5 Operads and operadic algebras in homotopy theory

by Michael A. Mandell

5.1 Introduction

Operads first appeared in the book *Geometry of iterated loop spaces* by J. P. May [194], though Boardman and Vogt had earlier implicitly defined a mathematically equivalent notion as a "PROP in standard form" [49, §2]. In those works, operads and operadic algebra structures provide a recognition principle and a delooping machine for n-fold loop spaces and infinite loop spaces. The basic idea is that an operad should encode the operations in some kind of homotopical algebraic structure. For example, an n-fold loop space $\Omega^n X$ comes with n different multiplications $(\Omega^n X)^2 \to \Omega^n X$, which can be iterated and generalized to a space of m-ary maps $\mathcal{C}_n(m)$ (from $(\Omega^n X)^m$ to $\Omega^n X$); here $\mathcal{C}_n$ is the Boardman–Vogt little n-cubes operad (see Construction 5.3.5 and Section 5.11 below). The content of the recognition theorem is that $\mathcal{C}_n$ specifies a structure that is essentially equivalent to the structure of an n-fold loop space for connected spaces. It was clear even at the time of introduction that operads were a big idea and in the almost 50 years since then, operads have found a wide range of other uses in a variety of areas of mathematics: a quick MathSciNet search for papers since 2015 with "operad" in the title comes up with papers in combinatorics, algebraic geometry, nonassociative algebra, geometric group theory, free probability, mathematical modeling, and physics, as well as in algebraic topology and homological algebra.

Even the topic of operads in algebraic topology is too broad to cover or even summarize in a single article. This expository article concentrates on what I view as the basic topics in the homotopy theory of operadic algebras: the definition of operads, the definition of algebras over operads, structural aspects of categories of algebras over operads, model structures on algebra categories, and comparison of algebra categories when changing operad or underlying category. In addition, it includes two applications of the theory: the original application to n-fold loop spaces, and an application to algebraic models of homotopy types (chosen purely on the basis of personal bias). This leaves out a long list of other topics that could also fit in this chapter, such as model structures on operads, Koszul duality, deformation theory and Quillen (co)homology, multiplicative structures in stable homotopy theory (for example, on Thom spectra, K-theory spectra, etc.), Deligne and Kontsevich conjectures, string topology, factorization homology, construction of moduli spaces, and Goodwillie calculus, just to name a few areas.

Notation and conventions

Although we concentrate on operads and operadic algebras in topology, much of the background applies very generally. Because of this and because we will want to discuss both the case of spaces and the case of spectra, we will use neutral notation: let $\mathscr{M}$ denote a symmetric monoidal category [145, §1.4], writing $\square$ for the monoidal product and $\mathbf{1}$ for the unit. (We will uniformly omit notation for associativity isomorphisms and typically omit notation for commutativity isomorphisms, but when necessary, we will write c_σ for the commutativity isomorphism associated to a permutation σ.) Usually, we will want $\mathscr{M}$ to have coproducts and sometimes more general colimits, which we will expect to commute with $\square$ on each side (keeping the other side fixed). This exactness of $\square$ is automatic if the monoidal structure is closed [145, §1.5], i.e., if for each fixed object X of $\mathscr{M}$, the functor $(-)\square X$ has a right adjoint; this is often convenient to assume, and when we do, we will use $F(X,-)$ for the right adjoint. The three basic classes of examples to keep in mind are:

(i) Convenient categories of topological spaces, including compactly generated weak Hausdorff spaces [206]; then $\square$ is the categorical product, $\mathbf{1}$ is the final object (one point space), and $F(X,Y)$ is the function space, often written Y^X.
(ii) Modern categories of spectra, including EKMM S-modules [94], symmetric spectra [133], and orthogonal spectra [178]; then $\square$ is the smash product, $\mathbf{1}$ is the sphere spectrum, and $F(-,-)$ is the function spectrum.
(iii) The category of chain complexes of modules over a commutative ring R; then $\square$ is the tensor product over R, $\mathbf{1}$ is the complex R concentrated in degree zero, and $F(-,-)$ is the Hom-complex $\mathrm{Hom}_R(-,-)$.

We now fix a convenient category of spaces and just call it "the category of spaces" and the objects in it "spaces", ignoring the classical category of topological spaces.

In the context of operadic algebras in spectra (i.e., (ii) above), it is often technically convenient to use operads of spaces. However, for uniformity of exposition, we have written this article in terms of operads internally in $\mathscr{M}$. The unreduced suspension functor $\Sigma^\infty_+(-)$ converts operads in spaces to operads in the given category of spectra.

Outline

The basic idea of an operad is that the pieces of it should parametrize a class of m-ary operations. From this perspective, the fundamental example of an operad is the ***endomorphism operad*** of an object X,

$$\mathcal{E}\mathrm{nd}_X(m) := F(X^{(m)},X), \qquad X^{(m)} := \underbrace{X\square\cdots\square X}_{m\text{ factors}},$$

which parametrizes all m-ary maps from X to itself. Abstracting the symmetry and composition properties leads to the definition of operad in [194]. We review this definition in Section 5.2.

Section 5.3 presents some basic examples of operads important in topology, including some A_∞ operads, E_∞ operads, and E_n operads.

May chose the term "operad" to match the term "monad" (see [191]), to show their close connection. Basically, a monad is an abstract way of defining some kind of structure on objects in a category, and an operad gives a very manageable kind of monad. Section 5.4 reviews the monad associated to an operad and defines algebras over an operad.

Section 5.5 gives the basic definition of a module over an operadic algebra and reviews the basics of the homotopy theory of module categories.

Section 5.6 discusses limits and colimits in categories of operadic algebras. It includes a general filtration construction that often provides the key tool to study pushouts of operadic algebras homotopically in terms of colimits in the underlying category. Section 5.7 discusses when categories of operadic algebras are enriched, and in the case of categories of algebras enriched over spaces, discusses the geometric realization of simplicial and cosimplicial algebras. Although these sections may seem less basic and more technical than the previous sections, the ideas here provide the tools necessary for further work with operadic algebras using the modern methods of homotopy theory.

Model structures on categories of operadic algebras provide a framework for proving comparison theorems and rectification theorems. Section 5.8 reviews some aspects of model category theory for categories of operadic algebras. In the terminology of this article, a ***comparison theorem*** is an equivalence of homotopy theories between categories of algebras over different operads that are equivalent in some sense (for example, between categories of algebras over different E_∞ operads) or between categories of algebras over equivalent base categories (for example, E_∞ algebras in spaces versus E_∞ algebras in simplicial sets). A ***rectification theorem*** is a comparison theorem where one of the operads is discrete in some sense: a comparison theorem for the category of algebras over an A_∞ operad and the category of associative algebras is an example of a rectification theorem, as is the comparison theorem for E_∞ algebras and commutative algebras in modern categories of spectra. Section 5.9 discusses these and other examples of comparison and rectification theorems. In both Sections 5.8 and 5.9, instead of stating theorems of maximal generality, we have chosen to provide "Example Theorems" that capture some examples of particular interest in homotopy theory and stable homotopy theory. Both the statements and the arguments provide examples: the arguments apply or can be adapted to apply in a wide range of generality.

The Moore space is an early rectification technique (predating operads and A_∞ monoids) for producing a genuine associative monoid version of the loop space; the construction applies generally to a little 1-cubes algebra to produce an associative algebra that we call the ***Moore algebra***. The concept of modules over an operadic algebra leads to another way of producing an associative algebra, called the ***enveloping***

algebra. Section 5.10 compares these constructions and the rectification of A_∞ algebras constructed in Section 5.9.

Sections 5.11 and 5.12 review two significant applications of the theory of operadic algebras. Section 5.11 reviews the original application: the theory of iterated loop spaces and the recognition principle in terms of E_n algebras. Section 5.12 reviews the equivalence between the rational and p-adic homotopy theory of spaces with the homotopy theory of E_∞ algebras.

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5.2 Operads and endomorphisms

We start with the definition of an operad. The collection of m-ary endomorphism objects $\mathcal{E}\mathrm{nd}_X(m) = F(X^{(m)}, X)$ provides the prototype for the definition, and we use its intrinsic structure to motivate and explain it. Although the endomorphism objects only make sense when the symmetric monoidal category is "closed" (which means that function objects exist), the definition of operad will not require or assume function objects, nor will the definition of operadic algebra in Section 5.4. To take in the picture, it might be best just to take $\mathscr{M}$ to be the category of spaces, the category of vector spaces over a field, or the category of sets on first introduction to this material.

In our basic classes of examples, and more generally as a principle of enriched category theory, function objects behave like sets of morphisms: the counit of the defining adjunction

$$F(X,Y) \mathbin{\square} X \to Y$$

is often called the ***evaluation map*** (and denoted ev). It allows "element-free" definition and study of composition: iterating evaluation maps

$$F(Y,Z) \mathbin{\square} F(X,Y) \mathbin{\square} X \to F(Y,Z) \mathbin{\square} Y \to Z$$

induces (by adjunction) a ***composition map***

$$\circ\colon F(Y,Z) \mathbin{\square} F(X,Y) \to F(X,Z).$$

One can check just using the basic properties of adjunctions that this composition is associative in the obvious sense. It is also unital: the identity element of $\mathscr{M}(X,X)$ specifies a map $1_X\colon \mathbf{1} \to F(X,X)$,

$$\mathrm{id}_X \in \mathscr{M}(X,X) \cong \mathscr{M}(\mathbf{1} \square X, X) \cong \mathscr{M}(\mathbf{1}, F(X,X)),$$

where the first isomorphism is induced by the unit isomorphism; essentially by construction, the composite

$$\mathbf{1} \square X \xrightarrow{1_X \square \mathrm{id}_X} F(X,X) \square X \xrightarrow{ev} X$$

is the unit isomorphism. It follows that the diagram

$$\begin{array}{ccccc}
\mathbf{1} \square F(X,Y) & \xrightarrow{\cong} & F(X,Y) & \xleftarrow{\cong} & F(X,Y) \square \mathbf{1} \\
{\scriptstyle 1_Y \square \mathrm{id}_{F(X,Y)}} \downarrow & & \| & & \downarrow {\scriptstyle \mathrm{id}_{F(X,Y)} \square 1_X} \\
F(Y,Y) \square F(X,Y) & \xrightarrow[\circ]{} & F(X,Y) & \xleftarrow[\circ]{} & F(X,Y) \square F(X,X)
\end{array}$$

commutes, where the top-level isomorphisms are the unit isomorphisms. More is true: the function objects enrich the category $\mathscr{M}$ over itself, and the $\square, F$ parametrized adjunction is itself enriched [145, §1.5–6].

In the case when $\mathscr{M}$ is the category of spaces, the evaluation map is just the map that evaluates functions on their arguments; thinking in these terms will make the formulas and checks clearer for the reader not used to working with adjunctions. Since in the category of spaces $\mathbf{1}$ is the one-point space, a map out of $\mathbf{1}$ just picks out an element of the target space and the map $\mathbf{1} \to F(X,X)$ is just the map that picks out the identity map of X.

The basic compositions above generalize to associative and unital m-ary compositions; now for simplicity and because it is the main case of interest here, we restrict to considering a fixed object X. The m-ary composition takes the form

$$F(X^{(m)}, X) \square (F(X^{(j_1)}, X) \square \cdots \square F(X^{(j_m)}, X)) \to F(X^{(j)}, X),$$

where $j = j_1 + \cdots + j_m$ and (as in the introduction) $X^{(m)}$ denotes the m-th $\square$ power of X; we think of the m-ary composition as plugging in the m j_i-ary maps into the first m-ary map; it is adjoint to the map

$$\begin{aligned}
&F(X^{(m)}, X) \square F(X^{(j_1)}, X) \square \cdots \square F(X^{(j_m)}, X) \square X^{(j)} \cong \\
&\qquad F(X^{(m)}, X) \square F(X^{(j_1)}, X) \square \cdots \square F(X^{(j_m)}, X) \square X^{(j_1)} \square \cdots \square X^{(j_m)} \to X
\end{aligned}$$

that does the evaluation map

$$F(X^{(j_i)}, X) \square X^{(j_i)} \to X,$$

then collects the resulting m factors of X and does the evaluation map

$$F(X^{(m)}, X) \square X^{(m)} \to X.$$

In this double evaluation, implicitly we have shuffled some of the factors of X past some of the endomorphism objects, but we take care not to permute factors of X

among themselves or the endomorphism objects among themselves. This defines a composition map

$$\Gamma^m_{j_1,\dots,j_m} \colon \mathcal{E}\mathrm{nd}_X(m) \,\Box\, \mathcal{E}\mathrm{nd}_X(j_1) \,\Box \cdots \Box\, \mathcal{E}\mathrm{nd}_X(j_m) \to \mathcal{E}\mathrm{nd}_X(j).$$

The composition is associative and unital in the obvious sense (which we write out in the definition of an operad, Definition 5.2.1, below).

We now begin systematically writing $\mathcal{E}\mathrm{nd}_X(m)$ for $F(X^{(m)},X)$. We observe that $\mathcal{E}\mathrm{nd}_X(m) = F(X^{(m)},X)$ has a right action by the symmetric group Σ_m induced by the left action of Σ_m on $X^{(m)}$ corresponding to permuting the $\Box$-factors. In general, for a permutation σ, we write c_σ for the map that permutes $\Box$-factors and a_σ for the action of σ on $\mathcal{E}\mathrm{nd}_X(m)$, i.e., the map that does c_σ on the domain of $\mathcal{E}\mathrm{nd}_X(m) = F(X^{(m)},X)$. We now study what happens when we permute the various factors in the formula for Γ above. (As these are a bit tricky, we do the formulas out here and repeat them below in the definition of an operad, Definition 5.2.1.)

First consider what happens when we permute the factors of X. We have nothing to say for an arbitrary permutation of the factors of X, but in the composition $\Gamma^m_{j_1,\dots,j_m}$, we can say something for a permutation that permutes the factors only within their given blocks of size $j_1,\dots,j_m$, i.e., when the overall permutation σ of all j factors is the block sum of permutations $\sigma_1 \oplus \cdots \oplus \sigma_m$ with σ_i in Σ_{j_i}. By extranaturality, performing the right action of σ_i on $\mathcal{E}\mathrm{nd}_X(j_i)$ and evaluating is the same as applying the left action of σ_i on $X^{(j_i)}$ and evaluating. It follows that the composition $\Gamma^m_{j_1,\dots,j_m}$ is $(\Sigma_{j_1} \times \cdots \times \Sigma_{j_m})$-equivariant where we use the Σ_{j_i}-actions on the $\mathcal{E}\mathrm{nd}_X(j_i)$'s in the source and block sum with the Σ_j-action on $\mathcal{E}\mathrm{nd}_X(j)$ on the target.

Permuting the endomorphism object factors is easier to understand when we also permute the corresponding factors of X. In the context of $\Gamma^m_{j_1,\dots,j_m}$, for σ in Σ_m, let $\sigma_{j_1,\dots,j_m} \in \Sigma_j$ permute the blocks $X^{(j_1)},\dots,X^{(j_m)}$ as σ permutes $1,\dots,m$. So, for example, if $m=3$, $j_1=1$, $j_2=3$, $j_3=2$, and $\sigma=(23)$, then $\sigma_{1,3,2}$ is the permutation

$$(23)_{1,3,2} = \left\{ \begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 5 & 6 & 2 & 3 & 4 \end{smallmatrix} \right\} = (25364).$$

In $\mathcal{E}\mathrm{nd}_X(j_1)\Box\cdots\Box\mathcal{E}\mathrm{nd}_X(j_m)\Box X^{(j)}$, if we apply σ to permute the endomorphism object factors and $\sigma_{j_1,\dots,j_m}$ to permute the X factors, then evaluation pairs the same factors as with no permutation and the diagram

$$\begin{array}{ccc}
(\mathcal{E}\mathrm{nd}_X(j_1) \,\Box \cdots \Box\, \mathcal{E}\mathrm{nd}_X(j_m)) \,\Box\, X^{(j)} & \xrightarrow{\ ev\ } & X^{(m)} \\
\Big\downarrow {\scriptstyle c_\sigma \Box c_{\sigma_{j_1,\dots,j_m}}} & & \Big\downarrow {\scriptstyle c_\sigma} \\
(\mathcal{E}\mathrm{nd}_X(j_{\sigma^{-1}(1)}) \,\Box \cdots \Box\, \mathcal{E}\mathrm{nd}_X(j_{\sigma^{-1}(m)})) \,\Box\, X^{(j)} & \xrightarrow[ev]{} & X^{(m)}
\end{array}$$

commutes. This now tells us what happens with $\Gamma^m_{j_1,\dots,j_m}$ and the permutation action on $\mathcal{E}\mathrm{nd}_X(n)$: the composite of the right action of σ on $\mathcal{E}\mathrm{nd}_X(m)$ with $\Gamma^m_{j_1,\dots,j_m}$,

$$\begin{aligned}
&\mathcal{E}\mathrm{nd}_X(m) \,\Box\, (\mathcal{E}\mathrm{nd}_X(j_1) \,\Box \cdots \Box\, \mathcal{E}\mathrm{nd}_X(j_m)) \\
&\qquad \xrightarrow{a_\sigma \Box \mathrm{id}} \mathcal{E}\mathrm{nd}_X(m) \,\Box\, (\mathcal{E}\mathrm{nd}_X(j_1) \,\Box \cdots \Box\, \mathcal{E}\mathrm{nd}_X(j_m)) \xrightarrow{\Gamma^m_{j_1,\dots,j_m}}, \mathcal{E}\mathrm{nd}_X(j),
\end{aligned}$$

is equal to the composite of the $\square$-permutation c_σ on the $\mathcal{E}\mathrm{nd}(j_i)$'s, the composition map $\Gamma^m_{j_{\sigma^{-1}(1)},\dots,j_{\sigma^{-1}(m)}}$, and the right action of $\sigma_{j_1,\dots,j_m}$ on $\mathcal{E}\mathrm{nd}_X(j)$:

$$\begin{aligned}
&\mathcal{E}\mathrm{nd}_X(m)\,\square\,(\mathcal{E}\mathrm{nd}_X(j_1)\,\square\cdots\square\,\mathcal{E}\mathrm{nd}_X(j_m))\\
&\qquad\xrightarrow{\mathrm{id}\,\square\, c_\sigma}\mathcal{E}\mathrm{nd}_X(m)\,\square\,(\mathcal{E}\mathrm{nd}_X(j_{\sigma^{-1}(1)})\,\square\cdots\square\,\mathcal{E}\mathrm{nd}_X(j_{\sigma^{-1}(m)}))\\
&\qquad\qquad\xrightarrow{\Gamma^m_{j_{\sigma^{-1}(1)},\dots,j_{\sigma^{-1}(m)}}}\mathcal{E}\mathrm{nd}_X(j)\xrightarrow{a_{\sigma_{j_1,\dots,j_m}}}\mathcal{E}\mathrm{nd}_X(j).
\end{aligned}$$

See Figure 5.2 on p. 191 for this equation written as a diagram.

Although we did not emphasize this above, we need to allow any of m, $j_1,\dots,j_m$, or j to be zero, where we understand empty $\square$-products to be the unit $\mathbf{1}$. The formulations above still work with this extension, using the unit isomorphism where necessary. The purpose of allowing these "zero-ary" operations is that it allows us to encode a unit object into the structure: For example, in the context of spaces $\mathbf{1}$ is the one point space $*$ and to describe the structure of a topological monoid, not only do we need the binary operation $X\times X\to X$, but we also need the zero-ary operation $*\to X$ for the unit.

Rewriting the properties of $\mathcal{E}\mathrm{nd}_X$ above as a definition, we get an element-free version of the definition of operad of May [194, 1.2].[1]

Definition 5.2.1. An operad in a symmetric monoidal category $\mathscr{M}$ consists of a sequence of objects $\mathcal{O}(m)$, $m=0,1,2,3,\dots$, together with

(a) a right action of the symmetric group Σ_m on $\mathcal{O}(m)$ for all m,
(b) a *unit map* $1\colon\mathbf{1}\to\mathcal{O}(1)$, and
(c) a *composition rule*

$$\Gamma^m_{j_1,\dots,j_m}\colon\mathcal{O}(m)\,\square\,\mathcal{O}(j_1)\,\square\cdots\square\,\mathcal{O}(j_m)\to\mathcal{O}(j)$$

for every m, $j_1,\dots,j_m$, where $j=j_1+\cdots+j_m$, typically written Γ when m and $j_1,\dots,j_m$ are understood or irrelevant,

satisfying the following conditions:

(i) The composition rule Γ is associative in the sense that for any m, $j_1,\dots,j_m$ and $k_1,\dots,k_j$, letting $j=j_1+\cdots+j_m$, $k=k_1+\cdots+k_j$, $t_i=j_1+\cdots+j_{i-1}$ (with $t_1=0$), and $s_i=k_{t_i+1}+\cdots+k_{t_i+j_i}$, the equation

$$\begin{aligned}
&\Gamma^j_{k_1,\dots,k_j}\circ(\Gamma^m_{j_1,\dots,j_m}\,\square\,\mathrm{id}_{\mathcal{O}(k_1)}\,\square\cdots\square\,\mathrm{id}_{\mathcal{O}(k_j)})\\
&\qquad\qquad=\Gamma^m_{s_1,\dots,s_m}\circ(\mathrm{id}_{\mathcal{O}(m)}\,\square\,\Gamma^{j_1}_{k_1,\dots,k_{j_1}}\,\square\cdots\square\,\Gamma^{j_m}_{k_{t_m+1},\dots,k_j})\circ c
\end{aligned}$$

holds in the set of maps

$$\mathcal{O}(m)\,\square\,\mathcal{O}(j_1)\,\square\cdots\square\,\mathcal{O}(j_m)\,\square\,\mathcal{O}(k_1)\,\square\cdots\square\,\mathcal{O}(k_j)\to\mathcal{O}(k),$$

[1] In the original definition, May required $\mathcal{O}(0)=\mathbf{1}$ in order to provide $\mathcal{O}$-algebras with units, which was desirable in the iterated loop space context, but standard convention has since dropped this requirement to allow non-unital algebras and other unit variants.

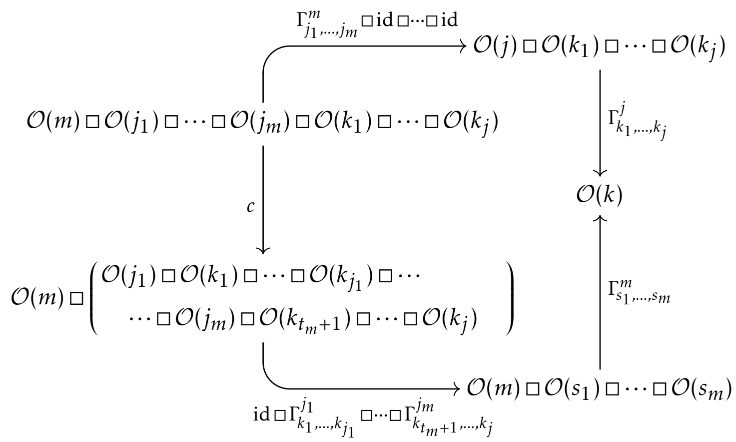


Figure 5.1 The diagram for 5.2.1(i). Here c is the $\square$-permutation that shuffles $\mathcal{O}(k_\ell)$'s past $\mathcal{O}(j_i)$'s as displayed, $j = j_1 + \cdots + j_m$, $t_i = j_1 + \cdots + j_{i-1}$ (with $t_1 = 0$), $s_i = k_{t_i+1} + \cdots + k_{t_i+j_i}$, and $k = k_1 + \cdots + k_j = s_1 + \cdots + s_m$.

where c is the $\square$-permutation

$$\mathcal{O}(m)\,\square\,\mathcal{O}(j_1)\,\square\cdots\square\,\mathcal{O}(j_m)\,\square\,\mathcal{O}(k_1)\,\square\cdots\square\,\mathcal{O}(k_j) \to$$
$$\mathcal{O}(m)\,\square\,(\mathcal{O}(j_1)\,\square\,\mathcal{O}(k_1)\,\square\cdots\square\,\mathcal{O}(k_{j_1}))\,\square\cdots\square\,(\mathcal{O}(j_m)\,\square\,\mathcal{O}(k_{t_m+1})\,\square\cdots\square\,\mathcal{O}(k_j)).$$

that shuffles the $\mathcal{O}(k_\ell)$'s and $\mathcal{O}(j_i)$'s as displayed (see Figure 5.1 for the diagram).

(ii) The unit map 1 is a left and right unit for the composition rule Γ in the sense that

$$\Gamma^1_m \circ (1\,\square\,\mathrm{id}) : \quad \mathbf{1}\,\square\,\mathcal{O}(m) \xrightarrow{1\square\mathrm{id}} \mathcal{O}(1)\,\square\,\mathcal{O}(m) \xrightarrow{\Gamma^1_m} \mathcal{O}(m)$$

is the unit isomorphism and

$$\Gamma^m_{1,\dots,1} \circ (\mathrm{id}\,\square\,1^{(m)}) : \quad \mathcal{O}(m)\,\square\,\mathbf{1}^{(m)} \xrightarrow{\mathrm{id}\square 1^{(m)}} \mathcal{O}(m)\,\square\,\mathcal{O}(1)^{(m)} \xrightarrow{\Gamma^m_{1,\dots,1}} \mathcal{O}(m)$$

is the iterated unit isomorphism for $\mathcal{O}(m)$ for all m.

(iii) The map $\Gamma^m_{j_1,\dots,j_m}$ is $(\Sigma_{j_1} \times \cdots \times \Sigma_{j_m})$-equivariant for the block sum inclusion of $\Sigma_{j_1} \times \cdots \times \Sigma_{j_m}$ in Σ_j.

(iv) For any m, $j_1, \dots, j_m$ and any $\sigma \in \Sigma_m$, the equation

$$\Gamma^m_{j_1,\dots,j_m} \circ (a_\sigma\,\square\,\mathrm{id}_{\mathcal{O}(j_1)}\,\square\cdots\square\,\mathrm{id}_{\mathcal{O}(j_m)}) = a_{\sigma_{j_1,\dots,j_m}} \circ \Gamma^m_{j_{\sigma^{-1}(1)},\dots,j_{\sigma^{-1}(m)}} \circ (\mathrm{id}_{\mathcal{O}(m)}\,\square\,c_\sigma)$$

holds in the set of maps

$$\mathcal{O}(m)\,\square\,\mathcal{O}(j_1)\,\square\cdots\square\,\mathcal{O}(j_m) \to \mathcal{O}(j),$$

where $\sigma_{j_1,\dots,j_m}$ denotes the block permutation in Σ_j corresponding to σ on the blocks of size $j_1, \dots, j_m$, a denotes the right action of (a), and c_σ denotes the $\square$-permutation corresponding to σ (see Figure 5.2 for the diagram).

$$\begin{array}{ccc}
\mathcal{O}(m)\,\square\,\mathcal{O}(j_1)\,\square\cdots\square\,\mathcal{O}(j_m) & \xrightarrow{a_\sigma\square\mathrm{id}\square\cdots\square\mathrm{id}} & \mathcal{O}(m)\,\square\,\mathcal{O}(j_1)\,\square\cdots\square\,\mathcal{O}(j_m) \\
 & & \downarrow{\scriptstyle \Gamma^m_{j_1,\dots,j_m}} \\
\downarrow{\scriptstyle \mathrm{id}\,\square\, c_\sigma} & & \mathcal{O}(j) \\
 & & \uparrow{\scriptstyle a_{\sigma_{j_1,\dots,j_m}}} \\
\mathcal{O}(m)\,\square\,\mathcal{O}(j_{\sigma^{-1}(1)})\,\square\cdots\square\,\mathcal{O}(j_{\sigma^{-1}(m)}) & \xrightarrow[\Gamma^m_{j_{\sigma^{-1}(1)},\dots,j_{\sigma^{-1}(m)}}]{} & \mathcal{O}(j)
\end{array}$$

Figure 5.2 The diagram for 5.2.1(iv). Here $\sigma \in \Sigma_m$, c_σ is the $\square$-permutation corresponding to σ, $\sigma_{j_1,\dots,j_m} \in \Sigma_j$ is the block permutation performing σ on blocks of sizes $j_1,\dots,j_m$, $j = j_1 + \cdots + j_m$, and a denotes the Σ_m action on $\mathcal{O}(m)$ and the Σ_j-action on $\mathcal{O}(j)$.

A map of operads consists of a map of each object that commutes with the structure:

Definition 5.2.2. A map of operads $(\{\mathcal{O}(m)\}, 1, \Gamma) \to (\{\mathcal{O}'(m)\}, 1', \Gamma')$ consists of Σ_m-equivariant maps $\phi_m \colon \mathcal{O}(m) \to \mathcal{O}'(m)$ for all m such that

$$\Gamma'^m_{j_1,\dots,j_m} \circ (\phi_m \,\square\, \phi_{j_1} \,\square \cdots \square\, \phi_{j_m}) = \phi_j \circ \Gamma^m_{j_1,\dots,j_m}$$

for all m, $j_1,\dots,j_m$ and $1' = \phi_1 \circ 1$; in commuting diagrams:

$$\begin{array}{ccc}
\mathcal{O}(m)\,\square\,\mathcal{O}(j_1)\,\square\cdots\square\,\mathcal{O}(j_m) & \xrightarrow{\Gamma^m_{j_1,\dots,j_m}} & \mathcal{O}(j) \\
\downarrow{\scriptstyle \phi_m\square\phi_{j_1}\square\cdots\square\phi_{j_m}} & & \downarrow{\scriptstyle \phi_j} \\
\mathcal{O}'(m)\,\square\,\mathcal{O}'(j_1)\,\square\cdots\square\,\mathcal{O}'(j_m) & \xrightarrow[\Gamma'^m_{j_1,\dots,j_m}]{} & \mathcal{O}'(j)
\end{array}
\qquad
\begin{array}{ccc}
 & \mathbf{1} & \\
{\scriptstyle 1}\swarrow & & \searrow{\scriptstyle 1'} \\
\mathcal{O}(1) & \xrightarrow[\phi_1]{} & \mathcal{O}'(1).
\end{array}$$

The endomorphism operad $\mathcal{E}\mathrm{nd}_X$ gives an example of an operad in any closed symmetric monoidal category (for any object X). Here are some additional important examples.

Example 5.2.3 (The identity operad). Assume the symmetric monoidal category $\mathscr{M}$ has an initial object $\emptyset$. If $\square$ preserves the initial object in each variable, $\emptyset \,\square\, (-) \cong \emptyset \cong (-) \,\square\, \emptyset$ (which is automatic in the closed case, i.e., when function objects exist), we also have the example of the ***identity operad*** $\mathcal{I}$, which has $\mathcal{I}(1) = \mathbf{1}$ (with 1 the identity) and $\mathcal{I}(m)$ the initial object for $m \neq 1$; this is the initial object in the category of operads.

Example 5.2.4 (The commutative algebra operad). The operad $\mathcal{C}\mathrm{om}$ exists in any symmetric monoidal category:

$$\mathcal{C}\mathrm{om}(m) = \mathbf{1}$$

for all m with the trivial symmetric group actions and composition law Γ given by the unit isomorphism; its category of algebras (see the next section) is isomorphic to the category of commutative monoids for $\square$ in $\mathscr{M}$ (defined in terms of the usual diagrams, i.e., [174, VII§3] plus commutativity); see Example 5.4.3.

Example 5.2.5 (The associative algebra operad). If $\mathscr{M}$ has finite coproducts and $\Box$ preserves finite coproducts in each variable, then we also have the operad $\mathcal{A}ss$:

$$\mathcal{A}ss(m) = \coprod_{\Sigma_m} \mathbf{1}$$

with symmetric group action induced by the natural (right) action of Σ_m on Σ_m and composition law Γ induced by block permutation and block sum of permutations,

$$\sigma \in \Sigma_m,\ \tau_1 \in \Sigma_{j_1}, \ldots, \tau_m \in \Sigma_{j_m} \mapsto \sigma_{j_1,\ldots,j_m} \circ (\tau_1 \oplus \cdots \oplus \tau_m) \in \Sigma_j.$$

Its category of algebras is isomorphic to the category of monoids for $\Box$ in $\mathscr{M}$; see Example 5.4.4.

For operads like $\mathcal{A}ss$, it is often useful to work in terms of ***non-symmetric operads***, which come without the permutation action.

Definition 5.2.6. A ***non-symmetric operad*** consists of a sequence of objects $\mathcal{O}(m)$, $m = 0, 1, 2, 3, \ldots$, together with a unit map and composition rule as in 5.2.1(b) and (c) satisfying the associativity and unit rules of 5.2.1(i) and (ii). A map of non-symmetric operads consists of a map of their object sequences that commutes with the unit map and the composition rule.

Forgetting the permutation action on $\mathcal{C}om$ gives a non-unital operad called $\overline{\mathcal{A}ss}$ that is the non-symmetric version of the operad $\mathcal{A}ss$. In general, under the finite coproduct assumption in Example 5.2.5, given a non-symmetric operad $\overline{\mathcal{O}}$, the product $\overline{\mathcal{O}} \Box \mathcal{A}ss$ has the canonical structure of an operad; it is the ***operad associated to*** $\overline{\mathcal{O}}$. In the category of spaces (or sets, but not in the category of abelian groups, the category of chain complexes, or the various categories of spectra), an operad $\mathcal{O}$ comes from a non-symmetric operad exactly when it admits a map to $\mathcal{A}ss$: the corresponding non-symmetric operad $\overline{\mathcal{O}}$ has $\overline{\mathcal{O}}(n)$ the subobject that maps to the identity permutation summand of $\mathcal{A}ss$, and there is a canonical isomorphism $\mathcal{O} \cong \overline{\mathcal{O}} \Box \mathcal{A}ss$ (which depends only on the original choice of map $\mathcal{O} \to \mathcal{A}ss$).

5.3 A_∞, E_∞, and E_n operads

This section reviews some of the most important classes of examples of operads in homotopy theory, the A_∞, E_∞, and E_n operads. We concentrate on the case of (unbased) spaces, with notes about the appropriate definition of such operads in other contexts. For example, in stable homotopy theory, the unbased suspension spectrum functor Σ^∞_+ converts model E_n operads into operads in the various modern categories of spectra. The universal role played by spaces in homotopy theory typically allows for reasonable definitions of these classes of operads in any homotopy theoretic setting.

The terminology of A_∞ space and the basic model of an A_∞ operad, due to Stasheff [282], preceded the definition of operad by several years.

Definition 5.3.1. An A_∞ operad in spaces is a non-symmetric operad whose m-th space is contractible for all m.

Informally, an operad (with symmetries) is A_∞ when there is an understood isomorphism to the operad associated to some A_∞ operad. The definition of A_∞ operad usually has a straightforward generalization to other symmetric monoidal categories with a notion of homotopy theory: contractibility corresponds to a weak equivalence with the unit $\mathbf{1}$ of the symmetric monoidal structure, and we should add the requirement that the non-symmetric operad composition rule should be a weak equivalence for all indexes (which is automatic in spaces). One wrinkle is that a flatness condition may be needed and should be imposed to ensure that the functor $\overline{\mathcal{O}}(m) \,\square\, X^{(m)}$ is weakly equivalent to $X^{(m)}$ (cf. Section 5.9); in the case of spaces, contractibility implicitly includes such a condition. In symmetric spectra and orthogonal spectra, a good flatness condition is to be homotopy equivalent to a cofibrant object; in EKMM S-modules, a good flatness condition is to be homotopy equivalent to a semi-cofibrant object (see [157, §6]).

We have already seen an example of an A_∞ operad: the operad $\overline{\mathcal{A}ss}$. The associahedra $\mathcal{K}(m)$ of Stasheff [282, I.§6] have the structure of a non-symmetric operad using the insertion maps [*ibid.*] for the composition rule, and this is an example of an A_∞ operad. The Boardman–Vogt little 1-cubes (non-symmetric) operad $\overline{\mathcal{C}}_1$ described below gives a third example.

Next we discuss E_∞ operads. Recall that a free Σ_m-cell complex is a space built by cells of the form $(\Sigma_m \times D^n, \Sigma_m \times S^{n-1})$, where D^n denotes the unit disk in $\mathbb{R}^n$. The definition of E_∞ operad asks for the constituent spaces to have the Σ_m-equivariant homotopy type of a free Σ_m-cell complex and the non-equivariant homotopy type of a point.

Definition 5.3.2. An operad $\mathcal{E}$ in spaces is an E_∞ operad when for each m, its m-th space is a universal Σ_m space: $\mathcal{E}(m)$ has the Σ_m-equivariant homotopy type of a free Σ_m-cell complex and is non-equivariantly contractible.

Unlike the A_∞ case, the operad $\mathcal{C}om$ is not E_∞ as its spaces do not have free actions. The Barratt–Eccles operad $\mathcal{E}\Sigma$ provides an example:

Example 5.3.3 (The Barratt–Eccles operad). Let $\mathcal{E}\Sigma(m)$ denote the nerve of the category $E\Sigma_m$ whose set of objects is Σ_m and which has a unique map between any two objects. The symmetric group Σ_m acts strictly on the category and the nerve $\mathcal{E}\Sigma(m)$ inherits a Σ_m-action; moreover, as the action of Σ_m on the simplices is free, the simplicial triangulation of $\mathcal{E}\Sigma(m)$ has the structure of a free Σ_m-cell complex. It is non-equivariantly contractible because every object of $E\Sigma_m$ is a zero object. The multiplication is induced by an operad structure on the sequence of categories using block sums of permutations as in the operad structure on $\mathcal{A}ss$. The resulting operad is called the *Barratt–Eccles operad*.

Boardman and Vogt [49, §2] defined another E_∞ operad, built out of linear isometries.

Example 5.3.4 (The linear isometries operad). The Boardman–Vogt linear isometries operad $\mathcal{L}$ has its m-th space the space of linear isometries

$$(\mathbb{R}^\infty)^m = \mathbb{R}^\infty \oplus \cdots \oplus \mathbb{R}^\infty \to \mathbb{R}^\infty$$

(where $\mathbb{R}^\infty = \bigcup \mathbb{R}^n$), with operad structure defined as in the example of an endomorphism operad. The topology comes from the identification

$$\mathcal{L}(m) = \lim_k \operatorname{colim}_n \mathcal{I}((\mathbb{R}^k)^m, \mathbb{R}^n)$$

for $\mathcal{I}((\mathbb{R}^k)^m, \mathbb{R}^n)$ the space of linear isometries $(\mathbb{R}^k)^m \to \mathbb{R}^n$ (with the usual manifold topology). The Σ_m-action induced by the action on the direct sum $(\mathbb{R}^\infty)^m$ is clearly free; each $\mathcal{I}((\mathbb{R}^k)^m, \mathbb{R}^n)$ is a Σ_m-manifold, and $\mathcal{L}(m)$ is homotopy equivalent to a free Σ_m-cell complex. Since $\mathcal{I}((\mathbb{R}^k)^m, \mathbb{R}^n)$ is $(n - km - 1)$-connected, it follows that $\mathcal{L}(m)$ is non-equivariantly contractible.

The Boardman–Vogt little ∞-cubes operad $\mathcal{C}_\infty$ described below gives a third example of an E_∞ operad.

The requirement for freeness derives from infinite loop space theory. As we review in Section 5.11, infinite loop spaces are algebras for the little ∞-cubes operad $\mathcal{C}_\infty$. As we review in Section 5.9, for any E_∞ operad $\mathcal{E}$ in spaces, the category of $\mathcal{E}$-algebras has an equivalent homotopy theory to the category of $\mathcal{C}_\infty$-algebras. On the other hand, any algebra in spaces for the operad $\mathcal{C}om$ must be a generalized Eilenberg–Mac Lane space, and the category of $\mathcal{C}om$-algebras does not have an equivalent homotopy theory to the category of $\mathcal{C}_\infty$-algebras. In generalizing the notion of E_∞ to other categories, getting the right category of algebras is key. For symmetric spectra, orthogonal spectra, and EKMM S-modules and for chain complexes of modules over a ring containing the rational numbers, it is harmless to allow $\mathcal{C}om$ to fit the definition of E_∞ operad (cf. Examples 5.9.3, 5.9.4); in spaces and chain complexes of modules over a finite field, some freeness condition is required. In general, the condition should be a flatness condition on $\mathcal{O}(m)$ for $(\mathcal{O}(m) \mathbin{\square} X^{(m)})/\Sigma_m$ as a functor of X (for suitable X) (cf. Definition 5.9.1).

Unlike the definition of E_∞ or A_∞ operad, which are defined in terms of homotopical conditions on the constituent spaces, the definition of E_n operads for other n depends on specific model operads first defined by Boardman–Vogt [49] and called the ***little n-cubes operads $\mathcal{C}_n$***.

Construction 5.3.5 (The little n-cubes operad). The m-th space $\mathcal{C}_n(m)$ of the little n-cubes operad is the space of m ordered almost disjoint parallel axis affine embeddings of the unit n-cube $[0,1]^n$ in itself. So $\mathcal{C}_n(0)$ is a single point representing the unique way to embed 0 unit n-cubes in the unit n-cube. A parallel axis affine embedding of the unit cube in itself is a map of the form

$$(t_1, \ldots, t_n) \in [0,1]^n \mapsto (x_1 + a_1 t_1, \ldots, x_n + a_n t_n) \in [0,1]^n$$

for some fixed $(x_1, \ldots, x_n)$ and $(a_1, \ldots, a_n)$ with each $a_i > 0$, $x_i \geq 0$, and $x_i + a_i \leq 1$; it is determined by the point $(x_1, \ldots, x_n)$ where it sends $(0, \ldots, 0)$ and the point

$$(y_1, \ldots, y_n) = (x_1 + a_1, \ldots, x_n + a_n)$$

where it sends $(1,\dots,1)$. So $\mathcal{C}_n(1)$ is homeomorphic to the subspace

$$\{((x_1,\dots,x_n),(y_1,\dots,y_n)) \in [0,1]^n \times [0,1]^n \mid x_1 < y_1, x_2 < y_2, \dots, x_n < y_n\}$$

of $[0,1]^n \times [0,1]^n$. For $m \geq 2$, almost disjoint means that the images of the open subcubes are disjoint (the embedded cubes only intersect on their boundaries), and $\mathcal{C}_n(m)$ is homeomorphic to a subset of $\mathcal{C}_n(1)^m$. The map 1 is specified by the element of $\mathcal{C}_n(1)$ that gives the identity embedding of the unit n-cube. The action of the symmetric group is to re-order the embeddings. The composition law $\Gamma^m_{j_1,\dots,j_m}$ composes the j_1 embeddings in $\mathcal{C}_n(j_1)$ with the first embedding in $\mathcal{C}_n(m)$, the j_2 embeddings in $\mathcal{C}_n(j_2)$ with the second embedding in $\mathcal{C}_n(m)$, etc., to give $j = j_1 + \cdots + j_m$ total embeddings. See Figure 5.3 for a picture in the case $n = 2$. Taking cartesian product with the identity map on $[0,1]$ takes a self-embedding of the unit n-cube to a self-embedding of the unit $(n+1)$-cube and induces maps of operads $\mathcal{C}_n \to \mathcal{C}_{n+1}$ that are closed inclusions of the underlying spaces. Let $\mathcal{C}_\infty(m) = \bigcup \mathcal{C}_n(m)$; the operad structures on the $\mathcal{C}_n$ fit together to define an operad structure on $\mathcal{C}_\infty$.

The space $\mathcal{C}_n(m)$ has the Σ_m-equivariant homotopy type of the configuration space $C(m,\mathbb{R}^n)$ of m (ordered) points in $\mathbb{R}^n$, or equivalently, $C(m,(0,1)^n)$ of m points in $(0,1)^n$. To see this, since both spaces are free Σ_m-manifolds (non-compact, and with boundary in the case of $\mathcal{C}_n(m)$), it is enough to show that they are non-equivariantly weakly equivalent, but it is in fact no harder to produce a Σ_m-equivariant homotopy equivalence explicitly. We have a Σ_m-equivariant map $\mathcal{C}_n(m) \to C(m,(0,1)^n)$ by taking the center point of each embedded subcube. It is easy to define a Σ_m-equivariant section of this map by continuously choosing cubes centered on the given configuration; one way to do this is to make them all have the same equal side length of 1/2 of the minimum of the distance between each of the points and the distance from each point to the boundary of $[0,1]^n$. A Σ_m-equivariant homotopy from the composite map on $\mathcal{C}_n(m)$ to the identity could (for example) first linearly shrink all sides that are bigger than their original length and then linearly expand all remaining sides to their original length. In particular, $\mathcal{C}_n(1)$ is always contractible and $\mathcal{C}_n(2)$ is Σ_2-equivariantly homotopy equivalent to the sphere S^{n-1} with the antipodal action. For

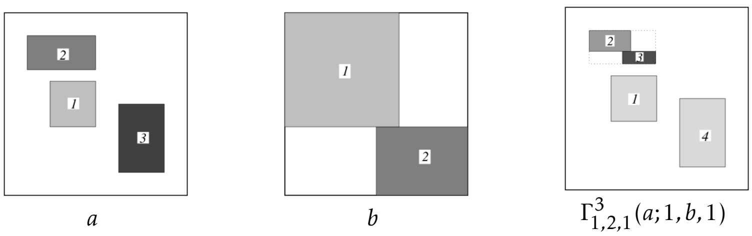


Figure 5.3 Composition of little 2-cubes. Shown is the composition

$$\Gamma^3_{1,2,1} : \mathcal{C}_2(3) \times \mathcal{C}_2(1) \times \mathcal{C}_2(2) \times \mathcal{C}_2(1) \to \mathcal{C}_2(4)$$

applied to elements $a \in \mathcal{C}_2(3)$, $1 \in \mathcal{C}_2(1)$, $b \in \mathcal{C}_2(2)$, $1 \in \mathcal{C}_2(1)$, with a and b as pictured.

$m > 2$, the configuration spaces can be described in terms of iterated fibrations, and their Borel homology was calculated by Cohen in [70] and [71, IV].

We can say more about the homotopy types in the cases $n = 1$, $n = 2$, and $n = \infty$. For $n = 1$, the natural order of the interval $[0, 1]$, gives a natural order to the embedded sub-intervals (1-cubes); let $\bar{\mathcal{C}}_1(m)$ denote the subspace of $\mathcal{C}_1(m)$ where the sub-intervals are numbered in their natural order. The spaces $\bar{\mathcal{C}}_1(m)$ are contractible and form a non-symmetric operad with $\mathcal{C}_1$ (canonically) isomorphic to the associated operad. In other words, the map of operads $\mathcal{C}_1 \to \mathcal{A}ss$ that takes a sequence of embeddings and just remembers the order they come in is a Σ_m-equivariant homotopy equivalence at each level. In particular $\mathcal{C}_1$ is an A_∞ operad.

For $n = 2$, the configuration space $C(m, \mathbb{R}^2)$ is easily seen to be an Eilenberg–Mac Lane space $K(A_m, 1)$, where A_m is the pure braid group (of braids with fixed endpoints) on m strands (see, for example, [194, §4]).

For $n = \infty$, $\mathcal{C}_\infty$ is an E_∞ operad; each $\mathcal{C}_\infty(m)$ is a universal Σ_m-space. To see this, it is easier to work with

$$C(m, \mathbb{R}^\infty) := \bigcup C(m, \mathbb{R}^n).$$

Choosing a homeomorphism $(0, 1) \cong \mathbb{R}$ that sends $1/2$ to 0, the induced homeomorphisms $\mathcal{C}_n(m) \to C(m, \mathbb{R}^n)$ are compatible with the inclusions $\mathcal{C}_n(m) \to \mathcal{C}_{n+1}(m)$ and $C(m, \mathbb{R}^n) \to C(m, \mathbb{R}^{n+1})$; as these inclusions are embeddings of closed submanifolds (with boundary in the case of $\mathcal{C}_n(m)$), the induced map

$$\mathcal{C}_\infty(m) = \bigcup \mathcal{C}_n(m) \to \bigcup C(m, \mathbb{R}^n) = C(m, \mathbb{R}^\infty)$$

remains a homotopy equivalence. One way to see that $C(m, \mathbb{R}^\infty)$ is non-equivariantly contractible is to start by choosing a homotopy though injective linear maps from the identity on $\mathbb{R}^\infty$ to the shift map that on basis elements sends e_i to e_{i+m}. We then homotope the configuration (which now starts with the first m coordinates all zero) so that the i-th point has i-th coordinate 1 and the remainder of the first m coordinates zero. Finally, we homotope the configuration to the configuration with i-th point at e_i.

We use the operads $\mathcal{C}_n$ to define E_n operads:

Definition 5.3.6. An operad $\mathcal{E}$ in spaces is an E_n operad when there is a zigzag of maps of operads relating it to $\mathcal{C}_n$, each of which is a Σ_m-equivariant homotopy equivalence on m-th spaces for all m.

This definition is standard, but a bit awkward, because it defines a property, whereas a better definition would define a structure and ask for at least a preferred equivalence class of zigzag.

As we review in Section 5.9, such maps induce equivalences of homotopy categories of algebras (indeed, Quillen equivalences). We have implicitly given two different definitions of E_∞ operad; the following proposition justifies this.

Proposition 5.3.7. *An operad $\mathcal{E}$ of spaces is E_∞ in the sense of Definition 5.3.2 if and only if it is E_∞ in the sense of Definition 5.3.6.*

Before reviewing the proof, we state a closely related proposition.

Proposition 5.3.8. *An operad $\mathcal{E}$ of spaces is E_1 if and only if it is isomorphic to the associated operad of an A_∞ operad.*

The previous two propositions (and their common proof) are the gist of the second half of §3 of May [194]. In each case one direction is clear, since $\mathcal{C}_1$ and $\mathcal{C}_\infty$ are A_∞ and E_∞ (respectively), and the conditions of Definitions 5.3.1 and 5.3.2 are preserved by the zigzags considered in Definition 5.3.6. The proof of the other direction is to exhibit an explicit zigzag:

Proof. Let $\mathcal{E}$ be the operad in question and assume it is either E_∞ in the sense of Definition 5.3.2 (for the first proposition) or A_∞ in the sense of Definition 5.3.1 ff. (for the second proposition). In the case of the first proposition, consider the product in the category of operads $\mathcal{C}_\infty \times \mathcal{E}$; it satisfies

$$(\mathcal{C}_\infty \times \mathcal{E})(m) = \mathcal{C}_\infty(m) \times \mathcal{E}(m)$$

with the diagonal Σ_m-action and the unit and composition maps the product of those for $\mathcal{C}_\infty$ and $\mathcal{E}$. The projections

$$\mathcal{C}_\infty \leftarrow \mathcal{C}_\infty \times \mathcal{E} \rightarrow \mathcal{E}$$

give a zigzag as required by Definition 5.3.6. For the second proposition, do the same trick with the non-symmetric operads $\bar{\mathcal{E}}$ and $\bar{\mathcal{C}}_1$ and then pass to the associated operads. □

Definitions 5.3.1 and 5.3.2 mean that identifying A_∞ and E_∞ operads is pretty straightforward. In unpublished work, Fiedorowicz [98] defines the notion of a ***braided operad***, which provides a good criterion for identifying E_2 operads. For $n > 2$ (finite), the spaces $\mathcal{C}_n(m)$ are not Eilenberg–Mac Lane spaces (for $m > 1$), and that makes identification of such operads much harder; however, Berger [36, 1.16] proves a theorem (which he attributes to Fiedorowicz) that gives a method to identify E_n operads that seems to work well in practice; see [205, §14], [37, §1.6].

The work of Dunn [85] and Fiedorowicz–Vogt [97] is the start of an abstract identification of E_n operads: The derived tensor product of n E_1 operads is an E_n operad. Here "tensor product" refers to the Boardman–Vogt tensor product of operads (or PROPs) in [48, 2§3], which is the universal pairing subject to "interchange", meaning that an $\mathcal{O} \otimes \mathcal{P}$-algebra structure consists of an $\mathcal{O}$-algebra and a $\mathcal{P}$-algebra structure on a space where the $\mathcal{O}$- and $\mathcal{P}$-structure maps commute (see ***ibid.*** for more details on the construction of the tensor product). This still essentially defines E_n operads in terms of reference models, though in principle, it gives a wide range of additional models. (I do not know an example where this is actually put to use, but [62] comes close.) The concept of interchange makes sense in any cartesian symmetric monoidal structure, so this also in principle tells how to extend the notion of E_n to other cartesian symmetric monoidal categories with a homotopy theory of operads for which the Boardman–Vogt tensor product is reasonably well-behaved. (Again, I know no examples where this is put to use, but perhaps work by Barwick (unpublished), Gepner (unpublished), and Lurie [164] on E_n structures is in a similar spirit.)

In categories suitably related to spaces, E_n algebras are defined by a reference model suitably related to $\mathcal{C}_n$. For example, in the context of simplicial sets, the total singular complex of the little n-cubes operad has the canonical structure of an operad of simplicial sets, and we define E_n operads in terms of this reference model. In symmetric spectra and orthogonal spectra, we have the reference model given by the unbased suspension spectrum functor: an operad is an E_n operad when it is related to $\Sigma^\infty_+\mathcal{C}_n$ by a zigzag of operad maps that are (non-equivariant) weak equivalences on m-th objects for all m. For categories of chain complexes, we use the singular chain complex of the little n-cubes operad to define the reference model. To make the singular chains an operad, we use the Eilenberg–Mac Lane shuffle map to relate tensor product of chains to chains on the cartesian product; the shuffle map is a lax symmetric monoidal natural transformation

$$C_*(X)\otimes C_*(Y)\to C_*(X\times Y),$$

meaning that it commutes strictly with the symmetry isomorphisms

$$C_*(X)\otimes C_*(Y)\cong C_*(Y)\otimes C_*(X) \quad \text{and} \quad C_*(X\times Y)\cong C_*(Y\times X)$$

and makes the following associativity diagram commute:

$$\begin{array}{ccc} C_*(X)\otimes C_*(Y)\otimes C_*(Z) & \longrightarrow & C_*(X\times Y)\otimes C_*(Z) \\ \downarrow & & \downarrow \\ C_*(X)\otimes C_*(Y\times Z) & \longrightarrow & C_*(X\times Y\times Z) \end{array}$$

See, for example, [200, §29].

The fact that E_n operads need to be defined in terms of a reference model is not entirely satisfactory, especially in homotopical contexts that are not topological. Nevertheless, the definition for spaces, simplicial sets, or chain complexes seems to suffice to cover all other contexts that arise in practice.[2]

5.4 Operadic algebras and monads

In the original context of iterated loop spaces and in many current contexts in homotopy theory and beyond, the main purpose of operads is to parametrize operations, which is to say, to define operadic algebras. For a closed symmetric monoidal category, there are three equivalent definitions, one in terms of operations, one in terms of endomorphism operads, and one in terms of monads. This section reviews the three definitions.

Viewing $\mathcal{O}(m)$ as parametrizing some m-ary operations on an object X means that we have an ***action map***

$$\mathcal{O}(m)\,\Box\, X^{(m)}\to X.$$

[2] In theory, the definition for simplicial sets should suffice for all homotopical contexts, but this may require changing models, which for a particular problem may be inconvenient or more complicated, or make it less concrete.

Since the right action of Σ_m on $\mathcal{O}(m)$ corresponds to reordering the arguments of the operations, applying $\sigma \in \Sigma_m$ to $\mathcal{O}(m)$ (and then performing the action map) should have the same effect as applying σ to permute the factors in $X^{(m)}$. A concise way of saying this is to say that the map is equivariant for the diagonal (left) action on the source $\mathcal{O}(m) \,\Box\, X^{(m)}$ and the trivial action on the target X (using the standard convention that the left action σ on $\mathcal{O}(m)$ is given by the right action of σ^{-1}). The action map should also respect the composition law Γ, making Γ correspond to composition of operations, and respect the identity 1, making 1 act by the identity operation. The following gives the precise definition:

Definition 5.4.1. Let $\mathscr{M}$ be a symmetric monoidal category and $\mathcal{O} = (\{\mathcal{O}(m)\}, \Gamma, 1)$ an operad in $\mathscr{M}$. An $\mathcal{O}$-algebra (in $\mathscr{M}$) consists of an object A in $\mathscr{M}$ together with *action maps*

$$\xi_m \colon \mathcal{O}(m) \,\Box\, A^{(m)} \to A$$

that are equivariant for the diagonal (left) Σ_m-action on the source and the trivial Σ_m-action on the target and that satisfy the following associativity and unit conditions:

(i) For all m, $j_1, \dots, j_m$,

$$\xi_m \circ (\mathrm{id}_{\mathcal{O}(m)} \Box \xi_{j_1} \Box \cdots \Box \xi_{j_m}) = \xi_j \circ (\Gamma^m_{j_1,\dots,j_m} \Box \mathrm{id}_A^{(j)}),$$

i.e., the diagram

$$\begin{array}{ccc} \mathcal{O}(m) \Box \mathcal{O}(j_1) \Box \cdots \Box \mathcal{O}(j_m) \Box A^{(j)} & \xrightarrow{\Gamma^m_{j_1,\dots,j_m} \Box \mathrm{id}_A^{(j)}} & \mathcal{O}(j) \Box A^{(j)} \\ \Big\downarrow {\scriptstyle \mathrm{id}_{\mathcal{O}(m)} \Box \xi_{j_1} \Box \cdots \Box \xi_{j_m}} & & \Big\downarrow {\scriptstyle \xi_j} \\ \mathcal{O}(m) \Box A^{(m)} & \xrightarrow[\xi_m]{} & A \end{array}$$

commutes.

(ii) The map $\xi_1 \circ (1 \Box \mathrm{id}_A) \colon \mathbf{1} \Box A \to A$ is the unit isomorphism for $\Box$.

A map of $\mathcal{O}$-algebras from $(A, \{\xi_m\})$ to $(A', \{\xi'_m\})$ consists of a map $f \colon A \to A'$ in $\mathscr{M}$ that commutes with the action maps, i.e., that make the diagrams

$$\begin{array}{ccc} \mathcal{O}(m) \Box A^{(m)} & \xrightarrow{\xi_m} & A \\ \Big\downarrow {\scriptstyle \mathrm{id}_{\mathcal{O}(m)} \Box f^{(m)}} & & \Big\downarrow {\scriptstyle f} \\ \mathcal{O}(m) \Box A'^{(m)} & \xrightarrow[\xi'_m]{} & A' \end{array}$$

commute for all m. We write $\mathscr{M}[\mathcal{O}]$ for the category of $\mathcal{O}$-algebras.

Example 5.4.2. When $\mathscr{M}$ has an initial object and $\Box$ preserves the initial object in each variable, the structure of an algebra over the identity operad $\mathcal{I}$ is no extra structure on an object of $\mathscr{M}$.

Per (ii) above and as illustrated in the previous example, the 1 in the structure of the operad corresponds to the identity operation. In some contexts algebras have units; when that happens, the unit is encoded in $\mathcal{O}(0)$ as in the examples of monoids and commutative monoids. Recall that a ***monoid object for*** $\Box$ ***in*** $\mathscr{M}$ (or $\Box$-*monoid* for short) consists of an object M together with a ***multiplication map*** $\mu\colon M \Box M \to M$ and ***unit map*** $\eta\colon \mathbf{1} \to M$ satisfying the following associativity and unit diagrams

$$\begin{array}{ccc} M \Box M \Box M & \xrightarrow{\mu \Box \mathrm{id}} & M \Box M \\ {\scriptstyle \mathrm{id} \Box \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ M \Box M & \xrightarrow[\mu]{} & M \end{array} \qquad \begin{array}{ccccc} \mathbf{1} \Box M & \xrightarrow{\eta \Box \mathrm{id}} & M \Box M & \xleftarrow{\mathrm{id} \Box \eta} & M \Box \mathbf{1} \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \cong} & \\ & & M & & \end{array}$$

(where the diagonal maps are the unit isomorphisms in $\mathscr{M}$). The ***opposite multiplication*** is the composite of the symmetry morphism $c\colon M \Box M \to M \Box M$ with μ, and a $\Box$-monoid is *commutative* when $\mu = \mu \circ c$.

Example 5.4.3. Given a $\mathcal{C}\mathrm{om}$-algebra A, defining η to be the action map ξ_0

$$\eta\colon \mathbf{1} = \mathcal{C}\mathrm{om}(0) \xrightarrow{\xi_0} A$$

and μ to be the composite of the (inverse) unit isomorphism and the action map ξ_2

$$\mu\colon A \Box A \cong \mathcal{C}\mathrm{om}(2) \Box A \Box A \xrightarrow{\xi_2} A$$

endows A with the structure of a commutative $\Box$-monoid: associativity follows from the fact that the maps $\Gamma^2_{1,2}$ and $\Gamma^2_{2,1}$ are both unit maps for $\Box$ so under the canonical isomorphisms

$$A \Box A \Box A \cong \mathcal{C}\mathrm{om}(2) \Box (\mathcal{C}\mathrm{om}(1) \Box \mathcal{C}\mathrm{om}(2)) \Box (A \Box A \Box A),$$
$$A \Box A \Box A \cong \mathcal{C}\mathrm{om}(2) \Box (\mathcal{C}\mathrm{om}(2) \Box \mathcal{C}\mathrm{om}(1)) \Box (A \Box A \Box A),$$

both maps induce the same map $A \Box A \Box A \to A$. Likewise, the unit condition follows from the fact that

$$\Gamma^2_{0,1}\colon \mathcal{C}\mathrm{om}(2) \Box (\mathcal{C}\mathrm{om}(0) \Box \mathcal{C}\mathrm{om}(1)) \to \mathcal{C}\mathrm{om}(1),$$
$$\Gamma^2_{1,0}\colon \mathcal{C}\mathrm{om}(2) \Box (\mathcal{C}\mathrm{om}(1) \Box \mathcal{C}\mathrm{om}(0)) \to \mathcal{C}\mathrm{om}(1)$$

are both unit maps. The multiplication is commutative because the action of the symmetry map on $\mathbf{1} = \mathcal{C}\mathrm{om}(2)$ is trivial. Conversely, we can convert a commutative $\Box$-monoid to a $\mathcal{C}\mathrm{om}$-algebra by taking ξ_0 to be the unit η, ξ_1 to be the unit isomorphism for $\Box$, ξ_2 to be induced by the unit isomorphism for $\Box$ and the multiplication, and all higher ξ_m's induced by the unit isomorphism for $\Box$ and (any) iterated multiplication. This defines a bijective correspondence between the set of commutative $\Box$-monoid structures and the set of $\mathcal{C}\mathrm{om}$-algebra structures on a fixed object and an isomorphism between the category of commutative $\Box$-monoids and the category of $\mathcal{C}\mathrm{om}$-algebras.

For a non-symmetric operad, defining an algebra in terms of the associated operad or in terms of the analogue of Definition 5.4.1 without the equivariance requirement produce the same structure.

Example 5.4.4. The constructions of Example 5.4.3 applied to the non-symmetric operad $\overline{\mathcal{A}\mathrm{ss}}$ give a bijective correspondence between the set of $\Box$-monoid structures and the set of $\mathcal{A}$ss-algebra structures on a fixed object and an isomorphism between the category of $\Box$-monoids and the category of $\mathcal{A}$ss-algebras.

The monoid and commutative monoid objects in the category of sets (with the usual symmetric monoidal structure given by cartesian product) are just the monoids and commutative monoids in the usual sense. Likewise, in spaces, they are the topological monoids and topological commutative monoids. In the category of abelian groups (with the usual symmetric monoidal structure given by the tensor product), the monoid objects are the rings and the commutative monoid objects are the commutative rings. In the category of chain complexes of R-modules for a commutative ring R (with the usual symmetric monoidal structure given by tensor product over R), the monoid objects are the differential graded R-algebras and the commutative monoid objects are the commutative differential graded R-algebras. In a modern category of spectra, the monoid objects are called *$\mathbb{S}$-algebras* or sometimes *strictly associative ring spectra*. Some authors take the term “ring spectrum” to be synonymous with $\mathbb{S}$-algebra, but others take it to mean the weaker notion of monoid object in the stable category (or even weaker notions). Work of Schwede–Shipley [265, 3.12.(3)] shows that the homotopy category of monoid objects in any modern category of spectra is equivalent to an appropriate full subcategory of the (mutually equivalent) homotopy category of monoid objects in EKMM S-modules, symmetric spectra, or orthogonal spectra (at least when “modern category of spectra” is used as a technical term to mean a model category with a preferred equivalence class of symmetric monoidal Quillen equivalence to the currently known modern categories of spectra); cf. Example Theorem 5.9.6 below. The analogous result does not hold for commutative monoid objects; see [151]. The term “commutative $\mathbb{S}$-algebra” is typically reserved for examples where the homotopy category of commutative monoid objects is equivalent to an appropriate full subcategory of the (mutually equivalent) homotopy category of commutative monoid objects in EKMM S-modules, symmetric spectra, or orthogonal spectra. See Chapter 6 of this volume for more on commutative ring spectra.

Returning to the discussion of operadic algebras, in the case when $\mathscr{M}$ is a closed symmetric monoidal category, adjoint to the action map

$$\xi_m \colon \mathcal{O}(m) \Box A^{(m)} \to A$$

is a map

$$\phi_m \colon \mathcal{O}(m) \to F(A^{(m)}, A) = \mathcal{E}\mathrm{nd}_A(m).$$

Equivariance for ξ_m is equivalent to equivariance for ϕ_m. Similarly, conditions (i) and (ii) in the definition of $\mathcal{O}$-algebra (Definition 5.4.1) are adjoint to the diagrams in the definition of map of operads (Definition 5.2.2). This proves the following proposition, which gives an alternative definition of $\mathcal{O}$-algebra.

Proposition 5.4.5. *Let $\mathscr{M}$ be a closed symmetric monoidal category, let $\mathcal{O}$ be an operad in $\mathscr{M}$, and let X be an object in $\mathscr{M}$. The adjunction rule $\xi_m \leftrightarrow \phi_m$ above defines a bijection between the set of $\mathcal{O}$-algebra structures on X and the set of maps of operads $\mathcal{O} \to \mathcal{E}\mathrm{nd}_X$.*

In the case when $\mathscr{M}$ is (countably) cocomplete (has (countable) colimits) and $\square$ preserves (countable) colimits in each variable (which includes the case when it is closed), algebras can also be defined in terms of a monad. The idea for the underlying functor is to gather the domains of all the action maps into a coproduct; since the action maps are equivariant with target having the trivial action, they factor through the coinvariants (quotient by the symmetric group action), and this goes into the definition.

Notation 5.4.6. Let $\mathscr{M}$ be a symmetric monoidal category with countable colimits, and let $\mathcal{O}$ be an operad in $\mathscr{M}$. Define the endofunctor $\mathbb{O}$ of $\mathscr{M}$ (i.e., a functor $\mathbb{O}\colon \mathscr{M} \to \mathscr{M}$) by

$$\mathbb{O}X = \coprod_{m=0}^{\infty} \mathcal{O}(m) \square_{\Sigma_m} X^{(m)},$$

where $\mathcal{O}(m) \square_{\Sigma_m} X^{(m)} := (\mathcal{O}(m) \square X^{(m)})/\Sigma_m$.

(When we use other letters for operads, we typically use the corresponding letters for the associated monad; for example, we write $\mathbb{A}$ for the monad associated to an operad $\mathcal{A}$, $\mathbb{B}$ for the monad associated to an operad $\mathcal{B}$, etc.)

The action maps for an $\mathcal{O}$-algebra A then specify a map $\xi : \mathbb{O}A \to A$; the conditions for defining an $\mathcal{O}$-structure also admit a formulation in terms of this map. The basic observation is that we have a canonical isomorphism

$$\begin{aligned}(\mathbb{O}X)^{(m)} &\cong \coprod_{j_1=0}^{\infty} \cdots \coprod_{j_m=0}^{\infty} (\mathcal{O}(j_1) \square_{\Sigma_{j_1}} X^{(j_1)}) \square \cdots \square (\mathcal{O}(j_m) \square_{\Sigma_{j_m}} X^{(j_m)}) \\ &\cong \coprod_{j=0}^{\infty} \coprod_{\substack{j_1,\dots,j_m \\ \sum j_i = j}} (\mathcal{O}(j_1) \square \cdots \mathcal{O}(j_m)) \square_{\Sigma_{j_1}\times\cdots\times\Sigma_{j_m}} X^{(j)},\end{aligned}$$

using the symmetry isomorphism to shuffle like factors without permuting them. We can use this isomorphism to give $\mathbb{O}X$ the canonical structure of an $\mathcal{O}$-algebra, defining the action map

$$\mu_m \colon \mathcal{O}(m) \square (\mathbb{O}X)^{(m)} \to \mathbb{O}X$$

by commuting the coproduct past $\square$, using the operad composition law, and passing to the quotient by the full permutation group:

$$\begin{aligned}\mathcal{O}(m) \square (\mathbb{O}X)^{(m)} &\cong \coprod_{j=0}^{\infty} \coprod_{\substack{j_1,\dots,j_m \\ \sum j_i = j}} \mathcal{O}(m) \square (\mathcal{O}(j_1) \square \cdots \mathcal{O}(j_m)) \square_{\Sigma_{j_1}\times\cdots\times\Sigma_{j_m}} X^{(j)} \\ &\xrightarrow{\coprod\coprod \Gamma^m_{j_1,\dots,j_m} \square \mathrm{id}_X^{(j)}} \coprod_{j=0}^{\infty} \mathcal{O}(j) \square_{\Sigma_{j_1}\times\cdots\times\Sigma_{j_m}} X^{(j)} \longrightarrow \coprod_{j=0}^{\infty} \mathcal{O}(j) \square_{\Sigma_j} X^{(j)} = \mathbb{O}X.\end{aligned}$$

The pictured map is well-defined because of the $(\Sigma_{j_1}\times\cdots\times\Sigma_{j_m})$-equivariance of $\Gamma^m_{j_1,\ldots,j_m}$ (5.2.1 (iii)). The other permutation rule (5.2.1 (iv)) implies that μ_m is Σ_m-equivariant. The remaining two parts of the definition of operad show that the μ_m define an $\mathcal{O}$-algebra structure map: 5.2.1 (i)–(ii) imply 5.4.1 (i)–(ii), respectively. This $\mathcal{O}$-algebra structure then defines a map

$$\mu\colon \mathbb{O}\mathbb{O}X \to \mathbb{O}X$$

as above, which is natural in X. The map $1\,\square\,\mathrm{id}_X$ also induces a natural transformation

$$\eta\colon X \to \mathbb{O}X.$$

These two maps together give $\mathbb{O}$ the structure of a monad.

Proposition 5.4.7. *Let $\mathcal{M}$ be a symmetric monoidal category with countable colimits and assume that $\square$ commutes with countable colimits in each variable. For an operad $\mathcal{O}$, the functor $\mathbb{O}$ and natural transformations μ, η form a monad: the diagrams*

$$\begin{array}{ccc} \mathbb{O}\mathbb{O}\mathbb{O}X & \xrightarrow{\mu} & \mathbb{O}\mathbb{O}X \\ {\scriptstyle \mathbb{O}\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathbb{O}\mathbb{O}X & \xrightarrow[\mu]{} & \mathbb{O}X \end{array} \qquad \begin{array}{ccc} \mathbb{O}X & \xrightarrow{\eta} & \mathbb{O}\mathbb{O}X \\ & \searrow^{=} & \downarrow{\scriptstyle \mu} \\ & & \mathbb{O}X \end{array}$$

commute (where the top map in the left-hand diagram is the map μ for the object $\mathbb{O}X$).

The proof is applying 5.4.1 (i)–(ii) for $\mathbb{O}X$.

Example 5.4.8. Under the hypotheses of the previous proposition, the monad associated to the identity operad $\mathcal{I}$ is canonically isomorphic (via the unit isomorphism) to the identity monad Id. The monad associated to the operad $\mathcal{C}$om is canonically isomorphic to the free commutative monoid monad

$$\mathbb{P}X = \coprod_{j=0}^{\infty} X^{(j)}/\Sigma_j.$$

The monad associated to the algebra $\mathcal{A}$ss is canonically isomorphic to the free monoid monad

$$\mathbb{T}X = \coprod_{j=0}^{\infty} X^{(j)}.$$

An algebra over the monad $\mathbb{O}$ consists of an object A and a map $\xi\colon \mathbb{O}A \to A$ such that the diagrams

$$\begin{array}{ccc} \mathbb{O}\mathbb{O}A & \xrightarrow{\mu} & \mathbb{O}A \\ {\scriptstyle \mathbb{O}\xi}\downarrow & & \downarrow{\scriptstyle \xi} \\ \mathbb{O}A & \xrightarrow[\xi]{} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta} & \mathbb{O}A \\ & \searrow^{=} & \downarrow{\scriptstyle \xi} \\ & & A \end{array}$$

commute. Given an $\mathcal{O}$-algebra $(A, \{\xi_m\})$, the map $\xi\colon \mathbb{O}A \to A$ constructed as the coproduct of the induced maps on coinvariants then is an $\mathbb{O}$-algebra action map. Conversely, given an $\mathbb{O}$-algebra (A,ξ), defining ξ_m to be the composite

$$\mathcal{O}(m)\,\square\, A^{(m)} \to \mathbb{O}A \xrightarrow{\xi} A,$$

the maps ξ_m make A an $\mathcal{O}$-algebra. This gives a second alternative definition of $\mathcal{O}$-algebra.

Proposition 5.4.9. *Under the hypotheses of Proposition 5.4.7, for X an object of $\mathscr{M}$, the correspondence $\{\xi_m\} \leftrightarrow \xi$ above defines a bijection between the set of $\mathcal{O}$-algebra structures on X and the set of $\mathbb{O}$-algebra structures on X and an isomorphism between the category of $\mathcal{O}$-algebras and the category of $\mathbb{O}$-algebras.*

5.5 Modules over operadic algebras

Just as an operad defines a category of algebras, an algebra defines a category of modules. Because this chapter concentrates on the theory of operadic algebras, we will only touch on the theory of modules. A complete discussion could fill a book and many of the aspects of the theory of operads we omit in this chapter (including Koszul duality, Quillen (co)homology, Deligne and Kontsevich conjectures) correspond to statements about categories of modules; even an overview could form its own chapter. We will just give a brief review of the definitions and the homotopy theory.

The original definition of modules over an operadic algebra seems to be due to Ginzburg and Kapranov [104, §1.6].

Definition 5.5.1. For an operad $\mathcal{O}$ and an $\mathcal{O}$-algebra A, an $(\mathcal{O},A)$-module (or just A-module when $\mathcal{O}$ is understood) consists of an object M of $\mathscr{M}$ and structure maps

$$\zeta_m \colon \mathcal{O}(m+1)\,\square\,(A^{(m)}\,\square\, M) \to M$$

for $m \geq 0$ such that the associativity, symmetry, and unit diagrams in Figure 5.4 commute. A map of A-modules is a map of the underlying objects of $\mathscr{M}$ that commutes with the structure maps.

Although the definition appears to favor A on the left, we obtain analogous right-hand structure maps

$$\mathcal{O}(m+1)\,\square\,(M\,\square\, A^{(m)}) \to M$$

satisfying the analogous right-hand version of the diagrams in Figure 5.4 by applying an appropriate permutation. Thus, an A-module structure can equally be regarded as either a left or right module structure. The following example illustrates this point.

Example 5.5.2. When $\mathcal{O} = \mathcal{A}ss$, the (symmetric) operad for associative algebras, and A is an $\mathcal{O}$-algebra (i.e., $\square$-monoid), an $(\mathcal{O},A)$-module in the sense of the previous definition is precisely an A-bimodule in the usual sense: it has structure maps

$$\lambda\colon A\,\square\, M \to M \qquad \text{and} \qquad \rho\colon M\,\square\, A \to M$$

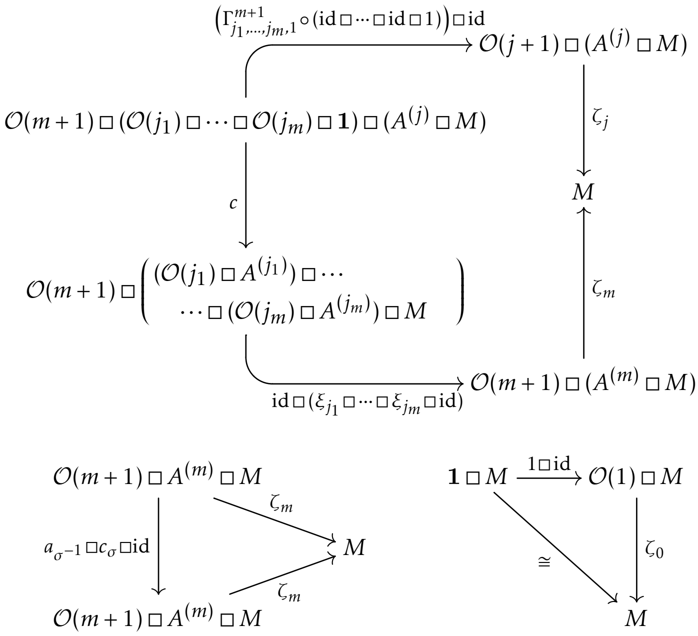
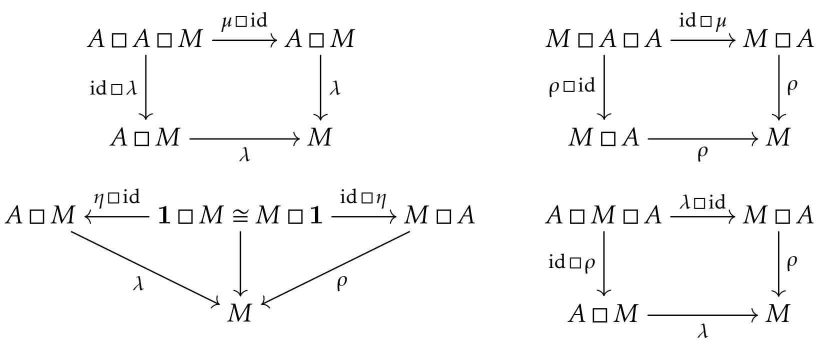


Figure 5.4 The diagrams for Definition 5.5.1. In the first diagram, $j = j_1 + \cdots + j_m$ and c is the $\square$-permutation that shuffles the $\mathcal{O}(j_i)$'s past the M and A's as displayed composed with the unit isomorphism for $\square$; ξ_i denote the $\mathcal{O}$-algebra structure maps for A. In the second diagram, σ is a permutation of $\{1,\dots,m\}$, permuting the factors of A, viewed as an element of Σ_{m+1} for permutation action on $\mathcal{O}(m+1)$. In the third diagram, the diagonal isomorphism is the unit isomorphism for $\square$.

satisfying the following associativity, unity, and interchange diagrams:



where μ denotes the multiplication and η the unit for A and the unlabeled arrow is the unit isomorphism for $\square$.

Obtaining a theory of modules closer to the idea of a left module (or right module) over an associative algebra requires working with non-symmetric operads.

Definition 5.5.3. Let $\overline{\mathcal{O}}$ be a non-symmetric operad and let A be an $\overline{\mathcal{O}}$-algebra. A left $(\overline{\mathcal{O}}, A)$-module (or just left A-module when $\overline{\mathcal{O}}$ is understood) consists of an object M of $\mathscr{M}$ and structure maps

$$\zeta_m \colon \overline{\mathcal{O}}(m+1) \mathbin{\square} (A^{(m)} \mathbin{\square} M) \to M$$

for $m \geq 0$ such that the associativity and unit diagrams in Figure 5.4 commute (with $\overline{\mathcal{O}}$ in place of $\mathcal{O}$). A map of left A-modules is a map of the underlying objects of $\mathscr{M}$ that commutes with the structure maps.

We also have the evident notion of a right A-module defined in terms of structure maps

$$\zeta_m \colon \overline{\mathcal{O}}(m+1) \mathbin{\square} (M \mathbin{\square} A^{(m)}) \to M$$

and the analogous right-hand associativity and unit diagrams.

Unlike in the case of operadic algebras, where working with a non-symmetric operad and its corresponding symmetric operad results in the same theory, in the case of modules, the results are very different.

Example 5.5.4. When $\overline{\mathcal{O}} = \overline{\mathcal{A}\mathrm{ss}}$, the non-symmetric operad for associative algebras, and A is an $\overline{\mathcal{O}}$-algebra (i.e., a $\square$-monoid), a left $(\overline{\mathcal{A}\mathrm{ss}}, A)$-module in the sense of the previous definition is precisely a left A-module in the usual sense defined in terms of an associative and unital left action map $A \mathbin{\square} M \to M$. Likewise, a right $(\overline{\mathcal{A}\mathrm{ss}}, A)$-module is precisely a right A-module in the usual sense.

Under mild hypotheses, the category of $(\mathcal{O}, A)$-modules is a category of modules for a $\square$-monoid called the enveloping algebra of A.

Construction 5.5.5 (The enveloping algebra). Assume that $\mathscr{M}$ admits countable colimits and $\square$ preserves countable colimits in each variable. For an operad $\mathcal{O}$ and an $\mathcal{O}$-algebra A, let $U^{\mathcal{O}}A$ (or UA when $\mathcal{O}$ is understood) be the coequalizer

$$\coprod_{m=0}^{\infty} \mathcal{O}(m+1) \mathbin{\square}_{\Sigma_m} (\mathbb{O}A)^{(m)} \rightrightarrows \coprod_{m=0}^{\infty} \mathcal{O}(m+1) \mathbin{\square}_{\Sigma_m} A^{(m)} \longrightarrow U^{\mathcal{O}}A,$$

where we regard Σ_m as the usual subgroup of Σ_{m+1} of permutations that fix $m+1$. Here one map is induced by the action map $\mathbb{O}A \to A$ and the other is induced by the operadic multiplication

$$\begin{aligned}
\mathcal{O}(m+1) \mathbin{\square} (\overline{\mathbb{O}}A)^{(m)} &\cong \coprod_{j_1,\dots,j_m} \mathcal{O}(m+1) \mathbin{\square} (\mathcal{O}(j_1) \mathbin{\square} A^{(j_1)}) \mathbin{\square} \cdots \mathbin{\square} (\mathcal{O}(j_m) \mathbin{\square} A^{(j_m)}) \\
&\cong \coprod_{j_1,\dots,j_m} \Big(\mathcal{O}(m+1) \mathbin{\square} (\mathcal{O}(j_1) \mathbin{\square} \cdots \mathbin{\square} \mathcal{O}(j_m) \mathbin{\square} \mathbf{1})\Big) \mathbin{\square} A^{(j)} \\
&\xrightarrow{\coprod \Gamma^{m+1}_{j_1,\dots,j_m,1} \mathbin{\square} \mathrm{id}} \mathcal{O}(j+1) \mathbin{\square} A^{(j)}
\end{aligned}$$

(where we have omitted writing $1 \colon \mathbf{1} \to \mathcal{O}(1)$ and as always $j = j_1 + \cdots + j_m$). Let

$\eta\colon \mathbf{1} \to UA$ be the map induced by $1\colon \mathbf{1} \to \mathcal{O}(1)$ and the inclusion of the $m = 0$ summand and let $\mu\colon UA \,\Box\, UA \to UA$ be the map induced from the maps

$$(\mathcal{O}(m+1)\,\Box\, A^{(m)})\,\Box\,(\mathcal{O}(n+1)\,\Box\, A^{(n)}) \to \mathcal{O}(m+n+1)\,\Box\, A^{(m+n)}$$

obtained from the map $\circ_{m+1}\colon \mathcal{O}(m+1)\,\Box\,\mathcal{O}(n+1) \to \mathcal{O}(m+n+1)$ defined as the composite

$$\mathcal{O}(m+1)\,\Box\,\mathcal{O}(n+1) \cong \mathcal{O}(m+1)\,\Box\,(\mathbf{1}\,\Box\cdots\Box\,\mathbf{1}\,\Box\,\mathcal{O}(n+1)) \xrightarrow{\Gamma^{m+1}_{1,\dots,1,n+1}} \mathcal{O}(m+n+1)$$

(where again we have omitted writing $1\colon \mathbf{1} \to \mathcal{O}(1)$). Associativity of the operad multiplication implies that η and μ give UA the structure of an associative monoid for $\Box$ and the resulting object is called the *enveloping algebra* of A over $\mathcal{O}$.

An easy argument from the definitions and universal property of the coequalizer proves the following proposition.

Proposition 5.5.6. *Assume $\mathscr{M}$ admits countable coproducts and $\Box$ preserves them in each variable. Let $\mathcal{O}$ be an operad and let A be an $\mathcal{O}$-algebra. For an object X of $\mathscr{M}$, $(\mathcal{O},A)$-module structures on X are in bijective correspondence with left $U^{\mathcal{O}}A$-module structures. In particular, the category of $(\mathcal{O},A)$-modules is isomorphic to the category of left $U^{\mathcal{O}}A$-modules.*

Similarly, in the case of non-symmetric operads, we can construct a left module enveloping algebra $\overline{U}^{\overline{\mathcal{O}}}A$ (denoted $\overline{U}A$ when $\overline{\mathcal{O}}$ is understood) as the coequalizer

$$\coprod_{m=0}^{\infty} \overline{\mathcal{O}}(m+1)\,\Box\,(\overline{\mathbb{O}}A)^{(m)} \rightrightarrows \coprod_{m=0}^{\infty} \overline{\mathcal{O}}(m+1)\,\Box\, A^{(m)} \longrightarrow \overline{U}^{\overline{\mathcal{O}}}A \tag{5.5.1}$$

with maps defined as in Construction 5.5.5. The analogous identification of module categories holds.

Proposition 5.5.7. *Assume $\mathscr{M}$ admits countable coproducts and $\Box$ preserves them in each variable. Let $\overline{\mathcal{O}}$ be a non-symmetric operad and let A be an $\overline{\mathcal{O}}$-algebra. For an object X of $\mathscr{M}$, left $(\overline{\mathcal{O}},A)$-module structures on X are in bijective correspondence with left $\overline{U}A$-module structures. In particular, the category of left $(\overline{\mathcal{O}},A)$-modules is isomorphic to the category of left $\overline{U}A$-modules.*

We develop some tools to study enveloping algebras in the next section. In the meantime, we can identify the enveloping algebra in some specific examples.

Example 5.5.8. For $\mathcal{O} = \mathcal{A}\mathrm{ss}$ and A an $\mathcal{A}\mathrm{ss}$-algebra (a $\Box$-monoid), $U^{\mathcal{A}\mathrm{ss}}A$ is $A\,\Box\,A^{\mathrm{op}}$, the usual enveloping algebra for a $\Box$-monoid. Viewing A as an $\overline{\mathcal{A}\mathrm{ss}}$-algebra, $\overline{U}^{\overline{\mathcal{A}\mathrm{ss}}}A$ is the $\Box$-monoid A. If A is a $\mathcal{C}\mathrm{om}$-algebra (a commutative $\Box$-monoid), then $U^{\mathcal{C}\mathrm{om}}A$ makes sense and is also the $\Box$-monoid A.

Example 5.5.9. Let $\mathcal{L}$ denote the Boardman–Vogt linear isometries operad of

Example 5.3.4. For an $\mathcal{L}$-algebra, the underlying space of $U^{\mathcal{L}}A$ is the pushout

$$\begin{array}{ccc}
\mathcal{L}(2)\times_{\mathcal{L}(1)}\mathcal{L}(0) & \xrightarrow{\mathrm{id}\times\xi_0} & \mathcal{L}(2)\times_{\mathcal{L}(1)}A \\
{\scriptstyle\circ_1}\downarrow & & \downarrow \\
\mathcal{L}(1) & \longrightarrow & U^{\mathcal{L}}A
\end{array}$$

where $\circ_1$ is the map induced by $1\colon * \to \mathcal{L}(1)$ and $\Gamma^2_{0,1}$ (as in Construction 5.5.5) and the right action on $\mathcal{L}(2)$ of $\mathcal{L}(1)\cong\mathcal{L}(1)\times *$ is via $\Gamma^2_{1,1}\circ(\mathrm{id}\times 1)$. The inclusions of the $m=0$ and $m=1$ summands induce the map from the pushout above to the coequalizer defining $U^{\mathcal{L}}A$; the inverse isomorphism uses the "Hopkins' Lemma" [94, I.5.4] isomorphism

$$\mathcal{L}(2)\times_{\mathcal{L}(1)\times\mathcal{L}(1)}(\mathcal{L}(i)\times\mathcal{L}(j))\cong\mathcal{L}(i+j) \tag{HL}$$

for $i,j\geq 1$. The pushout explicitly admits maps in from the $m=0$ and $m=1$ summands of the coequalizer, and for $m>1$ we have the map

$$\begin{aligned}
\mathcal{L}(m+1)\times_{\Sigma_m}A^{(m)} &\cong \mathcal{L}(m+1)\times_{\Sigma_m\times\mathcal{L}(1)}(A^{(m)}\times\mathcal{L}(1))\\
&\underset{\text{(HL)}}{\cong} \mathcal{L}(2)\times_{\mathcal{L}(1)\times\mathcal{L}(1)}((\mathcal{L}(m)\times_{\Sigma_m}A^{(m)})\times\mathcal{L}(1))\\
&\xrightarrow{\mathrm{id}\times(\xi_m\times\mathrm{id})} \mathcal{L}(2)\times_{\mathcal{L}(1)\times\mathcal{L}(1)}(A\times\mathcal{L}(1))\cong\mathcal{L}(2)\times_{\mathcal{L}(1)}A.
\end{aligned}$$

The previous display also indicates how the multiplication of $U^{\mathcal{L}}A$ works in the pushout description: it is induced by the map

$$\begin{aligned}
&(\mathcal{L}(2)\times A)\times(\mathcal{L}(2)\times A)\cong(\mathcal{L}(2)\times\mathcal{L}(2))\times A^{(2)}\\
&\qquad\xrightarrow{\circ_2\times\mathrm{id}}\mathcal{L}(3)\times A^{(2)}\to\mathcal{L}(3)\times_{\Sigma_2}A^{(2)}\to\mathcal{L}(2)\times_{\mathcal{L}(1)}A,
\end{aligned}$$

where the last map is the $m=2$ case of the map above. It turns out that the map $U^{\mathcal{L}}A\to A$ induced by the operadic algebra action maps is always a weak equivalence. (The proof is not obvious but uses the ideas from EKMM, especially [94, I.8.5, XI.3.1] in the context of the theory of $\mathcal{L}(1)$-spaces, as in for example [28, §6], [44, §4.6], or [45, §4.3].) If we forget the symmetries in $\mathcal{L}$ to create a non-symmetric operad $\mathcal{L}_{\not\Sigma}$, then $\overline{U}^{\mathcal{L}_{\not\Sigma}}A\cong U^{\mathcal{L}}A$. Even when A is just an $\mathcal{L}_{\not\Sigma}$-algebra, $\overline{U}^{\mathcal{L}_{\not\Sigma}}A$ can still be identified as the same pushout construction pictured above using the analogous comparison isomorphisms with the coequalizer definition (5.5.1). Analogous formulations also hold in the context of orthogonal spectra, symmetric spectra, and EKMM S-modules using the operad $\Sigma^\infty_+\mathcal{L}$ in the respective categories. In the context of Lewis–May spectra, these observations are closely related to the foundations of EKMM S-modules and the properties of the smash product ($\wedge_{\mathcal{L}}$, $\wedge$, and $\wedge_A$); this is the start of a much longer story on monoidal products and balanced products for A_∞ module categories (see, for example, [184] and [47, §17-18]).

Although in both previous examples we had an isomorphism of enveloping algebras for symmetric and non-symmetric constructions, this is not a general phenomenon, as can be seen, for example, by comparing $U^{\mathcal{A}\mathrm{ss}}$ and $\overline{U}^{\mathcal{A}\mathrm{ss}_{\not\Sigma}}$, where $\mathcal{A}\mathrm{ss}_{\not\Sigma}$ is the non-symmetric operad formed from $\mathcal{A}\mathrm{ss}$ by forgetting the symmetry. (In this style of notation, $\overline{\mathcal{A}\mathrm{ss}} = \mathcal{C}\mathrm{om}_{\not\Sigma}$.)

The left module enveloping algebra construction for the non-symmetric little 1-cubes operad, $\overline{U}^{\bar{\mathcal{C}}_1}(-)$, also admits a concrete description [184, §2], which we review in Section 5.10. It shares the feature with the previous two examples that for any $\bar{\mathcal{C}}_1$-algebra A, $\overline{U}^{\bar{\mathcal{C}}_1}A$ is weakly equivalent to A (see [184, 1.1] or Proposition 5.10.3).

Given Propositions 5.5.6 and 5.5.7, the homotopy theory of modules over operadic algebras reduces to (1) the homotopy theory of modules over $\Box$-monoids and (2) the homotopy theory of $U^{\mathcal{O}}A$ (or $\overline{U}^{\bar{\mathcal{O}}}A$) as a functor of $\mathcal{O}$ (or $\bar{\mathcal{O}}$) and A. The latter first requires a study of the homotopy theory of operadic algebras, which we review (in part) in the next few sections, before returning to this question in Corollary 5.9.7. On the other hand the homotopy theory of modules over $\Box$-monoids is very straightforward, and we give a short review of the main results in the remainder of this section. We discuss this in terms of closed model categories. (For an overview of closed model categories as a setting for homotopy theory, we refer the reader to [91]. See also Chapter 2 of this volume.) The following theorem gives a comprehensive result in some categories of primary interest.

Theorem 5.5.10. *Let $(\mathscr{M}, \Box, \mathbf{1})$ be the category of simplicial sets, spaces, symmetric spectra, orthogonal spectra, EKMM S-modules, simplicial abelian groups, chain complexes, or any category of modules over a commutative monoid object in one of these categories, with the usual monoidal product and one of the standard cofibrantly generated model structures. Let A be a monoid object in $\mathscr{M}$. The category of A-modules is a closed model category with weak equivalences and fibrations created in $\mathscr{M}$.*

The proof in all cases is much like the argument in [94, VII§4] or [267, 2.3]. Heuristically, whenever the small object argument applies and $\Box$ behaves well with respect to weak equivalences, pushouts, and sequential or filtered colimits, a version of the previous theorem should hold. For an example of a more general statement, see [267, 4.1].

A map of monoid objects $A \to B$ induces an obvious *restriction of scalars* functor from the category of B-modules to the category of A-modules. When $\mathscr{M}$ admits coequalizers and $\Box$ preserves coequalizers in each variable (as is the case in the examples in the previous theorem), the restriction of scalars functor admits a left adjoint *extension of scalars* functor $B \Box_A (-)$ which on the underlying objects is constructed as the coequalizer

$$B \Box A \Box M \rightrightarrows B \Box M \longrightarrow B \Box_A M,$$

where one map is induced by the A-action on M and the other by the A action on B (induced by the map of monoid objects). When the categories of modules have closed model structures with weak equivalences and fibrations created in the underlying category $\mathscr{M}$, this adjunction is automatically a Quillen adjunction, which implies a derived adjunction on homotopy categories. When the map $A \to B$ is a weak equivalence, we can often expect the Quillen adjunction to be a Quillen equivalence and induce an equivalence of homotopy categories; this is the case in the setting of the previous theorem.

Theorem 5.5.11. *Let $\mathscr{M}$ be one of the symmetric monoidal model categories of Theorem 5.5.10. A weak equivalence of monoid objects induces a Quillen equivalence on categories of modules.*

Again, significantly more general results hold; see, for example, [157], especially Theorem 8.3 and the subsection of Section 1 entitled “Extension of scalars”.

5.6 Limits and colimits in categories of operadic algebras

Before going on to the homotopy theory of categories of operadic algebras, we say a few words about certain constructions, limits and colimits in this section, and geometric realization in the next section. While limits of operadic algebras are pretty straightforward (as explained below), colimits tend to be more complicated and we take some space to describe in detail what certain colimits look like.

We start with limits. Let $D\colon \mathscr{D} \to \mathscr{M}[\mathcal{O}]$ be a diagram, i.e., a functor from a small category $\mathscr{D}$, where $\mathscr{M}$ is a symmetric monoidal category and $\mathcal{O}$ is an operad in $\mathscr{M}$. By neglect of structure, we can regard D as a diagram in $\mathscr{M}$, and suppose the limit L exists in $\mathscr{M}$. Then for each $d \in \mathscr{D}$, we have the canonical map $L \to D(d)$, and using the $\mathcal{O}$-algebra structure map for $D(d)$, we get a map

$$\mathcal{O}(m) \Box L^{(m)} \to \mathcal{O}(m) \Box D(d)^{(m)} \to D(d).$$

These maps satisfy the required compatibility to define a map

$$\mathcal{O}(m) \Box L^{(m)} \to L,$$

which together are easily verified to provide structure maps for an $\mathcal{O}$-algebra structure on L. This $\mathcal{O}$-algebra structure has the universal property for the limit of D in $\mathscr{M}[\mathcal{O}]$.

Proposition 5.6.1. *For any symmetric monoidal category $\mathscr{M}$, any operad $\mathcal{O}$ in $\mathscr{M}$, and any diagram of $\mathcal{O}$-algebras, if the limit exists in $\mathscr{M}$, then it has a canonical $\mathcal{O}$-algebra structure that gives the limit in $\mathscr{M}[\mathcal{O}]$.*

We cannot expect general colimits of operadic algebras to be formed in the underlying category, as can be seen from the examples of coproducts of $\Box$-monoids ($\mathcal{A}$ss-algebras) or of commutative $\Box$-monoids ($\mathcal{C}$om-algebras). The discussion of colimits simplifies if we assume that $\mathscr{M}$ has countable colimits and that $\Box$ preserves countable colimits in each variable, so that Proposition 5.4.9 holds and the category of $\mathcal{O}$-algebras is the category of algebras over the monad $\mathbb{O}$. The main technical tool in this case is the following proposition; because we have assumed in particular that $\Box$ preserves coequalizers in each variable, it follows that the m-th $\Box$-power functor preserves reflexive coequalizers (see [94, II.7.2] for a proof) and the filtered colimits that exist (by an easy cofinality argument).

Proposition 5.6.2. *If $\mathscr{M}$ satisfies the hypotheses of Proposition 5.4.7, then for any operad $\mathcal{O}$, the monad $\mathbb{O}$ preserves reflexive coequalizers in $\mathscr{M}$ and the filtered colimits that exist in $\mathscr{M}$.*

Recall that a reflexive coequalizer is a coequalizer

$$X \underset{b}{\overset{a}{\rightrightarrows}} Y \xrightarrow{c} C$$

where there exists a map $r\colon Y \to X$ such that $a \circ r = \mathrm{id}_Y$ and $b \circ r = \mathrm{id}_Y$; r is called a *reflexion*. The proposition says that if the above coequalizer exists in $\mathscr{M}$ and is reflexive then the diagram obtained by applying $\mathbb{O}$,

$$\mathbb{O}X \underset{\mathbb{O}b}{\overset{\mathbb{O}a}{\rightrightarrows}} \mathbb{O}Y \xrightarrow{\mathbb{O}c} \mathbb{O}C,$$

is also a (reflexive) coequalizer diagram in $\mathscr{M}$. Now suppose that a and b are maps of $\mathcal{O}$-algebras. Then the diagrams

$$\begin{array}{ccc} \mathbb{O}X & \xrightarrow{\mathbb{O}a} & \mathbb{O}Y \\ \downarrow & & \downarrow \\ X & \xrightarrow[a]{} & Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{O}X & \xrightarrow{\mathbb{O}b} & \mathbb{O}Y \\ \downarrow & & \downarrow \\ X & \xrightarrow[b]{} & Y \end{array}$$

commute (where the vertical maps are the $\mathcal{O}$-algebra structure maps) and we get an induced map

$$\mathbb{O}C \to C.$$

Repeating this for $\mathbb{O}X \rightrightarrows \mathbb{O}Y$ and the two maps $\mathbb{O}\mathbb{O}X \rightrightarrows \mathbb{O}\mathbb{O}Y$ to $\mathbb{O}X \rightrightarrows \mathbb{O}Y$, we see that the map $\mathbb{O}C \to C$ constructed above is an $\mathcal{O}$-algebra structure map and an easy check of universal properties shows that C with this $\mathcal{O}$-algebra structure is the coequalizer in $\mathscr{M}[\mathcal{O}]$. This shows that if a pair of parallel arrows in $\mathscr{M}[\mathcal{O}]$ has a reflexion in $\mathscr{M}$, then the coequalizer in $\mathscr{M}$ has the canonical structure of an $\mathcal{O}$-algebra and is the coequalizer in $\mathscr{M}[\mathcal{O}]$.

We can turn the observation in the previous paragraph into a construction of colimits of arbitrary shapes in $\mathscr{M}[\mathcal{O}]$. Given a diagram $D\colon \mathscr{D} \to \mathscr{M}[\mathcal{O}]$, assume that the colimit of the underlying functor to $\mathscr{M}$ exists, and denote it by $\operatorname{colim}^{\mathscr{M}} D$. If $\operatorname{colim}^{\mathscr{M}} \mathbb{O}D$ also exits, then we get a pair of parallel arrows

$$\mathbb{O}(\operatorname{colim}^{\mathscr{M}} \mathbb{O}D) \rightrightarrows \mathbb{O}(\operatorname{colim}^{\mathscr{M}} D)\,, \tag{5.6.1}$$

where one arrow is induced by the $\mathcal{O}$-algebra structure maps $\mathbb{O}D(d) \to D(d)$ and the other is the composite

$$\mathbb{O}(\operatorname{colim}^{\mathscr{M}} \mathbb{O}D) \xrightarrow{\mathbb{O}i} \mathbb{O}\mathbb{O}(\operatorname{colim}^{\mathscr{M}} D) \xrightarrow{\mu} \mathbb{O}(\operatorname{colim}^{\mathscr{M}} D),$$

where μ is the monadic multiplication $\mathbb{O}\mathbb{O} \to \mathbb{O}$ and

$$i\colon \operatorname{colim}^{\mathscr{M}} \mathbb{O}D \to \mathbb{O}(\operatorname{colim}^{\mathscr{M}} D)$$

is the map assembled from the maps $\mathbb{O}D(d) \to \mathbb{O}(\mathrm{colim}^{\mathscr{M}} D)$ induced by applying $\mathbb{O}$ to the canonical maps $D(d) \to \mathrm{colim}^{\mathscr{M}} D$. We also have a reflexion

$$\mathbb{O}(\mathrm{colim}^{\mathscr{M}} D) \to \mathbb{O}(\mathrm{colim}^{\mathscr{M}} \mathbb{O}D)$$

induced by the unit map $D(d) \to \mathbb{O}D(d)$. Thus, the coequalizer of (5.6.1) in $\mathscr{M}$ has the canonical structure of an $\mathcal{O}$-algebra which provides the coequalizer in $\mathscr{M}[\mathcal{O}]$; a check of universal properties shows that the coequalizer is the colimit in $\mathscr{M}[\mathcal{O}]$ of D.

Proposition 5.6.3. *Assume $\mathscr{M}$ satisfies the hypotheses of Proposition 5.4.7. For any operad $\mathcal{O}$ and any diagram $D\colon \mathscr{D} \to \mathscr{M}[\mathcal{O}]$, if the colimit of D and the colimit of $\mathbb{O}D$ exist in $\mathscr{M}$, then the colimit of D exists in $\mathscr{M}[\mathcal{O}]$ and is given by the coequalizer of the reflexive pair displayed in* (5.6.1).

For example, the coproduct $A \amalg^{\mathscr{M}[\mathcal{O}]} B$ in $\mathscr{M}[\mathcal{O}]$ can be constructed as the coequalizer

$$\mathbb{O}(\mathbb{O}A \amalg \mathbb{O}B) \rightrightarrows \mathbb{O}(A \amalg B) \longrightarrow A \amalg^{\mathscr{M}[\mathcal{O}]} B.$$

When $B = \mathbb{O}X$ for some X in $\mathscr{M}$, we can say more by recognizing that the category of $\mathcal{O}$-algebras under A is itself the category of algebras over an operad.

Construction 5.6.4 (The enveloping operad). For $m \geq 0$, define $\mathcal{U}^{\mathcal{O}}_A(m)$ by the coequalizer diagram

$$\coprod_{\ell=0}^{\infty} \mathcal{O}(\ell+m) \mathbin{\square}_{\Sigma_\ell} (\mathbb{O}A)^{(\ell)} \rightrightarrows \coprod_{\ell=0}^{\infty} \mathcal{O}(\ell+m) \mathbin{\square}_{\Sigma_\ell} A^{(\ell)} \longrightarrow \mathcal{U}^{\mathcal{O}}_A(m),$$

where one arrow is induced by the operadic multiplication

$$\Gamma^{\ell+m}_{j_1,\ldots,j_\ell,1,\ldots,1}\colon \mathcal{O}(\ell+m) \mathbin{\square} \mathcal{O}(j_1) \mathbin{\square} \cdots \mathbin{\square} \mathcal{O}(j_\ell) \mathbin{\square} \mathbf{1} \mathbin{\square} \cdots \mathbin{\square} \mathbf{1} \to \mathcal{O}(j+m)$$

and the other by the $\mathcal{O}$-algebra action map $\mathbb{O}A \to A$. We think of the ℓ factors of A (or $\mathbb{O}A$) as being associated with the first ℓ inputs of $\mathcal{O}(\ell+m)$, leaving the last m inputs open. We then have a Σ_m-action induced from the Σ_m-action on $\mathcal{O}(\ell+m)$ on the open inputs, unit map $1\colon \mathbf{1} \to \mathcal{U}^{\mathcal{O}}_A(1)$ induced by the unit map of $\mathcal{O}$ (on the summand $\ell = 0$), and operadic composition Γ induced by applying the operadic multiplication of $\mathcal{O}$ using the open inputs.

This operad is called the ***enveloping operad*** of A and generalizes the enveloping algebra $U^{\mathcal{O}}A$ of Construction 5.5.5: for $m = 1$, $\mathcal{U}^{\mathcal{O}}_A(1)$ is precisely the coequalizer defining $U^{\mathcal{O}}A$ and the operad unit and multiplication Γ^1_1 coincide with the $\square$-monoid unit and multiplication.

To return to the discussion of the category of $\mathcal{O}$-algebras under A, we note that for $m = 0$, the coequalizer in Construction 5.6.4 is

$$\mathbb{O}\mathbb{O}A \rightrightarrows \mathbb{O}A \longrightarrow \mathcal{U}^{\mathcal{O}}_A(0),$$

giving a canonical isomorphism $A \to \mathcal{U}^{\mathcal{O}}_A(0)$, and so a $\mathcal{U}^{\mathcal{O}}_A$-algebra T comes with a structure map $A \to T$. Looking at the summands with $\ell = 0$ above, we get a map of operads $\mathcal{O} \to \mathcal{U}^{\mathcal{O}}_A$, giving T an $\mathcal{O}$-algebra structure; the map $A \to T$ is a map of $\mathcal{O}$-algebras. On the other hand, given an $\mathcal{O}$-algebra B and a map of $\mathcal{O}$-algebras $A \to B$, we have maps

$$\mathcal{O}(\ell+m) \,\square\, A^{(\ell)} \,\square\, B^{(m)} \to \mathcal{O}(\ell+m) \,\square\, B^{(\ell)} \,\square\, B^{(m)} \to B$$

which together induce maps $\mathcal{U}^{\mathcal{O}}_A(m) \square B^{(m)} \to B$ that are easily checked to provide $\mathcal{U}^{\mathcal{O}}_A$-algebra structure maps. This gives a bijection between the structure of an $\mathcal{O}$-algebra under A and the structure of a $\mathcal{U}^{\mathcal{O}}_A$-algebra.

Proposition 5.6.5. *Let $\mathscr{M}$ satisfy the hypotheses of Proposition 5.4.7. For an object X of $\mathscr{M}$, the set of $\mathcal{U}^{\mathcal{O}}_A$-algebra structures on X is in bijective correspondence with the set of ordered pairs consisting of an $\mathcal{O}$-algebra structure on X and a map of $\mathcal{O}$-algebras $A \to X$ for that structure.*

As a consequence we have the following description of the coproduct of $\mathcal{O}$-algebras $A \amalg^{\mathscr{M}[\mathcal{O}]} \mathbb{O}X$, since $A \amalg^{\mathscr{M}[\mathcal{O}]} \mathbb{O}(-)$ is the left adjoint of the forgetful functor from $\mathcal{O}$-algebras under A to $\mathscr{M}$.

Proposition 5.6.6. *When $\mathscr{M}$ satisfies the hypotheses of Proposition 5.4.7,*

$$A \amalg^{\mathscr{M}[\mathcal{O}]} \mathbb{O}X \cong \mathbb{U}^{\mathcal{O}}_A X = \coprod_{m=0}^{\infty} \mathcal{U}^{\mathcal{O}}_A(m) \,\square_{\Sigma_m}\, X^{(m)}$$

(where the coproduct symbol undecorated by a category denotes coproduct in $\mathscr{M}$).

The decomposition above can be useful even without further information on $\mathcal{U}^{\mathcal{O}}_A$, but in fact we can be more concrete about what $\mathcal{U}^{\mathcal{O}}_A$ looks like in the case when A is built up iteratively from pushouts of free objects in $\mathscr{M}[\mathcal{O}]$. As a base case, an easy calculation gives

$$\mathcal{U}^{\mathcal{O}}_{\mathbb{O}X}(m) = \coprod_{\ell=0}^{\infty} \mathcal{O}(\ell+m) \,\square_{\Sigma_\ell}\, X^{(\ell)}.$$

Now suppose $A' = A \amalg^{\mathscr{M}[\mathcal{O}]}_{\mathbb{O}X} \mathbb{O}Y$ for some maps $X \to A$ and $X \to Y$ in $\mathscr{M}$; we can then describe $\mathcal{U}^{\mathcal{O}}_{A'}$ in terms of $\mathcal{U}^{\mathcal{O}}_A$ and pushouts in $\mathscr{M}[\mathcal{O}]$ as follows. (In particular, the calculation of $\mathcal{U}^{\mathcal{O}}_{A'}(0)$ describes A' in these terms and the calculation of $\mathcal{U}^{\mathcal{O}}_{A'}(1)$ describes UA' in these terms.) First, using the observations on colimits above, a little work shows that the coequalizer defining $\mathcal{U}^{\mathcal{O}}_{A'}$ simplifies in this case to

$$\coprod_{\ell=0}^{\infty} \mathcal{U}^{\mathcal{O}}_A(\ell+m) \,\square_{\Sigma_\ell}\, (X \amalg Y)^{(\ell)} \rightrightarrows \coprod_{\ell=0}^{\infty} \mathcal{U}^{\mathcal{O}}_A(\ell+m) \,\square_{\Sigma_\ell}\, Y^{(\ell)} \longrightarrow \mathcal{U}^{\mathcal{O}}_{A'}(m)$$

where one map is induced by the map $X \to A$ $(= \mathcal{U}^{\mathcal{O}}_A(0))$ and the other is induced by the map $X \to Y$. We then have a filtration on $\mathcal{U}^{\mathcal{O}}_{A'}(m)$ by powers of Y; specifically, define $F^k\mathcal{U}^{\mathcal{O}}_{A'}(m)$ by the coequalizer

$$\coprod_{\ell=0}^{k} \mathcal{U}^{\mathcal{O}}_A(\ell+m) \,\square_{\Sigma_\ell}\, (X \amalg Y)^{(\ell)} \rightrightarrows \coprod_{\ell=0}^{k} \mathcal{U}^{\mathcal{O}}_A(\ell+m) \,\square_{\Sigma_\ell}\, Y^{(\ell)} \longrightarrow F^k\mathcal{U}^{\mathcal{O}}_{A'}(m)$$

Then $\operatorname{colim}_k F^k \mathcal{U}^{\mathcal{O}}_{A'}(m) = \mathcal{U}^{\mathcal{O}}_{A'}(m)$. Comparing the universal properties for $F^{k-1}\mathcal{U}^{\mathcal{O}}_{A'}(m)$ and $F^k\mathcal{U}^{\mathcal{O}}_{A'}(m)$, we see that the following diagram is a pushout (in $\mathscr{M}$):

$$\begin{array}{ccc} \mathcal{U}^{\mathcal{O}}_{A}(k+m) \Box_{\Sigma_{k-1}} (X \Box Y^{(k-1)}) & \longrightarrow & \mathcal{U}^{\mathcal{O}}_{A}(k+m) \Box_{\Sigma_k} Y^{(k)} \\ \downarrow & & \downarrow \\ F^{k-1}\mathcal{U}^{\mathcal{O}}_{A'}(m) & \longrightarrow & F^k\mathcal{U}^{\mathcal{O}}_{A'}(m) \end{array}$$

This describes $\mathcal{U}^{\mathcal{O}}_{A'}$ in terms of iterated pushouts in $\mathscr{M}$, but we can do somewhat better, as can be seen in the example where $\mathscr{M}$ is the category of spaces and $X \to Y$ is a closed inclusion. In the pushout above, the top horizontal map comes from the map

$$\Sigma_k \times_{\Sigma_{k-1}} (X \times Y^{k-1}) \to Y^k$$

which fails to be an inclusion for $k > 1$ except in trivial cases; however, the image of this map can be described as an iterated pushout, starting with X^k and gluing in higher powers of Y. This works as well in the general case (which we now return to).

Let $Q^k_0(X \to Y) = X^{(k)}$, an object of $\mathscr{M}$ with a Σ_k-action and a Σ_k-equivariant map to $Y^{(k)}$. Inductively, for $i > 0$, define $Q^k_i(X \to Y)$ as the pushout

$$\begin{array}{ccc} \Sigma_k \times_{\Sigma_{k-i}\times\Sigma_i} (X^{(k-i)} \Box Q^i_{i-1}(X \to Y)) & \longrightarrow & \Sigma_k \times_{\Sigma_{k-i}\times\Sigma_i} (X^{(k-i)} \Box Y^{(i)}) \\ \downarrow & & \downarrow \\ Q^k_{i-1}(X \to Y) & \longrightarrow & Q^k_i(X \to Y) \end{array} \qquad (5.6.2)$$

with the evident Σ_k-action and Σ_k-equivariant map

$$Q^k_i(X \to Y) \to Y^{(k)}.$$

Then for all $j > 0$, we have a $(\Sigma_j \times \Sigma_k)$-equivariant map

$$X^{(j)} \Box Q^k_i(X \to Y) \to Q^{j+k}_i(X \to Y)$$

induced by the map

$$X^{(j)} \Box X^{(k-i)} \Box Y^{(i)} \cong X^{(j+k-i)} \Box Y^{(i)} \to Q^{j+k}_i(X \to Y)$$

and the compatible (inductively defined) map

$$X^{(j)} \Box Q^k_{i-1}(X \to Y) \to Q^{j+k}_{i-1}(X \to Y) \to Q^{j+k}_i(X \to Y),$$

which allows us to continue the induction. In the case when $\mathscr{M}$ is the category of topological spaces and $X \to Y$ is a closed inclusion, the maps

$$Q^k_0(X \to Y) \to \cdots \to Q^k_{k-1}(X \to Y) \to Y^{(k)}$$

are closed inclusions with $Q^k_i(X \to Y)$ the subspace of Y^k where at most i coordinates are in $Y \setminus X$. In the general case, an inductive argument shows that the map

$$\Sigma_k \times_{\Sigma_{k-i}\times\Sigma_i} (X^{(k-i)} \Box Y^{(i)}) \to Q^k_i(X \to Y)$$

is a categorical epimorphism and that the map

$$\mathcal{U}^{\mathcal{O}}_A(k+m)\,\square_{\Sigma_{k-1}}(X\,\square\,Y^{(k-1)})\to\mathcal{U}^{\mathcal{O}}_A(k+m)\,\square_{\Sigma_k}Q^k_{k-1}(X\to Y)$$

is a categorical epimorphism. Since this factors the map

$$\mathcal{U}^{\mathcal{O}}_A(k+m)\,\square_{\Sigma_{k-1}}(X\,\square\,Y^{(k-1)})\to\mathcal{U}^{\mathcal{O}}_A(k+m)\,\square_{\Sigma_k}Y^{(k)},$$

we get the following more sophisticated identification of $F^k\mathcal{U}^{\mathcal{O}}_{A'}(m)$ as a pushout:

$$\begin{array}{ccc}\mathcal{U}^{\mathcal{O}}_A(k+m)\,\square_{\Sigma_k}Q^k_{k-1}(X\to Y) & \longrightarrow & \mathcal{U}^{\mathcal{O}}_A(k+m)\,\square_{\Sigma_k}Y^{(k)}\\ \big\downarrow & & \big\downarrow\\ F^{k-1}\mathcal{U}^{\mathcal{O}}_{A'}(m) & \longrightarrow & F^k\mathcal{U}^{\mathcal{O}}_{A'}(m)\end{array}\tag{5.6.3}$$

In practice, the map $Q^k_{k-1}(X\to Y)\to Y^{(k)}$ is some kind of cofibration when $X\to Y$ is nice enough; the above formulation is then useful for deducing homotopical information in the presence of cofibrantly generated model category structures, as discussed in Section 5.8.

5.7 Enrichment and geometric realization

Categories of operadic algebras in spaces or spectra come with a canonical enrichment in spaces, i.e., they have mapping spaces and an intrinsic notion of homotopy. While more abstract notions of homotopy, for example, in terms of model structures, now play a more significant role in homotopy theory, the topological enrichment provides some powerful tools, including and especially geometric realization of simplicial objects.

We begin with a general discussion of enrichment of operadic algebra categories. When $\mathscr{M}$ satisfies the hypotheses of Proposition 5.4.7, Proposition 5.4.9 describes the maps in the category of $\mathcal{O}$-algebras as an equalizer

$$\mathscr{M}[\mathcal{O}](A,B)\longrightarrow\mathscr{M}(A,B)\rightrightarrows\mathscr{M}(\mathbb{O}A,B)\ ,$$

where one arrow $\mathscr{M}(A,B)\to\mathscr{M}(\mathbb{O}A,B)$ is induced by the action map $\mathbb{O}A\to A$ and the other is induced by applying the functor $\mathbb{O}\colon\mathscr{M}(A,B)\to\mathscr{M}(\mathbb{O}A,\mathbb{O}B)$ and then using the action map $\mathbb{O}B\to B$. When $\mathscr{M}$ is enriched over a complete symmetric monoidal category (for example, when the mapping sets of $\mathscr{M}$ are topologized or simplicial), then $\mathscr{M}[\mathcal{O}]$ becomes enriched exactly when $\mathbb{O}$ has the structure of an enriched functor, defining the enrichment of $\mathscr{M}[\mathcal{O}]$ by the equalizer above. Clearly it is not always possible for $\mathbb{O}$ to be enriched: if $\mathscr{M}$ is the category of abelian groups and $\mathcal{O}=\mathcal{A}$ss or $\mathcal{C}$om, then $\mathbb{O}$ is not an additive functor so cannot be enriched over abelian groups; this corresponds to the fact that the categories of rings and commutative rings are not enriched over abelian groups. On the other hand, enrichments over spaces and simplicial sets are usually inherited by algebra categories; the reason, as we now explain, derives from the fact that spaces and simplicial sets are cartesian.

For convenience, consider the case when $\mathscr{M}$ is a closed symmetric monoidal category. Let $\mathscr{E}, \times, *$ be a symmetric monoidal category (which we will eventually assume to be cartesian), and let $L\colon \mathscr{E} \to \mathscr{M}$ be a strong symmetric monoidal functor that is a left adjoint; let R denote its right adjoint. For formal reasons R is then lax symmetric monoidal and in particular RF provides an $\mathscr{E}$-enrichment of $\mathscr{M}$ (where, as always, F denotes the mapping object in $\mathscr{M}$). These hypotheses are not all necessary but avoid some review of enriched category theory and concisely state a lot of coherence data that more minimal hypotheses would force us to spell out. The iterated symmetric monoidal product in $\mathscr{M}$ then gives a multivariable enriched functor

$$RF(A_1, B_1) \times \cdots \times RF(A_m, B_m) \to RF(A_1 \square \cdots \square A_m, B_1 \square \cdots \square B_m).$$

Now assume that $\times$ is a cartesian monoidal product, meaning that it is the categorical product, the unit is the final object, and the symmetry and unit isomorphisms are the universal ones. With this assumption, we have a natural diagonal map $E \to E \times E$, which we can apply in particular to the object $RF(A, B)$ to get a natural map

$$RF(A, B) \to RF(A, B) \times \cdots \times RF(A, B) \to RF(A^{(m)}, B^{(m)}). \tag{5.7.1}$$

This makes the m-th $\square$-power into an $\mathscr{E}$-enriched functor for $m > 0$. In the case $m = 0$, we have the final map

$$RF(A, B) \to * \to R\mathbf{1} \xrightarrow{\cong} RF(A^{(0)}, B^{(0)}).$$

From here the rest is easy: the $\square, F$ adjunction gives a natural (and $\mathscr{E}$-natural) map

$$RF(A^{(m)}, B^{(m)}) \to RF(\mathcal{O}(m) \square A^{(m)}, \mathcal{O}(m) \square B^{(m)})$$

and the composite to $RF(\mathcal{O}(m)\square A^{(m)}, \mathcal{O}(m)\square_{\Sigma_m} B^{(m)})$ admits a canonical factorization

$$RF(A, B) \to RF(\mathcal{O}(m) \square_{\Sigma_m} A^{(m)}, \mathcal{O}(m) \square_{\Sigma_m} B^{(m)}),$$

since the target is a limit (in $\mathscr{E}$) that exists by right adjoint considerations when the quotient $\mathcal{O}(m) \square_{\Sigma_m} B^{(m)} = (\mathcal{O}(m) \square B^{(m)})/\Sigma_m$ in $\mathscr{M}$ exists. When we assume that $\mathscr{M}$ has countable coproducts, composing further into

$$RF(\mathcal{O}(m) \square_{\Sigma_m} A^{(m)}, \mathbb{O}B),$$

the countable categorical product over m exists, giving an $\mathscr{E}$-natural map

$$RF(A, B) \to RF(\mathbb{O}A, \mathbb{O}B)$$

which provides the $\mathscr{E}$-enrichment of $\mathbb{O}$. We state this as a theorem:

Theorem 5.7.1. *Let $\mathscr{M}$ be a closed symmetric monoidal category with countable colimits, and let $\mathcal{O}$ be an operad in $\mathscr{M}$. Let $\mathscr{E}$ be a cartesian monoidal category and let $\mathscr{E} \to \mathscr{M}$ be a strong symmetric monoidal functor with a right adjoint. Regarding $\mathscr{M}$ as $\mathscr{E}$-enriched over the right adjoint, the category $\mathscr{M}[\mathcal{O}]$ of $\mathcal{O}$-algebras has a canonical $\mathscr{E}$-enrichment for which the forgetful functor $\mathscr{M}[\mathcal{O}] \to \mathscr{M}$ is $\mathscr{E}$-enriched.*

We apply this now in the discussion of geometric realizations of (co)simplicial objects. Let $\mathscr{S}$ denote either the category of spaces or of simplicial sets, and write $C(-,-)$ for the internal mapping objects in $\mathscr{S}$. To avoid awkward circumlocutions, we will refer to objects of $\mathscr{S}$ as spaces in either case for the rest of the section. We now assume that $\mathscr{M}$ is closed symmetric monoidal and has countable coproducts and that we have a left adjoint symmetric monoidal functor L from $\mathscr{S}$ to $\mathscr{M}$, as above, so that Theorem 5.7.1 applies. We write R for the right adjoint to L as above, so that $RF(-,-)$ provides mapping spaces for $\mathscr{M}$. The category $\mathscr{M}$ then has ***tensors*** $X \otimes T$ and ***cotensors*** $T \pitchfork Y$, defined by the natural isomorphisms

$$RF(X \otimes T, -) \cong C(T, RF(X,-)) \qquad \text{(tensor)},$$
$$RF(-, T \pitchfork Y)) \cong C(T, RF(-,Y)) \qquad \text{(cotensor)},$$

for spaces T and objects X and Y of $\mathscr{M}$, constructed as follows.

Proposition 5.7.2. *In the context of Theorem 5.7.1, tensors and cotensors with spaces exist and are given by* $X \otimes T = X \square LT$ *and* $T \pitchfork Y = F(LT, Y)$ *for a space* T *and objects* X, Y *in* $\mathscr{M}$.

The proposition is an easy consequence of the formal isomorphism

$$RF(LT, X) \cong C(T, RX), \tag{5.7.2}$$

natural in spaces T and objects X of $\mathscr{M}$; the isomorphism in the forward direction is adjoint to the map

$$RF(LT,X) \times T \to RF(LT,X) \times RLT \to R(F(LT,X) \square LT) \to RX$$

and the isomorphism in the backwards direction is adjoint to the map $LC(T,RX) \to F(LT,X)$ adjoint to the map

$$LC(T,RX) \square LT \cong L(C(T,RX) \times T) \to LRX \to X.$$

Let $RF^{\mathscr{M}[\mathcal{O}]}(-,-)$ denote the mapping spaces constructed above for the category of $\mathcal{O}$-algebras; despite the suggestion of the notation, this is not typically a composite functor. For an $\mathcal{O}$-algebra A, $F(-,A)$ does not typically carry a canonical $\mathcal{O}$-algebra structure, but for a space T, $F(LT,A) = T \pitchfork A$ does: the structure map

$$\mathcal{O}(n) \square (T \pitchfork A)^{(n)} \to T \pitchfork A$$

is adjoint to the map

$$\mathcal{O}(n) \square (T \pitchfork A)^{(n)} \square LT = \mathcal{O}(n) \square (F(LT,A))^{(n)} \square LT \to A$$

constructed as the composite

$$\mathcal{O}(n) \square (F(LT,A))^{(n)} \square LT \to \mathcal{O}(n) \square (F(LT,A))^{(n)} \square (LT)^{(n)} \to \mathcal{O}(n) \square A^{(n)} \to A$$

using the diagonal map on the space T and the structure map on A. A check of universal properties then shows that $T \pitchfork A$ is the cotensor of A with T in the category of $\mathcal{O}$-algebras. Tensors in $\mathscr{M}[\mathcal{O}]$ can be constructed as reflexive coequalizers

$$\mathbb{O}(\mathbb{O}A \otimes T) \rightrightarrows \mathbb{O}(A \otimes T) \longrightarrow A \otimes^{\mathscr{M}[\mathcal{O}]} T.$$

Writing $\Delta[n]$ for the standard n-simplex, we then have the standard definition of geometric realization of simplicial objects in $\mathscr{M}$ and $\mathscr{M}[\mathcal{O}]$ (without additional assumptions) and geometric realization (often called "Tot") of cosimplicial objects in $\mathscr{M}$ and $\mathscr{M}[\mathcal{O}]$ when certain limits exist. Given a simplicial object $X_\bullet$ or a cosimplicial object $Y^\bullet$, the degeneracy subobject sX_n of X_n is defined as the colimit of the degeneracy maps and the degeneracy quotient object sY^n of Y^n is defined as the limit (if it exists) of the degeneracy maps. (In some literature, sX_n is called the "latching object" and sY^n the "matching object"; see [124, §15.2].) The geometric realization of $X_\bullet$ in $\mathscr{M}$ or $\mathscr{M}[\mathcal{O}]$ is then the sequential colimit of $|X_\bullet|_n$, where $|X_\bullet|_0 = X_0$ and $|X_\bullet|_n$ is defined inductively as the pushout

$$\begin{array}{ccc} (sX_n \otimes \Delta[n]) \cup_{(sX_n \otimes \partial\Delta[n])} (X_n \otimes \partial\Delta[n]) & \longrightarrow & X_n \otimes \Delta[n] \\ \downarrow & & \downarrow \\ |X_\bullet|_{n-1} & \longrightarrow & |X_\bullet|_n \end{array}$$

with both the tensor and the pushouts performed in $\mathscr{M}$ to define the geometric realization in $\mathscr{M}$ or performed in $\mathscr{M}[\mathcal{O}]$ to define the geometric realization in $\mathscr{M}[\mathcal{O}]$. The analogous, opposite construction defines the geometric realization of $Y^\bullet$ when all the limits exist. Because cotensors and limits (when they exist) coincide in $\mathscr{M}$ and $\mathscr{M}[\mathcal{O}]$, geometric realization of cosimplicial objects (when it exists) also coincides in $\mathscr{M}$ and $\mathscr{M}[\mathcal{O}]$. Because pushouts generally look very different in $\mathscr{M}$ than in $\mathscr{M}[\mathcal{O}]$, one might expect that geometric realization of simplicial objects in $\mathscr{M}$ and in $\mathscr{M}[\mathcal{O}]$ would also look very different; this turns out not to be the case.

Theorem 5.7.3. *Assume $\mathscr{M}$ satisfies the hypotheses of Theorem 5.7.1 for $\mathscr{E}$ either the category of spaces or the category of simplicial sets.*

(i) *Let $A^\bullet$ be a cosimplicial object in $\mathscr{M}[\mathcal{O}]$. If the limits defining the geometric realization* (Tot) *exist in $\mathscr{M}$, then that geometric realization has the canonical structure of an $\mathcal{O}$-algebra and is isomorphic to the geometric realization* Tot *in $\mathscr{M}[\mathcal{O}]$.*
(ii) *Let $A_\bullet$ be a simplicial object in $\mathscr{M}[\mathcal{O}]$. Then the geometric realization of $A_\bullet$ in $\mathscr{M}$ has the canonical structure of an $\mathcal{O}$-algebra and is isomorphic to the geometric realization of $A_\bullet$ in $\mathscr{M}$.*

As discussed above, only (ii) requires additional argument. For clarity in the argument for the theorem, we will write $|\cdot|$ for geometric realization in $\mathscr{M}$ and $|\cdot|^{\mathscr{M}[\mathcal{O}]}$ for geometric realization in $\mathscr{M}[\mathcal{O}]$. Here is the key fact:

Lemma 5.7.4. *For $\mathscr{M}$ as in the previous theorem, geometric realization in $\mathscr{M}$ is strong symmetric monoidal.*

Proof. Although we wrote a more constructive definition of geometric realization above, it can also be described as a coend

$$|X_\bullet| = \int^{\mathbf{\Delta}^{\mathrm{op}}} X_\bullet \otimes \Delta[\bullet],$$

where $\mathbf{\Delta}$ denotes the category of simplexes (the category with objects $[n] = \{0,\dots,n\}$ for $n = 0,1,2,\dots$, and maps the non-decreasing functions) and $\Delta[n]$ denotes the standard n-simplex in spaces or simplicial sets. Because the symmetric monoidal product $\square$ for $\mathscr{M}$ is assumed to commute with colimits in each variable, we can identify the product of geometric realizations also as a coend

$$|X_\bullet| \square |Y_\bullet| \cong \int^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}} (X_\bullet \square Y_\bullet) \otimes (\Delta[\bullet] \times \Delta[\bullet]).$$

On the other hand,

$$|X_\bullet \square Y_\bullet| = \int^{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{diag}(X_\bullet \square Y_\bullet) \otimes \Delta[\bullet].$$

Next, we need a purely formal observation, which is an adjoint form of the Yoneda lemma: if coproducts of appropriate cardinality exist in $\mathscr{C}$, then given a functor $F\colon \mathscr{C} \to \mathscr{D}$, functoriality of F induces a natural isomorphism

$$\int^{c\in\mathscr{C}} F(c) \times \mathscr{C}(c,-) \xrightarrow{\cong} F(-)$$

(where $\times$ denotes coproduct over the given set; this coend exists and the identification holds with no further hypotheses on $\mathscr{C}$ or $\mathscr{D}$). Applying this to

$$F((\bullet,\bullet)) = X_\bullet \square Y_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathscr{M}$$

and pre-composing with diag, we get an isomorphism

$$X_p \square Y_p \cong \int^{(m,n)\in\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}} (X_m \square Y_n) \times (\mathbf{\Delta}^{\mathrm{op}}(m,p) \times \mathbf{\Delta}(n,p))$$

of functors $p \in \mathbf{\Delta}^{\mathrm{op}} \to \mathscr{M}$. Commuting coends, we can reorganize the double coend

$$|X_\bullet \square Y_\bullet| \cong \int^{p\in\mathbf{\Delta}^{\mathrm{op}}} \left(\int^{(m,n)\in\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}} (X_m \square Y_n) \times (\mathbf{\Delta}^{\mathrm{op}}(m,p) \times \mathbf{\Delta}^{\mathrm{op}}(n,p)) \right) \otimes \Delta[p]$$

as

$$\int^{(m,n)\in\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}} (X_m \square Y_n) \otimes \left(\int^{p\in\mathbf{\Delta}^{\mathrm{op}}} (\mathbf{\Delta}^{\mathrm{op}}(m,p) \times \mathbf{\Delta}^{\mathrm{op}}(n,p)) \times \Delta[p] \right).$$

In the latter formula, the expression in parentheses is the coend formula for the geometric realization (in spaces) of the product of standard simplices (in simplicial sets) $\Delta[m]_\bullet \times \Delta[n]_\bullet$, which is $\Delta[m] \times \Delta[n]$ by the classic version of the lemma for geometric realization in spaces. This then constructs the natural isomorphism $|X_\bullet| \square |Y_\bullet| \cong |X_\bullet \square Y_\bullet|$, and a little more fiddling with coends shows that this natural transformation is symmetric monoidal. □

Because of the previous lemma, we have a natural isomorphism $\mathbb{O}|X_\bullet| \cong |\mathbb{O}X_\bullet|$ that makes the appropriate diagrams commute, so that the geometric realization (in $\mathscr{M}$) of a simplicial object $A_\bullet$ in $\mathscr{M}[\mathcal{O}]$ acquires the natural structure of an $\mathcal{O}$-algebra. Moreover, the canonical maps $A_n \otimes \Delta[n] \to |A_\bullet|$ induce maps of $\mathcal{O}$-algebras $A_n \otimes^{\mathscr{M}[\mathcal{O}]} \Delta[n] \to |A_\bullet|$ that assemble into a natural map of $\mathcal{O}$-algebras

$$|A_\bullet|^{\mathscr{M}[\mathcal{O}]} \to |A_\bullet|.$$

In the case when $A_\bullet = \mathbb{O}X_\bullet$, under the identification of colimits $|\mathbb{O}X_\bullet|^{\mathscr{M}[\mathcal{O}]} = \mathbb{O}|X_\bullet|$, this map is the isomorphism $\mathbb{O}|X_\bullet| \to |\mathbb{O}X_\bullet|$ above. To see that it is an isomorphism for arbitrary $A_\bullet$, write $A_\bullet$ as a (reflexive) coequalizer

$$\mathbb{O}\mathbb{O}A_\bullet \rightrightarrows \mathbb{O}A_\bullet \longrightarrow A_\bullet,$$

apply the functors, and compare diagrams.

5.8 Model structures for operadic algebras

The purpose of this section is to review the construction of model structures on some of the categories of operadic algebras that are of interest in homotopy theory; we use these in the next section in comparison theorems giving Quillen equivalences between some of these categories. Constructing model structures for algebras over operads is a special case of constructing model structures for algebras over monads; chapter VII of EKMM [94] seems to be an early reference for this kind of result, but it concentrates on the category of LMS-spectra and related categories. Schwede–Shipley [267] studies the general case of monads in cofibrantly generated monoidal model categories, which Spitzweck [280] specializes to the case of operads. Because less sharp results hold in the general case than in the special cases of interest, we state the results on model structures as a list of examples. This is an "example theorem" both in the sense that it gives a list of examples, but also in the sense that it fits into the general rubric of the kind of theorem that should hold very generally under appropriate technical hypotheses with essentially the same proof outline. Some terminology and notation is explained after the statement.

Example Theorem 5.8.1. *Let $\mathscr{M}$ be a symmetric monoidal category with a cofibrantly generated model structure and let $\mathcal{O}$ be an operad in $\mathscr{M}$ from one of the examples listed below. Then the category of $\mathcal{O}$-algebras in $\mathscr{M}$ is a closed model category with*

(i) *weak equivalences the underlying weak equivalences in $\mathscr{M}$,*
(ii) *fibrations the underlying fibrations in $\mathscr{M}$, and*
(iii) *cofibrations the retracts of regular $\mathbb{O}I$-cofibrations.*

This theorem holds in particular in the examples:

(a) *$\mathscr{M}$ is the category of symmetric spectra (of spaces or simplicial sets) with its positive stable model structure or orthogonal spectra with its positive stable model structure or* *the category of EKMM S-modules with its standard model structure (with $\square$ the smash product, $\mathbf{1}$ the sphere spectrum) and $\mathcal{O}$ is any operad in $\mathscr{M}$.* [68, 8.1]

(b) *$\mathscr{M}$ is the category of spaces or simplicial sets (with $\square = \times$, $\mathbf{1} = *$), or simplicial R-modules for some simplicial commutative ring R (with $\square = \otimes_R$, $\mathbf{1} = R$) and $\mathcal{O}$ is any operad.*

(c) *$\mathscr{M}$ is the category of (unbounded) chain complexes in R-modules for a commutative ring R (with $\square = \otimes_R$, $\mathbf{1} = R$) and either $R \supset \mathbb{Q}$ or $\mathcal{O}$ admits a map of operads $\mathcal{O} \to \mathcal{O} \otimes \mathcal{E}$ which is a section for the map $\mathcal{O} \otimes \mathcal{E} \to \mathcal{O} \otimes \mathcal{C}\mathrm{om} \cong \mathcal{O}$, where $\mathcal{E}$ is any E_∞ operad that naturally acts on the normalized cochains of simplicial sets.* [37, 3.1.3]

(d) *$\mathscr{M}$ is a monoidal model category in the sense of* [267, 3.1] *that satisfies the Monoid Axiom of* [267, 3.3] *and $\mathcal{O}$ is a cofibrant operad in the sense of* [280, §3]. [280, §4, Theorem 4]

The category of EKMM $\mathbb{L}$-spectra [94, I§4] also fits into example (a) if we allow $\mathscr{M}$ to be a "weak" symmetric monoidal category in the sense of [94, II.7.1]; the theorem then covers categories of operadic algebras in LMS spectra for operads over the linear isometries operad that have the form $\mathcal{O} \times \mathcal{L} \to \mathcal{L}$; see [68, 3.5].

In part (c), we note that for an operad that satisfies the section condition (or when $R \supset \mathbb{Q}$), the functor $\mathcal{O}(n) \times_{R[\Sigma_n]} (-)$ preserves preserve exactness of (homologically) bounded-below exact sequences of R-free $R[\Sigma_n]$-modules (for all n). For operads that satisfy this more general condition but not necessarily the section condition, the algebra category still has a theory of cofibrant objects and a good homotopy theory for those objects; see, for example, [181, §2].

It is beyond the scope of this chapter to do a full review of closed model category theory terminology, but we recall that a "cofibrantly generated model category" has a set I of "generating cofibrations" and a set J of "generating acyclic cofibrations" for which the Quillen small object argument can be done (perhaps transfinitely, but in the examples of (a), (b), and (c), sequences suffice). Then

$$\mathbb{O}I = \{\mathbb{O}f \mid f \in I\}$$

is the set of maps of $\mathcal{O}$-algebras obtained by applying $\mathbb{O}$ to each of the maps in I. The point of $\mathbb{O}I$ is that a map of $\mathcal{O}$-algebras has the left lifting property with respect to $\mathbb{O}I$ in $\mathcal{O}$-algebras exactly when the underlying map in $\mathscr{M}$ has the left lifting property with respect to I. The same definition and observations apply replacing I with J. The strategy for proving the previous theorem is to define the fibrations and weak equivalences of $\mathcal{O}$-algebras as in (i),(ii), and define cofibrations in terms of the left lifting property (obtaining the characterization in (iii) as a theorem). The advantage of this approach is that fibrations and acyclic fibrations are also characterized by lifting properties: a map of $\mathcal{O}$-algebras is a fibration if and only if it has the right lifting property with respect to $\mathbb{O}J$ and a map of $\mathcal{O}$-algebras is an acyclic fibration if and only if it has the right lifting property with respect to $\mathbb{O}I$. For these lifting properties, we can attempt the small object argument. We now outline the remaining steps in this approach.

Recall that a *regular $\mathbb{O}I$-cofibration* is a map formed as a (transfinite) composite of pushouts along coproducts of maps in $\mathbb{O}I$. This is the generalization of the notion of a *relative* $\mathbb{O}I$*-cell complex*, which is the colimit of a sequence of pushouts of coproducts of maps in $\mathbb{O}I$; in the case of examples (a), (b), and (c), in a regular $\mathbb{O}I$-cofibration the transfinite composite can always be replaced simply by a sequential composite and so a regular $\mathbb{O}I$-cofibration is a relative $\mathbb{O}I$-cell complex. The small object argument for I and J in $\mathscr{M}$ implies the small object argument for $\mathbb{O}I$ and $\mathbb{O}J$, which gives factorization in $\mathcal{O}$-algebras of a map as either a regular $\mathbb{O}I$-cofibration followed by an acyclic fibration or a regular $\mathbb{O}J$-cofibration followed by a fibration. (A small wrinkle comes up in going from the small object argument in $\mathscr{M}$ to the small object argument in $\mathscr{M}[\mathcal{O}]$ in the topological examples of (a) and (b): we need to check that regular $\mathbb{O}I$-cofibrations are nice maps, for example, closed inclusions on the constituent spaces; see the "Cofibration Hypothesis" of [94, VII§4] or [178, 5.3].)

This gets us most of the way to a model structure. Having defined a cofibration of $\mathcal{O}$-algebras as a map that has the left lifting property with respect to the acyclic fibrations, the free-forgetful adjunction shows that regular $\mathbb{O}I$-cofibrations are cofibrations; moreover, it follows formally that any cofibration is the retract of a regular $\mathbb{O}I$-cofibration: given a cofibration $f\colon A \to B$, factor it as $p \circ i$ for $i\colon A \to B'$ a regular $\mathbb{O}I$-cofibration and $p\colon B' \to B$ an acyclic fibration, then solving the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & B' \\ {\scriptstyle f}\downarrow & \overset{g}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\text{id}]{} & B \end{array}$$

to produce a map $g\colon B \to B'$ exhibits f as a retract of i.

$$\begin{array}{ccccc} A & \xrightarrow{\text{id}} & A & \xrightarrow{\text{id}} & A \\ {\scriptstyle f}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle f}\downarrow \\ B & \xrightarrow[g]{} & B' & \xrightarrow[p]{} & B \end{array}$$

We can try the same thing with regular $\mathbb{O}J$-cofibrations; they have the left lifting property with respect to all fibrations so are in particular cofibrations, but are they weak equivalences? This is the big question and what keeps us from having a fully general result for Theorem 5.8.1, especially in (c). If regular $\mathbb{O}J$-cofibrations are weak equivalences, then the trick in the previous argument shows that every acyclic cofibration is a retract of a regular $\mathbb{O}J$-cofibration, and the lifting property for acyclic cofibrations follows as does the other factorization, proving the model structure. (Conversely, if the model structure exists, because regular $\mathbb{O}J$-cofibrations have the left lifting property for all fibrations, it follows that they are weak equivalences.)

In many examples, including examples (a) and (b) in the theorem above, the homogeneous filtration on the pushout that we studied in Section 5.6 can be used to prove that regular $\mathbb{O}J$-cofibrations are weak equivalences. Specifically, for $X \to Y$ a map in J, taking $A' = A \amalg^{\mathscr{M}[\mathcal{O}]}_{\mathbb{O}X} \mathbb{O}Y$, the case $m = 0$ of the filtration on the enveloping operad for A gives a filtration on A' by objects of $\mathscr{M}$ starting from A. Now from the inductive definition of $Q^k_{k-1}(X \to Y)$ in (5.6.2), it can be checked in examples (a) and (b) that the map $Q^k_{k-1}(X \to Y) \to Y^{(k)}$ is an equivariant Hurewicz cofibration of the underlying spaces or a monomorphism of the underlying simplicial sets as well as being a weak equivalence. The pushout (5.6.3) then identifies the maps in the filtration of A' as weak equivalences as well. (This approach can also be used to prove versions of the "Cofibration Hypothesis" of [94, VII§4] or [178, 5.3] that regular $\mathbb{O}I$-cofibrations are closed inclusions on the constituent spaces.)

Example (d) is similar, except that it uses a filtration argument on the construction of a cofibrant operad; see [280, §4].

Example (c) fits into the case of the general theorem of Schwede–Shipley [267, 2.3], where every object is fibrant and has a path object. To complete the argument here, we need to show that every map $f: A \to B$ factors as a weak equivalence followed by a fibration:

$$A \xrightarrow{\simeq} A' \twoheadrightarrow B.$$

We then get the factorization of an acyclic cofibration followed by a fibration by using the factorization already established:

$$A \overset{\simeq}{\rightarrowtail} A'' \overset{\simeq}{\twoheadrightarrow} A' \twoheadrightarrow B.$$

In the case of (c) where we hypothesize a map of operads $\mathcal{O} \to \mathcal{O} \otimes \mathcal{E}$, this map gives a natural $\mathcal{O}$-algebra structure on $B \otimes C^*(-)$; the hypothesis that the composite map on $\mathcal{O}$ is the identity implies that the canonical isomorphism

$$B \cong B \otimes C^*(\Delta[0])$$

is an $\mathcal{O}$-algebra map. Looking at the maps between $\Delta[0]$ and $\Delta[1]$, we get maps of $\mathcal{O}$-algebras

$$B \to B \otimes C^*(\Delta[1]) \to B \times B$$

and the usual mapping path object construction

$$A \xrightarrow{\simeq} A \times_B (B \otimes C^*(\Delta[1])) \twoheadrightarrow B$$

consists of maps of $\mathcal{O}$-algebras and gives the factorization. In the case when $R \supset \mathbb{Q}$, the polynomial de Rham functor A^* reviewed in Section 5.12 is a functor from simplicial sets to commutative differential graded $\mathbb{Q}$-algebras, which can be used in the same way to construct a factorization

$$A \xrightarrow{\simeq} A \times_B (B \otimes_{\mathbb{Q}} A^*(\Delta[1])) \twoheadrightarrow B.$$

In the case of operadic algebras in spaces in example (b) and EKMM S-modules in example (a), we have another argument taking advantage of the topological enrichment. In these examples, the maps in J are deformation retractions, and so the maps in $\mathbb{O}J$ are deformation retractions in the category of $\mathcal{O}$-algebras. It follows that regular $\mathbb{O}J$-cofibrations are also deformation retractions and in particular homotopy equivalences. Since homotopy equivalences are weak equivalences, regular $\mathbb{O}J$-cofibrations are weak equivalences in examples where this argument can be made. The specific examples again fit into the case of [267, 2.3] where every object is fibrant and has a path object.

5.9 Comparison and rectification theorems for operadic algebras

This section discusses Quillen equivalences and Quillen adjunctions between the model categories in Example Theorem 5.8.1. When we change from simplicial sets to spaces or when we change the underlying symmetric monoidal category between the Quillen equivalent modern categories of spectra, we get Quillen equivalences of categories of operadic algebras under only mild technical hypotheses on the operad; this gives several comparison theorems. We also consider Quillen adjunctions and Quillen equivalences obtained by change of operads. In wide generality, the augmentation map $\mathcal{A} \to \mathcal{A}ss$ for an A_∞ operad induces a Quillen equivalence between categories of algebras. Likewise, in the case of modern categories of spectra, the augmentation map $\mathcal{E} \to \mathcal{C}om$ for an E_∞ operad induces a Quillen equivalence between categories of algebras. These comparison theorems are rectification theorems in that they show that a homotopical algebraic structure can be replaced up to weak equivalence with a strict algebraic structure.

We begin by reviewing the change of operad adjunction. Let $f\colon \mathcal{A} \to \mathcal{B}$ be a map of operads in a symmetric monoidal category $\mathscr{M}$. Such a map certainly gives a restriction functor U_f from $\mathcal{B}$-algebras to $\mathcal{A}$-algebras, and under mild hypothesis, this functor has a left adjoint. As in the discussion of colimits in Section 5.6, if we assume that $\mathscr{M}$ satisfies the hypotheses of Proposition 5.4.7 then we can define $P_f\colon \mathscr{M}[\mathcal{A}] \to \mathscr{M}[\mathcal{B}]$ by the reflexive coequalizer

$$\mathbb{B}(\mathbb{A}A) \rightrightarrows \mathbb{B}A \to P_f(A),$$

where $\mathbb{A}$ and $\mathbb{B}$ denote the monads associated to $\mathcal{A}$ and $\mathcal{B}$, one arrow is induced by the $\mathcal{A}$-algebra structure on A, and the other arrow is the composite $\mathbb{B}\mathbb{A} \to \mathbb{B}\mathbb{B} \to \mathbb{B}$ induced by the map of operads f and the monadic product on $\mathbb{B}$. As a side remark, not related to the rest of this section, we note that in this situation the category $\mathcal{B}$-algebras can be identified as the category of algebras for the monad $U_f P_f$ in $\mathscr{M}[\mathcal{A}]$ (for a general formal proof, see [94, II.6.6.1]).

Now suppose that $\mathscr{M}$ has a closed model structure and $\mathscr{M}[\mathcal{A}]$ and $\mathscr{M}[\mathcal{B}]$ are closed model categories with fibrations and weak equivalences created in $\mathscr{M}$. For a map of operads $f\colon \mathcal{A} \to \mathcal{B}$, we then get a Quillen adjunction

$$P_f\colon \mathscr{M}[\mathcal{A}] \rightleftarrows \mathscr{M}[\mathcal{B}] : U_f.$$

When can we expect it to be a Quillen equivalence? It is tempting to define an equivalence of operads in $\mathscr{M}$ to be a map f such that derived adjunction induces an equivalence of homotopy categories; then we have a tautological result that an equivalence of operads induces a Quillen equivalence of model structures. Instead we propose the following definition, which leads to a theorem with some substance

(Example Theorem 5.9.5). It is the condition used in practice in proving comparison and rectification theorems.

Definition 5.9.1. Let $\mathscr{M}$ be a closed model category with countable coproducts and with a symmetric monoidal product that preserves countable colimits in each variable. We say that a map $f\colon \mathcal{A} \to \mathcal{B}$ of operads in $\mathscr{M}$ is a *derived monad equivalence* if the induced map $\mathbb{A}Z \to \mathbb{B}Z$ is a weak equivalence for every cofibrant object Z in $\mathscr{M}$.

Though we have not put enough hypotheses on $\mathscr{M}$ to ensure it, in practice countable coproducts of reasonable objects in $\mathscr{M}$ will preserve and reflect weak equivalences and then f will be a derived monad equivalence if and only if each of the maps

$$\mathcal{A}(m) \,\square_{\Sigma_m} Z^{(m)} \to \mathcal{B}(m) \,\square_{\Sigma_m} Z^{(m)}$$

is a weak equivalence. In our examples of main interest, we have more intrinsic sufficient conditions for a map of operads to be a derived monad equivalence.

Example 5.9.2. In the category of spaces (or more generally, any topological or simplicial model category), a map of operads $f\colon \mathcal{A} \to \mathcal{B}$ that induces an equivariant homotopy equivalence $\mathcal{A}(m) \to \mathcal{B}(m)$ for all m is a derived monad equivalence. Indeed, the map $\mathbb{A}Z \to \mathbb{B}Z$ is a homotopy equivalence for all Z, and a homotopy equivalence in a topological or simplicial model category is a weak equivalence. As a special case, when $\overline{\mathcal{A}}$ is a non-symmetric operad with $\overline{\mathcal{A}}(m)$ contractible for all m, the map of operads $\mathcal{A} \to \mathcal{A}\mathrm{ss}$ is a derived monad equivalence.

Example 5.9.3. In the category of symmetric spectra (of spaces or simplicial sets) with its positive stable model structure or the category of orthogonal spectra with its positive model structure, a map of operads $f\colon \mathcal{A} \to \mathcal{B}$ that induces a (non-equivariant) weak equivalence $\mathcal{A}(n) \to \mathcal{B}(n)$ is a derived monad equivalence. This can be proved by generalizing the argument of [178, 15.5] (see [68, 8.3.(i)] for slightly more details). In the case of EKMM S-modules, if $f\colon \mathcal{A} \to \mathcal{B}$ is a map of operads of spaces that is a (non-equivariant) homotopy equivalence $\mathcal{A}(n) \to \mathcal{B}(n)$ for all n, then $\Sigma^\infty_+ f$ is a derived monad equivalence. This can be proved by generalizing the argument of [94, III.5.1]. (See [68, 8.3.(ii)] for a more general statement.) In particular, in these categories, the augmentation map $\mathcal{E} \to \mathcal{C}\mathrm{om}$ for an E_∞ operad (assumed to come from spaces in the EKMM S-module case) is a derived monad equivalence.

Example 5.9.4. In the context of chain complexes of R-modules, a map of operads $\mathcal{A} \to \mathcal{B}$ that is an $R[\Sigma_n]$-module chain homotopy equivalence $\mathcal{A}(n) \to \mathcal{B}(n)$ for all n is a derived monad equivalence. If the functors $\mathcal{A}(n) \otimes_{R[\Sigma_n]} (-)$ and $\mathcal{B}(n) \otimes_{R[\Sigma_n]} (-)$ preserve exactness of (homologically) bounded-below exact sequences of R-free $R[\Sigma_n]$-modules (for all n), then a weak equivalence $\mathcal{A} \to \mathcal{B}$ is a derived monad equivalence. This occurs in particular for part (c) of Example Theorem 5.8.1 when $\mathcal{A}$ and $\mathcal{B}$ both satisfy the stated operad hypotheses.

To go with these examples, we have the following example theorem.

Example Theorem 5.9.5. *Let $\mathscr{M}$ be a symmetric monoidal category and $f\colon \mathcal{A} \to \mathcal{B}$ a map of operads in $\mathscr{M}$, where $\mathscr{M}$, $\mathcal{A}$, and $\mathcal{B}$ fall into one of the examples of Example Theorem 5.8.1*(a)–(c). *If f is a derived monad equivalence then the Quillen adjunction $P_f\colon \mathscr{M}[\mathcal{A}] \rightleftarrows \mathscr{M}[\mathcal{B}] : U_f$ is a Quillen equivalence.*

Again, as in the previous section, this is an "example theorem" in that it gives an example of the kind of theorem that holds much more generally with a proof that can also be adapted to work much more generally. We outline the proof after the change of categories theorem below, as the arguments for both are quite similar.

In terms of change of categories, one should expect comparison theorems of the following form to hold quite generally:

Let $L\colon \mathscr{M} \rightleftarrows \mathscr{M}' : R$ be a Quillen equivalence between monoidal model categories with L strong symmetric monoidal, and let $\mathcal{O}$ be an operad in $\mathscr{M}$. With some technical hypotheses, the adjunction

$$L\colon \mathscr{M}[\mathcal{O}] \rightleftarrows \mathscr{M}'[L\mathcal{O}] : R$$

on operadic algebra categories is also a Quillen equivalence.

A minimal technical hypothesis is that $L\mathcal{O}$ be "the right thing" and an easy way to ensure this is to put some kind of cofibrancy condition on the objects $\mathcal{O}(n)$. In our cases of interest, we could certainly state such a theorem, but it would not cover the example in modern categories of spectra when $\mathcal{O}$ is the suspension spectrum functor applied to an operad of spaces; for such an operad, the spectra $\mathcal{O}(n)$ will not be cofibrant. On the other hand, in these examples the right adjoint preserves all weak equivalences and not just weak equivalences between fibrant objects; in this setup it seems more convenient to consider an operad $\mathcal{O}'$ in $\mathscr{M}'$ and a map of operads $\mathcal{O} \to R\mathcal{O}'$ (or equivalently, $L\mathcal{O} \to \mathcal{O}'$) that induces a weak equivalence

$$\mathbb{O}Z \to R(\mathbb{O}'LZ)$$

for all cofibrant objects Z of $\mathscr{M}$. We state such a theorem for our examples of interest.

Example Theorem 5.9.6. *Let $L\colon \mathscr{M} \rightleftarrows \mathscr{M}' : R$ be one of the Quillen adjunctions of symmetric monoidal categories listed below. Let $\mathcal{A}$ be an operad in $\mathscr{M}$, let $\mathcal{B}$ be an operad in $\mathscr{M}'$, and let $f\colon \mathcal{A} \to R\mathcal{B}$ be a map of operads that induces a weak equivalence*

$$\mathbb{A}Z \to R(\mathbb{B}LZ)$$

for all cofibrant objects Z of $\mathscr{M}$. Then the induced Quillen adjunction

$$P_{L,f}\colon \mathscr{M}[\mathcal{A}] \rightleftarrows \mathscr{M}'[\mathcal{B}] : U_{R,f}$$

is a Quillen equivalence. This theorem holds in particular in the examples:

(a) *$\mathscr{M}$ is the category of simplicial sets (with the usual model structure) or the category of symmetric spectra of simplicial sets, $\mathscr{M}'$ is the category of spaces or the category of symmetric spectra in spaces (respectively), and L, R is the geometric realization, singular simplicial set adjunction.*

(b) *$\mathscr{M}$ is the category of symmetric spectra, $\mathscr{M}'$ is the category of orthogonal spectra and L, R is the prolongation, restriction adjunction of* [178, p. 442].

(c) *$\mathscr{M}$ is the category of symmetric spectra or orthogonal spectra, $\mathscr{M}'$ is the category of EKMM S-modules, and L, R is the adjunction of* [263] *or* [177, I.1.1].

As indicated in the paragraph above the statement, the statement takes advantage of the fact that in the examples being considered in this section, the right adjoint preserves all weak equivalences; a general statement for other examples should use a fibrant replacement for $\mathbb{B}LZ$ in place of $\mathbb{B}LZ$. The proof sketch below also takes advantage of this property of the right adjoint. In generalizing the argument to the case when fibrant replacement is required in the statement, the fibrant replacement of the filtration can be performed in $\mathscr{M}'$.

The proof of the theorems above uses the homogeneous filtration on a pushout of the form $A' = A \amalg_{\mathbb{O}X}^{\mathscr{M}[\mathcal{O}]} \mathbb{O}Y$ studied in Section 5.6. This is the $m = 0$ case of the filtration on the enveloping operad $\mathcal{U}_{A'}^{\mathcal{O}}$, and we will need to use the filtration on the whole operad for an inductive argument even though we are only interested in the $m = 0$ case in the end. We will use uniform notation in the sketch proof that follows, taking $\mathscr{M}' = \mathscr{M}$ with adjoint functors L and R to be the identity in the case of Example Theorem 5.9.5. We use the notation I for the preferred set of generators for the cofibrations of $\mathscr{M}$ (as in Section 5.8).

Because fibrations and weak equivalences in the algebra categories are created in the underlying symmetric monoidal categories, the adjunction $P_{L,f}, U_{R,f}$ is automatically a Quillen adjunction (as indicated already in the statements), and we just have to prove that the unit of the adjunction

$$A \to U_{R,f}(P_{L,f}A) \tag{5.9.1}$$

is a weak equivalence for any cofibrant $\mathcal{A}$-algebra A. Every cofibrant $\mathcal{A}$-algebra is the retract of an $\mathbb{A}I$-cell $\mathcal{A}$-algebra, and so it suffices to consider the case when A is an $\mathbb{A}I$-cell $\mathcal{A}$-algebra; then $A = \operatorname{colim} A_n$ where $A_0 = \mathcal{A}(0)$ and each A_{n+1} is formed from A_n by cell attachment (of possibly an infinite coproduct of cells). As we shall see below, the underlying maps $A_n \to A_{n+1}$ are nice enough that A is the homotopy colimit (in $\mathscr{M}$ or $\mathscr{M}[\mathcal{A}]$) of the system of the finite stages A_n (this is the subject of the “Cofibration Hypothesis” of [94, VII§4] mentioned parenthetically in the previous section). Analogous observations apply for $P_{L,f}A$, which is a cell $\mathbb{B}LI$-algebra with stages $P_{L,f}A_n$. Thus, it will be enough to see that (5.9.1) is a weak equivalence for each A_n. By the hypothesis of the theorem, we know that this holds for A_0 (which is the free $\mathcal{A}$-algebra on the initial object of $\mathscr{M}$); moreover, as the enveloping operad of A_0 is $\mathcal{A}$ and the enveloping operad of $P_{L,f}A_0$ is $\mathcal{B}$, we can assume as an inductive hypothesis that

$$\mathbb{U}_{A_n}^{\mathcal{A}} Z \to \mathbb{U}_{P_{L,f}A_n}^{\mathcal{B}} LZ$$

is a weak equivalence for all cofibrant Z; in other words, we can assume by induction that the hypothesis of the theorem holds for the map of enveloping operads $\mathcal{U}_{A_n}^{\mathcal{A}} \to R(\mathcal{U}_{P_{L,f}A_n}^{\mathcal{B}})$. It then suffices to prove that the hypothesis of the theorem holds for the map of enveloping operads $\mathcal{U}^{\mathcal{A}}_{A_{n+1}} \to R(\mathcal{U}^{\mathcal{B}}_{P_{L,f}A_{n+1}})$; this is because in the categories $\mathscr{M}$ and $\mathscr{M}'$ of the examples, countable coproducts preserve and reflect weak equivalences and the unit map $A_{n+1} \to U_{R,f}(P_{L,f}A_{n+1})$ is the restriction of the map of monads to the homogeneous degree zero summand (at least in the homotopy category of $\mathscr{M}$).

To prove this, let $X \to Y$ be the coproduct of maps in I such that $A_{n+1} = A_n \amalg^{\mathscr{M}[\mathcal{A}]}_{\mathbb{A}X} \mathbb{A}Y$ and consider the filtration on $\mathcal{U}^{\mathcal{A}}_{A_{n+1}}(m)$ and $\mathcal{U}^{\mathcal{B}}_{P_{L,f}A_{n+1}}(m)$ studied in Section 5.6. We note that the induction hypothesis on A_n also implies that the map

$$\mathcal{U}^{\mathcal{A}}_{A_n}(m) \square_{\Sigma_{m_1}\times\cdots\times\Sigma_{m_i}} (Z_1^{(m_1)} \square \cdots \square Z_i^{(m_i)}) \\ \to R(\mathcal{U}^{\mathcal{B}}_{P_{L,f}A_n}(m) \square_{\Sigma_{m_1}\times\cdots\times\Sigma_{m_i}} (LZ_1^{(m_1)} \square \cdots \square LZ_i^{(m_i)}))$$

is a weak equivalence for all cofibrant objects $Z_1, \ldots, Z_i$ (where $m = m_1 + \cdots + m_i$) as this is a summand of the map

$$\mathcal{U}^{\mathcal{A}}_{A_n}(m) \square_{\Sigma_m} (Z_1 \amalg \cdots \amalg Z_i)^{(m)} \to R(\mathcal{U}^{\mathcal{B}}_{P_{L,f}A_n}(m) \square_{\Sigma_m} L(Z_1 \amalg \cdots \amalg Z_i)^{(m)}).$$

Looking at the pushout square (5.6.2) that inductively defines $Q^k_i(X \to Y)$, a bit of analysis shows that in our example categories the maps $Q^k_{i-1} \to Q^k_i$ are Σ_k-equivariant Hurewicz cofibrations (or in the simplicial categories, maps that geometrically realize to such). It follows that for any cofibrant object Z, the maps

$$\mathcal{U}^{\mathcal{A}}_{A_n}(k+m) \square_{\Sigma_k\times\Sigma_m} (Q^k_{i-1}(X \to Y) \square Z^{(m)}) \\ \to \mathcal{U}^{\mathcal{A}}_{A_n}(k+m) \square_{\Sigma_k\times\Sigma_m} (Q^k_i(X \to Y) \square Z^{(m)})$$

are (or geometrically realize to) Hurewicz cofibrations (likewise in $\mathscr{M}'$) and that the maps

$$\mathcal{U}^{\mathcal{A}}_{A_n}(k+m) \square_{\Sigma_k\times\Sigma_m} (Q^k_i(X \to Y) \square Z^{(m)}) \\ \to R(\mathcal{U}^{\mathcal{B}}_{P_{L,f}A_n}(k+m) \square_{\Sigma_k\times\Sigma_m} (Q^k_i(LX \to LY) \square LZ^{(m)}))$$

are weak equivalences. Now the pushout square (5.6.3) shows that for any cofibrant object Z, at each filtration level k, the map

$$F^{k-1}\mathcal{U}^{\mathcal{A}}_{A_{n+1}}(m) \square_{\Sigma_m} Z^{(m)} \to F^k\mathcal{U}^{\mathcal{A}}_{A_{n+1}}(m) \square_{\Sigma_m} Z^{(m)}$$

is (or geometrically realizes to) a Hurewicz cofibration (likewise in $\mathscr{M}'$) and that the maps

$$F^k\mathcal{U}^{\mathcal{A}}_{A_{n+1}}(m) \square_{\Sigma_m} Z^{(m)} \to R(F^k\mathcal{U}^{\mathcal{B}}_{P_{L,f}A_{n+1}}(m) \square_{\Sigma_m} LZ^{(m)})$$

are weak equivalences. The colimit is then weakly equivalent to the homotopy colimit and we get a weak equivalence

$$\mathcal{U}^{\mathcal{A}}_{A_{n+1}}(m) \square_{\Sigma_m} Z^{(m)} \to R(\mathcal{U}^{\mathcal{B}}_{P_{L,f}A_{n+1}}(m) \square_{\Sigma_m} LZ^{(m)}),$$

completing the induction and the sketch proof of Example Theorems 5.9.5 and 5.9.6.

The argument above proved the comparison theorems by proving equivalences of enveloping operads. Since the unary part of the enveloping operad is the enveloping algebra, we also get module category comparison results. We state this as the following corollary, which says that as long as the algebras are cofibrant, changing categories by Quillen equivalences and the algebras by derived monad equivalences results in Quillen equivalent categories of modules.

Corollary 5.9.7. *Let $L\colon \mathscr{M} \rightleftarrows \mathscr{M}' : R$ be one of the Quillen adjunctions of symmetric monoidal categories in Example Theorem 5.9.6 or the identity functor adjunction on one of the categories in Example Theorem 5.9.5. Let $f\colon \mathcal{A} \to R\mathcal{B}$ be a map of operads that induces a weak equivalence $\mathbb{A}Z \to R(\mathbb{B}LZ)$ for all cofibrant objects Z, and let $g\colon A \to RB$ be a weak equivalence of $\mathcal{A}$-algebras for an $\mathcal{A}$-algebra A and a $\mathcal{B}$-algebra B. If A and B are cofibrant (in $\mathscr{M}[\mathcal{A}]$ and $\mathscr{M}'[\mathcal{B}]$, respectively), then f and g induce a Quillen equivalence of the category of $(\mathcal{A}, A)$-modules and the category of $(\mathcal{B}, B)$-modules.*

Sketch proof. The argument above shows that under the given hypotheses, the map of $\Box$-monoids $U^{\mathcal{A}}A \to R(U^{\mathcal{B}}B)$ is a weak equivalence. The left and right adjoint functors in the Quillen adjunction on module categories are given by $U^{\overline{\mathcal{B}}}B \,\Box_{L(U^{\bar{\mathcal{A}}}A)} L(-)$ and R, respectively. These both preserve coproducts, homotopy cofiber sequences, and sequential homotopy colimits up to weak equivalence. It follows that the unit of the adjunction $X \to R(U^{\overline{\mathcal{B}}}B \,\Box_{L(U^{\bar{\mathcal{A}}}A)} LX)$ is a weak equivalence for every cofibrant A-module X. $\Box$

The analogous result also holds for modules over algebras on non-symmetric operads, proved by essentially the same filtration argument: we have a non-symmetric version $\overline{U}^{\overline{\mathcal{O}}}_A(m)$ of Construction 5.6.4. In this case, the resulting objects do not assemble into an operad; nevertheless, $\overline{U}^{\overline{\mathcal{O}}}_A(1)$ still has the structure of a $\Box$-monoid and coincides with the (non-symmetric) enveloping algebra $\overline{U}^{\overline{\mathcal{O}}}A$. The non-symmetric analogue of (5.6.3) holds, and the filtration argument (under the hypotheses of the previous corollary) proves that the map $\overline{U}^{\bar{\mathcal{A}}}A \to R(\overline{U}^{\overline{\mathcal{B}}}B)$ is a weak equivalence of $\Box$-monoids. We conclude that the unit map $X \to R(\overline{U}^{\overline{\mathcal{B}}}B \,\Box_{L\overline{U}^{\bar{\mathcal{A}}}A} LX)$ is a weak equivalence for every cofibrant A-module X.

5.10 Enveloping algebras, Moore algebras, and rectification

In the special case of Example 5.9.2, Example Theorem 5.9.5 gives an equivalence of the homotopy category of A_∞ algebras (over a given A_∞ operad) with the homotopy category of associative algebras, in particular constructing an associative algebra rectification of an A_∞ algebra. We know another way to construct an associative algebra from an A_∞ algebra, namely the (non-symmetric) enveloping algebra. In the case when the A_∞ operad is the operad of little 1-cubes $\overline{\mathcal{C}}_1$, there is also a classical rectification called the Moore algebra. The purpose of this section is to compare these constructions.

We first consider the rectification of Example Theorem 5.9.5 and the non-symmetric enveloping algebra. Let $\overline{\mathcal{O}}$ be a non-symmetric operad and $\epsilon\colon \overline{\mathcal{O}} \to \overline{\mathcal{A}\mathrm{ss}}$ a weak equivalence. Under the hypotheses of Example Theorem 5.9.5, the rectification (change of operads) functor P_ϵ associated to ϵ gives a $\Box$-monoid $P_\epsilon A$ and a map of $\bar{\mathcal{O}}$-algebras $A \to P_\epsilon A$ that is a weak equivalence when A is cofibrant. As part of the proof of Example Theorem 5.9.5, we get a weak equivalence of enveloping operads

$$\mathcal{U}^{\mathcal{O}}_A \to \mathcal{U}^{\mathcal{A}\mathrm{ss}}_{P_\epsilon A}.$$

As mentioned at the end of the previous section, the non-symmetric version of this argument also works to give a weak equivalence of $\Box$-monoids

$$\bar{U}^{\bar{\mathcal{O}}}A \to \bar{U}^{\overline{\mathcal{A}\mathrm{ss}}}(P_\epsilon A).$$

Moreover, in the case of the associative algebra operad $\overline{\mathcal{A}\mathrm{ss}}$, we have a natural isomorphism of $\Box$-monoids $\bar{U}^{\overline{\mathcal{A}\mathrm{ss}}}M \to M$ for any $\Box$-monoid M. Putting this together, we get:

Theorem 5.10.1. *Let $\mathscr{M}$ be a symmetric monoidal category and $\mathcal{O}$ an A_∞ operad that fall into one of the examples of Theorem 5.8.1*(a)–(c). *Write $\epsilon\colon \bar{\mathcal{O}} \to \overline{\mathcal{A}\mathrm{ss}}$ for the weak equivalence identifying $\mathcal{O}$ as an A_∞ operad. If A is a cofibrant $\mathcal{O}$-algebra then the natural maps*

$$A \to P_\epsilon A \cong \bar{U}^{\overline{\mathcal{A}\mathrm{ss}}}P_\epsilon A \leftarrow \bar{U}^{\bar{\mathcal{O}}}A$$

are weak equivalences of $\mathcal{O}$-algebras.

We now focus on A_∞ algebras for the little 1-cubes operad $\bar{\mathcal{C}}_1$, where we can describe results both more concretely and in much greater generality. For the rest of the section we work in the context of a symmetric monoidal category enriched over topological spaces as in Section 5.7: Let $\mathscr{M}$ be a closed symmetric monoidal category with countable colimits, and let $L\colon \mathscr{S} \to \mathscr{M}$ be strong symmetric monoidal left adjoint functor (whose right adjoint we denote as R). Then, by Theorem 5.7.1, $\mathscr{M}$ becomes enriched over topological spaces and we have a notion of homotopies and homotopy equivalences in $\mathscr{M}$, defined in terms of mapping spaces or equivalently in terms of tensor with the unit interval. We also have $L\bar{\mathcal{C}}_1$ as a non-symmetric operad in $\mathscr{M}$; for an $L\bar{\mathcal{C}}_1$-algebra A, we give a concrete construction of the enveloping algebra $\bar{U}A$, mostly following [184, §2]. We first write the formulas and then explain where they come from.

Construction 5.10.2. [184, §2] Let $\bar{D}$ be the space of subintervals of $[0,1]$ and let D be the subspace of $\bar{D}$ of those intervals that do not start at 0. We have a canonical isomorphism $\bar{D} \cong \bar{\mathcal{C}}_1(1)$ (sending a subinterval to the 1-tuple containing it) that we elide notation for. Under this isomorphism, the composition law Γ^1_1 defines a pairing $\gamma\colon \bar{D} \times \bar{D} \to \bar{D}$ that satisfies the formula

$$\gamma([x,y],[x',y']) = [x+(y-x)x',\, x+(y-x)y'].$$

We note that γ restricts to a pairing $D \times D \to D$, and that for formal reasons γ is associative:

$$\gamma(\gamma([x,y],[x',y']),[x'',y'']) = [x+(y-x)x'+(y-x)(y'-x')x'', x+(y-x)x'+(y-x)(y'-x')y''] = \gamma([x,y],\gamma([x',y'],[x'',y''])),$$

and unital:

$$\gamma([0,1],[x,y]) = [x,y] = \gamma([x,y],[0,1]),$$

making $\overline{D}$ a topological monoid and D a subsemigroup. Define $\alpha\colon D \times D \to \overline{\mathcal{C}}_1(2)$ by

$$\alpha([x,y],[x',y']) = \left(\left[0, \frac{x}{x+(y-x)x'}\right], \left[\frac{x}{x+(y-x)x'}, 1\right]\right).$$

Let DA be the object of $\mathscr{M}$ defined by the pushout diagram

$$\begin{array}{ccc} LD \,\square\, \mathbf{1} & \longrightarrow & LD \,\square\, A \\ \downarrow & & \downarrow \\ L\overline{D} \,\square\, \mathbf{1} & \longrightarrow & DA \end{array}$$

where the top map is induced by the composite of the isomorphism $\mathbf{1} \cong L\overline{\mathcal{C}}_1(0)$ (from the strong symmetric monoidal structure on L) and the $L\overline{\mathcal{C}}_1$-action map $L\overline{\mathcal{C}}_1(0) \to A$. We use γ and α to define a multiplication on DA as follows. We use the map

$$(LD \,\square\, A) \,\square\, (LD \,\square\, A) \to LD \,\square\, A \to DA$$

coming from the map

$$(LD \,\square\, A) \,\square\, (LD \,\square\, A) \cong L(D \times D) \,\square\, (A \,\square\, A) \to L(D \times \overline{\mathcal{C}}_1(2)) \,\square\, (A \,\square\, A) \cong LD \,\square\, (L\overline{\mathcal{C}}_1(2) \,\square\, (A \,\square\, A)) \to LD \,\square\, A$$

induced by the map $(\gamma, \alpha)\colon D \times D \to D \times \overline{\mathcal{C}}_1(2)$ and the $L\overline{\mathcal{C}}_1$-action map on A. We note that both associations

$$(LD \,\square\, A) \,\square\, (LD \,\square\, A) \,\square\, (LD \,\square\, A) \to LD \,\square\, A$$

coincide: both factor through the map

$$(LD \,\square\, A) \,\square\, (LD \,\square\, A) \,\square\, (LD \,\square\, A) \cong L(D \times D \times D) \,\square\, A^{(3)} \to L(D \times \overline{\mathcal{C}}_1(3)) \,\square\, A^{(3)}$$

induced by the map $D \times D \times D \to D \times \overline{\mathcal{C}}_1(3)$ given on the D factor as $\gamma \circ (\gamma \times \mathrm{id}) = \gamma \circ (1 \times \gamma)$ and on the $\overline{\mathcal{C}}_1(3)$ factor by the formula

$$[x,y],[x',y'],[x'',y''] \mapsto ([0,a],[a,b],[b,1]),$$

where

$$a = \frac{x}{x+(y-x)(x'+(y'-x')x'')}, \qquad b = \frac{x+(y-x)x'}{x+(y-x)(x'+(y'-x')x'')}.$$

When restricted to maps

$$(LD \,\square\, \mathbf{1}) \,\square\, (LD \,\square\, A), (LD \,\square\, A) \,\square\, (LD \,\square\, \mathbf{1}) \to DA,$$

this map coincides with the map induced by just γ and the unit isomorphism of $\mathscr{M}$ and so extends to compatible maps

$$(L\bar{D} \,\square\, \mathbf{1}) \,\square\, (L\bar{D} \,\square\, \mathbf{1}) \to DA,$$
$$(L\bar{D} \,\square\, \mathbf{1}) \,\square\, (LD \,\square\, A) \to DA,$$
$$(LD \,\square\, A) \,\square\, (L\bar{D} \,\square\, \mathbf{1}) \to DA,$$

and defines an associative multiplication on DA. The map $\mathbf{1} \to DA$ induced by the inclusion of the element $[0,1]$ of $\bar{D}$ is a unit for this multiplication.

To understand the construction, it is useful to think of D as a subspace of $\bar{\mathcal{C}}_1(2)$ rather than a subspace of $\bar{\mathcal{C}}_1(1)$, via the embedding

$$[x,y] \mapsto ([0,x],[x,y]).$$

Then we have a map $DA \to \bar{U}A$ sending $L\bar{D} \,\square\, \mathbf{1}$ and $LD \,\square\, A$ to the 0 and 1 summands

$$L\bar{D} \,\square\, \mathbf{1} \cong L\bar{\mathcal{C}}_1 \,\square\, A^{(0)} \qquad \text{and} \qquad LD \,\square\, A \to L\bar{\mathcal{C}}_1(2) \,\square\, A$$

in the coequalizer (5.5.1) for $\bar{U}A$. We also have a map back that sends the summand $L\bar{\mathcal{C}}_1(n+1) \,\square\, A^{(n)}$ (for $n \geq 1$) to $LD \,\square\, A$ by remembering just the last interval and using the rest to do the multiplication on A; specifically, for $[x_1,y_1],\dots,[x_{n+1},y_{n+1}]$, we use the element of $\bar{\mathcal{C}}_1(n)$ corresponding to

$$\left[\frac{x_1}{x_{n+1}}, \frac{y_1}{x_{n+1}}\right], \dots, \left[\frac{x_n}{x_{n+1}}, \frac{y_n}{x_{n+1}}\right]$$

for the map $A^{(n)} \to A$. It is straightforward to check that these maps give inverse isomorphisms of objects of $\mathscr{M}$; see [184, 2.5].

The isomorphism of the previous paragraph then forces the formula for the multiplication. Intuitively speaking, the first box in D (viewed as a subset of $\bar{\mathcal{C}}_1(2)$) holds the algebra (from the tensor) and the second box is a placeholder to plug in the module variable; the complement $\bar{D} \setminus D$ corresponds to the first box having length zero and then only the unit of the algebra can go there. For the composition, the right copy gets plugged into the second box of the left copy to give an element of $\bar{\mathcal{C}}_1(3)$ (i.e., the operadic composition $\ell \circ_2 r = \Gamma^2_{1,2}(\ell;1,r)$, where ℓ is the element of the left copy of D and r is the element of the right copy of D); the first and second boxes are on the one hand rescaled to an element of $\bar{\mathcal{C}}_1(2)$ that does the multiplication on the copies of A and on the other hand joined to give with the third box the new element of D, viewed as a subspace of $\bar{\mathcal{C}}_1(2)$. The associativity is straightforward to visualize in terms of plugging in boxes when written down on paper. (See Section 2 of [184].) When one of the elements comes from $\bar{D} \setminus D$, the corresponding copy of A is restricted to the unit $\mathbf{1}$ and the first box of zero length also works like a unit.

Using the isomorphism of $\square$-monoids $\bar{U}A \cong DA$, we have the following comparison theorem for the underlying objects of $\bar{U}A$ and A.

Proposition 5.10.3 ([184, 1.1]). *The map of $\bar{U}A$-modules $\bar{U}A \cong \mathbf{1} \,\square\, \bar{U}A \to A$ induced by the map $\mathbf{1} \cong L\bar{\mathcal{C}}_1(0) \to A$ is a homotopy equivalence of objects of $\mathscr{M}$.*

Proof. In concrete terms, the map in the statement is induced by the map

$$LD \,\Box\, A \to L\bar{\mathcal{C}}_1(1) \,\Box\, A \to A$$

for the map $D \to \bar{\mathcal{C}}_1(1)$ that sends $[x,y]$ to $([0,x])$, which is compatible with the map

$$L\bar{D} \,\Box\, \mathbf{1} \to \mathbf{1} \to A.$$

We can use any element of D to produce a map (in $\mathscr{M}$) from A to $\bar{U}A$; a path to the operad identity element 1 in $\bar{\mathcal{C}}_1(1)$ (which corresponds to $[0,1] \subseteq [0,1]$) then induces a homotopy of the composite map $A \to A$ to the identity map of A. We can construct a homotopy from the composite to the identity on $\bar{U}A$ using a homotopy of self-maps of $\bar{\mathcal{C}}_1(1)$ from the identity to the constant map on 1 (combined with the $\bar{\mathcal{C}}_1(1)$ action map on A) and a homotopy of self-maps of the pair $(\bar{D}, D)$ from the constant map (on the chosen element of D) to the identity map. For example, if the chosen element of D corresponds to the subinterval $[a,b]$ (with $a \neq 0$) then the linear homotopy

$$[x,y], t \mapsto [xt + a(1-t), yt + b(1-t)]$$

is such a homotopy of self-maps of the pair. □

In the context of spaces, J. C. Moore invented an associative version of the based loop space by parametrizing loops with arbitrary length intervals. This idea extends to the current context to give another even simpler construction of a $\Box$-monoid equivalent (in $\mathscr{M}$) to an $L\bar{\mathcal{C}}_1$-algebra A.

Construction 5.10.4. Define MA to be the object of $\mathscr{M}$ defined by the pushout diagram

$$\begin{array}{ccc} L\mathbb{R}^{>0} \,\Box\, \mathbf{1} & \longrightarrow & L\mathbb{R}^{>0} \,\Box\, A \\ \downarrow & & \downarrow \\ L\mathbb{R}^{\geq 0} \,\Box\, \mathbf{1} & \longrightarrow & MA \end{array}$$

(where $\mathbb{R}^{>0} \subset \mathbb{R}^{\geq 0}$ are the usual subspaces of positive and non-negative real numbers, respectively). We give this the structure of a $\Box$-monoid with the unit $\mathbf{1} \to MA$ induced by the inclusion of 0 in $\mathbb{R}^{\geq 0}$ and multiplication $MA \,\Box\, MA \to MA$ induced by the map

$$\begin{aligned} &(L\mathbb{R}^{>0} \,\Box\, A) \,\Box\, (L\mathbb{R}^{>0} \,\Box\, A) \cong L(\mathbb{R}^{>0} \times \mathbb{R}^{>0}) \,\Box\, (A \,\Box\, A) \\ &\qquad \to L(\mathbb{R}^{>0} \times \bar{\mathcal{C}}_1(2)) \,\Box\, (A \,\Box\, A) \cong L\mathbb{R}^{>0} \,\Box\, (L\bar{\mathcal{C}}_1(2) \,\Box\, (A \,\Box\, A)) \to L\mathbb{R}^{>0} \,\Box\, A \end{aligned}$$

induced by the $\bar{\mathcal{C}}_1$-action on A and the map

$$c\colon (r,s) \in \mathbb{R}^{>0} \times \mathbb{R}^{>0} \mapsto (r+s, ([0, \tfrac{r}{r+s}], [\tfrac{r}{r+s}, 1])) \in \mathbb{R}^{>0} \times \bar{\mathcal{C}}_1(2).$$

The idea is that the element of $\mathbb{R}^{>0}$ specifies a length (with the zero length only available for the unit) and the multiplication uses the proportionality of the two lengths to choose an element of $\bar{\mathcal{C}}_1(2)$ for the multiplication on A; the two lengths add to give the length in the result. In the case when $\mathscr{M}$ is the category of spaces and $A = \Omega X$ is the based loop space of a space X, MA is the Moore loop space. An element is specified by an element r of $\mathbb{R}^{\geq 0}$ together with an element of ΩX (which must be the basepoint when $r=0$) but can be visualized as a based loop parametrized by $[0,r]$ (or for $r=0$ the constant length zero loop at the basepoint). The multiplication concatenates loops by concatenating the parametrizations, an operation that is strictly associative and unital.

We can compare the $\square$-monoids MA and $\overline{U}A$ through a third $\square$-monoid NA constructed as follows. Let $N=\mathbb{R}^{>0}\times\mathbb{R}^{>0}\times\mathbb{R}^{\geq 0}$, let $\overline{N}=\mathbb{R}^{\geq 0}\times\mathbb{R}^{>0}\times\mathbb{R}^{\geq 0}$, and define NA by the pushout diagram

$$\begin{array}{ccc} LN\,\square\,\mathbf{1} & \longrightarrow & LN\,\square\,A \\ \downarrow & & \downarrow \\ L\overline{N}\,\square\,\mathbf{1} & \longrightarrow & NA \end{array}$$

We have maps $\overline{N}\times\overline{N}\to\overline{N}$ and $N\times N\to\bar{\mathcal{C}}_1(2)$ defined by

$$((r,s,t),(r',s',t'))\in\overline{N}\times\overline{N}\mapsto(r+sr',ss',st'+t)\in\overline{N},$$

$$((r,s,t),(r',s',t'))\in N\times N\mapsto c(t,st')=\left(\left[0,\tfrac{r}{r+sr'}\right],\left[\tfrac{r}{r+sr'},1\right]\right)\in\bar{\mathcal{C}}_1(2),$$

which we use to construct the multiplication on NA by the same scheme as above

$$(LN\,\square\,A)\,\square\,(LN\,\square\,A)\cong L(N\times N)\,\square\,(A\,\square\,A)\to L(N\times\bar{\mathcal{C}}_1(2))\,\square\,(A\,\square\,A)\to LN\,\square\,A.$$

The unit is the map $\mathbf{1}\to NA$ induced by the inclusion of $(0,1,0)$ in $\overline{N}$.

The parametrizing space $N=\{(r,s,t)\}$ generalizes D by allowing $[r,s]$ to be a subinterval of $[0,r+s+t]$ instead of $[0,1]$, or from another perspective, generalizes lengths in the definition on the Moore algebra by incorporating a scaling factor s and padding of length t. In other words, we have maps

$$[x,y]\in\overline{D}\mapsto(x,y-x,1-y)\in\overline{N},$$
$$r\in\mathbb{R}^{\geq 0}\mapsto(r,1,0)\in\overline{N}.$$

These maps induce maps of $\square$-monoids $\overline{U}A\cong DA\to NA$ and $MA\to NA$, respectively, and the argument of Proposition 5.10.3 shows that these maps are homotopy equivalences in $\mathscr{M}$. We state this as a theorem, repeating the conventions of this part of the section for easy reference.

Theorem 5.10.5. *Let $\mathscr{M}$ be a closed symmetric monoidal category admitting countable colimits and enriched over spaces via a strong symmetric monoidal left adjoint functor L. Then for algebras over the little 1-cubes operad ($L\bar{\mathcal{C}}_1$-algebras) the non-symmetric enveloping algebra $\overline{U}A$ and the Moore algebra MA fit in a natural zigzag of $\square$-monoids*

$$\overline{U}A\to NA\leftarrow MA,$$

where the maps are homotopy equivalences in $\mathscr{M}$. Moreover, the canonical maps $\overline{U}A\to A$ and $MA\to A$ are homotopy equivalences in $\mathscr{M}$.

To compare MA and A as A_∞ algebras, we use a new A_∞ operad $\bar{\mathcal{C}}^{\ell}$ defined as follows.

Construction 5.10.6. Let $\overline{\mathcal{C}}^{\ell}(0) = \mathbb{R}^{\geq 0}$ and for $m > 0$, let $\overline{\mathcal{C}}^{\ell}(m)$ be the set of ordered pairs (S, r) with r a positive real number and S a list of m almost non-overlapping closed subintervals of $[0, r]$ in their natural order, topologized analogously as in the definition of $\overline{\mathcal{C}}_1$ (as a semilinear submanifold of $\mathbb{R}^{2m+1}$). The operadic composition is defined by scaling and replacement of the subintervals: the basic composition

$$\Gamma^1_j(((([x,y]),r),(([x'_1,y'_1],\dots,[x'_j,y'_j]),r')) = (([x+ax'_1,x+ay'_1],\dots,[x+ax'_j,x+ay'_j]),r+a(r'-1))$$

(with $a := y - x$) scales the interval $[0, r']$ to length ar' and inserts that in place of $[x, y] \subset [0, r]$; the resulting final interval then has size $r - a + ar'$. The general composition $\Gamma^m_{j_1,\dots,j_m}$ does this operation on each of the m subintervals:

$$\Gamma^m_{j_1,\dots,j_m} \colon (([x^0_1,y^0_1],\dots,[x^0_m,y^0_m]),r_1), (([x^1_1,y^1_1],\dots,[x^1_{j_1},y^1_{j_1}]),r_1),\dots,(([x^m_1,y^m_1],\dots,[x^m_{j_m},y^m_{j_m}]),r_m),$$
$$\mapsto$$
$$(([x^0_1+a_1x^1_1,x^0_1+a_1y^1_1],\dots,[s_{m-1}+x^0_m+a_mx^m_{j_m},s_{m-1}+x^0_m+a_my^m_{j_m}]),r_0+s_m),$$

where $a_i := y^0_i - x^0_i$ and $s_i = a_1(r_1 - 1) + \cdots + a_i(r_i - 1)$. When one of the j_i is zero, that j_i contributes no subintervals but still scales the original subinterval $[x^0_i, y^0_i]$ to length $a_i r_i$ (or removes it when $r_i = 0$). The operad identity element is the element $(([0,1]),1) \in \overline{\mathcal{C}}^{\ell}(1)$.

The maps $\overline{\mathcal{C}}_1(m) \to \overline{\mathcal{C}}^{\ell}(m)$ that include $\overline{\mathcal{C}}_1(m)$ as the length 1 subspace assemble to a map of operads $i \colon \overline{\mathcal{C}}_1 \to \overline{\mathcal{C}}^{\ell}$. We also have a map of operads $j \colon \overline{\mathcal{A}\mathrm{ss}} \to \overline{\mathcal{C}}^{\ell}$ induced by sending the unique element of $\overline{\mathcal{A}\mathrm{ss}}(m)$ to the element

$$(([0,1],[1,2],\dots,[m-1,m]),m)$$

of $\overline{\mathcal{C}}^{\ell}(m)$. Using the map j, an $L\overline{\mathcal{C}}^{\ell}$-algebra has the underlying structure of a $\square$-monoid. A straightforward check of universal properties proves the following proposition.

Proposition 5.10.7. *The functor that takes a $\overline{\mathcal{C}}_1$-algebra A to its Moore algebra MA is naturally isomorphic to the functor that takes A to the underlying $\square$-monoid of the pushforward $P_{Li}A$ for the map of operads $Li \colon L\overline{\mathcal{C}}_1 \to L\overline{\mathcal{C}}^{\ell}$.*

The $\overline{\mathcal{C}}^{\ell}$-action map $L\overline{\mathcal{C}}^{\ell}(m) \square (MA)^{(m)} \to MA$ is induced by the map

$$\overline{\mathcal{C}}^{\ell}(m) \times (\mathbb{R}^{>0})^n \to \overline{\mathcal{C}}^{\ell}(m) \times \overline{\mathcal{C}}^{\ell}(1)^n \xrightarrow{\Gamma^m_{1,\dots,1}} \overline{\mathcal{C}}^{\ell}(m) \cong \mathbb{R}^{>0} \times \overline{\mathcal{C}}_1(m)$$

that includes $\mathbb{R}^{>0}$ in $\overline{\mathcal{C}}^{\ell}(1)$ by $r \mapsto (([0,r]),r)$, where the isomorphism is the map that takes an element $(([x_1,y_1],\dots,[x_m,y_m]),r)$ of $\overline{\mathcal{C}}^{\ell}(m)$ to the element

$$\left(r, \left(\left[\tfrac{x_1}{r}, \tfrac{y_1}{r}\right],\dots,\left[\tfrac{x_m}{r}, \tfrac{y_m}{r}\right]\right)\right)$$

of $\mathbb{R}^{>0} \times \overline{\mathcal{C}}_1(m)$.

The map of $\overline{\mathcal{C}}_1$-algebras that is the unit of the change of operads adjunction $A \to P_{Li}A$ is induced by the inclusion of 1 in $\mathbb{R}^{>0}$ and is a homotopy equivalence by a (simpler) version of the homotopy argument of Proposition 5.10.3. I do not see how to do a similar argument for the pushforward P_{Lj} from □-monoids to $\overline{\mathcal{C}}^{\ell}$-algebras, so we do not get a direct comparison of $\bar{\mathcal{C}}_1$-algebras between A (or $P_{Li}A$) and MA with the $\bar{\mathcal{C}}_1$-algebra structure inherited from its □-monoid structure without some kind of rectification result (such as Example Theorem 5.9.5) comparing the category of $L\overline{\mathcal{C}}^{\ell}$-algebras with the category of $\overline{\mathcal{A}}$ss-algebras.

The argument in [184, 2.5] that identifies $\overline{U}^{\bar{\mathcal{C}}_1}A$ as DA generalizes to identify $\overline{U}^{\overline{\mathcal{C}}^{\ell}}P_{Li}A$ as NA; the maps in Theorem 5.10.5 can then be viewed as the natural maps on enveloping algebras induced by maps of operads and maps of algebras.

5.11 E_n spaces and iterated loop space theory

The recognition principle for iterated loop spaces provided the first application for operads. Although the summary here has been spiced up with model category notions and terminology (in the adjoint functor formulation of [196, §8]), the mathematics has not changed significantly from the original treatment by May in [194], except for the improvements noted in the appendix to [71], which extend the results from connected to grouplike E_n spaces. (E_n spaces = E_n algebras in spaces.)

The original idea for the little n-cubes operads $\mathcal{C}_n$ and the start of the relationship between E_n spaces and n-fold loop spaces is the Boardman–Vogt observation that every n-fold loop space comes with the natural structure of a $\mathcal{C}_n$-algebra. The action map

$$\mathcal{C}_n(m) \times \Omega^n X \times \cdots \times \Omega^n X \to \Omega^n X$$

is defined as follows. We view S^n as $[0,1]^n/\partial$. Given an element $c \in \mathcal{C}_n(m)$, and elements $f_1, \ldots, f_m \colon S^n \to X$ of $\Omega^n X$, let $f_{c;f_1,\ldots,f_n} \colon S^n \to X$ be the function that sends a point x in S^n to the basepoint if x is not in one of the embedded cubes; the i-th embedded cube gets sent to X using the inverse of the embedding and the quotient map $[0,1]^n \to S^n$ followed by the map $f_i \colon S^n \to X$. This is a continuous based map $S^n \to X$ since the boundary of each embedded cube gets sent to the basepoint. Phrased another way, c defines a based map

$$S^n \to S^n \vee \cdots \vee S^n$$

with the i-th embedded cube mapping to the i-th wedge summand of S^n by collapsing all points not in an open cube to the basepoint and rescaling; we then apply $f_i \colon S^n \to X$ to the i-th summand to get a composite map $S^n \to X$.

The construction of the previous paragraph factors Ω^n as a functor from based spaces to $\mathcal{C}_n$-spaces (= $\mathcal{C}_n$-algebras in spaces). It is clear that not every $\mathcal{C}_n$-space arises as $\Omega^n X$ because $\pi_0 \Omega^n X$ is a group (for its canonical multiplication), whereas for the free $\mathcal{C}_n$-space $\mathbb{C}_n X$, $\pi_0 \mathbb{C}_n X$ is not a group unless X is the empty set; for example, $\pi_0 \mathbb{C}_n X \cong \mathbb{N}$ when X is path connected. We say that a $\mathcal{C}_n$-space A is ***grouplike*** when $\pi_0 A$ is a group (for its canonical multiplication). The following is the fundamental theorem of iterated loop space theory; it gives an equivalence of homotopy theories between n-fold loop spaces and grouplike $\mathcal{C}_n$-spaces.

Theorem 5.11.1 (May [194], Boardman–Vogt [48, §6]). *The functor Ω^n from based spaces to $\mathcal{C}_n$-spaces is a Quillen right adjoint. The unit of the derived adjunction*

$$A \to \Omega^n B^n A$$

is an isomorphism in the homotopy category of $\mathcal{C}_n$-spaces if (and only if) A is grouplike. The counit of the derived adjunction

$$B^n \Omega^n X \to X$$

is an isomorphism in the homotopy category of spaces if (and only if) X is $(n-1)$-connected; in general it is an $(n-1)$-connected cover.

We have written the derived functor of the left adjoint in Theorem 5.11.1 as B^n, suggesting an iterated bar construction. Although neither the point-set adjoint functor nor the model for its derived functor used in the argument of Theorem 5.11.1 is constructed iteratively, Dunn [86] shows that the derived functor is naturally equivalent to an iterated bar construction.

As a consequence of the statement of the theorem, the unit of the derived adjunction $A \to \Omega^n B^n A$ is the initial map in the homotopy category of $\mathcal{C}_n$-spaces from A to a grouplike $\mathcal{C}_n$-space and so deserves to be called "group completion". Group completion has various characterizations and for the purposes of sketching the ideas behind the proof of the theorem, it works best to choose one of them as the definition and state the property of the unit map as a theorem. One such characterization uses the classifying space construction, which we understand as the Eilenberg–Mac Lane bar construction (after converting the underlying $\mathcal{C}_1$-spaces to topological monoids) or the Stasheff bar construction (choosing compatible maps from the Stasheff associahedra into the spaces $\mathcal{C}_n(m)$).

Definition 5.11.2. A map $f\colon A \to G$ of $\mathcal{C}_n$-spaces is a *group completion* if G is grouplike and f induces a weak equivalence of classifying spaces.

In the case $n > 1$ (and under some hypotheses if $n = 1$), Quillen [227] gives a homological criterion for a map to be group completion: if G is grouplike, then a map $A \to G$ of $\mathcal{C}_n$-spaces is group completion if and only if

$$H_*(A)[(\pi_0 A)^{-1}] \to H_*(G)$$

is an isomorphism. Counterexamples exist in the case $n = 1$ (indeed, McDuff [208] gives a counterexample for every loop space homotopy type), but recent work of Braun, Chuang, and Lazarev [59] gives an analogous derived category criterion in terms of derived localization at the multiplicative set $\pi_0 A$. Using Definition 5.11.2 or any equivalent independent characterization of group completion, we have the following addendum to Theorem 5.11.1.

Addendum 5.11.3. *The unit of the derived adjunction in Theorem 5.11.1 is group completion.*

The homotopical heart of the proof of Theorem 5.11.1 is the May–Cohen–Segal Approximation Theorem ([194, §6–7], [70], [270]), which we now review. This theorem studies a version of the free $\mathcal{C}_n$-algebra functor $\tilde{\mathbb{C}}_n$ whose domain is the category of based spaces, where the basepoint becomes the identity element in the $\mathcal{C}_n$-algebra structure. This version of the free functor has the advantage that for a connected space X, $\tilde{\mathbb{C}}X$ is also a connected space; May's Approximation Theorem identifies $\tilde{\mathbb{C}}X$ in this case as a model for $\Omega^n\Sigma^n X$. Cohen (following conjectures of May) and Segal (working independently) then extended this to non-connected spaces: the group completion of $\tilde{\mathbb{C}}X$ is a model for $\Omega^n\Sigma^n X$.

For a based space X, $\tilde{\mathbb{C}}_n X$ is formed as a quotient of

$$\mathbb{C}X = \coprod \mathcal{C}_n(m) \times_{\Sigma_m} X^m$$

by the equivalence relation that identifies $(c,(x_1,\dots,x_i,*,\dots,*)) \in \mathcal{C}_n(m) \times X^m$ with $(c',(x_1,\dots,x_i)) \in \mathcal{C}_n(i) \times X^i$ for $c' = \Gamma(c;1,\dots,1,0,\dots,0)$ where 1 denotes the identity element in $\mathcal{C}_n(1)$ and 0 denotes the unique element in $\mathcal{C}_n(0)$. This is actually an instance of the operad pushforward construction: let $\mathcal{I}_{\mathrm{dbp}}$ be the operad with $\mathcal{I}_{\mathrm{dbp}}(0) = \mathcal{I}_{\mathrm{dbp}}(1) = *$ and $\mathcal{I}_{\mathrm{dbp}}(m) = \emptyset$ for $m > 1$. The functor associated to $\mathcal{I}_{\mathrm{dbp}}$ is the functor $(-)_+$ that adds a disjoint basepoint with the monad structure $((-)_+)_+ \to (-)_+$ that identifies the two disjoint basepoints; the category of algebras for this monad is the category of based spaces. The functor $\tilde{\mathbb{C}}_n$ from based spaces to $\mathcal{C}_n$-algebras is the pushforward P_f for f the unique map of operads $\mathcal{I}_{\mathrm{dbp}} \to \mathcal{C}_n$: formally P_f is the coequalizer described in Section 5.9, that in this case takes the form

$$\mathbb{C}_n(X_+) \rightrightarrows \mathbb{C}_n X \longrightarrow \tilde{\mathbb{C}}_n X.$$

As mentioned in an aside in that section (or as can be seen concretely here using the operad multiplication on $\mathcal{C}_n$ directly), the endofunctor $\tilde{\mathbb{C}}_n$ on based spaces (i.e., $U_f P_f$) has the structure of a monad, and we can identify the category of $\mathcal{C}_n$-spaces as the category of algebras over the monad $\tilde{\mathbb{C}}_n$.

The factorization of the functor Ω^n through $\mathcal{C}_n$-spaces has the formal consequence of producing a map of monads (in based spaces)

$$\tilde{\mathbb{C}}_n \to \Omega^n\Sigma^n.$$

Formally the map is induced by the composite

$$\tilde{\mathbb{C}}_n X \xrightarrow{\tilde{\mathbb{C}}_n\eta} \tilde{\mathbb{C}}_n\Omega^n\Sigma^n X \xrightarrow{\xi} \Omega^n\Sigma^n X,$$

where η is the unit of the Σ^n,Ω^n-adjunction and ξ is the $\mathcal{C}_n$-action map. This map has the following concrete description: an element $(c,(x_1,\dots,x_m)) \in \mathcal{C}_n(m) \times X^m$ maps to the element $\gamma\colon S^n \to \Sigma^n X$ of $\Omega^n\Sigma^n X$ given by the composite of the map

$$S^n \to S^n \vee \cdots \vee S^n$$

associated to c (as described above) and the map

$$S^n \cong \Sigma^n\{x_i\}_+ \subset \Sigma^n X$$

on the i-th factor of S^n. Either using this concrete description, or following diagrams in a formal categorical argument, it is straightforward to check that this defines a map of monads. We can now state the May–Cohen–Segal Approximation Theorem.

Theorem 5.11.4 (May–Cohen–Segal Approximation Theorem [194, 6.1], [70, 3.3], [270, Theorem 2]).
For any non-degenerately based space X, the map of $\mathcal{C}_n$-spaces $\tilde{\mathbb{C}}_n X \to \Omega^n\Sigma^n X$ is group completion.

("Non-degenerately based" means that the inclusion of the basepoint is a cofibration. Both $\tilde{\mathbb{C}}_n$ and $\Omega^n\Sigma^n$ preserve weak equivalences in non-degenerately based spaces, but for other spaces, either or both may have the wrong weak homotopy type.)

From here a sketch of the proof of Theorem 5.11.1 goes as follows. Since Ω^n as a functor from based spaces to based spaces has left adjoint Σ^n, a check of universal properties shows that the functor from $\mathcal{C}_n$-spaces to based spaces defined by the coequalizer

$$\Sigma^n\tilde{\mathbb{C}}_n A \rightrightarrows \Sigma^n A \longrightarrow \Sigma^n \otimes_{\mathbb{C}_n} A$$

is the left adjoint to Ω^n viewed as a functor from based spaces to $\mathcal{C}_n$-spaces. (In the coequalizer, one map is induced by the $\mathcal{C}_n$-action map on A and the other is adjoint to the map of monads $\tilde{\mathbb{C}} \to \Omega^n\Sigma^n$.) Because Ω^n preserves fibrations and weak equivalences, this is a Quillen adjunction.

The main tool to study the $\Sigma^n \otimes_{\mathbb{C}_n} (-), \Omega^n$-adjunction is the two-sided monadic bar construction, invented in [194, §9] for this purpose. Given a monad $\mathbb{T}$ and a right action of $\mathbb{T}$ on a functor F (say, to based spaces), the two-sided monadic bar construction is the functor on $\mathbb{T}$-algebras $B(F,\mathbb{T},-)$ defined as the geometric realization of the simplicial object

$$B_m(F,\mathbb{T},A) = F\underbrace{\mathbb{T}\cdots\mathbb{T}}_{m}A,$$

with face maps induced by the action map $F\mathbb{T} \to F$, the multiplication map $\mathbb{T}\mathbb{T} \to \mathbb{T}$ and the action map $\mathbb{T}A \to A$, and degeneracy maps induced by the unit map $\mathrm{Id} \to \mathbb{T}$. In the case when $F = \mathbb{T}$, the simplicial object $B_\bullet(\mathbb{T},\mathbb{T},A)$ has an extra degeneracy and the map from $B_\bullet(\mathbb{T},\mathbb{T},A)$ to the constant simplicial object on A is a simplicial homotopy equivalence (in the underlying category for $\mathbb{T}$, though not generally in the category of $\mathbb{T}$-algebras).

Because geometric realization commutes with colimits and finite cartesian products, we have a canonical isomorphism

$$\tilde{\mathbb{C}}_n B(\tilde{\mathbb{C}}_n,\tilde{\mathbb{C}}_n,A) \to B(\tilde{\mathbb{C}}_n\tilde{\mathbb{C}}_n,\tilde{\mathbb{C}}_n,A)$$

and the multiplication map $\tilde{\mathbb{C}}_n\tilde{\mathbb{C}}_n \to \tilde{\mathbb{C}}_n$ then gives $B(\tilde{\mathbb{C}}_n,\tilde{\mathbb{C}}_n,A)$ the natural structure of a $\mathcal{C}_n$-algebra. (See Section 5.7 for a more general discussion.) For the same reason, the canonical map

$$\Sigma^n \otimes_{\mathbb{C}_n} B(\tilde{\mathbb{C}}_n,\tilde{\mathbb{C}}_n,A) \to B(\Sigma^n \otimes_{\mathbb{C}_n} \tilde{\mathbb{C}}_n,\tilde{\mathbb{C}}_n,A) = B(\Sigma^n,\tilde{\mathbb{C}}_n,A)$$

is an isomorphism. The latter functor clearly[3] preserves weak equivalences of $\mathcal{C}_n$-spaces A whose underlying based spaces are non-degenerately based. (Besides being a hypothesis of the May–Cohen–Segal Approximation Theorem, non-degenerately based here also ensures that the inclusion of the degenerate subspace (or latching object) is a cofibration.) As a consequence of Theorem 5.7.3 it follows that when the underlying based space of A is cofibrant (which is the case in particular when A is cofibrant as a $\mathcal{C}_n$-space), then $B(\tilde{\mathbb{C}}_n, \tilde{\mathbb{C}}_n, A)$ is a cofibrant $\mathcal{C}_n$-space. Because $\Sigma^n \otimes_{\mathbb{C}_n} (-)$ is a Quillen left adjoint, it preserves weak equivalences between cofibrant objects, and looking at a cofibrant approximation $A' \xrightarrow{\sim} A$, we see from the weak equivalences

$$B(\Sigma^n, \tilde{\mathbb{C}}_n, A) \xleftarrow{\sim} B(\Sigma^n, \tilde{\mathbb{C}}_n, A') \cong \Sigma^n \otimes_{\mathbb{C}_n} B(\tilde{\mathbb{C}}_n, \tilde{\mathbb{C}}_n, A') \xrightarrow{\sim} \Sigma^n \otimes_{\mathbb{C}_n} A'$$

that $B(\Sigma^n, \tilde{\mathbb{C}}_n, A)$ models the derived functor $B^n A$ of $\Sigma^n \otimes_{\mathbb{C}_n} (-)$ whenever A is non-degenerately based.

To complete the argument, we need the theorem of [194, §12] that Ω^n commutes up to weak equivalence with geometric realization of (proper) simplicial spaces that are $(n-1)$-connected in each level. Then for A non-degenerately based, we have that the vertical maps are weak equivalences of $\mathcal{C}_n$-spaces

$$\begin{array}{ccc} B(\tilde{\mathbb{C}}_n, \tilde{\mathbb{C}}_n, A) & \longrightarrow & B(\Omega^n \Sigma^n, \tilde{\mathbb{C}}_n, A) \\ \downarrow & & \downarrow \\ A & & \Omega^n B(\Sigma^n, \tilde{\mathbb{C}}_n, A) \end{array}$$

while by the May–Cohen–Segal Approximation Theorem, the horizontal map is group completion. This proves that the unit of the derived adjunction is group completion.

For the counit of the derived adjunction, we have from the model above that B^n is always $(n-1)$-connected and the unit

$$\Omega^n X \to \Omega^n B^n \Omega^n X$$

on $\Omega^n X$ is a weak equivalence. Looking at Ω^n of the counit,

$$\Omega^n B^n \Omega^n X \to \Omega^n X,$$

the composite with the unit is the identity on $\Omega^n X$, and so it follows that Ω^n of the counit is a weak equivalence. Thus, the counit of the derived adjunction is an $(n-1)$-connected cover map.

5.12 E_∞ algebras in rational and p-adic homotopy theory

In the 1960's and 1970's, Quillen [228] and Sullivan [284, 286] showed that the rational homotopy theory of simply connected spaces (or simplicial sets) has an algebraic model

[3] At the time when May wrote the argument, this was far from clear: some of the first observations about when geometric realization of simplicial spaces preserves levelwise weak equivalences were developed in [194, §11] precisely for this argument.

in terms of rational differential graded commutative algebras or coalgebras. In the 1990's, I proved a mostly analogous theorem relating E_∞ differential graded algebras and p-adic homotopy theory and a bit later some results for using E_∞ differential graded algebras or E_∞ ring spectra to identify integral homotopy types. In this section, we summarize this theory following mostly the memoir of Bousfield–Gugenheim [57], and the papers [181][4] and [180]. In what follows k denotes a commutative ring, which is often further restricted to be a field.

In both the rational commutative differential graded algebra case and the E_∞ k-algebra case, the theory simplifies by working with simplicial sets instead of spaces, and the functor is some variant of the cochain complex. Sullivan's approach to rational homotopy theory uses a rational version of the de Rham complex, originally due to Thom (unpublished), consisting of forms that are polynomial on simplices and piecewise matched on faces:

Definition 5.12.1. The algebra $\nabla^*[n]$ of polynomial forms on the standard simplex $\Delta[n]$ is the rational commutative differential graded algebra free on generators $t_0,\dots,t_n$ (of degree zero), $dt_0,\dots,dt_n$ (of degree one) subject to the relations $t_0+\cdots+t_n=1$ and $dt_0+\cdots+dt_n=0$ (as well as the differential relation implicit in the notation).

Viewing $t_0,\dots,t_n$ as the barycentric coordinate functions on $\Delta[n]$ determines their behavior under face and degeneracy maps, making $\nabla^*[\bullet]$ a simplicial rational commutative differential graded algebra.

Definition 5.12.2. For a simplicial set X, the rational de Rham complex $A^*(X)$ is the rational graded commutative algebra of maps of simplicial sets from X to $\nabla^*[\bullet]$, or equivalently, the end over the simplex category

$$A^*(X) := \mathbf{\Delta}^{\mathrm{op}}\mathcal{S}\mathrm{et}(X,\nabla^*[\bullet]) = \int_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{S}\mathrm{et}(X_n,\nabla^*[n]) = \int_{\mathbf{\Delta}^{\mathrm{op}}} \prod_{X_n} \nabla^*[n]$$

(the last formula indicating how to regard $A^*(X)$ as a rational commutative differential graded algebra).

More concretely, $A^*(X)$ is the rational commutative differential graded algebra where an element of degree q consists of a choice of element of $\nabla^q[n]$ for each non-degenerate n-simplex of X (for all n) which agree under restriction by face maps, with multiplication and differential done on each simplex. (When X is a finite simplicial complex $A^*(X)$ also has a Stanley–Reisner ring style description; see [284, G.i)].) The simplicial differential graded $\mathbb{Q}$-module $\nabla^q[n]$ is a contractible Kan complex for each fixed q (the extension lemma [57, 1.1]) and is acyclic in the sense that the inclusion of the unit $\mathbb{Q}\to\nabla^*[n]$ is a chain homotopy equivalence for each fixed n (the Poincaré lemma [57, 1.3]). These formal properties imply that the cohomology of $A^*(X)$ is canonically naturally isomorphic to $H^*(X;\mathbb{Q})$, the rational cohomology of X (even

4 In the published version, in addition to several other unauthorized changes, the copy editors changed the typefaces with the result that the same symbols are used for multiple different objects or concepts; the preprint version available at https://pages.iu.edu/~mmandell/papers/einffinal.pdf does not have these changes and should be much more readable.

uniquely naturally isomorphic, relative to the canonical isomorphism $\mathbb{Q} \cong A^*(\Delta[0])$). The canonical isomorphism can be realized as a chain map to the normalized cochain complex $C^*(X;\mathbb{Q})$ defined in terms of integrating differential forms; see [57, 1.4,2.1,2.2].

In the p-adic case, we can use the normalized cochain complex $C^*(X;k)$ directly as it is naturally an E_∞ k-algebra. In the discussion below, we use the E_∞ k-algebra structure constructed by Berger–Fresse [37, §2.2] for the Barratt–Eccles operad $\mathcal{E}$ (the normalized chains of the Barratt–Eccles operad of categories or simplicial sets described in Example 5.3.3). Hinich–Schechtmann [123] and (independently) Smirnov [279] appear to have been the first to explicitly describe a natural operadic algebra structure on cochains; McClure–Smith [205] describes a natural E_∞ structure that generalizes classical $\cup_i$ product and bracket operations. The "cochain theory" theory of [179] shows that all these structures are equivalent in the sense that they give naturally quasi-isomorphic functors into a common category of E_∞ k-algebras, as does the polynomial de Rham complex functor A^* when $k = \mathbb{Q}$.

Both $A^*(X)$ and $C^*(X;k)$ fit into adjunctions of the contravariant type that send colimits to limits. Concretely, for a rational commutative differential graded algebra A and an E_∞ k-algebra E, define simplicial sets by the formulas

$$T(A) := \mathscr{C}_{\mathbb{Q}}(A, \nabla^*[\bullet]), \qquad U(E) := \mathscr{E}_k(E, C^*(\Delta[\bullet])),$$

where $\mathscr{C}_{\mathbb{Q}}$ denotes the category of rational commutative differential graded algebras and $\mathscr{E}_k$ denotes the category of E_∞ k-algebras (over the Barratt–Eccles operad). An easy formal argument shows that

$$A^*\colon \mathbf{\Delta}^{\mathrm{op}}\mathcal{S}\mathrm{et} \rightleftarrows \mathscr{C}_{\mathbb{Q}}^{\mathrm{op}} : T, \qquad C^*\colon \mathbf{\Delta}^{\mathrm{op}}\mathcal{S}\mathrm{et} \rightleftarrows \mathscr{E}_k^{\mathrm{op}} : U,$$

are adjunctions. As discussed in Section 5.8, both $\mathscr{C}_{\mathbb{Q}}$ and $\mathscr{E}_k$ have closed model structures with weak equivalences the quasi-isomorphisms and fibrations the surjections. Because both A^* and C^* preserve homology isomorphisms and convert injections to surjections, these are Quillen adjunctions. The main theorems of [57] and [181] then identify subcategories of the homotopy categories on which the adjunction restricts to an equivalence.

Before stating the theorems, first recall the $H_*(-;k)$-local model structure on simplicial sets: this has cofibrations the inclusions and weak equivalences the $H_*(-;k)$ homology isomorphisms. When k is a field, the weak equivalences depend only on the characteristic, and we also call this the *rational model structure* (in the case of characteristic zero) or the *p-adic model structure* (in the case of characteristic $p > 0$); we call the associated homotopy categories, the *rational homotopy category* and *p-adic homotopy category*, respectively. As with any localization, the local homotopy category is the homotopy category of local objects (that is to say, the fibrant objects): in the case of rational homotopy theory, the local objects are the Kan complexes of the homotopy type of rational spaces. In p-adic homotopy theory, the local objects are the Kan complexes that satisfy a p-completeness property described explicitly in [54, §5,7–8].

We say that a simplicial set X is *finite $H_*(-;k)$-type* (or *finite rational type* when k is a field of characteristic zero or *finite p-type* when k is a field of characteristic $p > 0$) when $H_*(X;k)$ is finitely generated over k in each degree (or, equivalently if

k is a field, when $H^*(X;k)$ is finite dimensional in each degree). Similarly a rational commutative differential graded algebra or E_∞ k-algebra A is ***finite type*** when its homology is finitely generated over k in each degree. It is *simply connected* when the inclusion of the unit induces an isomorphism $k \to H^0(A)$, $H^1(A) \cong 0$, and $H^n(A) \cong 0$ for $n < 0$ (with the usual cohomological grading convention that $H^n(A) := H_{-n}(A)$). With this terminology, the main theorem of [57] is the following:

Theorem 5.12.3 ([57, Section 8, Theorem 9.4]). *The polynomial de Rham complex functor, $A^*\colon \mathbf{\Delta}^{\mathrm{op}}\mathcal{S}\mathrm{et} \to \mathscr{C}_{\mathbb{Q}}^{\mathrm{op}}$, is a left Quillen adjoint for the rational model structure on simplicial sets. The left derived functor restricts to an equivalence of the full subcategory of the rational homotopy category consisting of the simply connected simplicial sets of finite rational type and the full subcategory of the homotopy category of rational commutative differential graded algebras consisting of the simply connected rational commutative differential graded algebras of finite type.*

For the p-adic version below, we need to take into account Steenrod operations. For $k = \mathbb{F}_p$, the Steenrod operations arise from the coherent homotopy commutativity of the p-fold multiplication, which is precisely encoded in the action of the E_∞ operad. Specifically, the p-th complex $\mathcal{E}(p)$ of the operad is a $k[\Sigma_p]$-free resolution of k, and by neglect of structure, we can regard it as a $k[C_p]$-free resolution of k where C_p denotes the cyclic group of order p. The operad action induces a map

$$\mathcal{E}(p) \otimes_{k[C_p]} (C^*(X;k))^{(p)} \to \mathcal{E}(p) \otimes_{k[\Sigma_p]} (C^*(X;k))^{(p)} \to C^*(X;k).$$

The homology of $\mathcal{E}(p) \otimes_{k[C_p]} (C^*(X;k))^{(p)}$ is a functor of the homology of $C^*(X;k)$ and the Steenrod operations P^s are precisely the images of certain classes under this map; see, for example, [198, 2.2]. This process works for any E_∞ k-algebra, not just the cochains on spaces, to give natural operations on the homology of $\mathcal{E}$-algebras, usually called Dyer–Lashof operations. The numbering conventions for these are opposite those of the Steenrod operations: on the cohomology of $C^*(X;\mathbb{F}_p)$, the Dyer–Lashof operation Q^s performs the Steenrod operation P^{-s}. If k is of characteristic p but not $\mathbb{F}_p$, the operations constructed this way are $\mathbb{F}_p$-linear but satisfy $Q^s(ax) = \phi(a)Q^s(x)$ for $a \in k$, where ϕ denotes the Frobenius automorphism of k.

The $\mathbb{F}_p$ cochain algebra of a space has the special property that the Steenrod operation $P^0 = Q^0$ is the identity operation on its cohomology; this is not true of the zeroth Dyer–Lashof operation in general. Indeed for a commutative $\mathbb{F}_p$-algebra regarded as E_∞ $\mathbb{F}_p$-algebra, Q^0 is the Frobenius. (That Q^0 is the identity for the $\mathbb{F}_p$-cochain algebra of a space is related to the fact that it comes from a cosimplicial $\mathbb{F}_p$-algebra where the Frobenius in each degree is the identity.) So when X is finite p-type, $C^*(X;k)$ in each degree has a basis that is fixed by Q^0. We say that a finite type E_∞ k-algebra is *spacelike* when in each degree its homology has a basis that is fixed by Q^0.

Theorem 5.12.4 ([181, Main Theorem, Theorem A.1]). *The cochain complex with coefficients in k, $C^*(-;k)\colon \mathbf{\Delta}^{\mathrm{op}}\mathcal{S}\mathrm{et} \to \mathscr{E}_k^{\mathrm{op}}$, is a left Quillen adjoint for the $H_*(-;k)$-local model structure on simplicial sets. If $k = \mathbb{Q}$ or k is characteristic p and $1 - \phi$ is surjective* *on k, then the left derived functor restricts to an equivalence of the full subcategory of the $H_*(-;k)$-local homotopy category consisting of the simply connected simplicial sets of finite $H_*(-;k)$-type and the full subcategory of the homotopy category of E_∞ k-algebras consisting of the spacelike simply connected E_∞ k-algebras of finite type.*

Given the Quillen equivalence between rational commutative differential graded algebras and E_∞ $\mathbb{Q}$-algebras (Theorem 5.9.5) and the natural quasi-isomorphism (zigzag) between $A^*(-)$ and $C^*(-;\mathbb{Q})$ [179, p. 549], the rational statement in Theorem 5.12.4 is equivalent to Theorem 5.12.3. The Sullivan theory in the latter often includes observations on *minimal models*. A simply connected finite type rational commutative differential graded algebra A has a cofibrant approximation $A' \to A$ whose underlying graded commutative algebra is free and such that the differential of every element is decomposable (i.e., is a sum of terms, all of which have word length greater than 1 in the generators); A' is called a minimal model and is unique up to isomorphism. As a consequence, simply connected simplicial sets of finite rational type are rationally equivalent if and only if their minimal models are isomorphic. The corresponding theory also works in the context of E_∞ $\mathbb{Q}$-algebras with the analogous definitions and proofs. The corresponding theory does not work in the context of E_∞ algebras in characteristic p for reasons closely related to the fact that unlike the rational homotopy groups, the p-adic homotopy groups of a simplicial set are not vector spaces.

The equivalences in Theorems 5.12.3 and 5.12.4 also extend to the nilpotent simplicial sets of finite type, but the corresponding category of E_∞ k-algebras does not have a known intrinsic description in the p-adic homotopy case; in the rational case, the corresponding algebraic category consists of the finite type algebras whose homology is zero in negative cohomological degrees and whose H^0 is isomorphic as a $\mathbb{Q}$-algebra to the cartesian product of copies of $\mathbb{Q}$ (cf. [182, §3]).

For other fields not addressed in the second part of Theorem 5.12.4, the adjunction does not necessarily restrict to the indicated subcategories and even when it does, it is never an equivalence. To be an equivalence, the unit of the derived adjunction would have to be an $H_*(-;k)$-isomorphism for simply connected simplicial sets of finite type. If $k \neq \mathbb{Q}$ is characteristic zero, then the right derived functor of U takes $C^*(S^2;k)$ to a simplicial set with π_2 isomorphic to k; if k is characteristic p, then the right derived functor of U takes $C^*(S^2;k)$ to a simplicial set with π_1 isomorphic to the cokernel of $1-\phi$. See [181, Appendix A] for more precise results. Because the algebraic closure of a field k of characteristic p does have $1-\phi$ surjective, even when $C^*(-;k)$ is not an equivalence, it can be used to detect p-adic equivalences. This kind of observation extends to the case $k = \mathbb{Z}$:

Theorem 5.12.5 ([180, Main Theorem]). *Finite type nilpotent spaces or simplicial sets X and Y are weakly equivalent if and only if $C^*(X;\mathbb{Z})$ and $C^*(Y;\mathbb{Z})$ are quasi-isomorphic as E_∞ $\mathbb{Z}$-algebras.*

Using the spectral version of Theorem 5.12.4 in [181, Appendix C], the proof of the previous theorem in [180] extends to show that when X and Y are finite nilpotent simplicial sets then X and Y are weakly equivalent if and only if their Spanier–

Whitehead dual spectra are weakly equivalent as E_∞ ring spectra. (This was the subject of a talk by the author at the Newton Institute in December 2002.)

We use the rest of the section to outline the argument for Theorems 5.12.3 and 5.12.4, using the notation of the latter. We fix a field k, which is either $\mathbb{Q}$ or is characteristic $p > 0$ and has $1 - \phi$ surjective. We write C^* for $C^*(-;k)$ or when $k = \mathbb{Q}$ and we are working in the context of Theorem 5.12.3, we understand C^* as A^*. We also use C^* to denote the derived functor and write $\mathbf{U}$ for the derived functor of its adjoint. The idea of the proof, going back to Sullivan, is to work with Postnikov towers, and so the first step is to find cofibrant approximations for $C^*(K(\pi, n))$. For $k = \mathbb{Q}$, this is easy since $H^*(K(\mathbb{Q}, n); \mathbb{Q})$ is the free graded commutative algebra on a generator in degree n.

Proposition 5.12.6. *If $k = \mathbb{Q}$ then $C^*(K(\mathbb{Q}, n))$ is quasi-isomorphic to the free (E_∞ or commutative differential graded) $\mathbb{Q}$-algebra on a generator in cohomological degree n.*

We use the notation $\mathbb{E}k[n]$ to denote the free E_∞ k-algebra on a generator in cohomological degree n. When k is characteristic p, there is a unique map in the homotopy category from $\mathbb{E}k[n] \to C^*(K(\mathbb{Z}/p, n))$ that sends the generator x_n to a class i_n representing the image of the tautological element of $H^n(K(\mathbb{Z}/p, n); \mathbb{Z}/p)$. Unlike the characteristic zero case, this is not a quasi-isomorphism since $Q^0[i_n] = [i_n]$ in $H^*(C^*(K(\mathbb{Z}/p, n)))$, but $Q^0[x_n] \neq [x_n]$ in $H^*(\mathbb{E}k[n])$. Let B_n be the homotopy pushout of a map $\mathbb{E}k[n] \to \mathbb{E}k[n]$ sending the generator to a class representing $[x_n] - Q^0[x_n]$ and the map $\mathbb{E}k[n] \to k$ sending the generator to 0. Then the map $\mathbb{E}k[n] \to C^*(K(\mathbb{Z}/p, n))$ factors through a map $B_n \to C^*(K(\mathbb{Z}/p, n))$. (The map in the homotopy category turns out to be independent of the choices.) The following is a key result of [181], whose proof derives from a calculation of the relationship between the Dyer–Lashof algebra and the Steenrod algebra.

Theorem 5.12.7 ([181, 6.2]). *Let k be a field of characteristic $p > 0$. Then*

$$B_n \to C^*(K(\mathbb{Z}/p, n))$$

is a cofibrant approximation.

(As suggested by the hypothesis, we do not need $1 - \phi$ to be surjective in the previous theorem; indeed, the easiest way to proceed is to prove it in the case $k = \mathbb{F}_p$ and it then follows easily for all fields of characteristic p by extension of scalars.)

The two previous results can be used to calculate $\mathbf{U}(C^*(K(\mathbb{Q}, n)))$ and $\mathbf{U}(C^*(K(\mathbb{Z}/p, n)))$. In the rational case,

$$\mathbf{U}(C^*(K(\mathbb{Q}, n))) \simeq U(\mathbb{E}\mathbb{Q}[n]) = Z(C^n(\Delta[\bullet])),$$

the simplicial set of n-cocycles of $C^*(\Delta[\bullet]; \mathbb{Q})$; this is the original model for $K(\mathbb{Q}, n)$, and a straightforward argument shows that the unit map $K(\mathbb{Q}, n) \to K(\mathbb{Q}, n)$ is a weak equivalence (the identity map with this model). In the context of Theorem 5.12.3, the same kind of argument is made in [57, 10.2]. In the p-adic case, we likewise have that $U(\mathbb{E}k[n])$ is the original model for $K(k, n)$, and so we get a fiber sequence

$$\Omega K(k, n) \to \mathbf{U}(K(\mathbb{Z}/p, n)) \to K(k, n) \to K(k, n).$$

The map $K(k,n) \to K(k,n)$ is calculated in [181, 6.3] to be the map that on π_n induces $1-\phi$. The kernel of $1-\phi$ is $\mathbb{F}_p$ and the unit map $K(\mathbb{Z}/p,n) \to \mathbf{U}(C^*(K,\mathbb{Z}/p,n))$ is an isomorphism on π_n. As a consequence, when $1-\phi$ is surjective (as we are assuming), the unit map is a weak equivalence for $K(\mathbb{Z}/p,n)$.

The game now is to show that for all finite type simply connected (or nilpotent) simplicial sets, the derived unit map $X \to \mathbf{U}C^*(X)$ is a rational or p-adic equivalence. The next result tells how to construct a cofibrant approximation for a homotopy pullback; it is not a formal consequence of the Quillen adjunction, but rather a version of the Eilenberg–Moore theorem.

Proposition 5.12.8 ([57, §3], [181, §3]). *Let*

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

be a homotopy fiber square of simplicial sets. If X,Y,Z are finite $H_*(-;k)$-type and X is simply connected, then

$$\begin{array}{ccc} C^*(X) & \longrightarrow & C^*(Y) \\ \downarrow & & \downarrow \\ C^*(Z) & \longrightarrow & C^*(W) \end{array}$$

is a homotopy pushout square of E_∞ k-algebras or rational commutative differential graded algebras.

Since we can write $K(\mathbb{Z}/p^m,n)$ as the homotopy fiber of a map

$$K(\mathbb{Z}/p^{m-1},n) \to K(\mathbb{Z}/p,n+1),$$

we see that the unit of the derived adjunction is a weak equivalence also for $K(\mathbb{Z}/p^m,n)$ (when k is characteristic p). Likewise, since products are homotopy pullbacks, we also get that the unit of the derived adjunction is a weak equivalence for $K(A,n)$ when A is a $\mathbb{Q}$ vector space (when $k=\mathbb{Q}$) or when A is a finite p-group (when k is characteristic p). Although also not a formal consequence of the adjunction, it is elementary to see that when a simplicial set X is the homotopy limit of a sequence X_j and the map $\operatorname{colim} H^*(X_j;k) \to H^*(X;k)$ is an isomorphism, then $C^*(X)$ is the homotopy colimit of $C^*(X_j)$ and $\mathbf{U}C^*(X)$ is the homotopy limit of $\mathbf{U}C^*(X_j)$. It follows that for $K(\mathbb{Z}^\wedge_p,n)$, the unit of the derived adjunction is a weak equivalence (when k is characteristic p). For any finitely generated abelian group, the map $K(A,n) \to K(A\otimes\mathbb{Q},n)$ is a rational equivalence and the map $K(A,n) \to K(A^\wedge_p,n)$ is a p-adic equivalence. Putting these results and tools all together, we see that the unit of the derived equivalence is an $H_*(-;k)$ equivalence for any X that can be built as a sequential homotopy limit $\operatorname{holim} X_j$ where $X_0 = *$, the connectivity of the map $X \to X_j$ goes to infinity, and each X_{j+1} is the homotopy fiber of a map $X_j \to K(\pi_{j+1},n)$ for π_{j+1} a finitely generated abelian group, or the rationalization (when $k=\mathbb{Q}$) or p-completion (when k is characteristic p) of a finitely generated abelian group. In particular, for a simply connected simplicial set, applying this to the Postnikov tower, we get the following result.

Theorem 5.12.9. *Assume $k = \mathbb{Q}$ or k is characteristic $p > 0$ and $1 - \phi$ is surjective. If X is a simply connected simplicial set of finite $H_*(-;k)$-type, then the unit of the derived adjunction $X \to \mathbf{U}C^*(X)$ is an $H_*(-;k)$-equivalence.*

The previous theorem formally implies that C^* induces an equivalence of the $H_*(-;k)$-local homotopy category of simply connected simplicial sets of finite $H_*(-;k)$-type with the full subcategory of the homotopy category E_∞ k-algebras or rational commutative differential graded algebras of objects in its image. The remainder of Theorems 5.12.3 and 5.12.4 is identifying this image subcategory. In the case when $k = \mathbb{Q}$, it is straightforward to see that a finite type simply connected algebra has a cofibrant approximation that $\mathbf{U}$ turns into a simply connected principal rational finite type Postnikov tower. The argument for k of characteristic p is analogous, but more complicated; see [181, §7].