defined to be the full subcategory of modules spanned by quasicoherent objects. It is presentable, stable, and symmetric monoidal (see [170, 2.2.4]).

For affine X, quasicoherent modules are just modules over the evident $\mathbb{E}_\infty$-ring.

Proposition 8.18.2. *If $X \approx \mathrm{Spét}\, A$ for some $A \in \mathrm{CAlg}$, then there is an equivalence*

$$\mathrm{QCoh}(X) \approx \mathrm{Mod}_A$$

of symmetric monoidal ∞-categories. The functor $\mathrm{QCoh}(X) \to \mathrm{Mod}_A$ sends a sheaf to its global sections; the functor $\mathrm{Mod}_A \to \mathrm{QCoh}(X)$ is $M \mapsto \mathrm{Spét}(A, M)$.

Remark 8.18.3. If $A \in \mathrm{CAlg}^\heartsuit$ is an ordinary ring, then

$$\mathrm{QCoh}(\mathrm{Spét}\, A) \approx \mathrm{Mod}_A \approx \mathrm{Ch}(\mathrm{Mod}_A^\heartsuit)[(\text{quasi-isos})^{-1}],$$

where $\mathrm{Mod}_A^\heartsuit \subseteq \mathrm{Mod}_A$ is the ordinary 1-category of A-modules.

There are other characterizations of quasicoherence. For instance, $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$ is quasicoherent if and only if the evident map

$$\mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \to \mathcal{F}(U)$$

is an isomorphism for all maps $U \to V$ between affine objects in $\mathcal{X}$ [170, 2.2.4.3].

There are pairs of adjoint functors

$$\mathrm{QCoh}(X) \rightleftarrows \mathrm{Mod}_{\mathcal{O}_X} \underset{\text{forget}}{\overset{\mathcal{O}_X \otimes -}{\rightleftarrows}} \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}).$$

The left adjoints of these pairs are symmetric monoidal, and preserve finite limits but not arbitrary limits in general.

Pullbacks and pushforwards of quasicoherent sheaves

Given a map $f\colon X \to Y$ of nonconnective spectral DM stacks, we have pairs of adjoint functors

$$\begin{array}{ccccc} \mathrm{QCoh}(X) & \rightleftarrows & \mathrm{Mod}_{\mathcal{O}_X} & \leftrightarrows & \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \\ {\scriptstyle f^*}\uparrow\downarrow{\scriptstyle f_*} & & {\scriptstyle f^*}\uparrow\downarrow{\scriptstyle f_*} & & {\scriptstyle f^*}\uparrow\downarrow{\scriptstyle f_*} \\ \mathrm{QCoh}(Y) & \rightleftarrows & \mathrm{Mod}_{\mathcal{O}_Y} & \leftrightarrows & \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{Y}) \end{array}$$

so that each functor labeled f^* is (strongly) symmetric monoidal, and such that the squares of *left adjoints* commute up to natural isomorphism, and the squares of *right adjoints* commute up to natural isomorphism. See [170, 2.5].

Descent for modules and quasicoherent sheaves

It turns out that the formation of categories of either modules or quasicoherent sheaves satisfies a version of descent. Given a nonconnective spectral DM stack $X = (\mathcal{X}, \mathcal{O}_X)$, we have a functor

$$U \mapsto X_U = (\mathcal{X}_{/U}, \mathcal{O}_X|_U)\colon \mathcal{X} \to \mathrm{SpDM},$$

whose colimit exists and is equivalent to X (8.8). For each $f\colon U \to V$ in $\mathcal{X}$ we have induced functors

$$f^*\colon \mathrm{Mod}_{\mathcal{O}_{X_V}} \to \mathrm{Mod}_{\mathcal{O}_{X_U}}, \qquad f^*\colon \mathrm{QCoh}(X_V) \to \mathrm{QCoh}(X_U),$$

which fit together to give functors $\mathcal{X}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$.

Proposition 8.18.4. *The functors* $\mathcal{X}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ *defined by* $U \mapsto \mathrm{Mod}_{\mathcal{O}_{X_U}}$ *and* $U \mapsto \mathrm{QCoh}(X_U)$ *are limit preserving.*

Proof. See [170, 2.1.0.5] and [170, proof of 2.2.4.1]. □

Thus, we may regard these constructions as defining sheaves of (presentable, stable, symmetric monoidal) ∞-categories on $\mathcal{X}$.

Quasicoherent sheaves on quasiaffine spectral DM stacks

We have seen that $\mathrm{QCoh}(X) \approx \mathrm{Mod}_A$ if $X = \mathrm{Sp\acute{e}t}\, A$. This generalizes to X which are *quasiaffine*.

A nonconnective spectral DM stack $X = (\mathcal{X}, \mathcal{O}_X)$ is **quasiaffine** if

1. the ∞-topos $\mathcal{X}$ is **quasicompact**, i.e., for any set $\{U_i\}$ of objects of $\mathcal{X}$ which is a cover, there is a finite subset $\{U_{i_k},\ k = 1, \dots, r\}$ which is a cover, and
2. it admits an *open immersion* into an affine, i.e., if there exists $A \in \mathrm{CAlg}$ and a (-1)-truncated object $U \in \mathrm{Shv}_A^{\text{ét}}$ such that $X \approx (\mathrm{Sp\acute{e}t}\, A)_U$.

Theorem 8.18.5. *If X is quasiaffine, then taking global sections defines an equivalence of categories* $\mathrm{QCoh}(X) \xrightarrow{\sim} \mathrm{Mod}_A$ *where* $A = \Gamma(\mathcal{X}, \mathcal{O}_X)$.

Proof. See [170, 2.4]. □

Example 8.18.6. Here is an example which illustrates both the theorem and its proof. Let $R = \mathbb{S}[x,y]$, and $X = \mathbf{A}^2 = \mathrm{Sp\acute{e}t}\, R = (\mathrm{Shv}_R^{\text{ét}}, \mathcal{O})$. Define $U \in \mathrm{Shv}_R^{\text{ét}} \subseteq \mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S})$ by

$$U(\mathbb{S}[x,y] \to B) := \begin{cases} * & \text{if } (x,y)\pi_0 B = \pi_0 B, \\ \varnothing & \text{if } (x,y)\pi_0 B \neq \pi_0 B. \end{cases}$$

Let $Y := X_U =$ "$\mathbf{A}^2 \smallsetminus \{0\}$". Clearly Y is quasiaffine.

We can write U as a colimit in $\mathrm{Shv}_R^{\text{ét}}$ of a diagram $U_x \leftarrow U_{xy} \to U_y$, where $U_x, U_y \subseteq U$ are the subobjects which are "inhabited" exactly at those $\mathbb{S}[x,y] \to B$ such that $x \in (\pi_0 B)^\times$ or $y \in (\pi_0 B)^\times$ respectively, and $U_{xy} = U_x \times_U U_y$. There is an equivalence of commutative squares

$$\begin{array}{ccc} X_{U_{xy}} & \longrightarrow & X_{U_y} \\ \downarrow & & \downarrow \\ X_{U_x} & \longrightarrow & X_U \end{array} \quad \approx \quad \begin{array}{ccc} \mathrm{Sp\acute{e}t}\, \mathbb{S}[x^\pm, y^\pm] & \longrightarrow & \mathrm{Sp\acute{e}t}\, \mathbb{S}[x, y^\pm] \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}\, \mathbb{S}[x^\pm, y] & \longrightarrow & Y \end{array}$$

which are pushout squares in SpDM by (8.8). Taking quasicoherent sheaves, we obtain a commutative square of ∞-categories

$$\begin{array}{ccc} \mathrm{Mod}_{\mathbb{S}[x^\pm, y^\pm]} & \longleftarrow & \mathrm{Mod}_{\mathbb{S}[x, y^\pm]} \\ \uparrow & & \uparrow \\ \mathrm{Mod}_{\mathbb{S}[x^\pm, y]} & \longleftarrow & \mathrm{QCoh}(Y) \end{array}$$

which is a pullback by descent.

On the other hand, consider the ring of global sections

$$\Gamma := \Gamma(\mathcal{X}_{/U}, \mathcal{O}_X|_U) \approx \lim\Big(\mathbb{S}[x^\pm, y] \to \mathbb{S}[x^\pm, y^\pm] \leftarrow \mathbb{S}[x, y^\pm]\Big).$$

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Mod}_{\mathbb{S}[x^\pm,y^\pm]} & \longleftarrow & \mathrm{Mod}_{\mathbb{S}[x,y^\pm]} \\ \uparrow & & \uparrow \\ \mathrm{Mod}_{\mathbb{S}[x^\pm,y]} & \longleftarrow & \mathrm{Mod}_\Gamma \end{array}$$

which is also seen to be a pullback of ∞-categories. The equivalence

$$\mathrm{Mod}_{\mathbb{S}[x^\pm,y]} \times_{\mathrm{Mod}_{\mathbb{S}[x^\pm,y^\pm]}} \mathrm{Mod}_{\mathbb{S}[x,y^\pm]} \to \mathrm{Mod}_\Gamma$$

is realized by a functor which sends "descent data"

$$\Big(M_x \in \mathrm{Mod}_{\mathbb{S}[x^\pm,y]},\ M_y \in \mathrm{Mod}_{\mathbb{S}[x,y^\pm]},\ \psi\colon M_x[y^{-1}] \xrightarrow{\sim} M_y[x^{-1}] \in \mathrm{Mod}_{\mathbb{S}[x^\pm,y^\pm]}\Big)$$

to the limit $\lim(M_x \to M_x[y^{-1}] \xrightarrow{\sim} M_y[x^{-1}] \leftarrow M_y)$ in Mod_Γ, while the inverse equivalence sends $N \in \mathrm{Mod}_\Gamma$ to $(\mathbb{S}[x^\pm,y] \otimes_\Gamma N,\ \mathbb{S}[x,y^\pm] \otimes_\Gamma N,\ \mathrm{id})$. The key observation for proving the equivalence is that both these functors preserve arbitrary colimits and finite limits, and are easy to evaluate on the "generating" objects $\Gamma \in \mathrm{Mod}_\Gamma$ and $(\mathbb{S}[x^\pm,y], \mathbb{S}[x,y^\pm], \mathrm{id})$ in the limit.

8.19 Elliptic cohomology and topological modular forms

I return to our motivating example of elliptic cohomology.

First, let us consider the moduli stack of (smooth) elliptic curves. This is an example of a "classical" Deligne–Mumford stack. However, according to (8.12) we can regard classical Deligne–Mumford stacks as a particular type of 0-truncated spectral DM stack, and since that is the language I have introduced in this paper, that is how I will generally talk about it.

The moduli stack of elliptic curves

The moduli stack of elliptic curves is a (classical) DM stack $\mathcal{M}_{\mathrm{Ell}} = (\mathcal{X}_{\mathrm{Ell}}, \mathcal{O})$ such that, for ordinary ring $A \in \mathrm{CAlg}^\heartsuit$, we have

$$\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\,A, \mathcal{M}_{\mathrm{Ell}}) \approx \big\{\text{elliptic curves over } \mathrm{Sp\acute{e}t}\,A\big\}. \tag{8.19.1}$$

The right-hand side of (8.19.1) represents the 1-groupoid of elliptic curves over $\mathrm{Sp\acute{e}t}\,A$ and isomorphisms between them. (Note that an isomorphism of elliptic curves is necessarily compatible with the distinguished sections e; we usually omit e from the notation.)

Remark 8.19.1. Here "elliptic curve" means a classical smooth elliptic curve, i.e., a proper and smooth morphism $\pi\colon C \to \mathrm{Sp\acute{e}t}\,A$ of schemes (i.e., of schematic DM stacks) whose geometric fibers are curves of genus 1, and which is equipped (as part of the data), with a section $e\colon \mathrm{Sp\acute{e}t}\,A \to C$ of π.

I will not review the theory of elliptic curves here. However, we should note that every elliptic curve is an ***abelian group scheme***; i.e., an elliptic curve $C \to \text{Spét}\, A$ is an abelian group object in the category of schemes over A. Furthermore, as it is 1-dimensional and smooth, the formal completion $C_e^{\wedge}$ at the identity section exists, and is an example of a 1-dimensional formal group over A.

That there exists such an object $\mathcal{M}_{\text{Ell}}$ is a theorem, which we will take as given.

Remark 8.19.2 (The étale site of $\mathcal{M}_{\text{Ell}}$). As a DM stack, and hence as a spectral DM stack, $\mathcal{M}_{\text{Ell}}$ is the colimit of a diagram whose objects are étale morphisms $\text{Spét}\, A \to \mathcal{M}_{\text{Ell}}$, and since $\mathcal{M}_{\text{Ell}}$ is 0-truncated the rings A which appear in this diagram will be ordinary rings. Thus, $\mathcal{M}_{\text{Ell}}$ can be reconstructed from the "étale site" of $\mathcal{M}_{\text{Ell}}$, i.e., the category $\mathcal{U}$ whose objects are elliptic curves $C \to \text{Spét}\, A$ represented by an étale map $\text{Spét}\, A \to \mathcal{M}_{\text{Ell}}$, and whose morphisms are commutative squares

$$\begin{array}{ccc} C & \longrightarrow & C' \\ \downarrow & & \downarrow \\ \text{Spét}\, A & \longrightarrow & \text{Spét}\, A' \end{array}$$

such that $C \to C' \times_{\text{Spét}\, A'} \text{Spét}\, A$ is an isomorphism of elliptic curves over $\text{Spét}\, A$.

It remains to characterize the objects of $\mathcal{U}$. Given elliptic curves $C \to S$ and $C' \to S'$, consider the functor

$$T \mapsto \text{Iso}_{C/S,C'/S'}(T) := \left\{(f\colon T \to S,\ f'\colon T \to S',\ \alpha\colon f^*C \xrightarrow{\sim} f'^*C')\right\}$$

which sends a scheme T to the set of tuples consisting of maps of schemes f and f', and a choice of isomorphism α of elliptic curves over T. It turns out that this functor is itself representable by a *scheme* $I_{C/S,C'/S'}$:

$$\text{Iso}_{C/S,C'/S'}(T) \approx \text{Map}_{\text{Sch}}(\text{Spec}\, T, I_{C/S,C'/S'}).$$

An elliptic curve $C \to S$ is represented by an étale morphism $S \to \mathcal{M}_{\text{Ell}}$ if and only if for every elliptic curve $C' \to S'$ the evident map $I_{C/S,C'/S'} \to S$ of schemes is étale.

See [144] for much more on the moduli stack of elliptic curves (although the word "stack" is rarely used there).

The theorem of Goerss–Hopkins–Miller

Let $\mathcal{U}$ denote the étale site of $\mathcal{M}_{\text{Ell}}$ as in (8.19.2). We note the functor

$$\mathcal{O}\colon \mathcal{U}^{\text{op}} \to \text{CAlg}^{\heartsuit}.$$

defined by $(C \to \text{Spét}\, A) \mapsto A$. Also recall that for each object $(C \to \text{Spét}\, A) \in \mathcal{U}$ we have 1-dimensional formal group law over A.

Question 8.19.3. Does there exist a functor $\mathcal{O}^{\text{top}}\colon \mathcal{U}^{\text{op}} \to \text{CAlg}$ $\mathcal{O}^{\text{top}}\colon \text{CAlg}$ sitting in a commutative diagram $\begin{array}{ccc} & & \mathrm{CAlg} \\ & \nearrow^{\mathcal{O}^{\mathrm{top}}} & \downarrow \pi_0 \\ \mathcal{U}^{\mathrm{op}} & \xrightarrow[\mathcal{O}]{} & \mathrm{CAlg}^{\heartsuit} \end{array}$ such that

1. for each object $C \to \mathrm{Spét}\,A$, the corresponding ring $R = \mathcal{O}^{\mathrm{top}}(C \to \mathrm{Spét}\,A)$ is weakly 2-periodic and has homotopy concentrated in even degrees, and hence is complex orientable; and
2. is equipped with natural isomorphisms $\mathrm{Spét}(R^0(\mathbb{CP}^\infty)) \approx C_e^\wedge$ of formal groups between the formal groups of $R = \mathcal{O}^{\mathrm{top}}(C \to \mathrm{Spét}\,A)$ and the formal completions $C_e^\wedge$ of elliptic curves

Remark 8.19.4. The formal groups $C_e^\wedge$ of elliptic curves $C \to \mathrm{Spét}\,A$ in the étale site $\mathcal{U}$ satisfy the Landweber condition (see [74, Ch. 4]), and thus for each such curve there we can certainly construct a homotopy-commutative ring spectrum R satisfying conditions (1) and (2). The point of the theorem is to rigidify this construction to an honest functor of ∞-categories, and while doing so lift it to a functor to structured commutative rings.

Theorem 8.19.5 (Goerss–Hopkins–Miller). *The answer to* (8.19.3) *is yes. Furthermore, the resulting functor $\mathcal{O}^{\mathrm{top}}$ defines a sheaf of $\mathbb{E}_\infty$-rings on the étale site of $\mathcal{M}_{\mathrm{Ell}}$.*

The pair $(\mathcal{X}_{\mathrm{Ell}}, \mathcal{O}^{\mathrm{top}})$ is an example of a nonconnective spectral DM stack, whose 0-truncation is the classical DM stack $\mathcal{M}_{\mathrm{Ell}}$. (That this is the case is because $\pi_0\mathcal{O}^{\mathrm{top}} \approx \mathcal{O}$, the structure sheaf on $\mathcal{M}_{\mathrm{Ell}}$.)

Given (8.19.5), we can now define

$$\mathrm{TMF} := \Gamma(\mathcal{X}_{\mathrm{Ell}}, \mathcal{O}^{\mathrm{top}}) \approx \lim_{(C \to \mathrm{Spét}\,A) \in \mathcal{U}} \mathcal{O}^{\mathrm{top}}(C \to \mathrm{Spét}\,A),$$

the periodic $\mathbb{E}_\infty$-ring of **topological modular forms**.

Remark 8.19.6. See [74] for more on (8.19.5), including details about the original proof, as well as more information on TMF.

8.20 The classifying stack for oriented elliptic curves

It turns out that the nonconnective spectral Deligne–Mumford stack $(\mathcal{X}_{\mathrm{Ell}}, \mathcal{O}^{\mathrm{top}})$ admits a modular interpretation in spectral algebraic geometry: it is the classifying object for *oriented elliptic curves.*

Elliptic curves in spectral geometry

A **variety** over an $\mathbb{E}_\infty$-ring R is a flat morphism $X \to \mathrm{Spét}\,R$ of nonconnective spectral DM stacks, such that the induced map $\tau_{\geq 0}X \to \mathrm{Spét}\,\tau_{\geq 0}R$ of spectral DM stacks is: proper, locally almost of finite presentation, geometrically reduced, and geometrically connected [166, 1.1], [170, 19.4.5]

Remark 8.20.1. We have not and will not describe all the adjectives in the above definition. See [170, 5.1] for proper, [170, 4.2] for locally almost of finite presentation, [170, 8.6] for geometrically reduced and geometrically connected.

An **abelian variety** over an $\mathbb{E}_\infty$-ring R is a variety X over R which is a commutative monoid object in $\mathrm{SpDM}_R^{\mathrm{nc}}$. It is an **elliptic curve** if it is of dimension 1.

Remark 8.20.2. "Commutative monoid object" is here taken in the sense of [168, 2.4.2]. In this case, it means that an abelian variety X over R represents a functor on $\mathrm{SpDM}_R^{\mathrm{nc}}$ which takes values in $\mathbb{E}_\infty$-spaces. In fact, one can show that every abelian variety in this sense is "grouplike", i.e., it actually represents a functor to grouplike $\mathbb{E}_\infty$-spaces [166, 1.4.4].

A **strict** abelian variety or elliptic curve is one in which the commutative monoid structure is equipped with a refinement to an abelian group structure; i.e., X represents a functor to $\mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}}$ (8.8.4).

Remark 8.20.3. Over an ordinary ring R, either notion of abelian variety reduces to the classical notion. In either case, the commutative monoid/abelian group structure coincides with the unique abelian group structure which exists on a classical abelian variety.

In the classical case, the underlying variety of an abelian variety admits a unique group structure compatible with a given identity section. In the spectral setting, this is no longer the case, and a group structure of some sort needs to be imposed.

There are ∞-categories $\mathrm{AbVar}(R)$ and $\mathrm{AbVar}^{\mathrm{s}}(R)$ of abelian varieties and strict abelian varieties; morphisms are maps of nonconnective spectral DM stacks over R which preserve the commutative monoid structure or abelian group structure as the case may be. We are going to be interested in $\mathrm{Ell}^{\mathrm{s}}(R) \subseteq \mathrm{AbVar}^{\mathrm{s}}(R)$, the full subcategory of strict elliptic curves.

Remark 8.20.4. Since abelian varieties over R are in particular flat morphisms, we see that $\mathrm{AbVar}(R) \approx \mathrm{AbVar}(\tau_{\geq 0}R)$ and $\mathrm{AbVar}^{\mathrm{s}}(R) \approx \mathrm{AbVar}^{\mathrm{s}}(\tau_{\geq 0}R)$ by (8.13.6).

There is a moduli stack of strict elliptic curves.

Theorem 8.20.5 (Lurie). *There exists a spectral DM stack $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{s}}$ such that*

$$\mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(\mathrm{Sp\acute{e}t}\,R, \mathcal{M}_{\mathrm{Ell}}^{\mathrm{s}}) \approx \mathrm{Ell}^{\mathrm{s}}(R)^{\simeq};$$

the right-hand side is the maximal ∞-groupoid inside $\mathrm{Ell}^{\mathrm{s}}(R)$. The underlying 0-truncated spectral DM stack of $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{s}}$ is equivalent to the classical moduli stack $\mathcal{M}_{\mathrm{Ell}}$.

This is proved in [166, 2], using the spectral version of the Artin Representability Theorem [170, 18.3]. That $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{s}}$ is a connective object (i.e., not nonconnective) is immediate from the fact that $\mathrm{Ell}^{\mathrm{s}}(R) \approx \mathrm{Ell}^{\mathrm{s}}(\tau_{\geq 0}R)$. That the underlying 0-truncated stack of $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{s}}$ is the classical one is a consequence of the fact that strict elliptic curves over ordinary rings are just classical elliptic curves.

Oriented elliptic curves

For any strict elliptic curve $C \to \mathrm{Spét}\, R$, we may consider the formal completion $C_e^\wedge$ along the identity section. It turns out that $C_e^\wedge$ is a 1-dimensional formal group over R [167, 7.1].

Thus, we define an **oriented** elliptic curve over R to be a pair (C, e) consisting of a strict elliptic curve $C \to \mathrm{Spét}\, R$ together with an orientation $e \in \mathrm{OrDat}(\widehat{C}_e)$ of its formal completion $\widehat{C}$ in the sense of (8.17). There is a corresponding ∞-category $\mathrm{Ell}^{\mathrm{or}}(R)$ of oriented elliptic curves: morphisms must preserve the orientation.

Theorem 8.20.6 (Lurie). *There exists a nonconnective spectral DM stack $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ such that*

$$\mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(\mathrm{Spét}\, R, \mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}) \approx \mathrm{Ell}^{\mathrm{or}}(R)^{\simeq}.$$

The map $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}} \to \mathcal{M}^{\mathrm{s}}_{\mathrm{Ell}}$ classifying the strict elliptic curve induces an equivalence of underlying classical DM stacks.

This is proved in [167, 7].

Remark 8.20.7. Taken together, we have maps of nonconnective spectral DM stacks

$$\mathcal{M}_{\mathrm{Ell}} \xrightarrow{i} \mathcal{M}^{\mathrm{s}}_{\mathrm{Ell}} \xleftarrow{p} \mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$$

in which is $\mathcal{M}^{\mathrm{s}}_{\mathrm{Ell}}$ is a spectral DM stack (i.e., is connective), and $\mathcal{M}_{\mathrm{Ell}}$ is a 0-truncated spectral DM stack (and in fact is a DM stack). The map i witnesses the fact that every classical elliptic curve is a strict elliptic curve, while the map p forgets about orientation. All of these objects have the same underlying DM stack (i.e., they have equivalent ∞-topoi and π_0 of their structure sheaves coincide); in the case of $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ this is a non-trivial observation.

Remark 8.20.8. Note that if $\mathrm{Spét}\, A \to \mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ is any map of nonconnective spectral DM stacks, then the theorem produces an oriented elliptic curve over A, and hence an oriented formal group over A. Thus (8.17.9) implies that the $\mathbb{E}_\infty$-ring A must be weakly 2-periodic and complex orientable.

In fact, the proof of the theorem shows a little more in the case that $\mathrm{Spét}\, A \to \mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ is étale. In this case, A is not merely weakly 2-periodic; it also has the property that $\pi_{\mathrm{odd}}(A) \approx 0$.

As the underlying classical DM stack of $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ is $\mathcal{M}_{\mathrm{Ell}}$, we have that the full subcategory $\mathcal{U}' \subseteq \mathrm{SpDM}^{\mathrm{nc}}_{/\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}}$ spanned by étale morphisms $\mathrm{Spét}\, A \to \mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$ is equivalent to the étale site of the classical stack $\mathcal{M}_{\mathrm{Ell}}$, which we called $\mathcal{U}$ in (8.19.2). Putting all this together, we see that we have functors

$$\mathcal{U}^{\mathrm{op}} \xleftarrow{\sim} \mathcal{U}'^{\mathrm{op}} \to \mathrm{CAlg}$$

given by

$$(\mathrm{Spét}\, \pi_0 A \to \mathcal{M}_{\mathrm{Ell}}) \leftarrow\!\shortmid (\mathrm{Spét}\, A \to \mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}) \mapsto A.$$

We see that the resulting functor $\mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}$ is precisely of the sort demanded by (8.19.3).