6 Commutative ring spectra

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6.1 Introduction

Since the 1990s we have had several symmetric monoidal categories of spectra at our disposal whose homotopy category is the stable homotopy category. The monoidal structure is usually denoted by $\wedge$ and is called the smash product of spectra. So since then we can talk about commutative monoids in any of these categories — these are commutative ring spectra. Even before such symmetric monoidal categories were constructed, the consequences of their existence were described. In [296, §2] Friedhelm Waldhausen outlines the role of "rings up to homotopy". He also coined the expression "brave new rings" in a 1988 talk at Northwestern University.

So what is the problem? Why don't we just write down nice commutative models of our favorite homotopy types and be done with it? Why does it make sense to have a whole chapter about this topic?

In algebra, if someone tells you to check whether a given ring is commutative, you can sit down and check the axiom for commutativity and you should be fine. In stable homotopy theory the problem is more involved, since strict commutativity may only be satisfied by some preferred point set level model of the underlying associative ring spectrum and the operadic incarnation of commutativity is an extra structure rather than a condition.

There is one class of commutative ring spectra that is easy to construct. If you take singular cohomology with coefficients in a commutative ring R, then this is represented by the Eilenberg–Mac Lane spectrum HR and this can be represented by a commutative ring spectrum.

So it would be nice if we could have explicit models for other homotopy types that come naturally equipped with a commutative ring structure. Sometimes this is possible. If you are interested in real (or complex) vector bundles over your space, then you want to understand real (or complex) topological K-theory, and Michael Joachim [136, 137] for instance has produced explicit analytically flavored models for periodic real and complex topological K-theory with commutative ring structures.

There are a few general constructions that produce commutative ring spectra for you. For instance, the construction of Thom spectra often gives rise to commutative ring spectra. We will discuss this important class of examples in Section 6.4. A classical construction due to Graeme Segal also produces small nice models of commutative ring spectra (see Section 6.5).

Quite often, however, the spectra that we like are constructed in a synthetic way: You have some commutative ring spectrum R and you kill a regular sequence of elements in its graded commutative ring of homotopy groups, $(x_1, x_2, \dots)$, $x_i \in \pi_*(R)$, and you consider a spectrum E with homotopy groups $\pi_*(E) \cong \pi_*(R)/(x_1, x_2, \dots)$. Then it is not clear that E is a commutative ring spectrum.

A notorious example is the Brown–Peterson spectrum, BP. Take the complex cobordism spectrum MU. Its homotopy groups are

$$\pi_*(MU) = \mathbb{Z}[x_1, x_2, \dots],$$

where each x_i is a generator in degree $2i$. If you fix a large even degree, then you have a lot of possible elements in that degree, so you might wish to consider a spectrum with sparser homotopy groups. Using the theory of (commutative, 1-dimensional) formal group laws you can do that: If you consider a prime p, then there is a spectrum, called the Brown–Peterson spectrum, that corresponds to p-typical formal group laws. It can be realized as the image of an idempotent on MU and satisfies

$$\pi_*(BP) \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots],$$

but now the algebraic generators are spread out in an exponential manner: The degree of v_i is $2p^i - 2$. You can actually choose the v_i as the x_{p^i-1}, so you can think of BP as a quotient of MU in the above sense. Since its birth in 1966 [60] its multiplicative properties have been an important issue. In [29], for instance, it was shown that BP has some partial coherence for homotopy commutativity, but in 2017 Tyler Lawson [152] finally showed that at the prime 2 BP is ***not*** a commutative ring spectrum! For the non-existence of E_∞-structures on BP at odd primes see [271].

There are even worse examples: If you take the sphere spectrum S and you try to kill the non-regular element $2 \in \pi_0(S)$ then you get the mod-2 Moore spectrum. That isn't even a ring spectrum up to homotopy. You can also kill all the generators $v_i \in \pi_*(BP)$ including $p = v_0$, leaving only one v_n alive. The resulting spectrum is the connective version of Morava K-theory, $k(n)$. At the prime 2 this isn't even homotopy commutative. In fact, Pazhitnov, Rudyak and Würgler show more [219, 303]: If π_0 of a homotopy commutative ring spectrum has characteristic two, then it is a generalized Eilenberg–Mac Lane spectrum. Recent work of Mathew, Naumann and Noel puts severe restrictions on finite E_∞-ring spectra [188].

Quite often, we end up working with ideals in the graded commutative ring of homotopy groups, but as we saw above, this is not a suitable notion of ideal. There is a notion of an ideal in the context of (commutative) ring spectra [131] due to Jeff Smith, but still several algebraic constructions do not have an analog in spectra.

So how can you determine whether a given spectrum is a commutative ring spectrum if you don't have a construction that tells you right away that it is commutative? This is where obstruction theory comes into the story.

There is an operadic notion of an E_∞-ring spectrum that goes back to Boardman–Vogt and May. Comparison theorems [178, 262] then tell you whether these more complicated objects are equivalent to commutative ring spectra. In the categories of symmetric spectra, orthogonal spectra and S-modules they are.

Obstruction theory might help you with a decision whether a spectrum carries a commutative monoid structure: One version [27] gives obstructions for lifting the ordinary Postnikov tower to a Postnikov tower that lives within the category of commutative ring spectra. The other kind finds some obstruction classes that tell you that you cannot extend some partial bits and pieces of a nice multiplication to a fully fledged structure of an E_∞-ring spectrum or that some homology or homotopy operation that you observe contradicts such a structure. This can be used for a negative result (as in [152]) or for positive statements: There are results by Robinson [246] and Goerss–Hopkins [106, 107] that tell you that you have a (sometimes even unique) E_∞-ring structure on your spectrum if all the obstruction ***groups*** vanish. Most notably Goerss and Hopkins used obstruction theory to prove that the Morava stabilizer groups acts on the corresponding Lubin–Tate spectrum via E_∞-morphisms [107].

The algebraic behavior on the level of homotopy groups can be quite deceiving: complexification turns a real vector bundle into a complex vector bundle. This induces a map $\pi_*(KO) \to \pi_*(KU)$ which can be realized as a map of commutative ring spectra $c\colon KO \to KU$. On homotopy groups we get

$$\pi_*(c)\colon \pi_*(KO) = \mathbb{Z}[\eta, y, \omega^{\pm 1}]/(2\eta, \eta^3, \eta y, y^2 - 4w) \to \mathbb{Z}[u^{\pm 1}] = \pi_*(KU). \tag{6.1.1}$$

Here the degrees are $|\eta| = 1$, $|y| = 4$, $|w| = 8$, $|u| = 2$ and y is sent to $2u^2$. So on the algebraic level c is horrible. But John Rognes showed that the conjugation action on KU turns the map $c\colon KO \to KU$ into a C_2-Galois extension of commutative ring spectra!

Even for ordinary rings, viewing a (commutative) ring R as a (commutative) ring spectrum via the Eilenberg–Mac Lane spectrum functor changes the situation completely. The ring R has a characteristic map $\chi\colon \mathbb{Z} \to R$ because the ring of integers is the initial ring. As a ring spectrum, $H\mathbb{Z}$ is far from being initial. The map $H\chi$ can be precomposed with the unit map of $H\mathbb{Z}$:

$$S \xrightarrow{\eta} H\mathbb{Z} \xrightarrow{H\chi} HR,$$

and the sphere spectrum S is the initial ring spectrum! Now there is a lot of space between the sphere and any ring. I will discuss two consequences that this has: There is actually algebraic geometry happening between the sphere spectrum and the prime field $\mathbb{F}_p$: There is a Galois extension of commutative ring spectra (see 6.8.1) $A \to H\mathbb{F}_p$!

Another feature is that there exist differential graded algebras A_* and B_* that are not quasi-isomorphic, but whose associated algebra spectra over an Eilenberg–Mac Lane spectrum [275] are equivalent as ring spectra [80]. Similar phenomena happen if you consider differential graded E_∞-algebras: there are non quasi-isomorphic ones whose associated commutative algebras over an Eilenberg–Mac Lane spectrum [236] are equivalent as commutative ring spectra [33].

Content

The structure of this overview is as follows: We start with some basic features of commutative ring spectra and their model category structures in Section 6.2. The most basic way to relate classical algebra to brave new algebra is via the Eilenberg–Mac Lane spectrum functor. We study chain algebras and algebras over Eilenberg–Mac Lane ring spectra in Section 6.3. As you can study the group of units of a ring we consider units of ring spectra and Thom spectra in Section 6.4. In Section 6.5 we present a construction going back to Segal. Plugging in a bipermutative category yields a commutative ring spectrum.

In Section 6.6 we introduce topological Hochschild homology and some of its variants and topological André–Quillen homology. In Section 6.7 we discuss some versions of obstruction theory that tell you whether a given multiplicative cohomology theory can be represented by a strict commutative model.

Some concepts from algebra translate directly to spectra but some others don't. We discuss the different concepts of étale maps for commutative algebra spectra in Section 6.8. Picard and Brauer groups for commutative ring spectra are important invariants and feature in Section 6.9.

Disclaimers

For more than 30 years, the phrase *commutative ring spectrum* meant a commutative monoid in the homotopy category of spectra. Since the 1990s this has changed. At the beginning of this new era people were careful not to use this name, in order to avoid confusion with the homotopy version. In this paper we reserve the phrase *commutative ring spectrum* for a commutative monoid in some symmetric monoidal category of spectra.

The second disclaimer is that for this paper a space is always compactly generated weak Hausdorff. I denote the corresponding category just by Top.

Last but not least: Of course, this overview is not complete. I had to omit important aspects of the field due to space constraints. Most prominently probably is the omission of topological cyclic homology and its wonderful applications to algebraic K-theory.

I try to give adequate references, but often it was just not feasible to describe the whole development of a topic and much worse, I probably have forgotten to cite important contributions. If you read this and it affects you, then I can only apologize.

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6.2 Features of commutative ring spectra

Some basics

Before we actually start with model structures, we state some basic facts about commutative ring spectra.

Let R be a commutative ring spectrum. Then the category of R-module spectra is closed symmetric monoidal: For two such R-module spectra M, N the smash product over R, $M \wedge_R N$, is again an R-module and the usual axioms of a symmetric monoidal category are satisfied. There is an R-module spectrum $F_R(M, N)$, the function spectrum of R-module maps from M to N.

We denote the category of R-module spectra by $\mathcal{M}_R$. The category of commutative R-algebras is the category of commutative monoids in $\mathcal{M}_R$ and we denote it by $\mathcal{C}_R$.

By definition, every object A of $\mathcal{C}_R$ receives a unit map from R and hence R is initial in $\mathcal{C}_R$. In particular, the sphere spectrum is the initial commutative ring spectrum. Every discrete ring is a $\mathbb{Z}$-algebra; similarly, every (commutative) ring spectrum is a (commutative) S-algebra. If R is a commutative ring spectrum, then the category of commutative R-algebras is isomorphic to the category of commutative ring spectra under R, i.e., the category of commutative ring spectra A with a distinguished map $\eta\colon R \to A$ in that category.

We allow the trivial R-algebra corresponding to the one-point spectrum $*$ and this spectrum is a terminal object in $\mathcal{C}_R$.

In any symmetric monoidal category $(\mathcal{C}, \otimes, 1, \tau)$ the coproduct of two commutative monoids A and B in $\mathcal{C}$ is $A \otimes B$. So, for two commutative R-algebras A and B, their coproduct is $A \wedge_R B$.

Model structures on commutative monoids

I will assume that you are familiar with the concept of model categories and that you have seen some examples and read Chapter 3 in this book. Good general references are Hovey's [130] and Hirschhorn's [124] books. You could also just skip this section and have in mind that there are some serious model category issues lurking in the dark.

For this section I will mainly focus on two models for spectra: symmetric spectra [133] and S-modules [94]. They are different concerning their model structures. In the model structure in [133] on symmetric spectra the sphere spectrum is cofibrant, whereas in the one for S-modules it is not, but all objects are fibrant.

The model structures on commutative monoids in either of the categories [94, 133] are special cases of a ***right induced model structure***: We have a functor P_R from R-module spectra to commutative R-algebra spectra assigning the free commutative R-algebra spectrum on M to any R-module spectrum M: explicitly,

$$P_R(M) = \bigvee_{n \geqslant 0} M^{\wedge_R n}/\Sigma_n.$$

The symbol P_R should remind you of a polynomial algebra. This functor has a right adjoint, the forgetful functor U. In a right-induced model structure one determines the fibrations and weak equivalences by the right adjoint functor. In our cases, a map of commutative R-algebra spectra is a fibration or a weak equivalence if it is one in the underlying category of R-module spectra. Note that establishing right induced model structures on commutative monoids in some model category does not always work. The standard example is the category of $\mathbb{F}_p$-chain complexes (say p is an odd prime). Then the chain complex $\mathbb{D}^2$ is acyclic, having $\mathbb{F}_p$ in degrees 1 and 2 with the identity map as differential, but the free graded commutative monoid generated by it is $\Lambda_{\mathbb{F}_p}(x_1) \otimes \mathbb{F}_p[x_2]$ with $|x_i| = i$ and the induced differential is determined by $d(x_2) = x_1$ and the Leibniz rule. But then $d(x_2^p)$ is a cycle that is not a boundary, so the resulting object is not acyclic.

If R is a commutative S-algebra in the setting of EKMM [94], then the categories of associative R-algebras and of commutative R-algebras possess a right induced model structure [94, Corollary VII.4.10]. The existence of the model structure for commutative monoids is a special case of the existence of right-induced model structures for $\mathbb{T}$-algebras [94, Theorem VII.4.9], where $\mathbb{T}$ is a continuous monad on the category of R-module spectra that preserves reflexive coequalizers and satisfies the cofibration hypothesis [94, VII.4]. The category of commutative S-algebras is identified [94, Proposition II.4.5] with the category of algebras for the monad P_S as above on the category of S-modules.

In diagram categories such as symmetric spectra and orthogonal spectra the situation is different: In the standard model structures on these categories the sphere spectrum is cofibrant. If one were to take a right-induced model structure on the category of commutative monoids, i.e., the model structure such that a map of commutative ring spectra $f: A \to B$ is a fibration or weak equivalence if it is one in the underlying category, then the sphere would still be cofibrant. If we take a fibrant replacement of the sphere $S \to S^{\mathrm{fib}}$, then in particular S^{fib} would be fibrant in the model category of symmetric spectra; hence it would be an Ω-spectrum and its zeroth level would be a strictly commutative model for QS^0. However, Moore shows [212, Theorem 3.29] that this would imply that QS^0 has the homotopy type of a product of Eilenberg–Mac Lane spaces — but this is false.

The usual way to avoid this problem is to consider a positive model structure on Sp^Σ (see [178, Definition 6.1] for the general approach). Here the positive level fibrations (weak equivalences) are maps $f \in \mathsf{Sp}^\Sigma(X, Y)$ such that $f(n)$ is a fibration (weak equivalence) for all levels $n \geqslant 1$. The positive cofibrations are then cofibrations in Sp^Σ that are isomorphisms in level zero. The positive stable model category is then obtained by a Bousfield localization that forces the stable equivalences to be the weak equivalences and the right-induced model structure on the commutative monoids in Sp^Σ then has the desired properties.

There is another nice model for connective spectra, given by Γ-spaces [268, 172]. This category is built out of functors from finite pointed sets to spaces, so it is a very hands-on category with explicit constructions. It is also a symmetric monoidal category with a suitable model structure. We refer to [172, 264] for background on this. Its (commutative) monoids are called (commutative) Γ-rings. Beware that commutative Γ-rings, however, do *not* model all connective commutative ring spectra. Tyler Lawson proves in [151] that commutative Γ-rings satisfy a vanishing condition for Dyer–Lashof operations of positive degree on classes in their zeroth mod-p-homology (for all primes p) and that for instance the free E_∞-ring spectrum generated by $\mathbb{S}^0$ cannot be modeled as a commutative Γ-ring.

Behavior of the underlying modules

In the setting of EKMM it is shown that the underlying R-modules of cofibrant commutative R-algebras have a well-behaved smash product in the derived category of R-modules:

Theorem 6.2.1 [94, Theorem VII.6.7]. *If A and B are two cofibrant commutative R-algebras, and if $\varphi_A\colon \Gamma A \xrightarrow{\sim} A$ and $\varphi_B\colon \Gamma B \xrightarrow{\sim} B$ are chosen cell R-module spectra approximations then*

$$\varphi_A \wedge_R \varphi_B\colon \Gamma A \wedge_R \Gamma B \to A \wedge_R B$$

is a weak equivalence.

Brooke Shipley developed a model structure for commutative symmetric ring spectra in [274] in which the underlying symmetric spectrum of a cofibrant commutative ring spectrum is also cofibrant as a symmetric spectrum [274, Corollary 4.3].

She starts with introducing a different model structure on symmetric spectra. Let M denote the class of monomorphisms of symmetric sequences in pointed simplicial sets and let $S \otimes M$ denote the set $\{S \otimes f, f \in M\}$, where $\otimes$ denotes the tensor product of symmetric sequences. An S*-cofibration* is a morphism in $(S \otimes M)$-cof, i.e., a morphism in Sp^Σ that has the left lifting property with respect to maps that have the right lifting property with respect to $S \otimes M$. She shows that the classes of S-cofibrations and stable equivalences determine a model structure with the S-fibrations being the class of morphisms with the right lifting property with respect to S-cofibrations that are also stable equivalences [274, Theorem 2.4]. This model structure was already mentioned in [133, 5.3.6]. Shipley proves that this model structure is cofibrantly generated, is monoidal and satisfies the monoid axiom [274, 2.4, 2.5].

Note that symmetric spectra are S-modules in symmetric sequences. This allows for a version of an R*-model structure* for every associative symmetric ring spectrum R with R-cofibrations, R-fibrations and stable equivalences [274, Theorem 2.6]. In the positive variant of this model structure the positive R-cofibrations are R-cofibrations that are isomorphisms in level zero. Together with the stable equivalences this determines the *positive R-model structure.*

The corresponding right induced model structure on commutative R-algebra spectra for a commutative symmetric ring spectrum R is then the *convenient* model structure: The weak equivalences are stable equivalences, the fibrations are positive R-fibrations and the cofibrations are determined by the structure.

She then shows a remarkable property of this model structure on commutative R-algebra spectra:

Theorem 6.2.2 [274, Corollary 4.3]. *If A is cofibrant as a commutative R-algebra then A is R-cofibrant in the R-model structure. If A is fibrant as a commutative R-algebra, then A is fibrant in the positive R-model structure on R-module spectra.*

The positive R-model structure ensures that R is ***not*** cofibrant; hence cofibrant commutative R-algebras will not be positively R-cofibrant!

Comparison, rigidification and E_n-structures

Stefan Schwede proves [262, Theorem 5.1] that the homotopy category of commutative S-algebras from [94] is equivalent to the homotopy category of commutative symmetric ring spectra by establishing a Quillen equivalence between the corresponding model categories. In [178, Theorem 0.7] the analogous comparison result is proven for commutative orthogonal ring spectra and commutative symmetric ring spectra.

Even before any symmetric monoidal category of spectra was constructed, the notion of operadically defined E_∞-ring spectra [199] was available. An E_∞-structure on a spectrum is a multiplication that is homotopy commutative in a coherent way. See Chapter 5 of this book for background on operads and their role in the study of spectra with additional structure.

There is an explicit comparison of the good old E_∞-ring spectra and commutative ring spectra, see [94, Proposition II.4.5] or [178, Remark 0.14]; in particular, every E_∞-ring spectrum $\tilde{R}$ can be rigidified to a commutative ring spectrum R in such a way that the homotopy type is preserved.

There are several popular E_∞-operads that will show up later: for instance the linear isometries operad (see (6.4.3)) and the Barratt–Eccles operad. The n-ary part of the latter is easy to describe: You take $\mathcal{O}(n) = E\Sigma_n$, a contractible space with free Σ_n-action. For compatibility reasons it is advisable to take the realization of the standard simplicial model of $E\Sigma_n$ whose set of q-simplices is $(\Sigma_n)^{q+1}$.

An operad with a nice geometric description is the little m-cubes operad, that in arity n consists of the space of n-tuples of linearly embedded m-cubes in the standard m-cube with disjoint interiors and with axes parallel to that of the ambient cube [49, Example 5]. We call this (and every equivalent) operad in spaces E_m. For $m = 1$ this operad parametrizes A_∞-structures and the colimit is an E_∞-operad. Hence the intermediate E_m's for $1 < m < \infty$ interpolate between these structures; they give A_∞-structures with homotopy-commutative multiplications that are coherent up to some order.

Power operations

The extra structure of an E_∞-ring spectrum gives homology operations. The general setting allows for H_∞-ring spectra [63]; for simplicity we assume that E and R are two E_∞-ring spectra whose structure is given by the Barratt–Eccles operad, i.e., there are structure maps

$$\xi_R^n\colon (E\Sigma_n)_+ \wedge_{\Sigma_n} R^{\wedge n} \to R \tag{6.2.1}$$

for R and also for E. McClure describes the general setting of power operations in [63, IX §1]. Fix a prime p and abbreviate $(E\Sigma_p)_+ \wedge_{\Sigma_p} R^{\wedge p}$ by $D_p(R)$; one often calls D_pR the ***p-th extended power construction on** R*. A ***power operation*** assigns to every class $[x] \in E_nR$ and every class $e \in E_m(D_pS^n)$ a class $Q^e[x] \in E_mR$; hence we can view Q^e as a map

$$Q^e\colon E_nR \to E_mR.$$

The construction is as follows. Take a representative $x\colon S^n \to E \wedge R$ of $[x]$ and $e \in E_m(D_pS^n)$ and apply the following composition to e:

$$\begin{tikzcd}
E_m(D_pS^n) \arrow[r, "E_m(D_px)"] \arrow[rrdddd, dotted] & E_m(D_p(E \wedge R)) \arrow[r, "\delta"] & E_m(D_pE \wedge D_pR) \arrow[d, "E_m(\xi_E^p \wedge \mathrm{id})"] \\
& & E_m(E \wedge D_pR) \arrow[d, "\mu_*"] \\
& & E_m(D_pR) \arrow[d, "E_m(\xi_R^p)"] \\
& & E_m(R).
\end{tikzcd} \tag{6.2.2}$$

Here,

$$\delta\colon (E\Sigma_p)_+ \wedge_{\Sigma_p} (E \wedge R)^{\wedge p} \to (E\Sigma_p)_+ \wedge_{\Sigma_p} E^{\wedge p} \wedge (E\Sigma_p)_+ \wedge_{\Sigma_p} R^{\wedge p}$$

is the canonical map induced by the diagonal on the space $E\Sigma_p$ and μ denotes the multiplication in E, so it induces

$$\mu_*\colon \pi_*(E \wedge E \wedge D_pR) \to \pi_*(E \wedge D_pR).$$

There are several important special cases of this construction:

1. For E the sphere spectrum one obtains operations on the homotopy groups of an E_∞-ring spectrum; see [63, IV §7].
2. For $E = H\mathbb{F}_p$ the power operations for certain classes $e_i \in H_i(\Sigma_p; \mathbb{F}_p)$ are often called (Araki–Kudo–)Dyer–Lashof operations. These are natural homomorphisms

$$Q^i\colon (H\mathbb{F}_p)_n(R) \to (H\mathbb{F}_p)_{n+2i(p-1)}(R) \tag{6.2.3}$$

   for odd primes and $Q^i\colon (H\mathbb{F}_2)_n(R) \to (H\mathbb{F}_2)_{n+i}(R)$ at the prime 2 that satisfy a list of axioms [63, Theorem III.1.1] and compatibility relations with the homology Bockstein and the dual Steenrod operations.
3. There are also important $K(n)$-local versions of such operations and we will encounter them later.

6.3 Chain algebras and algebras over Eilenberg–Mac Lane spectra

The derived category of a ring is an important object in many subjects. The initial ring is the ring of integers. Every ring R has an associated Eilenberg–Mac Lane spectrum, HR.

HR-module and algebra spectra

We collect some results that compare the category of chain complexes of R-modules with the category of module spectra over HR. We start with additive statements and move to comparison results for flavors of differential graded R-algebras. For an overview of algebraic applications of these equivalences see for instance [111].

In the 1980s, so before any strict symmetric monoidal category of spectra was constructed, Alan Robinson developed the notion of the derived category, $\mathcal{D}(E)$, of right E-module spectra for every A_∞-ring spectrum E. He showed the following result.

Theorem 6.3.1 [249, Theorem 3.1]. *For every associative ring R there is an equivalence of categories between the derived category of R, $\mathcal{D}(R)$, and the derived category of the associated Eilenberg–Mac Lane spectrum, $\mathcal{D}(HR)$.*

Later, in the context of S-modules this corresponds to [94, IV, Theorem 2.4]. Work of Schwede and Shipley strengthened the result to a Quillen equivalence of the corresponding model categories:

Theorem 6.3.2 [266, Theorem 5.1.6]. *The model category of unbounded chain complexes of R-modules is Quillen equivalent to the model category of HR-module spectra .*

Stefan Schwede uses the setting of Γ-spaces [264] to embed simplicial rings and modules into the stable world: He constructs a lax symmetric monoidal Eilenberg–Mac Lane functor H from simplicial abelian groups to Γ-spaces together with a linearization functor L in the opposite direction and proves the following comparison result:

Theorem 6.3.3 [264, Theorems 4.4 and 4.5]. *If R is a simplicial ring, then the adjoint functors H and L constitute a Quillen equivalence between the categories of simplicial R-modules and HR-module spectra. If R is in addition commutative, then H and L induce a Quillen equivalence between the categories of simplicial R-algebras and HR-algebra spectra.*

Here, the functor L is left inverse to H and induces an isomorphism of Γ-spaces

$$\mathsf{Hom}(HA, HB) \cong H(\mathsf{Hom}_{\mathsf{sAb}}(A, B))$$

[264, Lemma 2.1]; thus H embeds algebra into brave new algebra.

Brooke Shipley extends this equivalence to corresponding categories of monoids in the differential graded setting:

Theorem 6.3.4 [275, Theorem 1.1]. *For any commutative ring R, the model categories of unbounded differential graded R-algebras and HR-algebra spectra are Quillen equivalent.*

Dugger and Shipley show in [80] that there are examples of HR-algebras that are weakly equivalent as S-algebras, but that are not quasi-isomorphic. A concrete example is the differential graded ring A_* which is generated by an element in degree 1, e_1, and has $d(e_1) = 2$ and satisfies $e_1^4 = 0$. The corresponding $H\mathbb{Z}$-algebra spectrum is equivalent as a ring spectrum to the one on the exterior algebra $B_* = \Lambda_{\mathbb{F}_2}(x_2)$ (with $|x_2| = 2$) but A_* and B_* are *not* quasi-isomorphic. You find more examples and proofs in [80, §§4,5].

We cannot expect that commutative HR-algebra spectra correspond to commutative differential graded R-algebras unless R is of characteristic zero, because of cohomology operations, but we get the following result:

Theorem 6.3.5 [236, Corollary 8.3]. *If R is a commutative ring, then there is a chain of Quillen equivalences between the model category of commutative HR-algebra spectra and E_∞-monoids in the category of unbounded R-chain complexes.*

Haldun Özgür Bayındır shows [33] that one can find E_∞-differential graded algebras that are not quasi-isomorphic, but whose corresponding commutative HR-algebra spectra are equivalent as commutative ring spectra.

Cochain algebras

A prominent class of examples of commutative HR-algebra spectra consists of function spectra $F(X_+, HR)$. Here, X is an arbitrary space and R is a commutative ring. The diagonal $\Delta\colon X \to X \times X$ and the multiplication on HR, μ_{HR}, induce a multiplication

$$\begin{tikzcd} F(X_+, HR) \wedge F(X_+, HR) \arrow[r] & F(X_+ \wedge X_+, HR \wedge HR) \cong F((X \times X)_+, HR \wedge HR) \arrow[d, "\Delta^*{,}\mu_{HR}"] \\ & F(X_+, HR) \end{tikzcd}$$

that turns $F(X_+, HR)$ into a HR-algebra spectrum. As the diagonal is cocommutative and as μ_{HR} is commutative, the resulting multiplication is commutative.

These function spectra are models for the singular cochains of a space X with coefficients in R:

$$\pi_*(F(X_+, HR)) \cong H^{-*}(X; R).$$

Beware that the homotopy groups of $F(X_+, HR)$ are concentrated in non-positive degrees — i.e., $F(X_+, HR)$ is coconnective.

Studying the singular cochains of a space $S^*(X; R)$ as a differential graded R-module is not enough in order to recover the homotopy type of X. If we work over the rational numbers, then Quillen showed that rational homotopy theory is algebraic in the sense that one can use rational differential graded Lie algebras or coalgebras as models for rational homotopy theory [228]. Sullivan [286] constructed a functor, assigning a rational differential graded commutative algebra to a space, that is closely related to the singular cochain functor with rational coefficients. He used this to classify rational homotopy types.

For a general commutative ring R, the singular cochains are an E_∞-algebra. Mike Mandell proves [181, Main Theorem] that the singular cochain functor with coefficients in an algebraic closure of $\mathbb{F}_p$, $\overline{\mathbb{F}}_p$, induces an equivalence between the homotopy category of connected p-complete nilpotent spaces of finite p-type and a full subcategory of the homotopy category of E_∞-$\overline{\mathbb{F}}_p$-algebras. He also characterizes those E_∞-$\overline{\mathbb{F}}_p$-algebras that arise as cochain algebras of 1-connected p-complete spaces of finite p-type explicitly [181, Characterization Theorem]. There is also an integral version of this result, stating that finite type nilpotent spaces are weakly equivalent if and only if their E_∞-algebras of integral cochains are quasi-isomorphic [180, Main Theorem].

A strictly commutative integral model of the E_∞-algebra of cochains on a space is constructed in [235] using chain complexes indexed by the category of finite sets and injections.

6.4 Units of ring spectra and Thom spectra

One construction that can give rise to highly structured multiplications on a spectrum is the Thom spectrum construction: For instance, complex bordism, MU, obtains a commutative ring structure this way. Mahowald emphasized [175] early on that multiplicative properties of the structure maps for Thom spectra translate to multiplicative structures on the resulting Thom spectra. Their properties and the corresponding orientation theory is systematically studied in [199, 196]. There is the following general result by Lewis:

Theorem 6.4.1 [155, Theorem IX.7.1 and Remark IX.7.2]. *Assume that f is a map of spaces from X to the classifying space for stable spherical fibrations, BG, that is a C-map for some operad C over the linear isometries operad. Then the Thom spectrum $M(f)$ associated to f carries a C-structure. In particular, infinite loop maps from X to BG give rise to E_∞-ring spectra.*

Note that BG is the classifying space of the units of the sphere spectrum, $GL_1(S)$. A seemingly naive definition of $GL_1(S)$ is given by the pullback of the diagram

$$\begin{tikzcd} GL_1 S \arrow[r, dotted] \arrow[d, dotted] & \Omega^\infty S \arrow[d] \\ \pi_0(S)^\times = \{\pm 1\} \arrow[r] & \pi_0(S) = \mathbb{Z} \end{tikzcd} \tag{6.4.1}$$

so by the components of QS^0 corresponding to $\pm 1 \in \mathbb{Z}$.

We next give a short overview of Thom spectra that arise in a more general context, where the target of the map is the space of units, $GL_1(R)$, for a commutative ring spectrum R. The first idea is to define the space $GL_1(R)$ as the space that represents the functor that sends a space X to the units in $R^0(X)$. Copying the definition from (6.4.1) above with S replaced by R gives a valid definition of $GL_1(R)$ and it was shown in [199] that for commutative R this model is an E_∞-space.

In the approaches [6] and [31], the idea is to replace the above model of $GL_1(R)$ with its E_∞-structure with a strictly commutative model. As spaces with an E_∞-structure are not equivalent to strictly commutative spaces (that's the problem again that then QS^0 would be a product of Eilenberg–Mac Lane spaces [212]), one has to find a different category with the property that there is a Quillen equivalence between commutative monoids in that category and E_∞-monoids in spaces and such that there are models of $\Omega^\infty(R)$ and $GL_1(R)$ in this category.

In [6] the authors work with $*$-modules and in [31] the authors use Schlichtkrull's model of $GL_1(R)$ in commutative I-spaces, where I is the skeleton of the category of finite sets and injections.

The idea is to construct a spectrum version of the assembly map for discrete rings: If R is a discrete ring and if $R^\times$ is its group of units, then there is a canonical map

$$\mathbb{Z}[R^\times] \to R \tag{6.4.2}$$

from the group ring $\mathbb{Z}[R^\times]$ to R that takes an element $\sum_{i=1}^n a_i r_i$ of $\mathbb{Z}[R^\times]$ (with $a_i \in \mathbb{Z}$ and $r_i \in R^\times$) to the same sum, but now we use the ring structure of R to convert the formal sum into an actual sum $\sum_{i=1}^n a_i r_i \in R$. Note that $R^\times$ is an abelian group if R is a commutative ring.

We will sketch both constructions of Thom spectra and briefly discuss the application in [6] to the question of when a Thom spectrum allows for an E_∞-map to some other E_∞-ring spectrum: for instance, whether one can realize an E_∞-version of the string orientation $MO\langle 8\rangle \to \mathsf{tmf}$ [7] or an E_∞-version of a complex orientation [127].

The focus in [31] is on multiplicative properties of the Thom spectrum functor and on applications to topological Hochschild homology. We present the results about multiplicative structures and discuss their results on THH of Thom spectra in Section 6.6. We'll also describe how the concept of I-spaces can be generalized to a setting in which the units can be adapted to non-connective ring spectra.

Thom spectra via $\mathbb{L}$-spaces and orientations

Fix a countably infinite-dimensional real vector space U and consider

$$\mathbb{L} = \mathcal{L}(1) = \mathcal{L}(U, U),$$

the space of linear isometries from U to itself. The notation $\mathcal{L}(1)$ is due to the fact that $\mathcal{L}(1)$ is the 1-ary part of the famous linear isometries operad [49, §1] whose term of arity n is

$$\mathcal{L}(n) = \mathcal{L}(U^n, U). \tag{6.4.3}$$

See [49] or [6] for details. Note that $\mathbb{L}$ is a monoid with respect to composition.

Definition 6.4.2. The *category of* $\mathbb{L}$*-spaces,* $\mathsf{Top}[\mathbb{L}]$, is the category of spaces with a left action of the monoid $\mathbb{L}$.

Using the 2-ary part of the linear isometries operad, one can manufacture a product on $\mathsf{Top}[\mathbb{L}]$: For objects X, Y of $\mathsf{Top}[\mathbb{L}]$ their product $X \times_{\mathbb{L}} Y$ is the coequalizer

$$\mathcal{L}(2) \times (\mathcal{L}(1) \times \mathcal{L}(1)) \times X \times Y \rightrightarrows \mathcal{L}(2) \times X \times Y \dashrightarrow X \times_{\mathbb{L}} Y.$$

Here, one map uses the $\mathcal{L}(1)$-action on the spaces X and Y and the other map uses the operad product $\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1) \to \mathcal{L}(2)$.

As $\mathcal{L}(2) = \mathcal{L}(U^2, U)$ has a left $\mathcal{L}(1)$-action, $X \times_{\mathbb{L}} Y$ is an $\mathcal{L}(1)$-space. The product is associative and has a symmetry, but it is only weakly unital. See [45, §4] for a careful discussion.

By [45, Proposition 4.7] there is an isomorphism of categories between commutative monoids with respect to $\times_{\mathbb{L}}$ and E_∞-spaces whose E_∞-structure is parametrized by the linear isometries operad.

For strict unitality, one restricts to the full subcategory $\mathcal{M}_*$ of objects of $\mathsf{Top}[\mathbb{L}]$ for which the unit map is a homeomorphism. Such objects are called $*$*-modules.* The commutative monoids in $\mathcal{M}_*$ again model E_∞-spaces [45, Proposition 4.11].

For an associative ring spectrum R, there is a strictly associative model in $\mathcal{M}_*$ of the space of units $GL_1(R)$ and the functor GL_1 is right adjoint to the inclusion of grouplike objects. One can form a bar construction, $B_{\times_{\mathbb{L}}}$, of a cofibrant replacement, $GL_1(R)^c$, of $GL_1(R)$ with respect to the monoidal product $\times_{\mathbb{L}}$, where $B_{\times_{\mathbb{L}}}(GL_1(R)^c)$ is the geometric realization of the simplicial $\mathcal{M}_*$ object

$$[n] \mapsto * \times_{\mathbb{L}} \underbrace{GL_1^c(R) \times_{\mathbb{L}} \ldots \times_{\mathbb{L}} GL_1^c(R)}_{n} \times_{\mathbb{L}} *.$$

Similarly, $E_{\times_{\mathbb{L}}}(GL_1(R)^c)$ is constructed out of the simplicial object

$$[n] \mapsto * \times_{\mathbb{L}} \underbrace{GL_1^c(R) \times_{\mathbb{L}} \ldots \times_{\mathbb{L}} GL_1^c(R)}_{n+1}.$$

Adapted to the situation there are suspension spectrum and underlying infinite loop space functors [159, Lemma 7.5]

$$\mathcal{M}_* \underset{\Omega_S^\infty}{\overset{(\Sigma_{\mathbb{L}}^\infty)_+}{\rightleftarrows}} \mathcal{M}_S \tag{6.4.4}$$

that are a Quillen adjoint pair of functors. Here, the suspension functor is strong symmetric monoidal and the underlying loop space functor is lax symmetric monoidal.

The spectrum version of the assembly map from (6.4.2) is

$$(\Sigma_{\mathbb{L}}^\infty)_+(GL_1^c(R)) \to (\Sigma_{\mathbb{L}}^\infty)_+(GL_1(R)) \to R,$$

where the first map comes from the cofibrant replacement of the units and the second one is the counit of an adjunction [6, (3.1)].

Definition 6.4.3 [6, Definition 3.12]. Given a map $f\colon X \to B_{\times_{\mathbb{L}}}(GL_1^c(R))$, the *Thom spectrum for f in $\mathcal{M}_*$* is the R-module spectrum (in the world of [94])

$$M(f) = (\Sigma^\infty_{\mathbb{L}})_+ P^c \wedge_{(\Sigma^\infty_{\mathbb{L}})_+ GL_1^c(R)} R. \tag{6.4.5}$$

Here, P^c is a cofibrant replacement as a right $GL_1^c(R)$-module of the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & E_{\times_{\mathbb{L}}}(GL_1^c(R)) \arrow[d] \\ X \arrow[r] & B_{\times_{\mathbb{L}}}(GL_1^c(R)) \end{tikzcd}$$

Remark 6.4.4. Because of the cofibrancy of P^c, the smash product in (6.4.5) is actually a derived smash product. See [6, §3] for the necessary background on the model structures involved.

In the commutative case, [6, §4, §5] is set in the classical framework of E_∞-ring spectra and E_∞-spaces as in [199]. For an E_∞-ring spectrum R, the space $\Omega^\infty R$ is actually an *E_∞-ring space* [197, Corollary 7.5]; this is a space on which a pair of E_∞-operads acts: one codifying the additive structure that is present in every spectrum and one encoding the multiplicative structure [197, §1]. Actually more is true. Call an E_∞-ring space ring-like if its π_0 is actually a ring and not just a *rig* — a ri**n**g without **n**egatives. The homotopy category of ring-like E_∞-ring spaces is equivalent to the homotopy category of connective E_∞-ring spectra [197, Theorem 9.12].

If R is a commutative ring spectrum or an E_∞-ring spectrum then the space of units, $GL_1(R)$, is a group-like E_∞-space and hence is an infinite loop space that has an associated connective spectrum, $gl_1(R)$, with $\Omega^\infty gl_1(R) = GL_1(R)$.

The crucial ingredient in this case is the pair of functors $(\Sigma^\infty_+\Omega^\infty, gl_1)$ that is an adjunction between the homotopy category of connective spectra and the homotopy category of E_∞-ring spectra in the sense of Lewis–May–Steinberger.

In particular, one gets a version of the assembly map from (6.4.2):

$$\Sigma^\infty_+\Omega^\infty(gl_1(R)) \to R$$

for every E_∞-ring spectrum. By [94] one can replace E_∞-ring spectra with commutative S-algebras, i.e., with commutative ring spectra. This simplifies the discussion of pushouts and allows us to replace $\Sigma^\infty_+\Omega^\infty$ by $(\Sigma^\infty_{\mathbb{L}})_+\Omega^\infty_S$ from (6.4.4) to get

$$(\Sigma^\infty_{\mathbb{L}})_+\Omega^\infty_S(gl_1(R)) \to R.$$

Note that a map of infinite loop spaces $f\colon B \to BGL_1(R)$ encodes the same data as a map of spectra $f\colon b \to bgl_1(R)$, where the lowercase letters denote the associated connective spectra. As before we consider the pullback p:

$$\begin{tikzcd} p \arrow[r] \arrow[d] & egl_1(R) \arrow[d] \\ b \arrow[r, "f"] & bgl_1(R) \end{tikzcd}$$

and form the corresponding derived smash product:

Definition 6.4.5. Let $f\colon b \to bgl_1(R)$ be a map of connective spectra. The ***Thom spectrum associated to*** f, $M(f)$, is the homotopy pushout in the category of commutative S-algebras

$$M(f) = (R \wedge (\Sigma^\infty_{\mathbb{L}})_+ \Omega^\infty_S p) \wedge^L_{R \wedge (\Sigma^\infty_{\mathbb{L}})_+ \Omega^\infty_S gl_1(R)} R.$$

As the (homotopy) pushout is the (derived) smash product, this resembles the construction from (6.4.5).

In the commutative ring spectrum setting the question about orientations is the following problem: Assume that there is a map of commutative ring spectra $\alpha\colon R \to A$, then A is a commutative R-algebra spectrum. For a map of spectra $f\colon b \to bgl_1(R)$ as above we can ask whether there is a morphism of commutative R-algebra spectra from $M(f)$ to A. As $M(f)$ is defined as a (homotopy) pushout, we get a condition that says that we need maps from the ingredients of the derived smash product. As we start with a map α from R to A, we get an induced map

$$gl_1(\alpha)\colon gl_1(R) \to gl_1(A).$$

So what is missing is a map

$$(\Sigma^\infty_{\mathbb{L}})_+ \Omega^\infty_S p \to A$$

that is compatible with the map $(\Sigma^\infty_{\mathbb{L}})_+ \Omega^\infty_S gl_1(R) \to A$. With the help of the adjunction this means that we need a map

$$p \to gl_1(A)$$

whose precomposition with the map $gl_1(R) \to p$ gives $gl_1(\alpha)$. This argument can be turned into a proof for the following result:

Theorem 6.4.6 [6, Theorem 4.6]. ***The derived mapping space of commutative*** R***-algebras from*** $M(f)$ ***to*** A, $\boldsymbol{Map}_{\mathcal{C}_R}(M(f), A)$, ***is weakly equivalent to the fiber in the map between derived mapping spaces***

$$\boldsymbol{Map}_{\mathcal{M}_S}(p, gl_1(A)) \to \boldsymbol{Map}_{\mathcal{M}_S}(gl_1(R), gl_1(A))$$

at the basepoint $gl_1(\alpha)$ ***of*** $\boldsymbol{Map}_{\mathcal{M}_S}(gl_1(R), gl_1(A))$.

An important example is the question of the string orientation of the spectrum of topological modular forms, tmf. For background on tmf and its variants see [74], whose Chapter 10 contains André Henriques' notes of Mike Hopkins' lecture on the string orientation. Let $BO\langle 8\rangle$ be the 7-connected cover of BO and let $bo\langle 8\rangle$ be the associated spectrum with the canonical map $f\colon bo\langle 8\rangle \to bgl_1(S)$. So we are in the situation where $R = S$ and we take $A = \mathsf{tmf}$. Ando, Hopkins and Rezk [7] establish the existence of an E_∞-map

$$M\mathrm{String} = MO\langle 8\rangle \to \mathsf{tmf}$$

by showing a fiber property as above.

An approach to orientations of the form $MU \to E$ is described in [127]: You start with an E_∞-ring spectrum E and an ordinary complex orientation of E [234, §6.1] and want to know whether you can refine this to an E_∞-map $MU \to E$. Hopkins and Lawson establish a filtration of MU by E_∞-Thom spectra

$$S \to MX_1 \to MX_2 \to \cdots \to MU$$

and for a given E_∞-map $MX_n \to E$ they identify the space of extensions to an E_∞-map $MX_{n+1} \to E$ [127, Theorem 1].

Remark 6.4.7. In [6] the authors present a different approach to Thom spectra and questions about orientations that uses ∞-categorical techniques. In certain cases it is unrealistic to hope for E_∞-maps out of Thom spectra, for instance if one doesn't know that the target spectrum carries an E_∞ structure. The space of E_n-maps out of Thom spectra is described in [68, Theorem 4.2] and [11, Corollary 3.18].

Thom spectra via I-spaces

Let I be the skeleton of the category of finite sets and injective maps. As objects we choose the sets $\mathbf{n} = \{1, \ldots, n\}$ for $n \geqslant 0$ with the convention that $\mathbf{0}$ denotes the empty set. A morphism $f \in I(\mathbf{n}, \mathbf{m})$ is an injective function from $\mathbf{n}$ to $\mathbf{m}$. Hence $\mathbf{0}$ is an initial object of I and the permutation group Σ_n is the group of automorphisms of $\mathbf{n}$ in I. The category I is symmetric monoidal with respect to the disjoint union: $\mathbf{n} \sqcup \mathbf{m} = \mathbf{n} + \mathbf{m}$ with unit $\mathbf{0}$ and non-trivial symmetry $\mathbf{n} + \mathbf{m} \to \mathbf{m} + \mathbf{n}$ given by the shuffle permutation that moves the first n elements to the positions $m+1, \ldots, m+n$.

The functor category of I-spaces, Top^I, i.e., the category of functors $X\colon I \to \mathsf{Top}$ together with natural transformations as morphisms, inherits a symmetric monoidal structure from I and Top via the Day convolution product. Explicitly, one gets:

Definition 6.4.8. The product $X \boxtimes Y$ of two I-spaces X, Y is the I-space given by

$$(X \boxtimes Y)(\mathbf{n}) = \operatorname{colim}_{\mathbf{p} \sqcup \mathbf{q} \to \mathbf{n}} X(\mathbf{p}) \times Y(\mathbf{q}).$$

The unit 1_I is the discrete I-space $\mathbf{n} \mapsto I(\mathbf{0}, \mathbf{n})$.

As $\mathbf{0}$ is initial, the unit 1_I is the terminal object in Top^I. Commutative monoids in Top^I are called *commutative I-space monoids* in [31] and their category is denoted by $C(\mathsf{Top}^I)$. A general fact about Day convolution products is that commutative monoids correspond to lax symmetric monoidal functors.

For an I-space X let's denote by X_{hI} the Bousfield–Kan homotopy colimit of X.

Definition 6.4.9 [31, Definition 2.2]. A map of I-spaces $f\colon X \to Y$ is *an I-equivalence* if the induced map on homotopy colimits $f_{hI}\colon X_{hI} \to Y_{hI}$ is a weak homotopy equivalence in Top.

With the corresponding I-model structure the category of I-spaces is actually Quillen equivalent to the category of spaces [256, Theorem 3.3], but there is a *positive flat model structure* on I-spaces (see [31, §2]) that lifts to a right-induced model structure on $C(\mathsf{Top}^I)$ that makes it Quillen equivalent to E_∞-spaces.

Let Sp^Σ denote the category of symmetric spectra. There is a canonical Quillen adjoint functor pair

$$\mathsf{Top}^I \underset{\Omega^I}{\overset{\mathbb{S}^I}{\rightleftarrows}} \mathsf{Sp}^\Sigma \tag{6.4.6}$$

modeling the suspension spectrum functor and the underlying infinite loop space functor with

$$\mathbb{S}^I X(n) = \mathbb{S}^n \wedge X(\mathbf{n}), \quad \Omega^I(E)(\mathbf{n}) = \Omega^n E_n,$$

where $\mathbb{S}^n$ is the n-fold smash product of the 1-sphere with Σ_n-action given by permutation of the smash factors.

Stable equivalences in symmetric spectra do not in general agree with stable homotopy equivalences, but there is a notion of ***semistable*** symmetric spectra that has the feature that a map $f: E \to F$ between two semistable symmetric spectra is a stable equivalence if and only if it is a stable homotopy equivalence. See [133, §5.6] for details and other characterizations.

Definition 6.4.10. For a commutative semistable symmetric ring spectrum R the commutative I-space monoid of units, $GL_1^I(R)$, has as $GL_1^I(R)(\mathbf{n})$ those components of the commutative I-space monoid $\Omega^I(R)(\mathbf{n}) = \Omega^n R_n$ that represent units in $\pi_0(R)$.

The adjunction from (6.4.6) gives a version of the assembly map from (6.4.2) as

$$\mathbb{S}^I(GL_1^I(R)) \to \mathbb{S}^I\Omega^I(R) \to R.$$

For technical reasons one has to work with a cofibrant replacement of $GL_1^I(R)$, $G \to GL_1^I(R)$ in the positive flat model structure on $C(\mathsf{Top}^I)$. The construction of a Thom spectrum associated to a map $f: X \to BG$ is now similar to the approach in [6]; one defines BG and EG via two sided-bar constructions and takes a suitable pushout:

Definition 6.4.11 [31, Definitions 2.10, 2.12, 3.6].

- Let $BG = B_\boxtimes(1_I, G, 1_I)$ and let EG be defined via a functorial factorization

$$B_\boxtimes(1_I, G, G) \overset{\sim}{\rightarrowtail} EG \twoheadrightarrow BG.$$

- For any I-space X over BG define $U(X)$ as the I-space with G-action given by the pullback

$$\begin{tikzcd} U(X) \arrow[r, dotted] \arrow[d, dotted] & X \arrow[d] \\ EG \arrow[r] & BG. \end{tikzcd}$$

  Here, X and BG are considered as I-spaces with trivial G-action.
- Let R be a semistable commutative symmetric ring spectrum that is S-cofibrant.

  The *Thom spectrum associated with a map of* I*-spaces* $f: X \to BG$ is

$$M^I(f) = B_\boxtimes(\mathbb{S}^I(UX), \mathbb{S}^I G, R). \tag{6.4.7}$$

You should think of this two-sided bar construction as

$$\mathbb{S}^I(UX) \boxtimes^L_{\mathbb{S}^I G} R$$

and then you have to admit that this looks very similar to (6.4.5). This Thom spectrum functor is homotopically meaningful (see [31, Proposition 3.8]). Concerning multiplicative structures one obtains the following result.

Proposition 6.4.12 [31, Proposition 3.10, Corollary 3.11]. *The functor $M^I(-)$ is lax symmetric monoidal and if $\mathcal{D}$ is an operad in spaces, then it sends $\mathcal{D}$-algebras in* Top^I *over BG to $\mathcal{D}$-algebras in R-modules in symmetric spectra over $M^I GL_1(R) := B_\boxtimes(\mathbb{S}^I(EG), \mathbb{S}^I(G), R)$.*

If you dislike diagram categories for some reason, there is also an *I-spacification functor* [31, §4.1] that transforms a map of topological spaces

$$f\colon X \to BG_{hI} \tag{6.4.8}$$

into a map of I-spaces over BG, so you can associate a Thom spectrum to such a map as well. By abuse of notation, we will still denote this Thom spectrum by $M^I(f)$. This construction respects actions of operads augmented over the Barratt–Eccles operad and hence it also provides an E_∞ Thom spectrum functor.

An important question is: Can a given ring spectrum A be realized as a Thom spectrum with respect to a loop map, i.e., in the setting of [31] is A equivalent to $M^I(f)$ with f a loop map to BG_{hI}? A striking result is that one can identify certain quotients as such Thom spectra!

Theorem 6.4.13 [31, Theorem 5.6]. *Let R be a commutative ring spectrum whose homotopy groups are concentrated in even degrees and let $u_i \in \pi_{2i}(R)$ be arbitrary elements with $1 \leqslant i \leqslant n-1$. Then the iterative cofiber $R/(u_1, \ldots, u_{n-1})$ of the multiplication maps by the u_i's can be realized as the Thom spectrum of a loop map from $SU(n)$ to BG_{hI}. In particular, $R/(u_1, \ldots, u_{n-1})$ can be realized as an associative ring spectrum.*

An example of such a quotient is $R = ku \to ku/u = H\mathbb{Z}$. Note that there is no assumption on the regularity of the elements u_i in the above statement. For periodic ring spectra the assumptions on the degree of the elements can be relaxed and the two-periodic version of Morava K-theory can be constructed as a Thom spectrum relative to $R = E_n$, the n-th Morava E-theory or Lubin–Tate spectrum [31, Corollary 5.7]. A related but different construction of quotients of Lubin–Tate spectra modeling versions of Morava K-theory is carried out in [128, §3].

Graded units

There is one problem with the constructions of spaces and spectra of units as above. As they are constructed from the underlying infinite loop space of a spectrum and just take into account the units in π_0, they ignore graded units coming from periodicity elements in the homotopy groups of a spectrum. So for instance, the Bott class $u \in \pi_2(KU)$ is not represented in $GL_1(KU)$ or $GL_1^I(KU)$.

There is a construction of *graded units*. We'll sketch the construction and mention two of its applications: graded Thom spectra and logarithmic ring spectra.

Definition 6.4.14 [256, Definition 4.2]. The category J has as objects pairs of objects of I. A morphism in $J((\mathbf{n}_1,\mathbf{n}_2),(\mathbf{m}_1,\mathbf{m}_2))$ is a triple (α,β,σ) where $\alpha \in I(\mathbf{n}_1,\mathbf{m}_1)$, $\beta \in I(\mathbf{n}_2,\mathbf{m}_2)$ and σ is a bijection

$$\sigma\colon \mathbf{m}_1 \setminus \alpha(\mathbf{n}_1) \to \mathbf{m}_2 \setminus \beta(\mathbf{n}_2).$$

For another morphism $(\gamma,\delta,\xi) \in J((\mathbf{m}_1,\mathbf{m}_2),(\mathbf{l}_1,\mathbf{l}_2))$ the composition is the morphism $(\gamma\circ\alpha,\delta\circ\beta,\tau(\xi,\sigma))$ where $\tau(\xi,\sigma)$ is the permutation

$$\tau(\xi,\sigma)(s) = \begin{cases} \xi(s) & \text{if } s \in \mathbf{l}_1 \setminus \gamma(\mathbf{m}_1), \\ \delta(\sigma(t)) & \text{if } s = \gamma(t) \in \gamma(\mathbf{m}_1 \setminus \alpha(\mathbf{n}_1)). \end{cases}$$

Note that $\mathbf{l}_1 \setminus \gamma(\alpha(\mathbf{n}_1))$ is the disjoint union of $\mathbf{l}_1 \setminus \gamma(\mathbf{m}_1)$ and $\gamma(\mathbf{m}_1 \setminus \alpha(\mathbf{n}_1))$.

With these definitions J is actually a category and it inherits a symmetric monoidal structure from I via componentwise disjoint union [256, Proposition 4.3]. In particular, the category of J-spaces, Top^J, is symmetric monoidal with the Day convolution product. Note, however, that the unit for the monoidal structure $\boxtimes_J$ is $J((\mathbf{0},\mathbf{0}),(-,-))$; this is not a constant functor, but $J((\mathbf{0},\mathbf{0}),(\mathbf{n},\mathbf{n}))$ can be identified with the symmetric group Σ_n!

Proposition 6.4.15 [256, 4.4, 4.5]. *For every J-space X the homotopy colimit, X_{hJ}, is a space over QS^0.*

Proof. It is not hard to see that J is isomorphic to Quillen's category $\Sigma^{-1}\Sigma$ [256, Proposition 4.4] and its classifying space is QS^0 by the Barratt–Priddy–Quillen result. Therefore BJ is QS^0. Every J-space has a map to the terminal J-space that is the constant J-diagram on a point and this induces a map

$$X_{hJ} \to *_{hJ} = BJ \simeq QS^0. \qquad \square$$

For any I-space X we also get that X_{hI} is a space over BI, but as I has an initial object this just gives a map to $BI \simeq *$, the terminal object.

Let $C(\mathsf{Top}^J)$ denote the category of commutative J-space monoids, i.e., commutative monoids in Top^J. The following result is crucial:

Theorem 6.4.16 [256, Theorem 4.11]. *There is a model structure on $C(\mathsf{Top}^J)$ such that there is a Quillen equivalence between $C(\mathsf{Top}^J)$ and the category of E_∞-spaces over BJ.*

Here, the E_∞-structure is parametrized by the Barratt–Eccles operad.

For a (commutative) J-space monoid, one can associate units:

Definition 6.4.17 [256, §4]. Let A be a J-space monoid. Then let $A^\times$ be the J-space monoid with $A^\times(\mathbf{n}_1,\mathbf{n}_2)$ being the union of those components of $A(\mathbf{n}_1,\mathbf{n}_2)$ that represent units in $\pi_0(A_{hJ})$.

So now one has to construct a functor from spectra to J-spaces that sees all the homotopy groups, not just the ones in non-negative degrees:

Definition 6.4.18 [256, (4.5)].

- Let Ω^J be the functor from symmetric spectra to J-spaces that takes a symmetric spectrum E and sends it to the J-space with

$$\Omega^J(E)(\mathbf{n}_1, \mathbf{n}_2) = \Omega^{n_2} E_{n_1}.$$

- If R is a symmetric ring spectrum, then its ***J-space of units*** is

$$GL_1^J(R) = (\Omega^J(R))^\times.$$

Sagave and Schlichtkrull show that this is homotopically meaningful and that for a commutative symmetric ring spectrum R, the units $GL_1^J(R)$ are actually in $C(\mathsf{Top}^J)$ [256, §4]. Most importantly, the inclusion $GL_1^J(R) \hookrightarrow \Omega^J(R)$ realizes the inclusion of graded units $\pi_*(R)^\times$ into $\pi_*(R)$ for positively fibrant R.

Hence, for instance $GL_1^I(KU)$ (and any other model of the "usual" units) only detects the units ± 1 in $\pi_0(KU)$ whereas $GL_1^J(KU)$ also detects the Bott class.

Remark 6.4.19.

1. John Rognes developed the concept of logarithmic ring spectra and in [255] and [253] this concept is fully explored with the help of graded units. The idea is that you want a spectrum that sits between a commutative ring spectrum like ku and its localization KU, so you remember the Bott class as the extra datum of a logarithmic structure. This concept has its origin in algebraic geometry and is useful in stable homotopy theory, for instance for obtaining localization sequences in topological Hochschild homology [253].
2. In [257] Sagave and Schlichtkrull use graded units adapted to the setting of orthogonal spectra, GL_1^W, to construct ***graded Thom spectra*** associated to virtual vector bundles, i.e., associated to a map $f\colon X \to \mathbb{Z} \times BO$ in such a way that uses the E_∞-structure on $\mathbb{Z} \times BO$. They use this for orientation theory and relate GL_1^W-orientations to logarithmic structures. They provide an E_∞-Thom isomorphism that allows to compute the homology of spectra appearing in connection with logarithmic ring spectra [257, §§ 7,8].

6.5 Constructing commutative ring spectra from bipermutative categories

In section 6.4 we saw that Thom spectra give rise to commutative ring spectra. Algebraic K-theory is another machine that takes a commutative ring (spectrum) R and produces a commutative ring spectrum $K(R)$. In this section we focus on a classical construction that takes a small bipermutative category $\mathcal{R}$ and turns it into a commutative ring spectrum. This construction goes back to Segal [268]; its multiplicative properties were investigated by May [199, 195, 192, 193], Shimada–Shimakawa [273], Woolfson [302] and Elmendorf–Mandell [95].

We sketch a simplified version of the construction, present some important examples and refer to [95] for a discussion of the multiplicative properties.

Definition 6.5.1. A ***permutative category*** $(\mathcal{C},\oplus,0,\tau)$ is a category $\mathcal{C}$ together with an object 0 of $\mathcal{C}$, a functor $\oplus\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and a natural isomorphism $\tau_{C_1,C_2}\colon C_1\oplus C_2\to C_2\oplus C_1$ for all objects C_1,C_2 of $\mathcal{C}$ such that

- $\oplus$ is strictly associative, i.e., for all objects C_1,C_2,C_3 of $\mathcal{C}$

$$C_1\oplus(C_2\oplus C_3)=(C_1\oplus C_2)\oplus C_3.$$

- 0 is a strict unit, i.e., for all objects C of $\mathcal{C}$: $C\oplus 0=C=0\oplus C$.
- τ^2 is the identity, i.e., for all objects C_1,C_2 of $\mathcal{C}$ the composite

$$C_1\oplus C_2\xrightarrow{\tau_{C_1,C_2}}C_2\oplus C_1\xrightarrow{\tau_{C_2,C_1}}C_1\oplus C_2$$

  is the identity on $C_1\oplus C_2$.
- The diagrams

$$\begin{tikzcd} C\oplus 0 \arrow[rr, "\tau_{C,0}"] \arrow[dr, equal] & & 0\oplus C \arrow[dl, equal] \\ & C, & \end{tikzcd} \qquad \begin{tikzcd} C_1\oplus C_2\oplus C_3 \arrow[rr, "\tau_{C_1\oplus C_2,C_3}"] \arrow[dr, "\mathrm{id}_{C_1}\oplus\tau_{C_2,C_3}"'] & & C_3\oplus C_1\oplus C_2 \\ & C_1\oplus C_3\oplus C_2 \arrow[ur, "\tau_{C_1,C_3}\oplus\mathrm{id}_{C_2}"'] & \end{tikzcd}$$

  commute for all objects C,C_1,C_2,C_3 of $\mathcal{C}$.

We work with small permutative categories, i.e., we require that the objects of $\mathcal{C}$ form a set (and not a proper class). We recall Segal's construction from [268, §2]:

Definition 6.5.2. Let $\mathcal{C}$ be a small permutative category and let X be a finite set with basepoint $+\in X$. Let $\mathcal{C}(X)$ be the category whose objects are families $(C_S,\varrho_{S,T})$ such that:

- $S\subset X$ and $+\notin S$;
- S and T are pairs of such subsets that are disjoint;
- the C_S are objects of $\mathcal{C}$ and $\varrho_{S,T}$ is an isomorphism in $\mathcal{C}$:

$$\varrho_{S,T}\colon C_S\oplus C_T\to C_{S\cup T};$$

- $C_\varnothing=0$ and $\varrho_{\varnothing,T}=\mathrm{id}_{C_T}$ for all T; and
- for pairwise disjoint S,T,U that don't contain + the following diagrams commute:

$$\begin{tikzcd} C_S\oplus C_T \arrow[r, "\varrho_{S,T}"] \arrow[d, "\tau"'] & C_{S\cup T} \arrow[d, equal] \\ C_T\oplus C_S \arrow[r, "\varrho_{T,S}"] & C_{T\cup S}, \end{tikzcd} \qquad \begin{tikzcd} C_S\oplus C_T\oplus C_U \arrow[r, "\varrho_{S,T}\oplus\mathrm{id}_{C_U}"] \arrow[d, "\mathrm{id}_{C_S}\oplus\varrho_{T,U}"'] & C_{S\cup T}\oplus C_U \arrow[d, "\varrho_{S\cup T,U}"] \\ C_S\oplus C_{T\cup U} \arrow[r, "\varrho_{S,T\cup U}"] & C_{S\cup T\cup U}. \end{tikzcd}$$

Morphisms $\alpha\colon(C_S,\varrho_{S,T})\to(C'_S,\varrho'_{S,T})$ consist of a family of morphisms $\alpha_S\in\mathcal{C}(C_S,C'_S)$

for all $S \subset X$ with $+ \notin S$ such that $\alpha_\varnothing = \mathrm{id}_0$ and for all $S, T \in X$ with $+ \notin S, T$ and $S \cap T = \varnothing$ the diagram

$$\begin{tikzcd} C_S \oplus C_T \arrow[r, "\rho_{S,T}"] \arrow[d, "\alpha_S \oplus \alpha_T"'] & C_{S\cup T} \arrow[d, "\alpha_{S\cup T}"] \\ C'_S \oplus C'_T \arrow[r, "\rho'_{S,T}"] & C'_{S\cup T} \end{tikzcd}$$

commutes.

So up to isomorphism, every object C_S for $S = \{x_1, \ldots, x_n\}$ can be decomposed as

$$C_S \cong C_{\{x_1\}} \oplus \cdots \oplus C_{\{x_n\}}$$

by an iterated application of the isomorphisms ρ, but these isomorphisms are part of the data. Segal shows [268, Corollary 2.2] that this construction gives rise to a so-called Γ-space (see [268, Definition 1.2] for a definition) that sends a finite pointed set X to the classifying space of $\mathcal{C}(X)$. Every Γ-space gives rise to a spectrum, and we denote the spectrum associated to $\mathcal{C}$ by $H\mathcal{C}$.

Remark 6.5.3. Segal's construction actually works for symmetric monoidal categories and it produces a spectrum whose associated infinite loop space is the group completion of the classifying space of the category $\mathcal{C}$, $B\mathcal{C}$, and the latter is the geometric realization of the nerve of $\mathcal{C}$.

Definition 6.5.4. A ***bipermutative category*** $\mathcal{R}$ is a category with two permutative category structures, $(\mathcal{R}, \oplus, 0_\mathcal{R}, \tau_\oplus)$ and $(\mathcal{R}, \otimes, 1_\mathcal{R}, \tau_\otimes)$, that are compatible in the following sense:

1. $$0_\mathcal{R} \otimes C = 0_\mathcal{R} = C \otimes 0_\mathcal{R}$$

   for all objects C of $\mathcal{R}$.
2. For all objects A, B, C we have an equality between $(A \oplus B) \otimes C$ and $A \otimes C \oplus B \otimes C$, and the diagram

   $$\begin{tikzcd} (A \oplus B) \otimes C \arrow[r, equal] \arrow[d, "\tau_\oplus \otimes \mathrm{id}"'] & A\otimes C \oplus B\otimes C \arrow[d, "\tau_\oplus"] \\ (B \oplus A) \otimes C \arrow[r, equal] & B\otimes C \oplus A\otimes C \end{tikzcd}$$

   commutes.
3. We define the distributivity isomorphism $d_\ell \colon A \otimes (B \oplus C) \to A\otimes B \oplus A\otimes C$ for all A, B, C in $\mathcal{R}$ via the diagram

   $$\begin{tikzcd} A \otimes (B \oplus C) \arrow[r, "\tau_\otimes"] \arrow[d, dotted, "d_\ell"'] & (B \oplus C) \otimes A \arrow[d, equal] \\ A\otimes B \oplus A\otimes C & B\otimes A \oplus C\otimes A \arrow[l, "\tau_\otimes \oplus \tau_\otimes"] \end{tikzcd}$$

Then the diagram

$$\begin{tikzcd}
(A \oplus B) \otimes (C \oplus D) \arrow[r, "d_\ell"] \arrow[d, equal] & (A \oplus B) \otimes C \oplus (A \oplus B) \otimes D \arrow[d, equal] \\
A \otimes (C \oplus D) \oplus B \otimes (C \oplus D) \arrow[d, "d_\ell \oplus d_\ell"'] & A \otimes C \oplus B \otimes C \oplus A \otimes D \oplus B \otimes D \\
A \otimes C \oplus A \otimes D \oplus B \otimes C \oplus B \otimes D \arrow[ur, "\mathrm{id} \oplus \tau_\oplus \oplus \mathrm{id}"] &
\end{tikzcd}$$

commutes.

This definition is taken from [199, Definition VI.3.3, p. 154]. The definition in [95] is less strict, but bipermutative categories in the above sense are also bipermutative in the sense of [95, Definition 3.6]. We refer to Elmendorf and Mandell for a proof that for a bipermutative category $\mathcal{R}$, one actually obtains a commutative ring spectrum $H\mathcal{R}$:

Theorem 6.5.5 [95, Corollary 3.9]. *If $\mathcal{R}$ is a bipermutative category, then $H\mathcal{R}$ is equivalent to a strictly commutative symmetric ring spectrum.*

There is an alternative construction of an E_∞-ring spectrum from a bipermutative category in [193]: May first constructs an E_∞-ring space associated to a bipermutative category and then builds the corresponding E_∞-ring spectrum.

Segal's construction enables us to find small and explicit models for certain connective commutative ring spectra. Famous examples of bipermutative categories and their associated commutative ring spectra are the following:

1. If R is a commutative discrete ring, then the category $\mathcal{R}_R$ which has the elements of R as objects and only identity morphisms is a bipermutative category with the addition in the ring being $\oplus$ and the multiplication being $\otimes$. The associated spectrum, $H\mathcal{R}_R$ is the Eilenberg–Mac Lane spectrum of the ring R, HR.
2. Let $\mathcal{E}$ denote the bipermutative category of finite sets whose objects are the finite sets $\mathbf{n} = \{1, \ldots, n\}$ for $n \in \mathbb{N}_0$. By convention $\mathbf{0}$ is the empty set. The morphisms in $\mathcal{E}$ are
$$\mathcal{E}(\mathbf{n}, \mathbf{m}) = \begin{cases} \varnothing & \text{for } n \neq m, \\ \Sigma_n & \text{for } n = m. \end{cases}$$
For the full structure see [199, VI, Example 5.1]. Here $H\mathcal{E}$ is the sphere spectrum, S.
3. The bipermutative category of complex vector spaces, $\mathcal{V}_{\mathbb{C}}$, with objects the natural numbers with zero and morphisms
$$\mathcal{V}_{\mathbb{C}}(n, m) = \begin{cases} \varnothing & \text{for } n \neq m, \\ U(n) & \text{for } n = m, \end{cases}$$
is bipermutative. On objects we set $n \oplus m = n + m$ and $n \otimes m = nm$ and on morphisms we use the block sum and the tensor product of matrices. The associated spectrum is $H\mathcal{V}_{\mathbb{C}} = ku$, the connective version of topological complex K-theory. Its real analog, $\mathcal{V}_{\mathbb{R}}$, gives a model for connective topological real K-theory, ko. You

can also work with the general linear group instead of the unitary or orthogonal group.

4. If R is a discrete commutative ring, then we define the category F_R as the one with objects $\mathbb{N}_0$ again. As morphisms we have
$$F_R(n,m) = \begin{cases} \varnothing & \text{for } n \neq m, \\ GL_n(R) & \text{for } n = m. \end{cases}$$
This category is often called *the small category of free R-modules.* Its spectrum is the *free algebraic K-theory of* R, $K^f(R)$. Its homotopy groups agree with the algebraic K-groups of R from degree 1 on.

6.6 From topological Hochschild to topological André–Quillen homology

For rings and algebras Hochschild homology contains a lot of information. For commutative rings and algebras André–Quillen homology is the adequate tool. There are spectrum level versions of these homology theories: topological Hochschild homology, THH, and topological André–Quillen homology, TAQ.

We can determine classes in the algebraic K-theory of a ring spectrum using the trace to topological Hochschild homology or to topological cyclic homology:
$$\mathrm{tr}\colon K(R) \to \mathsf{THH}(R). \tag{6.6.1}$$
For instance the trace from $K(\mathbb{Z})$ to $\mathsf{THH}(\mathbb{Z})$ detects important classes. Bökstedt, Madsen and Rognes [50, 252] show for instance that
$$\mathrm{tr}\colon K_{2p-1}(\mathbb{Z}) \to \mathsf{THH}_{2p-1}(\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$$
is surjective for all primes p.

We give a construction of topological Hochschild homology and, more generally, for commutative ring spectra R we define $X \otimes R$ for X a finite pointed simplicial set. We give some examples of calculations of such X-homology groups of R and tell you about topological Hochschild cohomology as a derived center of an algebra spectrum. We define topological André–Quillen homology and we will see applications to Postnikov towers for commutative ring spectra later in Section 6.7.

THH and friends

Let X be a finite pointed simplicial set and let R be a cofibrant commutative ring spectrum.

Definition 6.6.1. We denote by $X \otimes R$ the simplicial spectrum with
$$(X \otimes R)_n = \bigwedge_{x \in X_n} R.$$
By slight abuse of notation we will use the same symbol for the geometric realization of $X \otimes R$.

Remarks 6.6.2. – As the smash product is the coproduct in $\mathcal{C}_S$, the simplicial structure maps of $X \otimes R$ are induced from the ones on X.
– As X is pointed, $X \otimes R$ comes with maps

$$R \to X \otimes R \to R$$

whose composition is the identity.
– The commutative multiplication on R induces a commutative multiplication on $X \otimes R$; hence $X \otimes R$ is an augmented commutative (simplicial) R-algebra spectrum.
– One could also use the fact that the spectra of [94] are tensored over topological spaces or, similarly, that symmetric spectra [133] in topological spaces are enriched over simplicial sets and over topological spaces. This gives an equivalent situation. It is for instance shown in [94, Corollary VII.3.4] that $|X \otimes A| \simeq |X| \otimes A$ for simplicial spaces X and commutative R-algebra spectra A.
– The above definition can be extended to tensoring with an arbitrary pointed simplicial set by expressing such a simplicial set as the colimit of its finite pointed simplicial subcomplexes.

There are many important special cases of this construction.

Definition 6.6.3.

1. For the simplicial 1-sphere $X = \mathbb{S}^1$ the commutative R-algebra spectrum $\mathbb{S}^1 \otimes R$ is the ***topological Hochschild homology of*** R and is denoted by $\mathsf{THH}(R)$.
2. More generally, for an n-sphere, we denote by $\mathsf{THH}^{[n]}(R)$ the spectrum $\mathbb{S}^n \otimes R$; this is called ***topological Hochschild homology of order*** n.
3. If $\mathbb{T}^n$ denotes the torus $(\mathbb{S}^1)^n$, then $\mathbb{T}^n \otimes R$ is the n***-torus homology of*** R.

For the small model of the simplicial 1-sphere with just one non-degenerate 0- and 1-simplex we have $(\mathbb{S}^1)_n = \{0, 1, \dots, n\}$ and the simplicial spectrum $\mathbb{S}^1 \otimes R$ is precisely the cyclic bar construction on R:

$$R \leftleftarrows R \wedge R \;\substack{\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow}\; R \wedge R \wedge R \;\substack{\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow}\; \cdots,$$

where the degeneracy map $s_i \colon R^{n+1} \to R^{n+2}$ inserts the unit map $\eta \colon S \to R$ after the i-th factor of R and the face maps $d_i \colon R^{n+1} \to R^n$ for $0 \leqslant i < n$ are given by the multiplication in R of the i-th and $(i+1)$-st smash factor. The last face map d_n cyclically permutes the smash factors to bring the last one to the front and then it multiplies the former factors numbered n and 0.

As for Hochschild homology you should think about this as a genuine cyclic object:

$$\begin{array}{ccccc} & & R & & \\ R & \wedge & & \wedge & \ddots \\ \wedge & & & & \\ R & & & & \\ & \wedge & \ldots & & \end{array}$$

The original definition of THH is due to Marcel Bökstedt [52]. McClure, Schwänzl and Vogt [203] show that for an E_∞-ring spectrum R, $\mathsf{THH}(R)$ is equivalent to tensoring R with the topological 1-sphere. Kuhn systematically studies constructions like $X \otimes R$ in a reduced setting [150] for pointed spaces X. So the above definition is an unreduced variant of this that uses simplicial sets instead of topological spaces.

Lemma 6.6.4. *Let X and Y be finite simplicial pointed sets. Then*

$$(X \times Y) \otimes R \simeq X \otimes (Y \otimes R).$$

Proof. Observe that

$$((X \times Y) \otimes R)_n = \bigwedge_{(x,y) \in X_n \times Y_n} R \cong \bigwedge_{x \in X_n} \left(\bigwedge_{y \in Y_n} R \right)$$

and this is the diagonal of the bisimplicial spectrum

$$([m], [\ell]) \mapsto (X \otimes ((Y \otimes R)_\ell))_m$$

in degree n. □

One of the important features of $\mathsf{THH}(R)$ is that it receives a trace map from algebraic K-theory (see (6.6.1)), which we can now write as

$$\mathrm{tr}\colon K(R) \to \mathbb{S}^1 \otimes R.$$

Taking higher-dimensional tori gives targets for iterated trace maps. Algebraic K-theory of a commutative ring spectrum is again a commutative ring spectrum and the trace map is a map of commutative ring spectra; hence one can iterate the process of forming K-theory and traces. If we denote by $K^n(R)$ the n-fold iteration, then, since we have the product formula from Lemma 6.6.4, we get an iterated trace to $\mathbb{T}^n \otimes R$. Explicitly, for $n = 2$ this is

$$K(K(R)) \to \mathbb{S}^1 \otimes (\mathbb{S}^1 \otimes R) \simeq (\mathbb{S}^1 \times \mathbb{S}^1) \otimes R = \mathbb{T}^2 \otimes R.$$

There are variants of Definition 6.6.1: As we work with pointed simplicial sets, we can glue an R-module to the base point and use the R-module structure for the face maps. A second variant is to work relative to some commutative ring spectrum R: in Definition 6.6.1 the smash products were over the sphere spectrum, but if we work with a commutative R-algebra spectrum A, then we can take smash products over R instead of S. Recall that $\wedge_R$ is the coproduct in the category of commutative R-algebra spectra, $\mathcal{C}_R$.

Definition 6.6.5. Let R be a cofibrant commutative ring spectrum, A a cofibrant commutative R-algebra spectrum, M an A-module spectrum over R and let X be a finite pointed simplicial set. We denote by $\mathcal{L}^R_X(A;M)$ the simplicial spectrum with

$$\mathcal{L}^R_X(A;M)_n = M \wedge_R \bigwedge^R_{x \in X_n \setminus *} A.$$

We call $\mathcal{L}^R_X(A;M)$ the *Loday construction of A over R with coefficients in M*.

As M is just an A-module spectrum, the resulting simplicial spectrum and also its realization carries an A-module structure over R, but no multiplicative structure in general. However, if we place a commutative A-algebra C at the basepoint, then the resulting spectrum is an augmented commutative C-algebra spectrum.

We will see in Section 6.8 that for instance

$$\mathsf{THH}^R(A) := \mathcal{L}^R_{\mathbb{S}^1}(A)$$

measures properties of A as a commutative R-algebra spectrum. The case of $X = \mathbb{S}^0$ gives

$$\mathcal{L}^R_{\mathbb{S}^0}(A) = A \wedge_R A,$$

so there is a Künneth spectral sequence [94, IV.4.1] for calculating its homotopy groups.

An important example of a Loday construction is Pirashvili's construction of *higher-order Hochschild homology.* He works with discrete commutative k-algebras A and A-modules M and defines $\mathsf{HH}^k_X(A;M)$ [223, §5.1]. For $X = \mathbb{S}^n$ this is his notion of higher-order Hochschild homology (in his notation $H^{[n]}(A;M)$). In our setting this corresponds to $\mathcal{L}^{Hk}_X(HA;HM)$ if A is flat over k.

Examples

1. A classical example of a THH-calculation is the one of $H\mathbb{Z}$ and $H\mathbb{F}_p$ by Marcel Bökstedt ([51]; see [161, Chapter 13] and the references for published accounts of these results):

Proposition 6.6.6. $$\mathsf{THH}_*(H\mathbb{F}_p) \cong \mathbb{F}_p[\mu], \quad |\mu| = 2.$$

$$\mathsf{THH}_i(H\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z}/j\mathbb{Z} & \text{if } i = 2j-1, \\ 0 & \text{otherwise.} \end{cases}$$

A crucial ingredient for these and many other calculations of THH is Bökstedt's spectral sequence: If R is a commutative ring spectrum and if E_* is a homotopy commutative ring spectrum such that $E_*(R)$ is flat over E_* then there is a multiplicative spectral sequence

$$E^2_{p,q} = \mathsf{HH}^{E_*}_{p,q}(E_*(R)) \Rightarrow E_{p+q}\mathsf{THH}(R).$$

Here $\mathsf{HH}_{p,q}$ denotes Hochschild homology in homological degree p and internal degree q ([51], [94, Theorem IV.1.9]).

2. If we apply THH to Eilenberg–Mac Lane spectra of number rings, Lindenstrauss and Madsen show that THH detects arithmetic properties:

Proposition 6.6.7 [160, Theorem 1.1]. *Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. Then*

$$\mathsf{THH}_n(H\mathcal{O}_K) = \begin{cases} \mathcal{O}_K & \text{if } n = 0, \\ \mathcal{D}^{-1}_{\mathcal{O}_K}/\ell\mathcal{O}_K & \text{if } n = 2\ell-1, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\mathcal{D}^{-1}_{\mathcal{O}_K}$ is the inverse different. This is the set of those $x \in K$ such that the trace $\mathrm{tr}(xy)$ is an integer for all $y \in \mathcal{O}_K$. The inverse different detects ramified primes.

Dundas, Lindenstrauss and I calculate higher-order THH of number rings with reduced coefficients in [82, Theorem 4.3].

3. For a suspension spectrum on a based (Moore) loop space, $\Sigma^\infty_+\Omega_M X$, the cyclic bar construction reduces to the suspension spectrum of the cyclic bar construction on $\Omega_M X$ and Goodwillie [109, Proof of Theorem V.1.1] identifies the latter with the free loop space on X, LX. Hence one obtains

$$\mathsf{THH}(\Sigma^\infty_+\Omega_M X) \simeq \Sigma^\infty_+ LX.$$

4. Let R be a ring spectrum and let Π be a pointed monoid. Hesselholt and Madsen show that $\mathsf{THH}(R[\Pi])$ splits as

$$\mathsf{THH}(R[\Pi]) \simeq \mathsf{THH}(R) \wedge |N^{cy}\Pi|,$$

where $|N^{cy}\Pi|$ denotes the cyclic nerve of Π [119, Theorem 7.1].

5. As a sample calculation for second order THH Dundas, Lindenstrauss and I get [83, Theorem 2.1]:

$$\mathsf{THH}^{[2]}_*(H\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[x_1, x_2, \dots]/(p^n x_n, x_n^p - p x_{n+1}, n \geqslant 1) \tag{6.6.2}$$

with $|x_1| = 2p$.

6. At an odd prime $KU_{(p)}$ splits as

$$KU_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} L.$$

Here, L is the Adams summand of $KU_{(p)}$ with $\pi_*(L) \cong \mathbb{Z}_{(p)}[v_1^{\pm 1}]$ and $|v_1| = 2p-2$. For consistency we set $L = KU_{(2)}$ at the prime 2. We denote by ku, ℓ and ko the connective covers of KU, L and KO.

McClure and Staffeldt determine the mod p-homotopy of $\mathsf{THH}(\ell)$ at odd primes [202] and they show that $\mathsf{THH}(L)_p \simeq L_p \vee (\Sigma L_p)_{\mathbb{Q}}$ [202, Corollary 7.2, Theorem 8.1].

Ausoni [13] determines the mod p and mod v_1 homotopy of $\mathsf{THH}(ku)$ as an input for his work on $K(ku)$.

Angeltveit, Hill and Lawson show [9, Theorem 2.6] that for all primes,

$$\mathsf{THH}_*(\ell) \cong \ell_* \oplus \Sigma^{2p-1} F \oplus T$$

as ℓ_*-modules, where F is a torsionfree summand and T is an infinite direct sum of torsion modules concentrated in even degrees. They describe F explicitly using a rational calculation. Determining the torsion is way more involved [9, Theorem 2.8]. The calculation of $\mathsf{THH}_*(\ell)$ uses the method of *dueling Bockstein spectral sequences* for the Bockstein spectral sequences associated to

$$\begin{tikzcd} \ell \arrow[r] \arrow[d] & \ell/p \arrow[d] \\ \ell/v_1 = H\mathbb{Z}_{(p)} \arrow[r] & H\mathbb{F}_p = \ell/(p, v_1) \end{tikzcd}$$

They describe the 2-local homotopy groups of $\mathsf{THH}(ko)$ [9, §7] by first determining $\mathsf{THH}_*(ko; ku)$ and then using the Bockstein spectral sequence associated to the cofiber sequence $\Sigma ko \to ko \to ku$.

Again, things are way easier for the periodic versions (see [13, Proposition 7.13] and [9, Corollary 7.9]):

$$\mathsf{THH}(KO) \simeq KO \vee \Sigma KO_{\mathbb{Q}}, \quad \mathsf{THH}(KU) \simeq KU \vee \Sigma KU_{\mathbb{Q}}.$$

7. John Greenlees uses a generalization of the concept of Gorenstein maps of commutative rings to the spectral world in order to determine Gorenstein descent properties for cofiber sequences of connective commutative ring spectra [110, Theorem 7.4].

Topological Hochschild homology of Thom spectra

We start with a general statement about $X \otimes M^I(f)$ if $M^I(f)$ is a Thom spectrum associated to an E_∞-map to BG_{hI} with BG_{hI} as in (6.4.8) with $R = S$; hence G is a cofibrant replacement of $GL_1^I(S)$.

Theorem 6.6.8 [259, Theorem 1.1]. *For any pointed simplicial set X and any map of grouplike E_∞-spaces $f: A \to BG_{hI}$ there is an equivalence of E_∞-ring spectra,*

$$X \otimes M^I(f) \simeq M^I(f) \wedge \Omega^\infty(a \wedge |X|)_+,$$

where a is the spectrum associated to A with $\Omega^\infty a = A$.

This result generalizes [45], where the case of $X = \mathbb{S}^1$ is covered. In general, for $X = \mathbb{S}^n$ Theorem 6.6.8 determines the higher-order topological Hochschild homology of $M^I(f)$ [259, (1.6)] as

$$\mathsf{THH}^{[n]}(M^I(f)) \simeq M^I(f) \wedge B^n A_+.$$

As an example, for the canonical map $f: BU \to BG_{hI}$ one obtains

$$X \otimes MU \simeq MU \wedge \Omega^\infty(bu \wedge |X|),$$
$$\mathsf{THH}^{[n]}(MU) \simeq MU \wedge \Omega^\infty(bu \wedge \mathbb{S}^n) \simeq MU \wedge B^n BU_+.$$

There is also a statement about THH of Thom spectra associated to single loop maps in [45, Theorem 1]. We state the relative version of this, so in the following G is a cofibrant replacement of $GL_1^I(R)$.

Theorem 6.6.9 [31, Theorem 6.6]. *Assume that R is a commutative symmetric ring* *spectrum that is semistable and S-cofibrant. Let $M^I(f)$ be a Thom spectrum associated to a map $f: M \to BG_{hI}$ of topological monoids, where M is grouplike and well-pointed. Then*

$$\mathsf{THH}^R(M^I(f)) \simeq M^I(L^\eta(B(f))).$$

Here, $M^I(L^\eta(B(f)))$ is the Thom spectrum associated to the map

$$\begin{tikzcd} L(B(M)) \arrow[r, "L(B(f))"] \arrow[d, dotted, "L^\eta(B(f))"'] & LBBG_{hI} \simeq BG_{hI} \times BBG_{hI} \arrow[d, "\mathrm{id}\times\eta"] \\ BG_{hI} & BG_{hI} \times BG_{hI} \arrow[l, "\mu"'] \end{tikzcd}$$

Note that BBG_{hI} is an H-group, so we can split the free loop space $LBBG_{hI}$ into the base space and the based loops

$$LBBG_{hI} \simeq BBG_{hI} \times \Omega BBG_{hI}$$

and the second factor is equivalent to BG_{hI}. As usual, η denotes the Hopf map $\eta\colon \mathbb{S}^3 \to \mathbb{S}^2$ and it induces a map $\eta\colon BBG_{hI} \to BG_{hI}$ as above via

$$BBG_{hI} \simeq \Omega^2 B^4 G_{hI} \to \Omega^3 B^4 G_{hI} \simeq BG_{hI}$$

by reducing the loop coordinates by precomposition.

For quotient spectra, this result gives a new way of calculating $\mathsf{THH}^R(R/I)$. For related results see [8] and in the case where R/I is commutative see [83, §7].

A second example comes from viewing $H\mathbb{Z}_{(p)}$ as a Thom spectrum associated to a 2-fold loop map $\Omega^2(\mathbb{S}^3\langle 3\rangle) \to BG_{hI}$, which allows for a determination of $\mathsf{THH}(H\mathbb{Z}_{(p)})$ as $H\mathbb{Z}_{(p)} \wedge \Omega(\mathbb{S}^3\langle 3\rangle)_+$ [45, Theorem 3.8] and an additive equivalence

$$\mathsf{THH}^{[2]}(H\mathbb{Z}_{(p)}) \simeq H\mathbb{Z}_{(p)} \wedge \mathbb{S}^3\langle 3\rangle_+.$$

This gives a geometric interpretation of (6.6.2), but without an identification of the multiplicative structure. See also [149, §4], where Klang presents related results, using the framework of factorization homology.

Topological Hochschild cohomology as a derived center

In the discrete case, i.e., for a commutative ring k and a k-algebra A one can describe the center of A,

$$Z(A) = \{b \in A, ab = ba \text{ for all } a \in A\},$$

as the set of A-bimodule maps from A to A. If f is such a map, $f: A \to A$ with $f(cad) = cf(a)d$ for all $a, c, d \in A$, then f is determined by $f(1) =: b$ and this b satisfies

$$ab = af(1) = f(a \cdot 1) = f(a) = f(1 \cdot a) = f(1)a = ba,$$

so the set of such morphisms gives rise to an element in the center; conversely, for any $b \in Z(A)$ we get such an f by setting $f(1) = b$.

Hochschild cohomology of A over k can be described as

$$\mathsf{HH}^*_k(A) = \mathsf{Ext}^*_{A \otimes_k A^o}(A, A)$$

if A is k-projective. Hence $\mathsf{HH}^0_k(A) = Z(A)$ and the Hochschild cohomology of A is the *derived center of A*. Hochschild cohomology has a graded commutative algebra structure via a cup product, but the solved Deligne conjecture [204] says that the Hochschild cochain complex is in general not a differential graded commutative algebra, but that it has an E_2-algebra structure.

For ring spectra there is no homotopically meaningful definition of a center: requiring equality translates to an equalizer diagram and this wouldn't be homotopy invariant. For a commutative ring spectrum R and an R-algebra spectrum A this equalizer corresponds precisely to taking not just R-module endomorphisms but A-bimodule endomorphisms. So a homotopy invariant version is as follows.

Definition 6.6.10. For a commutative ring spectrum R and an R-algebra spectrum A, the *topological Hochschild cohomology groups of A over R* are

$$\mathsf{THH}^*_R(A) = \pi_* \mathsf{Ext}_{A \wedge_R A^o}(A, A)$$

and the *derived center of A over R* is

$$\mathsf{THH}_R(A) = \mathsf{Ext}_{A \wedge_R A^o}(A, A).$$

Here, $\mathsf{Ext}_{A \wedge_R A^o}(A, A)$ denotes the derived endomorphism spectrum of A as an A-bimodule [94, IV §1].

McClure and Smith's proof of the Deligne conjecture also provides a spectrum version for topological Hochschild cohomology, giving the derived center an E_2-structure:

Theorem 6.6.11 [204]. *If A is an associative R-algebra spectrum, then $\mathsf{THH}_R(A)$ is an E_2-ring spectrum.*

An important example of a calculation of such a derived center is Angeltveit's calculation of $\mathsf{THH}_{E_n}(K_n)$. Here E_n denotes Morava E-theory with

$$\pi_*(E_n) \cong W(\mathbb{F}_q)[[u_1, \ldots, u_{n-1}]][u^{\pm 1}],$$

where the u_i are deformation parameters for the height-n Honda formal group law with $|u_i| = 0$ and u is a periodicity element with $|u| = 2$. The sequence of elements $(p, u_1, \ldots, u_{n-1})$ is a regular sequence and K_n is the 2-periodic version of Morava K-theory:

$$K_n = E_n/(p, u_1, \ldots, u_{n-1}), \quad (K_n)_* = \mathbb{F}_q[u^{\pm 1}].$$

Angeltveit shows that the derived center of K_n over E_n depends on the chosen A_∞-algebra structure of K_n over E_n:

Theorem 6.6.12 [8, Theorems 5.21, 5.22]. 1. *For any prime p and any $n \geqslant 1$ there is an A_∞-structure on K_n such that $\mathsf{THH}_{E_n}(K_n) \simeq E_n$.*

2. ***For*** $n = 1$ ***and any*** d ***with*** $1 \leqslant d < p - 1$ ***and any*** a ***with*** $1 \leqslant a \leqslant p - 1$ ***there is an*** A_∞***-structure on*** K_1 ***with***

$$\mathsf{THH}^*_{E_1}(K_1) \cong \pi_*(E_1)[[q]]/(p + a(uq)^d).$$

Here, the structure in statement 1 occurs as the one coming from the ***least commutative*** A_∞***-structure*** on K_n (see [8, Theorem 5.8] for a precise statement). The case $n = 1, p = 2$ of statement 1 is due to Baker and Lazarev [16, Proof of Theorem 3.1] who show that at the prime 2

$$\mathsf{THH}_{KU_2}(K(1)) \simeq KU_2.$$

Topological André–Quillen homology

We will first sketch the definition of ordinary André–Quillen homology. See [225] for the original account and [134] for a very readable modern introduction.

Definition 6.6.13. Let k be a commutative ring with unit and let A be a commutative k-algebra. The A***-module of Kähler differentials of*** A ***over*** k is the A-module generated by elements $d(a)$ for $a \in A$ subject to the relations that d is k-linear and satisfies the Leibniz rule:

$$d(ab) = d(a)b + ad(b).$$

This A-module is denoted by $\Omega^1_{A|k}$.

The conditions imply $d(1) = d(1 \cdot 1) = 2d(1)$ and hence $d(1) = 0$. For a polynomial algebra $A = k[x_1, \ldots, x_n]$ the A-module $\Omega^1_{k[x_1,\ldots,x_n]|k}$ is freely generated by $dx_1, \ldots, dx_n$. By induction one shows $d(x_i^m) = mx_i^{m-1}d(x_i)$ for all $m \geqslant 2$.

Consider for instance the $\mathbb{F}_p$-algebra $\mathbb{F}_p[x]/(x^p - x)$. Then the module of Kähler differentials is generated by $d(x)$. However, as we are in characteristic p we get

$$d(x) = d(x^p) = px^{p-1}d(x) = 0$$

and hence $\Omega^1_{\mathbb{F}_p[x]/(x^p-x)|\mathbb{F}_p} = 0$.

Remark 6.6.14. For a commutative k-algebra A there is an isomorphism between $\Omega^1_{A|k}$ and the first Hochschild homology group of A over k: Every $a \otimes b$ in Hochschild chain degree one is a cycle and if you send $a \otimes b$ to $ad(b)$ then this gives a well-defined map modulo Hochschild boundaries and it induces an isomorphism $\mathsf{HH}^k_1(A) \cong \Omega^1_{A|k}$ [161, Proposition 1.1.10].

Definition 6.6.15. Let M be an A-module. A k***-linear derivation from*** A ***to*** M is a k-linear map $\delta \colon A \to M$ which satisfies the Leibniz rule.

The set of all such derivations, $\mathsf{Der}_k(A, M)$, is an A-submodule of the A-module of all k-linear maps. The symbol d in the definition of $\Omega^1_{A|k}$ satisfies the conditions of a derivation; hence the map

$$d \colon A \to \Omega^1_{A|k}, \quad a \mapsto da$$

is a derivation, in fact, it is the ***universal derivation***:

Proposition 6.6.16 [134]. *For all A-modules M the canonical map*

$$\mathsf{Hom}_A(\Omega^1_{A|k}, M) \to \mathsf{Der}_k(A, M), \quad f \mapsto f \circ d,$$

is an A-linear isomorphism.

There is another crucial reformulation of the above isomorphism: $\mathsf{Der}_k(A, M)$ can also be identified with the morphisms of commutative k-algebras over A from A to the square-zero extension $A \oplus M$. The latter is the commutative augmented A-algebra with underlying module $A \oplus M$ with multiplication

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + a_2 m_1), \quad a_1, a_2 \in A,\ m_1, m_2 \in M.$$

A derivation $\delta\colon A \to M$ corresponds to the map into the second component of $A \oplus M$.

The idea of André–Quillen homology is to take the derived functor of $A \mapsto M \otimes_A \Omega^1_{A|k}$. But in which sense? As A is a commutative algebra, we need a resolution of A as such an algebra. The category of differential graded commutative k-algebras in general doesn't have a (right-induced) model structure, so instead one works with ***simplicial resolutions***. The category of simplicial commutative k-algebras ***does*** have a nice model structure. Let $P_\bullet \to A$ be a cofibrant resolution. Each P_n can be chosen to be a polynomial algebra [134, §4].

Definition 6.6.17. The ***André–Quillen homology of** A **over** k **with coefficients in** M* is

$$\mathsf{AQ}_*(A|k : M) = \pi_*(M \otimes_{P_\bullet} \Omega^1_{P_\bullet|k}).$$

A definition of $\Omega^1_{A|k}$ in terms of generators and relations is not suitable for a generalization to commutative ring spectra. Instead we use the following description:

Lemma 6.6.18. *Denote by I the kernel of the multiplication map $\mu\colon A \otimes_k A \to A$. Then $\Omega^1_{A|k}$ is isomorphic to I/I^2.*

Proof. The ideal I is generated by elements of the form $a \otimes 1 - 1 \otimes a$. Such an element is identified with $d(a)$. Taking the quotient by I^2 corresponds to the Leibniz rule for d. □

The ideal I can also be viewed as a non-unital k-algebra and I/I^2 is the ***module of indecomposables of** I*. This definition translates to brave new commutative rings. Basterra's work is formulated in the setting of [94]:

Definition 6.6.19. Let A be a commutative R-algebra spectrum.

- We define $I(A \wedge_R A)$ as the pullback

$$\begin{tikzcd} I(A \wedge_R A) \arrow[r, dotted] \arrow[d, dotted] & A \wedge_R A \arrow[d, "\mu"] \\ * \arrow[r] & A \end{tikzcd}$$

– If N is a non-unital commutative R-algebra spectrum, then its R*-module of indecomposables*, $Q(N)$, is defined as the pushout

$$\begin{tikzcd} N \wedge_R N \arrow[r] \arrow[d, "\mu_N"'] & * \arrow[d, dotted] \\ N \arrow[r, dotted] & Q(N) \end{tikzcd}$$

– For an A-module spectrum M we define the *topological André–Quillen homology of* A *over* R *with coefficients in* M as

$$\mathsf{TAQ}(A|R;M) = \mathbf{L}Q(\mathbf{R}I(A \wedge_R A)) \tag{6.6.3}$$

and denote its homotopy groups as $\mathsf{TAQ}_*(A|R;M)$. We use the abbreviation $\Omega_{A|R}$ for $\mathbf{L}Q(\mathbf{R}I(A \wedge_R A))$.

Thus for $\Omega_{A|R}$ we take homotopy invariant versions of the kernel of the multiplication map followed by taking indecomposables by applying the right derived functor of I and the left derived functor of Q.

Definition 6.6.20. Dually, *topological André–Quillen cohomology of* A *over* R *with coefficients in* M is $F_A(\Omega_{A|R}, M)$ and we set $\mathsf{TAQ}^n(A|R;M) = \pi_{-n}F_A(\Omega_{A|R}, M)$.

Basterra proves [27, Proposition 3.2] that maps from $\Omega_{A|R}$ to M in the homotopy category of A-modules correspond to maps in the homotopy category of commutative R-algebra maps over A from A to $A \vee M$, where $A \vee M$ carries the square-zero multiplication.

For example, if $f\colon B \to BGL_1(S)$ is an infinite loop map and $M(f)$ is the associated Thom spectrum, then Basterra and Mandell show [28, Theorem 5 and Corollary] that

$$\mathsf{TAQ}(M(f)) \simeq M(f) \wedge b,$$

where $\Omega^\infty b \simeq B$. In the case of an E_∞-space B the spherical group ring $\Sigma^\infty_+ B$ has

$$\mathsf{TAQ}(\Sigma^\infty_+ B) \simeq \Sigma_+ B \wedge b.$$

6.7 How do we recognize ring spectra as being (non) commutative?

If you have a concrete model of a homotopy type, say in symmetric spectra, then you can be lucky and this model possesses a commutative structure and you should be able to check this by hand. Of course you could also try to disprove commutativity by showing that your spectrum doesn't have power operations as in (6.2.2) and this has been done in many cases, but sometimes you might need a different approach.

Obstructions via filtrations and resolutions

An obstruction theory for A_∞-structures on homotopy ring spectra was developed as early as 1989 [247] by Alan Robinson. Obstruction theories for E_∞-structures came much later: Goerss–Hopkins [107] and Robinson [246] independently developed one with obstruction groups that later turned out to be isomorphic [30]. The idea is to use a filtration or resolution of an operad such that the corresponding filtration quotients or the corresponding spectral sequence give rise to obstruction groups that contain obstructions for lifting a partial structure to a full E_∞-ring structure ([246, Theorem 5.6] and [107, Corollary 5.9]). The Goerss–Hopkins approach also allows one to calculate the homotopy groups of the derived E_∞ mapping space between two such E_∞-ring spectra [107, Theorem 4.5].

The obstruction groups have as input the algebra of cooperations E_*E of a spectrum E and they compute André–Quillen cohomology groups of the graded commutative E_*-algebra E_*E in the setting of differential graded (or simplicial) E_∞-algebras. See [185] or [106, §2.4] for background on these cohomology groups and see [30, §2] for the comparison results. In Robinson's setting these groups are called Γ-cohomology. The obstruction groups vanish if for instance E_*E is étale as an E_*-algebra.

If you prefer to work with explicit chain complexes, then there are several equivalent ones computing Γ-cohomology groups in Robinson's setting (see [246, §2.5], [250, §6], [222, §2]) and therefore, by the comparison result from [30, Theorem 2.6], computing the obstruction groups in the Goerss–Hopkins setting as well.

There is another version of obstruction theory for promoting a homotopy T-algebra structure to an actual one, where T is a monad, by Johnson and Noel [139]. This includes obstructions for operadic structures on spectra but also includes for instance group actions. Noel shows that in certain situations the obstruction theory [139] can be compared to the one of [107].

We list some important applications:

1. The development of the Hopkins–Miller and Goerss–Hopkins obstruction theory was motivated by the Morava-E-theory spectra E_n, also known as Lubin–Tate spectra, and their variants. These are Landweber exact cohomology theories that govern the deformation theory of height n formal group laws. In [234] an obstruction theory was established leading to a proof that the E_n are A_∞-spectra and that the Morava stabilizer group $\mathbb{G}_n$ acts on E_n via maps of A_∞-spectra. In [107] the corresponding obstruction theory for E_∞-structures was developed and [107, Corollaries 7.6, 7.7] shows that the $\mathbb{G}_n$-action is via E_∞-maps.

2. It was known that KU and KO are E_∞-spectra and it was also known that the p-completed Adams summand L_p is E_∞. In [18] Andy Baker and I use Robinson's version of the E_∞-obstruction theory to show that these E_∞-structures are unique and that the p-local Adams summand also has a unique E_∞-structure. Uniqueness also holds for the connective covers [19]. It is important to have uniqueness results for E_∞-structures because calculations can depend on a choice of such a structure.

3. For an E_∞-ring spectrum R there is a θ-algebra structure on its p-adic K-theory,

$\pi_* L_{K(1)}(KU_p \wedge R)$ [106, Theorem 2.2.4], and in good cases

$$\pi_* L_{K(1)}(KU_p \wedge R) \cong \lim_k (KU_p)_*(R \wedge M(p^k)),$$

where $M(p^k)$ is the mod-p^k Moore spectrum. The study of such structures was initiated by McClure in [63, Chapter IX]. There is a variant of the Goerss–Hopkins obstruction theory for realizing for instance a θ-algebra (see [106, §2.4.4] and [153, Theorem 5.14]) as a $K(1)$-local E_∞-ring spectrum.

There is one for realizing an E_∞-Hk-algebra spectrum with a fixed Dyer–Lashof structure on its homotopy [217, Proposition 2.2] (for k a field of characteristic p). Other variants can be found in the literature.

The θ-algebra version was successfully applied by Lawson and Naumann [153] to show that $BP\langle 2\rangle$ at 2 has an E_∞-structure. By a different method Hill and Lawson [122, Theorem 4.2] find a commutative model for $BP\langle 2\rangle$ at the prime 3.

4. Mathew, Naumann and Noel use operations in Morava-E-theory to prove May's nilpotence conjecture:

Theorem 6.7.1 [188, Theorem A]. *If R is an H_∞-ring spectrum and if $x \in \pi_*(R)$ is in the kernel of the Hurewicz homomorphism $\pi_*(R) \to H_*(R;\mathbb{Z})$, then x is nilpotent.*

They use this — among many other applications — for the following result about E_∞-ring spectra:

Theorem 6.7.2 [188, Proposition 4.2]. *If R is an E_∞-ring spectrum and if there is an $m \in \mathbb{Z}$, $m \neq 0$ with $m \cdot 1 = 0 \in \pi_0(R)$, then, for all primes p and all $n \geqslant 1$,*

$$K(n)_*(R) \cong 0.$$

Lawson observed that using $K(n)$-techniques (see [231] for background) this implies that for finite E_∞-ring spectra R either the rational homology is non-trivial or R is weakly contractible, because if $H_*(R;\mathbb{Q}) \cong 0$, then by the above result all the Morava K-theories also vanish on R, but then the finiteness of R implies weak contractibility (see [188, Remark 4.3] for the full argument).

The Dyer–Lashof variant is for instance important when one wants to decide whether a given H_∞-map can be upgraded to an E_∞-map: roughly speaking, an H_∞-spectrum is like an E_∞-spectrum in the homotopy category. You can find applications of this approach for instance in Noel's work [217] and in [139].

Other spectra, such as BP, come with homology operations just because they sit in the right place: analyzing the maps $MU \to BP \to H\mathbb{F}_p$ gives [63, p. 63] that $(H\mathbb{F}_p)_*(BP)$ embeds into the dual of the Steenrod algebra such that $(H\mathbb{F}_p)_*(BP)$ is closed under the action of the Dyer–Lashof algebra — even without establishing a structured multiplication on BP. This led Lawson [152] to look for the right obstructions for an E_∞-structure of BP at 2 via secondary operations (see Theorem 6.7.5).

Obstructions via Postnikov towers

A different approach to obstruction theory is to consider Postnikov towers in the world of commutative ring spectra [27] or in the setting of E_n-algebras [29].

To this end Basterra uses TAQ-cohomology to lift ordinary k-invariants of a connective commutative ring spectrum to k-invariants in a multiplicative Postnikov tower:

Assume that R is a connective commutative ring spectrum. Then there is a map of commutative ring spectra

$$p_0 \colon R \to H(\pi_0(R))$$

and without loss of generality we can assume that p_0 is a cofibration of commutative ring spectra that realizes the identity on π_0, i.e., $\pi_0(p_0) = \mathrm{id}_{\pi_0(R)}$.

If we abbreviate $\pi_0(R)$ to B and if M is a B-module, an element in $\mathsf{TAQ}^n(A|R;HM)$ corresponds to a morphism $\varphi \colon A \to A \vee \Sigma^n HM$ in the homotopy category of R-algebra spectra over A and we can form the pullback of

$$\begin{tikzcd} & A \arrow[d, "i_A"] \\ A \arrow[r, "\varphi"] & A \vee \Sigma^n HM \end{tikzcd}$$

If we postcompose φ with the projection map to $\Sigma^n HM$

$$A \xrightarrow{\varphi} A \vee \Sigma^n HM \longrightarrow \Sigma^n HM \tag{6.7.1}$$

such a TAQ-class forgets to an Ext-class in $\mathsf{Ext}^n_R(A;HM)$, specifically to an ordinary cohomology class if R is the sphere spectrum. Basterra shows that this projection maps k-invariants in the world of commutative ring spectra to ordinary k-invariants of the underlying spectrum.

Theorem 6.7.3 [27, Theorem 8.1]. *For any connective commutative ring spectrum A there is a sequence of commutative ring spectra A_i, $\pi_0(A)$-modules M_i and elements*

$$\tilde{k}_i \in \mathsf{TAQ}^{i+2}(A_i|S;HM_{i+1})$$

such that

- *$A_0 = H\pi_0(A)$ and A_{i+1} is the pullback of A_i with respect to $\tilde{k}_i$,*
- *$\pi_j A_i = 0$ for all $j > i$,*
- *there are maps of commutative ring spectra $\lambda_i \colon A \to A_i$ which induce an isomorphism in homotopy groups up to degree i such that the diagram*

$$\begin{tikzcd} & A_{i+1} \arrow[d] \\ A \arrow[ur, "\lambda_{i+1}"] \arrow[r, "\lambda_i"] & A_i \end{tikzcd}$$

commutes in the homotopy category of commutative ring spectra.

You start with $A_0 = H\pi_0(A)$ and then you have to find a suitable map $A_0 \to A_0 \vee \Sigma^2 H(\pi_1(A))$ as a starting point of the multiplicative Postnikov tower.

Basterra's result can be used as an obstruction theory as follows. If A is a connective spectrum then it has an ordinary Postnikov tower with k-invariants living in ordinary cohomology groups

$$k_i \in H^{i+2}(A_i; \pi_{i+1}(A)).$$

You can then investigate whether it is possible to find multiplicative k-invariants

$$\tilde{k}_i \in \mathsf{TAQ}^{i+2}(A_i|S; H\pi_{i+1}(A))$$

that forget to the k_i's under the map from (6.7.1).

Using Postnikov towers for E_n-algebra spectra, Basterra and Mandell show:

Theorem 6.7.4 [29, Theorem 1.1]. *The Brown Peterson spectrum, BP, has an E_4-structure at every prime.*

This ensures by the main result of [184] that the derived category of BP-module spectra has a symmetric monoidal smash product. Tyler Lawson, however, showed that there are certain secondary operations in the $\mathbb{F}_2$-homology of every such spectrum with an E_{12}-structure and he could show that these are not present in the $\mathbb{F}_2$-homology of BP at 2. Let $BP\langle n\rangle$ denote the spectrum $BP/(v_{n+1}, v_{n+2}, \dots)$.

Theorem 6.7.5 [152, Theorem 1.1.2]. *The Brown–Peterson spectrum at the prime 2 does not possess an E_n-structure for any n with $12 \leqslant n \leqslant \infty$. The truncated Brown–Peterson spectrum $BP\langle n\rangle$ for $n \geqslant 4$ cannot have an E_n-structure for any n with $12 \leqslant n \leqslant \infty$.*

See [271] for the corresponding results at odd primes.

Realization of E_∞-spectra via derived algebraic geometry

There is a completely different important and highly successful approach to realization problems, using *derived algebraic geometry*, for which see Chapter 8 of this volume.

6.8 What are étale maps?

We first recall the algebraic notion of an étale k-algebra from [161, E.1]: Let k be a commutative ring and let A be a finitely generated commutative k-algebra. Then A is *étale* if A is flat over k and if the module of Kähler differentials $\Omega^1_{A|k}$ is trivial. If $\Omega^1_{A|k} = 0$, then $k \to A$ is called *unramified.* A k-algebra B (not necessarily commutative) is called *separable* if the multiplication map

$$B \otimes_k B^o \to B$$

has a section as a B-bimodule map. In algebra, a commutative separable algebra has Hochschild homology concentrated in homological degree zero, in particular the module of Kähler differentials is trivial.

Rognes' Galois extensions of commutative ring spectra

Definition 6.8.1 [251, Definition 4.1.3]. Let $A \to B$ be a map of commutative ring spectra and let G be a finite group acting on B via commutative A-algebra maps. Assume that $S \to A \to B$ is a sequence of cofibrations in the model structure on commutative ring spectra of [94, Corollary VII.4.10]. Then $A \to B$ is a G-Galois extension if

1. the canonical map $\iota\colon A \to B^{hG}$ is a weak equivalence and
2. $$h\colon B \wedge_A B \to \prod_G B \tag{6.8.1}$$
is a weak equivalence.

The first condition is the familiar fixed points condition from classical Galois theory of fields. The map ι comes from taking the adjoint of the map

$$A \wedge EG_+ \xrightarrow{\mathrm{id}\wedge p} A \wedge S^0 \cong A \longrightarrow B,$$

where $p\colon EG_+ \to S^0$ collapses EG to the non-base point of S^0.

The map h is adjoint to the composite

$$B \wedge_A B \wedge G_+ \to B \wedge_A B \to B$$

that comes from the G-action on the right factor of $B \wedge_A B$ followed by the multiplication in B. (Informally, if smashes were tensors, then $h(b_1 \otimes b_2) = (b_1 \cdot g(b_2))_{g\in G}$.) Note that $\prod_G B$ is isomorphic to $F(G_+, B)$, so we could rewrite the condition in (6.8.1) as the requirement that

$$h\colon B \wedge_A B \to F(G_+, B)$$

is a weak equivalence.

The condition that the map h from (6.8.1) is a weak equivalence is crucial. It is also necessary for Galois extensions of discrete commutative rings in order to ensure that the extension is unramified. For instance, $\mathbb{Z} \subset \mathbb{Z}[i]$ satisfies $\mathbb{Z}[i]^{C_2} = \mathbb{Z}$, but $h\colon \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Z}[i] \to \mathbb{Z}[i] \times \mathbb{Z}[i]$ is not surjective: h detects the ramification at the prime 2. Therefore $\mathbb{Z} \to \mathbb{Z}[i]$ is ***not*** a C_2-Galois extension but $\mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}, i]$ is C_2-Galois.

Galois extensions of commutative ring spectra can have rather bad properties as modules. So the following definition is actually an additional assumption (this does not happen in the discrete setting).

Definition 6.8.2 [251, Definition 4.3.1]. A Galois extension $A \to B$ is ***faithful*** if it is faithful as an A-module, i.e., for every A-module M with $M \wedge_A B \simeq *$ we have $M \simeq *$.

Important examples of Galois extensions of commutative ring spectra are the following. By C_n we denote the cyclic group of order n.

1. The concept of Galois extensions of commutative ring spectra corresponds to the one for commutative rings via the Eilenberg–Mac Lane spectrum functor [251, Proposition 4.2]: Let $R \to T$ be a homomorphism of discrete commutative rings and

let G be a finite group acting on T via R-algebra homomorphisms. Then $R \to T$ is a G-Galois extension of commutative rings if and only if $HR \to HT$ is a G-Galois extension of commutative ring spectra.

2. The complexification of real vector bundles gives rise to a map of commutative ring spectra $KO \to KU$ from real to complex topological K-theory. There is a C_2-action on KU corresponding to complex conjugation of complex vector bundles. Rognes shows [251, Proposition 5.3.1] that this turns $KO \to KU$ into a C_2-Galois extension.

3. At an odd prime p there is a p-adic Adams operation on KU_p that gives rise to a C_{p-1}-action on KU_p such that $L_p \to KU_p$ is a C_{p-1}-Galois extension [251, §5.5.4].

4. There is a notion of pro-Galois extensions of commutative ring spectra and $L_{K(n)}S \to E_n$ is a $K(n)$-local pro-Galois extension with the extended Morava stabilizer group as the Galois group [251, Theorem 5.4.4].

5. Let p be an arbitrary prime. The projection map $\pi\colon EC_p \to BC_p$ induces a map on function spectra

$$F(\pi_+, H\mathbb{F}_p)\colon F((BC_p)_+, H\mathbb{F}_p) \to F((EC_p)_+, H\mathbb{F}_p) \sim H\mathbb{F}_p$$

which identifies $H\mathbb{F}_p$ as a C_p-Galois extension over $F((BC_p)_+, H\mathbb{F}_p)$ [251, Proposition 5.6.3]. Hence in the world of commutative ring spectra group cohomology sits between S and $H\mathbb{F}_p$ as the base of a Galois extension! Beware, this Galois extension is not faithful. This observation is due to Ben Wieland: the Tate construction $H\mathbb{F}_p^{tC_p}$ isn't trivial and it is actually killed by the Galois extension (in the spectral sequence you augment a Laurent generator to zero).

6. Studying elliptic curves with level structures gives C_2-Galois extensions $\mathsf{TMF}_0(3) \to \mathsf{TMF}_1(3)$ and $\mathsf{Tmf}_0(3) \to \mathsf{Tmf}_1(3)$ [187, Theorems 7.6, 7.12]. For $\mathsf{TMF}_1(3)$ and $\mathsf{Tmf}_1(3)$ you consider elliptic curves with one chosen point of exact order 3 and for $\mathsf{TMF}_0(3)$ and $\mathsf{Tmf}_0(3)$ you only remember a subgroup of order 3. As $C_2 \cong \mathbb{Z}/3\mathbb{Z}^\times$ this gives a C_2-action. This can be made rigorous; see [121, 122, 187].

Notions of étale morphisms

Weibel–Geller [298] show that for an étale extension of commutative rings $\varphi\colon A \to B$ Hochschild homology satisfies *étale descent*: The map $\mathsf{HH}(\varphi)_*$ induces an isomorphism

$$B \otimes_A \mathsf{HH}_*(A) \cong \mathsf{HH}_*(B) \tag{6.8.2}$$

and for finite G-Galois extensions $\varphi\colon A \to B$ one obtains ***Galois descent***:

$$\mathsf{HH}_*(A) \cong \mathsf{HH}_*(B)^G. \tag{6.8.3}$$

It is easy to see that for a G-Galois extension of discrete commutative rings $\varphi\colon A \to B$ with finite G, the induced extension of graded commutative rings $\mathsf{HH}_*(\varphi)\colon \mathsf{HH}_*(A) \to \mathsf{HH}_*(B)$ is again G-Galois. In addition to having the right fixed-point property it satisfies

$$\begin{aligned}\mathsf{HH}_*(B)\otimes_{\mathsf{HH}_*(A)}\mathsf{HH}_*(B) &\cong B\otimes_A \mathsf{HH}_*(A)\otimes_{\mathsf{HH}_*(A)} B\otimes_A \mathsf{HH}_*(A)\\ &\cong B\otimes_A B\otimes_A \mathsf{HH}_*(A)\\ &\cong \prod_G B\otimes_A \mathsf{HH}_*(A)\\ &\cong \prod_G \mathsf{HH}_*(B).\end{aligned}$$

If $\varphi\colon A\to B$ is étale, then the module of Kähler differentials $\Omega^1_{B|A}$ is trivial and it can be easily seen that the map $B\to \mathsf{HH}^A_*(B)$ is an isomorphism and that André–Quillen homology of B over A is trivial, because étale algebras are smooth.

For commutative ring spectra the situation is different. There are several non-equivalent notions of étale maps:

Definition 6.8.3. Let $\varphi\colon A\to B$ be a morphism of commutative ring spectra.

1. [168, Definition 7.5.1.4] We call φ ***Lurie-étale*** if $\pi_0(\varphi)\colon \pi_0(A)\to\pi_0(B)$ is an étale map of commutative rings and if the canonical map

$$\pi_*(A)\otimes_{\pi_0(A)}\pi_0(B)\to\pi_*(B)$$

is an isomorphism. In Chapter 8, this will be the only notion of étale map considered, and the adjective “Lurie” will be dropped.
2. [201, Definiton 3.2], [251, Definition 9.2.1] The morphism φ is ***(formally)*** THH***-étale*** if $B\to\mathsf{THH}^A(B)$ is a weak equivalence.
3. [201, Definiton 3.2], [251, Definition 9.4.1] We define φ to be ***(formally)*** TAQ***-étale*** if $\mathsf{TAQ}(B|A)$ is weakly equivalent to $*$.

Remark 6.8.4.
- Rognes [251] reserves the labels THH-étale and TAQ-étale for maps that, in addition to the conditions above, identify B as a dualizable A-module.
- The condition of being Lurie-étale is strong and is a very algebraic one. It is for instance not satisfied by the C_2-Galois extension $KO\to KU$ because on the level of homotopy groups this extension is rather appalling, compare (6.1.1).
- McCarthy and Minasian show that THH-étale implies TAQ-étale and they show that for $n>1$ the map $H\mathbb{F}_p\to F(K(\mathbb{Z}/p\mathbb{Z},n)_+,H\mathbb{F}_p)$ is a TAQ-étale morphism that is not THH-étale. They attribute this example to Mandell [201, Example 3.5]. Minasian [211, Corollary 2.8] proves that both notions are equivalent for morphisms between connective commutative ring spectra.
- For connective spectra, the notion of Lurie-étaleness has good features [168, §7.5] and Mathew shows in [186, Corollary 3.1] that one can use [165, Lemma 8.9] to show that under some finiteness condition TAQ-étaleness implies Lurie-étaleness in the connective case.

Definition 6.8.5 [251, Definition 9.1.1]. Let C be a cofibrant associative A-algebra spectrum. Then C is ***separable*** if the multiplication map $\mu\colon C\wedge_A C^o\to C$ has a section in the homotopy category of C-bimodule spectra.

Proposition 6.8.6 [251, Lemma 9.2.6]. *If C is a commutative separable A-algebra spectrum, then C is* THH*-étale.*

Proof. Recall from Remark 6.6.2 that $\mathsf{THH}^A(C)$ is an augmented commutative C-algebra spectrum, so the composite of the unit map $C \to \mathsf{THH}^A(C)$ with the augmentation

$$C \to \mathsf{THH}^A(C) \to C$$

is the identity. We also get a splitting in the homotopy category of C-bimodule spectra,

$$C \xrightarrow{s} C \wedge_A C \xrightarrow{\mu} C,,$$

i.e., the above composite is the identity on C. Taking the derived smash product $C \wedge^L_{C\wedge_A C} (-)$ of the above sequence gives the sequence

$$\mathsf{THH}^A(C) \to C \to \mathsf{THH}^A(C),$$

in which the last map is equivalent to the unit map of $\mathsf{THH}^A(C)$ and whose composite is the identity. So the unit map $C \to \mathsf{THH}^A(C)$ has a right and a left inverse in the homotopy category of C-module spectra. $\square$

Definition 6.8.7. Let $A \to B$ be a map of commutative ring spectra and let G be a finite group acting on B via maps of commutative A-algebra spectra. Assume that $S \to A \to B$ is a sequence of cofibrations in the model structure on commutative ring spectra of [94, Corollary VII.4.10]. Then $A \to B$ is *unramified* if

$$h\colon B \wedge_A B \to \prod_G B$$

is a weak equivalence.

Proposition 6.8.8 (compare [251, Lemma 9.1.2]). *If $A \to B$ is unramified, then B is separable over A.*

Proof. The canonical inclusion map $i\colon B \to F(G_+, B)$ can be modeled by the pointed map from G_+ to S^0 that sends the neutral element $e \in G$ to the non-basepoint of S^0 and sends all other elements to the basepoint. We define a section to the multiplication map of B to be

$$B \xrightarrow{i} F(G_+, B) \xleftarrow{h,\sim} B \wedge_A B.$$

Note that h is not a B-bimodule map, but we are only interested in its e-component of $F(G_+, B)$. $\square$

Thus we can conclude that unramified maps of commutative ring spectra are THH-étale and that the failure of the map $B \to \mathsf{THH}^A(B)$ to be a weak equivalence detects ramification. This idea was exploited in [83] in order to show that the inclusion of the Adams summand $\ell \to ku_{(p)}$ is tamely ramified [83, Theorem 4.1]. Sagave also identifies this map as being log-étale [255, Theorem 1.6].

Versions of étale descent

Transferring the Geller–Weibel result to the setting of commutative ring spectra, it seems natural to define two versions of descent:

Definition 6.8.9. In the following $\varphi\colon A \to B$ is a cofibration and A is cofibrant.

- The morphism $\varphi\colon A \to B$ satisfies ***étale descent*** if the canonical morphism

$$B \wedge_A \mathsf{THH}(A) \to \mathsf{THH}(B) \tag{6.8.4}$$

  is a weak equivalence.
- If $\varphi\colon A \to B$ is a map of commutative ring spectra and if G is a finite group acting on B via commutative A-algebra maps, then we say that φ satisfies ***Galois descent*** if the map

$$\mathsf{THH}(A) \to \mathsf{THH}(B)^{hG} \tag{6.8.5}$$

  is a weak equivalence.

Akhil Mathew clarifies the relationship between the different notions of étale morphisms and the notions of descent. He proves that Lurie-étale morphisms satisfy étale descent [186, Theorem 1.3] and that for a faithful G-Galois extension with finite Galois group G, both descent properties are equivalent [186, Proposition 4.3] and they are in turn equivalent to the property that $\mathsf{THH}(A) \to \mathsf{THH}(B)$ is again a G-Galois extension.

Moreover, he shows that the morphism

$$\varphi\colon F(\mathbb{S}^1_+, H\mathbb{F}_p) \to F(\mathbb{S}^1_+, H\mathbb{F}_p)$$

that is induced by the degree-p map on $\mathbb{S}^1$ is a faithful C_p-Galois extension, but that it does *not* satisfy étale descent [186, Theorem 2.1] and hence it doesn't satisfy Galois descent.

The Hopf fibration $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$ is a principal $\mathbb{S}^1$-bundle. The corresponding morphism of commutative $H\mathbb{Q}$-algebra spectra of cochains

$$F(\eta, H\mathbb{Q})\colon F(\mathbb{S}^2_+, H\mathbb{Q}) \to F(\mathbb{S}^3_+, H\mathbb{Q})$$

is therefore an $\mathbb{S}^1$-Galois extension [251, Proposition 5.6.3].

In joint work with Christian Ausoni we show that the morphism $F(\eta, H\mathbb{Q})$ does not satisfy Galois descent, i.e.,

$$\mathsf{THH}(F(\mathbb{S}^2_+, H\mathbb{Q})) \not\simeq \mathsf{THH}(F(\mathbb{S}^3_+, H\mathbb{Q}))^{h\mathbb{S}^1}.$$

Indeed, the homotopy groups of $\mathsf{THH}(F(\mathbb{S}^2_+, H\mathbb{Q}))$ contain an element in degree -1 that is not present in $\pi_*(\mathsf{THH}(F(\mathbb{S}^3_+, H\mathbb{Q}))^{h\mathbb{S}^1})$.

Mathew identifies the problem with étale descent of finite faithful Galois extensions for THH as being caused by the non-trivial fundamental group of $\mathbb{S}^1$. He shows the following result.

Theorem 6.8.10 [186, Proposition 5.2]. *Let X be a simply connected pointed space and let $A \to B$ be a faithful G-Galois extension of commutative ring spectra with finite G. Then the map*

$$B \wedge_A (X \otimes A) \to X \otimes B$$

is a weak equivalence.

In particular, higher-order topological Hochschild homology, $\mathsf{THH}^{[n]}$ for $n \geqslant 2$, *does* satisfy étale descent for faithful finite Galois extensions. However, étale descent remains for instance an issue for torus homology.

Sometimes THH *does* satisfy descent, even for ramified maps of commutative ring spectra. For instance, Ausoni shows in [13, Theorem 10.2] that $\mathsf{THH}(\ell_p)$ is p-adically equivalent to $\mathsf{THH}(ku_p)^{hC_{p-1}}$ and even that $K(\ell_p)$ is p-adically equivalent to $K(ku_p)^{hC_{p-1}}$.

Remark 6.8.11. In [69] Clausen, Mathew, Naumann and Noel prove far-reaching Galois descent results for topological Hochschild homology and algebraic K-theory; in particular they confirm a Galois descent conjecture for algebraic K-theory by Ausoni and Rognes in many important cases. They identify THH as a *weakly additive invariant* (see [69, Definition 3.10]) and prove descent in the form of [69, Theorems 5.1 and 5.6].

6.9 Picard and Brauer groups

Picard groups in the setting of a symmetric monoidal category

Let $(\mathcal{C}, \otimes, 1, \tau)$ be a symmetric monoidal category. An important class of objects in $\mathcal{C}$ are those objects C that have an inverse with respect to $\otimes$, i.e., such that there is an object C' of $\mathcal{C}$ satisfying

$$C \otimes C' \cong 1.$$

One wants to gather such objects in a category and build a space and spectrum out of them:

Definition 6.9.1. The *Picard groupoid of $\mathcal{C}$*, $\mathsf{Picard}(\mathcal{C})$, is the category whose objects are the invertible objects of $\mathcal{C}$ and whose morphisms are isomorphisms between invertible objects.

If C_1 and C_2 are objects of $\mathsf{Picard}(\mathcal{C})$, then so is $C_1 \otimes C_2$; in fact, $\mathsf{Picard}(\mathcal{C})$ is itself a symmetric monoidal category. But in general, this category might not be small.

Definition 6.9.2. Let $\mathcal{C}$ be as above and assume that $\mathsf{Picard}(\mathcal{C})$ is small. Then $\mathsf{PIC}(\mathcal{C})$ is the classifying space of the symmetric monoidal category $\mathsf{Picard}(\mathcal{C})$ and let $\mathsf{pic}(\mathcal{C})$ denote the connective spectrum associated to the infinite loop space associated to $\mathsf{PIC}(\mathcal{C})$. The *Picard group of $\mathcal{C}$*, $\mathsf{Pic}(\mathcal{C})$, is $\pi_0\mathsf{PIC}(\mathcal{C})$.

If the Picard groupoid of $\mathcal{C}$ is small, then the Picard group can also be described as the set of isomorphism classes of invertible objects of $\mathcal{C}$ with the product

$$[C_1] \otimes [C_2] := [C_1 \otimes C_2].$$

The neutral element is the isomorphism class of the unit, $[1]$.

Definition 6.9.3. Let R be a (discrete) commutative ring; we denote by $\mathsf{Pic}(R)$ the Picard group of the symmetric monoidal category of the category of R-modules and by $\mathsf{PIC}(R)$ (and $\mathsf{pic}(R)$) the Picard space (and Picard spectrum) of this category.

For instance the Picard group of a ring of integers in a number ring is its ideal class group.

Picard group for commutative ring spectra

For commutative ring spectra R, the above definition of $\mathsf{PIC}(R)$ and $\mathsf{pic}(R)$ would either be much too rigid (if one chose $\mathcal{C}$ to be the category of R-module spectra and isomorphisms) or not strict enough (if one took $\mathcal{C}$ to be the homotopy category of R-module spectra). See [189, §2] for an adequate background for a suitable definition and see [103, §4] for a dictionary how to pass from a commutative ring spectrum R and its category of modules to the ∞-categorical setting. Instead of working with symmetric monoidal categories, one uses presentable symmetric monoidal ∞-categories $\mathcal{C}$. Then the Picard ∞-groupoid of $\mathcal{C}$ is the maximal subgroupoid of the underlying ∞-category of $\mathcal{C}$ spanned by the invertible objects. This groupoid is equivalent to a grouplike E_∞-space $\mathsf{PIC}(\mathcal{C})$ and hence there is a connective ring spectrum, $\mathsf{pic}(\mathcal{C})$, associated to $\mathcal{C}$ [103, §5].

Let R be a commutative ring spectrum. The operadic nerve of the category of cofibrant-fibrant R-modules is a stable presentable symmetric monoidal ∞-category [168, Proposition 4.1.3.10] and we will abbreviate this as the ∞-category of R-modules, Rmod.

Definition 6.9.4. The ***Picard group of a commutative ring spectrum*** R, $\mathsf{Pic}(R)$, is the group $\pi_0(\mathsf{PIC}(R\mathrm{mod}))$.

Again, these Picard groups can also be described as the set of isomorphism classes of invertible R-modules in the homotopy category of R-module spectra.

The Picard space $\mathsf{PIC}(R)$ is a delooping of the units of R ([189, §2.2], [289, §5]): There is an equivalence

$$\mathsf{PIC}(R) \simeq \mathsf{Pic}(R) \times BGL_1(R).$$

Remark 6.9.5. There is a map $\mathsf{Pic}(\pi_* R) \to \mathsf{Pic}(R)$ that realizes an element in the algebraic Picard group of invertible graded $\pi_* R$-modules as a module over R and in many cases this map is an isomorphism [17, Theorem 43]. In this case we call $\mathsf{Pic}(R)$ ***algebraic***. A notable exception comes from Galois extensions of ring spectra: As in algebra, if $A \to B$ is a G-Galois extension of commutative ring spectra with abelian

Galois group G, then $[B] \in \mathsf{Pic}(A[G])$ [251, Proposition 6.5.2]. But for instance $[KU_*]$ is certainly ***not*** an element in the algebraic Picard group $\mathsf{Pic}(KO_*[C_2])$; see (6.1.1).

The equivalence classes of suspensions of R are always in $\mathsf{Pic}(R)$, but if R is periodic, these suspensions don't generate a free abelian group. Let us mention some crucial examples of Picard groups of commutative ring spectra:

- The Picard group of the initial commutative ring spectrum S is $\mathsf{Pic}(S) \cong \mathbb{Z}$, where $n \in \mathbb{Z}$ corresponds to the class of S^n [129].
- For connective commutative ring spectra the Picard group of R is algebraic; see [17, Theorem 21], [189, Theorem 2.4.4].
- For periodic real and complex K-theory the Picard groups just notice the suspensions of the ground ring: the Picard group of KU is algebraic, with $\mathsf{Pic}(KU) \cong \mathbb{Z}/2\mathbb{Z}$, and $\mathsf{Pic}(KO) \cong \mathbb{Z}/8\mathbb{Z}$ (Hopkins, [189, Example 7.1.1] and [103, §7]).
- The same applies to the periodic version of the spectrum of topological modular forms: $\mathsf{Pic}(\mathsf{TMF}) \cong \mathbb{Z}/576\mathbb{Z}$ [189, Theorem A]. But for Tmf, the spectrum of topological forms that mediates between TMF and its connective version tmf, one gets [189, Theorem B]
$$\mathsf{Pic}(\mathsf{Tmf}) \cong \mathbb{Z} \oplus \mathbb{Z}/24\mathbb{Z},$$
where the copy of the integers comes from the suspensions of Tmf and the generator of the $\mathbb{Z}/24\mathbb{Z}$-summand is described in [189, Construction 8.4.2].
- Using Galois descent techniques for pic, Heard, Mathew and Stojanoska prove in [116, Theorem 1.5] that, for any odd prime and any finite subgroup G of the full Morava stabilizer group $\mathbb{G}_{p-1}$, the Picard group of E_{p-1}^{hG} is a cyclic group generated by the suspension of E_{p-1}^{hG}.

A Picard group that contains more elements than just the ones coming from suspensions of the commutative ring spectrum says that there are more self-equivalences of the homotopy category of R-modules than the standard suspensions. One might view these as twisted suspensions. Gepner and Lawson explore the concept of having a Picard grading on the category of R-module spectra and they develop a Pic-resolution model category structure in the sense of Bousfield [103, §3.2].

Descent method and local versions

A crucial method for calculating Picard groups is Galois descent. If $A \to B$ is a G-Galois extension (for G finite), then for the Picard spectra and spaces the following equivalences hold [103, 189]:
$$\mathsf{pic}(A) \simeq \tau_{\geqslant 0}\mathsf{pic}(B)^{hG}, \quad \mathsf{PIC}(A) \simeq \mathsf{PIC}(B)^{hG}. \tag{6.9.1}$$
Here, $\tau_{\geqslant 0}$ denotes the connective cover of a spectrum. In general, the extension B is easier to understand than A; for instance, in the case of the C_2-Galois extension $KO \to KU$, one obtains information about $\mathsf{pic}(A)$ using the homotopy fixed point spectral sequence
$$H^{-s}(G; \pi_t \mathsf{pic}(B)) \Rightarrow \pi_{t-s}(\mathsf{pic}(B)^{hG}).$$

In [121, §6], for instance, Hill and Meier use Galois descent to determine the Picard groups of $\mathsf{TMF}_0(3)$ and $\mathsf{Tmf}_0(3)$:

Theorem 6.9.6 [121, Theorems 6.9, 6.12].

$$\mathsf{Pic}(\mathsf{TMF}_0(3)) \cong \mathbb{Z}/48\mathbb{Z}, \quad \mathsf{Pic}(\mathsf{Tmf}_0(3)) \cong \mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}.$$

Hopkins–Mahowald–Sadofsky started the investigation of the Picard groups of the $K(n)$-local homotopy categories for varying n [129]. They denote these Picard groups by Pic_n. Note that the relevant symmetric monoidal product for fixed n is

$$X \otimes Y = L_{K(n)}(X \wedge Y)$$

for $K(n)$-local X and Y. They determined Pic_1 for all primes p:

Theorem 6.9.7 [129, Theorem 3.3, Proposition 2.7].

- *At the prime* 2, $\mathsf{Pic}_1 \cong \mathbb{Z}_2^\times \times \mathbb{Z}/4\mathbb{Z}$.
- *For all odd primes* p, $\mathsf{Pic}_1 \cong \mathbb{Z}_p \times \mathbb{Z}/q\mathbb{Z}$ *with* $q = 2p-2$.

In the $K(n)$-local setting the notion of algebraic elements in Pic_n is slightly more involved. Hopkins, Mahowald and Sadofsky show [129] (see also [108, Theorem 2.4]) that a $K(n)$-local spectrum X is $K(n)$-locally invertible if and only if $\pi_*(L_{K(n)}(E_n \wedge X))$ is a free $(E_n)_*$-module of rank one and if and only if $\pi_*(L_{K(n)}(E_n \wedge X))$ is invertible as a continuous module over the completed group ring $(E_n)_*[[\mathbb{G}_n]]$. Here, $\mathbb{G}_n$ is the full Morava-stabilizer group. Hence applying $\pi_*(L_{K(n)}(E_n \wedge -))$ gives a map from Pic_n to the Picard group of continuous $(E_n)_*[[\mathbb{G}_n]]$-modules and this group is called $\mathsf{Pic}_n^{\mathrm{alg}}$. The kernel of the map, κ_n, collects the exotic elements in Pic_n:

$$0 \to \kappa_n \to \mathsf{Pic}_n \to \mathsf{Pic}_n^{\mathrm{alg}}.$$

For odd primes, all elements in Pic_1 can be detected algebraically but for $p = 2$ one has a non-trivial element in κ_1. See [108] for Pic_2 at $p = 3$ and a general overview. There is ongoing work on Pic_2 at $p = 2$ by Agnès Beaudry, Irina Bobkova, Paul Goerss and Hans-Werner Henn.

Brauer groups of commutative rings

Probably most of you will know the definition of the Brauer group of a field. But as for many features that we want to transfer to the spectral world we need to consider algebraic concepts developed for commutative rings (not fields).

Azumaya started to think about general Brauer groups [14] in the setting of local rings. A general definition of the Brauer group of a commutative ring R was given by Auslander and Goldman [12] as Morita equivalence classes of Azumaya algebras. The Brauer group was then globalized to schemes by Grothendieck [114]. He also shows that the Brauer group of the initial ring $\mathbb{Z}$ is trivial; this is a byproduct of his identification of Brauer groups of number rings in [114, III, Proposition (2.4)].

Brave new Brauer groups

Baker and Lazarev define in [16] what an Azumaya algebra spectrum is. We use one version of this definition in [20] to develop Brauer groups for commutative ring spectra. Related concepts can be found in [138] and [291].

Fix a cofibrant commutative ring spectrum R.

Definition 6.9.8. A cofibrant associative R-algebra A is called an *Azumaya R-algebra spectrum* if A is dualizable and faithful as an R-module spectrum and if the canonical map

$$A \wedge_R A^o \to F_R(A, A)$$

is a weak equivalence.

We list some crucial properties of Azumaya algebra spectra. For the first property recall the discussion of derived centers from Definition 6.6.10.

Proposition 6.9.9.

1. [16, Proposition 2.3] *If A is an Azumaya R-algebra spectrum, then A is homotopically central over R, i.e., $R \to \mathsf{THH}_R(A)$ is a weak equivalence.*
2. [20, Proposition 1.5] *If A is Azumaya over R and if C is a cofibrant commutative R-algebra then $A \wedge_R C$ is Azumaya over C. Conversely, if C is as above and dualizable and faithful as an R-module, then $A \wedge_R C$ being Azumaya over C implies that A is Azumaya over R.*

   If A and B are Azumaya over R, then $A \wedge_R B$ is also Azumaya over R.
3. [20, 2.2] *If M is a faithful, dualizable, cofibrant R–module, then (a cofibrant replacement of) $F_R(M, M)$ is an R-Azumaya algebra spectrum.*

Thus the endomorphism Azumaya algebras are the ones that are always there and you want to ignore them.

Definition 6.9.10. Let A and B be two Azumaya R-algebra spectra. We call them *Brauer equivalent* if there are dualizable, faithful R-modules N and M such that there is an R-algebra equivalence

$$A \wedge_R F_R(M, M) \simeq B \wedge_R F_R(N, N).$$

We denote by $Br(R)$ the set of Brauer equivalence classes of R-Azumaya algebra spectra.

Note that $Br(R)$ is an abelian group with multiplication induced by the smash product over R. Johnson shows [138, Lemma 5.7] that one can reduce the above relation to what he calls *Eilenberg–Watts equivalence*. This implies that one can still think about the Brauer group of a commutative ring spectrum as the Morita equivalence classes of Azumaya algebra spectra.

We showed a Galois descent result [20, Proposition 3.3], saying that under a natural condition you can descent an Azumaya algebra C over B to an Azumaya algebra C^{hG} over A if $A \to B$ is a faithful G-Galois extension with finite Galois group G.

Examples of Brauer groups

As we know that $Br(\mathbb{Z}) = 0$, we conjectured [20] that the Brauer group of the initial ring spectrum is also trivial. This conjecture was proven in [10, Corollary 7.17]. The authors actually showed a much stronger result:

Theorem 6.9.11 [10, Theorem 7.16]. *If R is a connective commutative ring spectrum such that $\pi_0(R)$ is either $\mathbb{Z}$ or the Witt vectors $W(\mathbb{F}_q)$, then the Brauer group of R is trivial.*

Different approaches — see [10, Definition 7.1], [103, §5], and [289] — can be used to construct a ***Brauer space***, Br_R, for a commutative ring spectrum R and to show that this space is a delooping of the Picard space, PIC

$$\Omega Br_R \simeq \mathsf{PIC}(R)$$

with $\pi_0(Br_R) \cong Br(R)$.

An important question in the classical context of Brauer groups of schemes is to which extent these groups can be controlled by the second étale cohomology group. See the introduction of [291] for a nice overview. Toën shows that for quasi-compact and quasi-separated schemes X one can identify the *derived Brauer group of* X with $H^1_{\text{ét}}(X;\mathbb{G}_m) \times H^2_{\text{ét}}(X;\mathbb{G}_m)$. The work of Antieau and Gepner [10, §7.4] relates Brauer groups of connective commutative ring spectra to étale cohomology groups by establishing a spectral sequence starting from étale cohomology groups for étale sheaves over a connective commutative ring spectrum converging to the homotopy groups of the Brauer space [10, Theorem 7.12].

The integral version of the quaternions gives a non-trivial element in $Br(S[\frac{1}{2}])$ [20, Proposition 6.3]. Antieau and Gepner show in [10, Corollary 7.18]

$$Br(S[\tfrac{1}{p}]) \cong \mathbb{Z}/2\mathbb{Z} \quad \text{for all primes } p$$

and they prove the existence of a short exact sequence

$$0 \to Br(S_{(p)}) \to \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{q \neq p} \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$$

by applying [10, Corollary 7.13], where they calculate the homotopy groups of the Brauer space of any connective commutative ring spectrum R in terms of étale cohomology groups and the homotopy groups of R.

They use the classical exact sequence for the Brauer group of the rationals [114, §2] coming from the Albert–Brauer–Hasse–Noether theorem:

$$0 \to Br(\mathbb{Q}) \to \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{p \text{ prime}} Br(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z} \to 0,$$

with $Br(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$. This determines $Br(\mathbb{Z}[\frac{1}{p}])$ and $Br(\mathbb{Z}_{(p)})$ and this in turn gives the above result for the sphere spectra with p inverted or localized at p.

In [20, Theorem 10.1] we show that the $K(n)$-local Brauer group of the $K(n)$-local sphere is non-trivial at least for odd primes and $n > 1$.

Gepner and Lawson prove a version of Galois descent for a suitable ∞-category of Azumaya algebras:

Theorem 6.9.12 [103, Theorem 6.15]. *There is an equivalence of symmetric monoidal ∞-categories*

$$Az_A \to (Az_B)^{hG}$$

for every G-Galois extension $A \to B$ with finite G.

They also construct a map of ∞-groupoids $Az_R \to Br_R$ for any commutative ring spectrum R and show that this map is essentially surjective, so that equality in $\pi_0(Br_R)$ corresponds precisely to Morita equivalence. They investigate the algebraic Brauer groups (i.e., the Morita classes of Azumaya algebras over the coefficients) [103, §7.1] of 2-periodic commutative ring spectra with vanishing odd homotopy groups, such as KU or E_n, by relating them to the classical Brauer–Wall group of π_0 of the ring spectrum and they identify a non-trivial Morita class of a quaternion KO-algebra that becomes Morita-trivial over KU.

There is recent work by Hopkins and Lurie [128] who identify the $K(n)$-local Brauer group of a Lubin–Tate spectrum E at all primes. For odd primes they obtain:

Theorem 6.9.13 [128, Theorem 1.0.11]. *The $K(n)$-local Brauer group of E is the product of the Brauer–Wall group of the residue field $\pi_0(E)/m$ and a group $Br'(E)$ which in turn can be expressed as an inverse limit of abelian groups Br'_ℓ such that the kernel of $Br'_\ell \to Br'_{\ell-1}$ is non-canonically isomorphic to $m^{\ell+2}/m^{\ell+3}$.*

One ingredient is their construction of *atomic E-algebra spectra* [128, Definition 1.0.2] via a Thom spectrum construction relative to E for polarizations of lattices [128, Definition 3.2.1] using the machinery from [6, 5]. Here, the starting point is a lattice Λ of finite rank together with a *polarization map*

$$Q\colon K(\Lambda, 1) \to \mathsf{PIC}(E) \simeq \mathsf{Pic}(E) \times BGL_1(E).$$