4 Stable homotopy theory via ∞-categories

by Clark Barwick

The task before us is to investigate stable homotopy theory — and stable homotopy theories more generally — through the lens of ∞-category theory. Of necessity, this chapter is somewhat ahistorical; we refer the reader to the historical discussions outlined in Chapter 3 for background on the development of modern categories of spectra. However, the reader who is familiar with that story will have a keen appreciation for the foundational problems that become much cleaner in this framework.

Let us assume familiarity with elementary ∞-category theory as presented by Jacob Lurie in [169] — most particularly, the theory of limits, colimits, adjunctions, and presentability. In particular, Chapter 5 of [169] will be frequently cited, but this is the upper limit: nothing of the later chapters or of any more advanced text will be needed here. We have tried to be systematic in our citations.

Our exposition is largely a gentle introduction to some of the material in [168] and subsequent papers, and of course much of our understanding of spectra was informed by this remarkable and beautiful text. We hope that this presentation will appeal to mathematicians both within and without homotopy theory.

I offer my sincere thanks to Andrew Blumberg for his enormous assistance in making my writing palatable.

4.1 Spectra

Let X be a pointed simplicial set. An old observation of Dan Kan provides a simple way to extract the reduced homology of the geometric realisation $|X|$ from X. Namely, we let $\widetilde{\mathbb{Z}}\{X\}$ be the simplicial abelian group in which $\widetilde{\mathbb{Z}}\{X\}_n = \widetilde{\mathbb{Z}}\{X_n\}$ is freely generated by the ***pointed*** set X_n (so that the point of X_n becomes the zero element of $\widetilde{\mathbb{Z}}\{X\}_n$). One then has

$$\widetilde{H}_n(|X|, \mathbb{Z}) \cong \pi_n \widetilde{\mathbb{Z}}\{X\}.$$

More precisely, the simplicial abelian group $\widetilde{\mathbb{Z}}\{X\}$ corresponds, under Dold–Kan, to the chain complex $\widetilde{C}_*(|X|, \mathbb{Z})$.

Let us disregard the abelian group structure and regard $\widetilde{\mathbb{Z}}\{X\}$ merely as a pointed simplicial set. In fact, the functor $X \mapsto \widetilde{\mathbb{Z}}\{X\}$ preserves weak equivalences of pointed spaces, so we are entitled to think of this assignment as a functor from the ∞-category of pointed spaces to itself. We can also deduce the following properties:

1. The functor $X \mapsto \widetilde{\mathbb{Z}}\{X\}$ is ***reduced.*** That is, if X is contractible, then so is the simplicial abelian group $\widetilde{\mathbb{Z}}\{X\}$. Thus $\widetilde{H}_i(*) = 0$.
2. The functor $X \mapsto \widetilde{\mathbb{Z}}\{X\}$ is ***unital.*** In other words, $\widetilde{\mathbb{Z}}\{S^0\}$ is the constant simplicial set with value $\mathbb{Z}$; under the Dold–Kan correspondence, it corresponds to the complex $\mathbb{Z}[0]$ concentrated in degree 0. Thus $\widetilde{H}_0(S^0) = \mathbb{Z}$, and $\widetilde{H}_i(S^0) = 0$ for $i > 0$.
3. The functor $X \mapsto \widetilde{\mathbb{Z}}\{X\}$ is ***excisive:*** for any homotopy pushout
$$\begin{array}{ccc} U & \xrightarrow{i} & V \\ \downarrow & & \downarrow \\ W & \longrightarrow & X \end{array}$$
(e.g., any "honest" pushout in which i is a monomorphism of simplicial sets), the square
$$\begin{array}{ccc} \widetilde{\mathbb{Z}}\{U\} & \longrightarrow & \widetilde{\mathbb{Z}}\{V\} \\ \downarrow & & \downarrow \\ \widetilde{\mathbb{Z}}\{W\} & \longrightarrow & \widetilde{\mathbb{Z}}\{X\} \end{array}$$
is homotopy cartesian, so that one obtains a long exact sequence
$$\cdots \to \widetilde{H}_n(|U|,\mathbb{Z}) \to \widetilde{H}_n(|V|,\mathbb{Z}) \oplus \widetilde{H}_n(|W|,\mathbb{Z}) \to \widetilde{H}_n(|X|,\mathbb{Z}) \to \widetilde{H}_{n-1}(|U|,\mathbb{Z}) \to \cdots.$$
One can prove this by reducing to the case in which i is an inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ and verifying this case explicitly.
4. Finally, the functor $X \mapsto \widetilde{\mathbb{Z}}\{X\}$ is ***of finite presentation***, in that it preserves filtered colimits.

These four properties actually identify the functor $X \mapsto \widetilde{\mathbb{Z}}\{X\}$ uniquely, up to canonical natural equivalence. This is the ***uniqueness of homology.***

We may regard $X \mapsto \widetilde{\mathbb{Z}}\{X\}$ as a kind of categorified version of ***a line of slope*** 1. The first two conditions describe the values of the functor on two objects — the one-point space $*$, which (as the unit for $\vee$) is our analogue of 0, and S^0, which (as the unit for $\wedge$) is our analogue of 1. Under our analogy, we have insisted that $f(0) = 0$ and $f(1) = 1$. The other two axioms declare that $X \mapsto \widetilde{\mathbb{Z}}\{X\}$ is ***linear.*** This linearity now determines the values of this functor on all other objects.

Spectra

If we merely eliminate the "slope 1" condition (unitality), we arrive at the notion of a ***spectrum.*** Here is the definition.

Definition 4.1.1. Write $\mathbb{S}_*$ for the ∞-category of pointed spaces (the full subcategory of $\mathrm{Fun}(\Delta^1, \mathbb{S})$ spanned by those objects $X \to Y$ in which X is contractible).

Then a functor $E\colon \mathbb{S}_* \to \mathbb{S}_*$ is called a ***linear functor*** or a ***spectrum*** if E is reduced, excisive, and of finite presentation. That is:

1. The functor E is *reduced*: for any contractible pointed space P, the pointed space $E(P)$ is also contractible.
2. The functor E is *excisive*: for any pushout square

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ W & \longrightarrow & X \end{array}$$

   in $\mathbb{S}_*$, the induced square

$$\begin{array}{ccc} E(U) & \longrightarrow & E(V) \\ \downarrow & & \downarrow \\ E(W) & \longrightarrow & E(X) \end{array}$$

   is a pullback in $\mathbb{S}_*$.
3. The functor E is *of finite presentation*: for any filtered diagram $\alpha \mapsto X_\alpha$ in $\mathbb{S}_*$, the natural map

$$\mathrm{colim}_\alpha E(X_\alpha) \to E(\mathrm{colim}_\alpha X_\alpha)$$

   is an equivalence.

This makes precise the sense in which spectra are said to “be” generalised homology theories. But in order to come to grips with this definition, we must do some work to unpack the axioms in turn.

Reduced functors

Reducedness is nothing profound. If one has a functor $F\colon \mathbb{S}_* \to \mathbb{S}_*$ that isn't reduced, one may “repair” it by passing to the reduction F^{red}, which carries a pointed space X to the cofiber of the map $F(*) \to F(X)$.

Finitely presented functors

Finite presentability is also relatively straightforward. We say that a pointed space X is *finite* if it can be expressed as a finite colimit of contractible pointed spaces. Every pointed space is the filtered colimit of the finite spaces that map to it, so a functor $F\colon \mathbb{S}_* \to \mathbb{S}_*$ is of finite presentation if and only if it is left Kan extended from its restriction to the ∞-category $\mathbb{S}_*^{fin}$ of finite spaces.

So a finitely presented functor $\mathbb{S}_* \to \mathbb{S}_*$ is uniquely determined by its restriction to $\mathbb{S}_*^{fin}$. That is, the ∞-category of finitely presented functors $\mathbb{S}_* \to \mathbb{S}_*$ is equivalent to the ∞-category of (arbitrary) functors $\mathbb{S}_*^{fin} \to \mathbb{S}_*$.

Excisive functors

The excision condition is where the rubber meets the road. An important special case of a square in $\mathcal{S}_*$ is when the corners are contractible spaces:

$$\begin{array}{ccc} Y & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array} \tag{4.1.1}$$

If (4.1.1) is a pushout square, then X is the ***suspension*** ΣY; since the forgetful functor $\mathcal{S}_* \to \mathcal{S}$ preserves pushouts, the pointing of Y is irrelevant. Dually, if (4.1.1) is a pullback square in $\mathcal{S}_*$, then Y is the ***loopspace*** ΩX; here, the pointing of X is important.

4.1.2. Suspension is left adjoint to loopspace:

$$\Sigma\colon \mathcal{S}_* \rightleftarrows \mathcal{S}_* : \Omega.$$

In particular, we have the unit $\varepsilon\colon \mathit{id} \to \Omega\Sigma$ and the counit $\eta\colon \Sigma\Omega \to \mathit{id}$.

Now if $F\colon \mathcal{S}_* \to \mathcal{S}_*$ is a reduced functor, then we may apply it to the pushout square

$$\begin{array}{ccc} Y & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma Y \end{array}$$

to obtain a canonical map

$$\sigma_Y\colon FY \to \Omega F \Sigma Y.$$

We may also write σ_Y^F whenever disambiguation is called for. It is clear that if F is excisive, then the natural transformation σ_Y is an equivalence. However, it is relatively surprising that this condition ***suffices*** to ensure the excisiveness of F.

Lemma 4.1.3. *Let $F\colon \mathcal{S}_* \to \mathcal{S}_*$ be a reduced functor. Then F is excisive if and only if, for any pointed space Y, the map*

$$\sigma_Y\colon FY \to \Omega F \Sigma Y$$

is an equivalence.

Proof. The "only if" direction is trivial, so we focus on the "if" direction. For this, suppose

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ W & \longrightarrow & X \end{array}$$

is a pushout square in $\mathcal{S}_*$. We expand this square into a diagram

$$\begin{array}{ccccccc}
U & \longrightarrow & V & \longrightarrow & * & & \\
\downarrow & & \downarrow & & \downarrow & & \\
W & \longrightarrow & X & \longrightarrow & W/U & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & V/U & \longrightarrow & \Sigma U & \longrightarrow & \Sigma W \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & * & \longrightarrow & \Sigma V & \longrightarrow & \Sigma X
\end{array} \tag{4.1.2}$$

of $\mathcal{S}_*$ in which every square is a pushout. When we apply F to this diagram, we obtain a solid arrow square

$$\begin{array}{ccc}
F(U) & \longrightarrow & \Omega F(\Sigma U) \\
\downarrow & \overset{\lambda}{\nearrow} & \downarrow \\
F(W) \times_{F(X)} F(V) & \longrightarrow & \Omega F(\Sigma V) \times_{\Omega F(\Sigma X)} \Omega F(\Sigma W)
\end{array} \tag{4.1.3}$$

in which both horizontal maps are, by assumption, equivalences. The universal property of $\Omega F(\Sigma U)$ supplies us with a dotted lift λ, and it follows that every map in this square is an equivalence. ☐

Notation 4.1.4. We write $\mathrm{Fun}^{fp,red}(\mathcal{S}_*, \mathcal{S}_*)$ for the full subcategory of the ∞-category $\mathrm{Fun}(\mathcal{S}_*, \mathcal{S}_*)$ spanned by the reduced functors F of finite presentation, and we write $\mathbf{Sp} \subset \mathrm{Fun}^{fp,red}(\mathcal{S}_*, \mathcal{S}_*)$ for the full subcategory spanned by the spectra.

4.1.5. Let E be a spectrum. Then we obtain a sequence of spaces

$$\{X_n = E(S^n)\}_{n \geq 0}$$

along with equivalences $\{X_n \xrightarrow{\sim} \Omega X_{n+1}\}_{n \geq 0}$. Thus a spectrum gives rise to what we might call a ***sequential spectrum***. We will show that these are in fact equivalent homotopy theories.

Exercise 4.1.6. Show that $\mathrm{Fun}^{fp,red}(\mathcal{S}_*, \mathcal{S}_*)$ is a presentable ∞-category:

For any finite pointed space X, denote by $h^X : \mathcal{S}_* \to \mathcal{S}_*$ the functor corepresented by X, so that $h^X(Y) \simeq \mathrm{Map}_{\mathcal{S}_*}(X, Y)$. Observe that h^X is a reduced functor of finite presentation. For any pointed space Y, the counit $\Sigma\Omega \to \boldsymbol{id}$ induces a map

$$d_X(Y) \colon \Sigma \mathrm{Map}_{\mathcal{S}_*}(\Sigma X, Y) \simeq \Sigma\Omega \mathrm{Map}_{\mathcal{S}_*}(X, Y) \to \mathrm{Map}_{\mathcal{S}_*}(X, Y);$$

this is functorial in Y, whence we obtain a natural transformation $d_X \colon \Sigma \circ h^{\Sigma X} \to h^X$. Show that a reduced finitely presented functor F is a spectrum if and only if it is local with respect to the set of maps $\{d_X : X \in \mathcal{S}_*^{fin}\}$. Thus $\mathbf{Sp}$ is the accessible localisation of $\mathrm{Fun}^{fp,red}(\mathcal{S}_*, \mathcal{S}_*)$, and the class of morphisms that are inverted by the localisation is exactly the saturated class generated by $\{d_X : X \in \mathcal{S}_*^{fin}\}$.

Deduce that $\mathbf{Sp}$ is a presentable ∞-category, and the fully faithful inclusion functor $\mathbf{Sp} \hookrightarrow \mathrm{Fun}^{fp,red}(\mathcal{S}_*, \mathcal{S}_*)$ preserves limits and filtered colimits.

Shifting

There is a nontrivial auto-equivalence of the loopspace or suspension of any space. If X is a pointed space, then the universal property of the kernel product provides an endomorphism

$$(-1)\colon \Omega X = * \times_X * \to * \times_X * = \Omega X$$

obtained by exchanging the roles of the two points. The map (-1) is clearly an auto-equivalence, and it is a nontrivial one, because it *reverses* the direction of the (implicit) homotopy in the square

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array}$$

The map (-1) is a natural auto-equivalence on the functor Ω. It is not homotopic to *id*, but it is an involution in the sense that its composition $(-1)^2$ with itself is homotopic (in a canonical fashion) to *id*. This goes some way to justifying the notation.

Geometrically, each point of ΩX corresponds to a parametrised loop, and (-1) takes each point to the point representing the same loop, parametrised in the reverse direction. On $\pi_1 X = \pi_0 \Omega X$, the auto-equivalence (-1) induces the assignment $\gamma \mapsto \gamma^{-1}$.

In precisely the same manner, we obtain an involution

$$(-1)\colon \Sigma \to \Sigma$$

of the suspension functor.

These two involutions are compatible under the adjunction between suspension and loopspace. Indeed, in a square

$$\begin{array}{ccc} Y & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array}$$

reversing the direction of the implicit homotopy is at once tantamount to the composition of the map $Y \to \Omega X$ with (-1) and to the composition of (-1) with the map $\Sigma Y \to X$. (This point, silly as it is, is the origin of virtually *all* the signs throughout stable homotopy theory and homological algebra.)

Warning 4.1.7. There are two ways to iterate the suspension maps σ_Y, and they are not homotopic; they differ by a sign. For any reduced functor $F\colon \mathcal{S}_* \to \mathcal{S}_*$, one has a natural homotopy

$$-\Omega\sigma^F\Sigma \simeq \sigma^{\Omega F\Sigma} \tag{4.1.4}$$

between the two functors $\Omega F \Sigma \to \Omega^2 F \Sigma^2$.

Exercise 4.1.8. Construct the homotopy (4.1.4) by contemplating the diagrams (4.1.2) and (4.1.3) in the case in which both V and W are contractible.

Notation 4.1.9. Let $E\colon \mathbb{S}_* \to \mathbb{S}_*$ be a reduced excisive functor. Then for any natural numbers $a \leq b$, the natural transformation

$$\sigma^{\Omega^{b-1}E\Sigma^{b-1}} \cdots \sigma^{\Omega^{a+1}E\Sigma^{a+1}} \sigma^{\Omega^{a}E\Sigma^{a}} : \Omega^a E \Sigma^a \to \Omega^b E \Sigma^b$$

is an equivalence, which induces, for any pointed space X, a natural isomorphism

$$\pi_{n+a} E\Sigma^a X \cong \pi_{n+b} E \Sigma^b X$$

for any integer n such that $n \geq -a$. Consequently, we may define, for any integer n, an abelian group

$$E_n X = \pi_{n+a} E \Sigma^a X$$

for some a such that $a \geq \max\{2, -n\}$, secure in our knowledge that this abelian group is canonically independent of the choice of a. We thus obtain a functor

$$E_*\colon \mathbb{S}_* \to \mathbf{Ab}^{\mathbb{Z}},$$

where the target is the 1-category of $\mathbb{Z}$-graded abelian groups. This is the ***E-homology*** functor.

If E is a spectrum, then we may define the ***suspension*** or ***shift by*** 1 of E as the spectrum $E[1] = E \circ \Sigma$. In the other direction, we may define the ***loop*** or ***shift by*** -1 of E as the spectrum $E[-1] = \Omega \circ E$. Iterating these, we obtain shifts $E[m]$ for any $m \in \mathbb{Z}$, and we note that on homology theories,

$$E[m]_n \cong E_{n-m}.$$

It is quite common in the literature to see E_n as a shorthand for the ***group*** $E_n(S^0)$.

Homology and cohomology

Let X be a pointed space, and let E be a spectrum. Then we define the ***E-homology and E-cohomology*** of X as the groups

$$E_n(X) = \pi_n E(X) \qquad \text{and} \qquad E^n(X) = \pi_{-n} \operatorname{Map}(X, E(S^0)).$$

4.2 Examples

Eilenberg–Mac Lane spectra

Our motivation for the definition of a spectrum was our contemplation of ordinary homology. We therefore already have one class of examples in hand:

Example 4.2.1. The functor $X \mapsto \widetilde{\mathbb{Z}}\{X\}$ is a spectrum $H\mathbb{Z}\colon \mathbb{S}_* \to \mathbb{S}_*$. This is the ***Eilenberg–Mac Lane spectrum of*** $\mathbb{Z}$. The groups $(H\mathbb{Z})_*(X)$ are zero in negative degrees, and in nonnegative degrees, they are the reduced homology groups $\widetilde{H}_*(X, \mathbb{Z})$ of X.

More generally, for any abelian group A, let us contemplate the functor $X \mapsto \tilde{A}\{X\}$, which as a functor on pointed simplicial sets carries X to the pointed simplicial set $\tilde{A}\{X\}_* = \widetilde{\mathbb{Z}}\{X\}_* \otimes A$. This is a spectrum $HA\colon \mathbb{S}_* \to \mathbb{S}_*$, called the ***Eilenberg–Mac Lane*** *spectrum of* A. The groups $(HA)_*(X)$ are zero in negative degrees, and in nonnegative degrees, they are the reduced homology groups $\widetilde{H}_*(X, A)$ of X.

The derivative

Just as one may often find a best linear approximation of a general (differentiable) function by forming the derivative, we can construct the best linear approximation of a general (reduced and finitely presented) functor. This provides us with a few more useful examples.

Indeed, we have already seen that the full subcategory $\mathbf{Sp} \subseteq \mathrm{Fun}^{fp,red}(\mathcal{S}_*, \mathcal{S}_*)$ is a localisation; that is, the inclusion admits a left adjoint D, which we call the *derivative.* The bonus good news is that we can write a convenient formula for this D.

Construction 4.2.2. Let $F\colon \mathcal{S}_* \to \mathcal{S}_*$ be a reduced functor of finite presentation. Then we may look at the sequence of reduced functors of finite presentation

$$F \xrightarrow{\sigma^F} \Omega F \Sigma \xrightarrow{\sigma^{\Omega F \Sigma}} \Omega^2 F \Sigma^2 \xrightarrow{\sigma^{\Omega^2 F \Sigma^2}} \cdots,$$

which is indexed on the natural numbers $\mathbb{N}$. We write $DF = \mathrm{colim}_n \Omega^n F \Sigma^n$ for the colimit of this diagram of functors.

The assignment $F \mapsto DF$ comes equipped with a natural transformation $\alpha\colon id \to D$.

Lemma 4.2.3. *For any reduced functor*

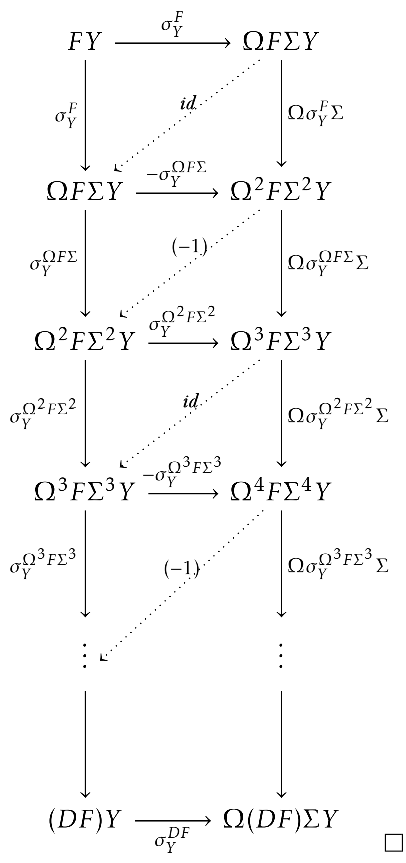
$$F\colon \mathcal{S}_* \to \mathcal{S}_*$$

of finite presentation, the functor $DF\colon \mathcal{S}_* \to \mathcal{S}_*$ *is a spectrum.*

Proof. Let Y be a pointed space, and consider the morphism

$$\sigma_Y^{DF}\colon (DF)Y \to \Omega(DF)\Sigma Y.$$

We represent σ_Y^{DF} as the filtered colimit of the solid arrow sequence of morphisms, shown on the diagram to the side. The dotted arrows are all equivalences that make this diagram commute, and thus in the colimit they define an inverse to σ_Y^{DF}.



□

We note that if F is already a spectrum, then $\alpha_F\colon F \to DF$ is in fact already an equivalence. In fact, we now show that DF is the universal linear approximation to F.

Proposition 4.2.4. *The natural transformation* α *exhibits* D *as a localisation functor on* $\mathrm{Fun}^{fp,red}(\mathcal{S}_*, \mathcal{S}_*)$ *whose essential image is precisely* $\mathbf{Sp}$.

Proof. If E is a spectrum, then $\alpha_E\colon E \to DE$ is an equivalence. Thus **Sp** is the essential image of D, and for any reduced functor F of finite presentation, $\alpha_{DF}\colon DF \to DDF$ is an equivalence. We must also check that $D\alpha_F\colon DF \to DDF$ is an equivalence; for this, it suffices to note that $D\sigma^F\colon DF \to D(\Omega F\Sigma)$ is an equivalence. □

Suspension spectra

Now let us use this construction to define some interesting examples of spectra.

Construction 4.2.5. Let $\mathbb{S}^0 = \Sigma^\infty S^0$ be the spectrum $D\mathit{id}$. That is, for any point space Y, we have

$$(\Sigma^\infty S^0)Y = \operatorname{colim}_n \Omega^n \Sigma^n Y.$$

This is the ***sphere spectrum***, which represents stable homotopy:

$$\pi_m^s(Y) = (\Sigma^\infty S^0)_m Y \cong \operatorname{colim}_n \pi_{m+n} \Sigma^n Y;$$

by Freudenthal, one has $\pi_m^s(Y) \cong \pi_{2m+2}(\Sigma^{m+2} Y)$.

More generally, for any space X, consider the reduced, finitely presented functor $s_X\colon \mathbb{S}_* \to \mathbb{S}_*$ given by the assignment $Y \mapsto X \wedge Y$. We define

$$\Sigma^\infty X = Ds_X;$$

this is the ***suspension spectrum of*** X. We therefore obtain

$$(\Sigma^\infty X)Y = \operatorname{colim}_n \Omega^n (X \wedge \Sigma^n Y) \simeq \operatorname{colim}_n \Omega^n \Sigma^n (X \wedge Y) \simeq (\mathbb{S}^0)(X \wedge Y).$$

When X is a sphere, we write $\mathbb{S}^n = \Sigma^\infty S^n$, and we observe that

$$\mathbb{S}^n \simeq \mathbb{S}^0[n].$$

The suspension spectrum is a functor $\Sigma^\infty\colon \mathbb{S}_* \to \mathbf{Sp}$. In the other direction, we have a functor $\Omega^\infty\colon \mathbf{Sp} \to \mathbb{S}_*$ that carries a spectrum E to the value $E(S^0)$. They are related in the following manner:

Proposition 4.2.6. *The functor* Σ^∞ *is left adjoint to the functor* Ω^∞.

Exercise 4.2.7. Verify this.

4.2.8. With the suspension functor in hand, we may define the ***E-cohomology*** of a pointed space X as

$$E^n(X) = \pi_{-n} \operatorname{Map}_{\mathbf{Sp}}(\Sigma^\infty X, E)$$

Spanier–Whitehead duals

It is also possible to generalise the sphere spectrum $\mathbb{S}^0$ in a dual manner. We will study the phenomenon of Spanier–Whitehead duality in a structured manner soon.

Example 4.2.9. Let X be a finite pointed space, and let $h^X : \mathbb{S}_* \to \mathbb{S}_*$ be the functor corepresented by X; that is, $h^X(Y) = \mathrm{Map}_{\mathbb{S}_*}(X, Y)$. Since X is finite, h^X is finitely presented, and we obtain

$$(Dh^X)Y = \mathrm{colim}_n \Omega^n \mathrm{Map}_{\mathbb{S}_*}(X, \Sigma^n Y) \simeq \mathrm{Map}_{\mathbb{S}_*}(X, \mathrm{colim}_n \Omega^n \Sigma^n Y)$$
$$\simeq \mathrm{Map}_{\mathbb{S}_*}(X, (\Sigma^\infty S^0)(Y)).$$

The spectrum $(\Sigma^\infty X)^\vee = Dh^X$ is the ***Spanier–Whitehead dual of*** X.

The assignment $X \mapsto (\Sigma^\infty X)^\vee$ is a *contravariant* functor from pointed finite spaces to spectra.

Also, the Spanier–Whitehead dual of a finite pointed space may well have negative homotopy groups. For example, when X is a sphere, we obtain an identification

$$(\Sigma^\infty S^n)^\vee \simeq \mathbb{S}^0[-n],$$

whence we are compelled to define $\mathbb{S}^{-n} = (\Sigma^\infty S^n)^\vee$.

Exercise 4.2.10. For any spectrum E and any finite pointed space X, exhibit a homotopy equivalence

$$\mathrm{Map}_{\mathbf{Sp}}((\Sigma^\infty X)^\vee, E) \simeq E(X).$$

Thom spectra

Definition 4.2.11. Let X be a space. Then a ***local system of spectra on*** X is a functor $X^{op} \to \mathbf{Sp}$; we write $\mathbf{Sp}_X = \mathrm{Fun}(X^{op}, \mathbf{Sp})$ for the ∞-category of local systems of spectra.

4.2.12. If we unpack the definitions a bit, a local system of spectra on a space X is a functor $X^{op} \times \mathbb{S}_* \to \mathbb{S}_*$, written $(x, T) \mapsto \zeta(x)(T)$, such that for any point $x \in X$, the functor $\zeta(x) \colon \mathbb{S}_* \to \mathbb{S}_*$ is a spectrum.

Example 4.2.13. For any spectrum E, we have a ***constant local system*** E_X at E.

Example 4.2.14 (The J homomorphism). To any finite-dimensional real vector space V we can attach the one-point compactification S^V. This is a topologically enriched functor from finite-dimensional real vector spaces and isomorphisms to topological spaces. After passing to the attached ∞-categories, we may compose this functor with the suspension functor to obtain a local system

$$\coprod_{n \geq 0} BO(n) \to \mathbf{Sp}.$$

This functor factors through the group completion $\mathbb{Z} \times BO \to \mathbf{Sp}$. The ***J homomorphism*** is then the restricted map

$$J_O \colon BO \simeq \{0\} \times BO \subset \mathbb{Z} \times BO \to \mathbf{Sp},$$

which is a local system over BO. If X is a topological space with a real vector bundle $\nu \colon X \to BO$, one obtains a local system of spectra by composition with J_O.

In the same manner, we obtain a local system

$$J_U : BU \to \mathbf{Sp},$$

and if X is a topological space with a complex vector bundle $\nu: X \to BU$, one obtains a local system of spectra by composition with J_U.

Definition 4.2.15. Let X be a space. A *stable spherical fibration over* X is a local system of spectra $\zeta: X^{op} \to \mathbf{Sp}$ such that, for each point $x \in X$, the spectrum $\zeta(x)$ is (abstractly) equivalent to the sphere spectrum $\mathbb{S}^0$.

The *Thom spectrum* X^{ζ} of a stable spherical fibration ζ is the colimit of the diagram

$$\zeta: X^{op} \to \mathbf{Sp}.$$

4.2.16. For any stable spherical fibration ζ over X, the Thom spectrum enjoys the following universal property: for any spectrum E, we have a natural weak homotopy equivalence

$$\mathrm{Map}_{\mathbf{Sp}}(X^{\zeta}, E) \simeq \mathrm{Map}_{\mathbf{Sp}_X}(\zeta, E_X).$$

As a functor $\mathbb{S}_* \to \mathbb{S}_*$, the Thom spectrum X^{ζ} carries a space T to the space

$$\mathrm{colim}_{n\to+\infty}\,\Omega^n\left(\mathrm{colim}_{x\in X}\,\Sigma^n\zeta(x)(T)\right).$$

Example 4.2.17. If $\zeta: X^{op} \to \mathbf{Sp}$ is a constant spherical fibration, then the Thom spectrum X^{ζ} is nothing more than $\Sigma^{\infty}X_+$.

Example 4.2.18 (Cobordism). By taking the Thom spectra attached to the J homomorphism, we obtain

$$MO = (BO)^{J_O} \qquad \text{and} \qquad MU = (BU)^{J_U}.$$

These spectra are the *real* and *complex cobordism spectra*, respectively.

The homotopy of MO and MU are known — the former by Thom and the latter by Milnor:

$$\pi_*MO \cong \mathbb{F}_2[x_n : n \geq 2, n \neq 2^j - 1, |x_n| = n];$$
$$\pi_*MU \cong \mathbb{Z}[z_n : n \geq 1, |z_n| = 2n].$$

Example 4.2.19. If X is a topological space with a real vector bundle ν, then we may abuse notation slightly and write X^{ν} for the Thom spectrum $X^{J_O\nu}$. We may define the Thom spectrum of a complex vector bundle in the same manner.

Example 4.2.20 (Atiyah duality). Let X be a compact manifold. For a sufficiently general embedding of X into $\mathbb{R}^n$, the Spanier–Whitehead dual $(\Sigma^{\infty}X_+)^{\vee}$ is naturally equivalent to $\Sigma^{\infty}(\mathbb{R}^n/(\mathbb{R}^n - X))[-n]$, which in turn can be identified with the Thom spectrum of the stable normal bundle of X.

4.3 Smash products

One of the most important aspects of the theory of spectra is the presence of the smash product, which provides **Sp** with a symmetric monoidal structure. We won't dive headlong into the details of the theory of symmetric monoidal structures on ∞-categories, but the setup of higher categories makes it possible to characterize the smash product of spectra with a homotopy-coherent universal property.

Day convolution

Let $E_1, \ldots, E_n$ be a finite collection of reduced functors of finite presentation. Since these can be regarded as functors $\mathcal{S}_*^{fin} \to \mathcal{S}_*$, and since both source and target are endowed with the smash product symmetric monoidal structure, we may form their *Day convolution*: this is the functor

$$E_1 \star \cdots \star E_n \colon \mathcal{S}_*^{fin} \to \mathcal{S}_*$$

defined as the left Kan extension of the functor $(K_1, \ldots, K_n) \mapsto E_1 K_1 \wedge \cdots \wedge E_n K_n$ along the functor $(K_1, \ldots, K_n) \mapsto K_1 \wedge \cdots \wedge K_n$. In other words, we have, for any finite pointed space Y, the formula

$$(E_1 \star \cdots \star E_n) Y = \operatorname{colim}_{K_1 \wedge \cdots \wedge K_n \to Y} E_1 K_1 \wedge \cdots \wedge E_n K_n,$$

where the colimit is taken over the ∞-category

$$(\mathcal{S}_*^{fin} \times \cdots \times \mathcal{S}_*^{fin}) \times_{\mathcal{S}_*^{fin}} (\mathcal{S}_*^{fin})_{/Y}.$$

It is immediate from this formula that $E_1 \star \cdots \star E_n$ is a reduced functor. When $n = 0$, it's immediate that the unit is the inclusion functor $\mathcal{S}_*^{fin} \hookrightarrow \mathcal{S}_*$.

4.3.1. The Day convolution actually defines a symmetric monoidal structure on the ∞-category $\operatorname{Fun}^{fp,red}(\mathcal{S}_*, \mathcal{S}_*)$, but we won't concern ourselves with that now. For now, we simply observe that $*$ is associative and symmetric up to homotopy in the most naïve sense possible.

Let us note that, for any finite collection $X_1, \ldots, X_n$ of finite pointed spaces, the natural morphism

$$h^{X_1} \star \cdots \star h^{X_n} \to h^{X_1 \wedge \cdots \wedge X_n}$$

is an equivalence, and the Day convolution $(E_1, \ldots, E_n) \mapsto E_1 \star \cdots \star E_n$ preserves colimits separately in each variable.

The point here is that, since every reduced functor of finite presentation is a colimit of corepresentables, the Day convolution is controlled by its behaviour on the corepresentables, where it mirrors the smash product of pointed spaces.

Smash product

Now if $E_1, \ldots, E_n$ are spectra, we may form the *smash product*

$$E_1 \wedge \cdots \wedge E_n = D(E_1 \star \cdots \star E_n).$$

This gives us the explicit (but, in all honesty, not tremendously useful) formula

$$(E_1 \wedge \cdots \wedge E_n)Y = \mathrm{colim}_m \Omega^m(\mathrm{colim}_{K_1 \wedge \cdots \wedge K_n \to \Sigma^m Y} E_1 K_1 \wedge \cdots \wedge E_n K_n).$$

This is only a reasonable definition because of the following technical lemma, which expresses a compatibility of the Day convolution with the derivative D. Here recall the collection of natural transformations $\{d_X : X \in \mathbb{S}_*^{fin}\}$ from 4.1.6.

Lemma 4.3.2. *For any finite space X and any finitely presented reduced functor F, the natural transformation $d_X \colon \Sigma h^{\Sigma X} \to h^X$ induces a morphism*

$$d_X * id \colon \Sigma h^{\Sigma X} * F \to h^X * F$$

that lies in the strongly saturated class of morphisms of $\mathrm{Fun}^{fp,red}(\mathbb{S}_*, \mathbb{S}_*)$ generated by $\{d_X : X \in \mathbb{S}_*^{fin}\}$.

Proof. Any finitely presented reduced functor F is a colimit of functors of the form h^Y for Y a finite pointed space, so it suffices to assume that $F = h^Y$. In that case, $d_X * id$ is homotopic to the natural transformation

$$d_{X \wedge Y} \colon \Sigma h^{\Sigma X \wedge Y} \to h^{X \wedge Y}. \qquad \square$$

This lemma will actually imply that **Sp** is symmetric monoidal under the smash product, and the derivative D is symmetric monoidal. For now, we will make do with the following less structured assertion:

Proposition 4.3.3. *The smash product preserves colimits separately in each variable. Additionally, D carries the convolution product to the smash product in the sense that if $E_1, \ldots, E_n$ are reduced functors $\mathbb{S}_*^{fin} \to \mathbb{S}_*$, then the canonical natural transformation on Day convolutions $\alpha_{E_1} \star \cdots \star \alpha_{E_n} \colon E_1 \star \cdots \star E_n \to DE_1 \star \cdots \star DE_n$ induces an equivalence*

$$D(E_1 \star \cdots \star E_n) \simeq DE_1 \wedge \cdots \wedge DE_n.$$

In particular, the sphere spectrum $\mathbb{S}^0$ is a unit for the smash product.

Proof. The first claim is formal. For the second, we observe that by the previous lemma, $\alpha_{E_1} \star \cdots \star \alpha_{E_n}$ lies in the saturated class generated by $\{d_X : X \in \mathbb{S}_*^{fin}\}$. $\square$

Many of the spectra we've been contemplating so far are obtained via the derivative. This result shows that when we are smashing derivatives, we may delay the application of the derivative to the last possible moment. We deduce the following pleasant corollary:

Corollary 4.3.4. *If $X_1, \ldots, X_n$ are pointed spaces, the natural map is an equivalence:*

$$(\Sigma^\infty X_1) \wedge \cdots \wedge (\Sigma^\infty X_n) \simeq \Sigma^\infty(X_1 \wedge \cdots \wedge X_n).$$

4.3.5. The smash product also appears in the classical formula for the value of an excisive functor. If $E\colon \mathbb{S}_* \to \mathbb{S}_*$ is a spectrum, then any map $K_1 \wedge K_2 \to S^0$ induces a map $E(K_1) \wedge K_2 \wedge X \to E(K_1 \wedge K_2 \wedge X) \to E(X)$, natural in X; together, these define an equivalence

$$\Omega^\infty(\Sigma^\infty X \wedge E) \xrightarrow{\sim} E(X),$$

natural in X.

Function spectra and duality

For any spectrum E, the functor $E'' \mapsto E'' \wedge E$ preserves colimits, and since **Sp** is presentable, it follows that there exists a right adjoint $E' \mapsto \mathbb{F}(E, E')$ thereto. This is the ***function spectrum*** from E to E'. As a functor on pointed spaces, it is given by the assignment

$$X \mapsto \mathrm{Map}_{\mathbf{Sp}}((\Sigma^\infty X)^\vee \wedge E, E').$$

For any pointed finite space X and for any map $f : K_1 \wedge K_2 \to Y$ of pointed finite spaces, evaluation defines a map

$$f \circ (ev \wedge id)\colon \mathrm{Map}(X, K_1) \wedge X \wedge K_2 \to K_1 \wedge K_2 \to Y.$$

Letting f and Y vary, we obtain a natural transformation $h^X \star s_X \to h^{S^0}$. Applying D, we obtain a morphism of spectra

$$(\Sigma^\infty X)^\vee \wedge \Sigma^\infty X \to \mathbb{S}^0,$$

which in turn specifies a map

$$\delta_X\colon (\Sigma^\infty X)^\vee \to \mathbb{F}(\Sigma^\infty X, \mathbb{S}^0),$$

which turns out to be an equivalence. We therefore take this as motivation for the following definition.

Definition 4.3.6. For any spectrum E, the ***dual*** of E is the spectrum

$$E^\vee = \mathbb{F}(E, \mathbb{S}^0).$$

4.3.7. If E is a spectrum, then there is a morphism of spectra

$$E \wedge E^\vee \simeq E^\vee \wedge E \to \mathbb{S}^0,$$

which corresponds to a morphism $E \to E^{\vee\vee}$.

Let's classify the finite objects of **Sp**. It turns out that finiteness in **Sp** is a far simpler matter than in $\mathbb{S}$; in effect, problems that the Wall finiteness obstruction catches in $\mathbb{S}$ are finessed in **Sp**:

Theorem 4.3.8. *Let E be a spectrum. The following are equivalent.*

1. *There exists a finite pointed space X, an integer $n \in \mathbb{Z}$, and an equivalence $E \simeq (\Sigma^\infty X)[n]$.*

2. *The spectrum E can be expressed as a finite colimit of spectra of the form $\mathbb{S}^n$ for $n \in \mathbb{Z}$.*
3. *The spectrum E is compact as an object of* **Sp**.
4. *The natural morphism $E \to E^{\vee\vee}$ is an equivalence.*

Definition 4.3.9. A spectrum is said to be *finite* if it satisfies the conditions of 4.3.8. We write $\mathbf{Sp}^{fin} \subset \mathbf{Sp}$ for the full subcategory spanned by the finite spectra.

4.4 Stable ∞-categories

A stable ∞-category is much like an abelian category, except that what is asked of monomorphisms or epimorphisms in an abelian category is asked of *all* morphisms of a stable ∞-category. In an abelian category, every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel. The definition of stable ∞-category is rigged so that *every morphism* of a stable ∞-category is both the kernel of its cokernel and the cokernel of its kernel.

Another way of thinking about stable ∞-categories is in relation to triangulated categories. A central theme in modern mathematics is the idea of encoding geometric structure in terms of a triangulated category of modules of some sort, such as the derived category of a scheme, the stable module category of a finite group, or the Fukaya category of a symplectic manifold. A lot of work (e.g., see [214]) permits the use of triangulated categories as a setting for abstract stable homotopy theory. This is explained by the connection to stable ∞-categories. The structure of a triangulated category is, in a precise sense, the shadow of the structure of a stable ∞-category: the homotopy category of a stable ∞-category is a triangulated category. However, stable ∞-categories are much easier to work with. For one thing, the definition is considerably more concise, as the axioms of a triangulated category immediately become basic computations with kernels and cokernels. For another, a variety of problems go away — notably, the formation of cokernels is functorial in a stable ∞-category, but it is almost never so in a triangulated category. As a result, there are important invariants that require functorial cokernels, like algebraic K-theory, that really only make sense for an ∞-category: they are capable of distinguishing two stable ∞-categories with triangulated-equivalent homotopy categories (e.g., see [258]).

Definition 4.4.1. An ∞-category A is said to be *stable* if the following conditions obtain.

1. There is a *zero object* — that is, an object that is both initial and terminal — in A.
2. The ∞-category A has all finite limits and all finite colimits.
3. A square

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ W & \longrightarrow & X \end{array}$$

is a pushout if and only if it is a pullback.

If A and B are stable ∞-categories, a functor $f\colon A \to B$ is left exact (i.e., finite-limit-preserving) if and only if it is right exact (i.e., finite-colimit-preserving). In this case, we simply call f *exact*. The subcategory of $\mathbf{Cat}_\infty$ whose objects are stable ∞-categories and whose morphisms are exact functors is denoted $\mathbf{Stab}_\infty$.

Example 4.4.2. Naturally, **Sp** is stable, as is $\mathbf{Sp}^{fin}$. On the other hand, although $\mathcal{S}_*$ and $\mathcal{S}_*^{fin}$ have zero objects, finite limits, and finite colimits, they certainly aren't stable.

Example 4.4.3. For any small ∞-category C and any stable ∞-category A, the ∞-category $\mathrm{Fun}(C, A)$ is stable. In particular, for any space X, the ∞-category $\mathbf{Sp}_X$ of local systems of spectra on X is stable.

Example 4.4.4. If A is a stable ∞-category, then so is A^{op}.

Exercise 4.4.5. Show that if A is a small stable ∞-category, then so is $\mathrm{Ind}(E)$.

Kernels and cokernels

Let A be an ∞-category with a zero object 0, and let $f\colon X \to Y$ be a morphism thereof. We can form the *kernel*[1] or *fibre* or *cocone* $i\colon K \to X$ of f, which is the pullback

$$\begin{array}{ccc} K & \longrightarrow & 0 \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

and the *cokernel* or *cofibre* or *cone* $p\colon Y \to C$ of f, which is the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ 0 & \longrightarrow & C \end{array}$$

In a stable ∞-category, pullback squares and pushout squares coincide, so f is both the cokernel of i and the kernel of p. We can keep pushing and pulling with the aid of the loopspace and the suspension:

Construction 4.4.6. If A has all finite colimits, then we have the endofunctors

$$X \mapsto \Sigma X = 0 \cup^X 0 \quad \text{and} \quad X \mapsto \Omega X = 0 \times_X 0.$$

Note that the functors Σ and Ω each admit an involution -1 given by swapping the zero objects.

These functors are adjoint, but if A is stable, they are also inverse to each other;

[1] We have opted to keep the terms "kernel" and "cokernel" in circulation — even though this is uncommon lingo in stable ∞-category literature — because we think the parallel to abelian categories is highlighted clearly this way.

that is, the unit $id \to \Omega\Sigma$ and the counit $\Sigma\Omega \to id$ are each equivalences. In that case, we also may write

$$X[1] = \Sigma X \quad \text{and} \quad X[-1] = \Omega X,$$

and we call these the ***shift*** functors. In particular, for any object X, there is an endomorphism $-1\colon X \to X$ that arises from thinking of X as $X[1][-1]$ or $X[-1][1]$.

4.4.7. When A is stable, the kernel of the cokernel of our morphism f is f again, and the cokernel of the kernel of f is f again. The kernel of the kernel of f is the morphism $-\Omega p\colon Y[-1] \to C[-1] \simeq F$, and the cokernel of the cokernel of f is the morphism $-\Sigma i\colon C \simeq F[1] \to X[1]$. If we continue to form kernels and cokernels, we obtain a diagram

$$\begin{array}{ccccccc}
X[-1] & \xrightarrow{-\Omega f} & Y[-1] & \longrightarrow & 0 & & \\
\downarrow & & \downarrow{\scriptstyle -\Omega p} & & \downarrow & & \\
0 & \longrightarrow & F & \xrightarrow{i} & X & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \\
 & & 0 & \longrightarrow & Y & \xrightarrow[p]{} & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow{\scriptstyle -\Sigma i} & & \downarrow \\
 & & & & 0 & \longrightarrow & X[1] & \xrightarrow[-\Sigma f]{} & Y[1]
\end{array} \tag{4.4.1}$$

in which every square is both a pushout and a pullback. In such a diagram, a shift of a morphism changes sign precisely when it turns from horizontal to vertical or vice versa.

4.4.8. If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of a stable ∞-category A, and if η is a ***nullhomotopy*** of gf, that is, a homotopy between gf and the ***zero morphism*** $0\colon X \to Z$, which is the composite of the unique morphisms $X \to 0$ and $0 \to Z$, then we can ask whether η exhibits g is the cokernel of f. If it does, then there is a further morphism $h\colon Z \to X[1]$, and one calls the sequence $X \to Y \to Z \to X[1]$ a ***distinguished triangle*** or a ***fibre/cofibre sequence.*** The "triangle" here is the diagram

$$\begin{array}{ccccc}
 & & Y & & \\
 & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\
X & & \xleftarrow[h]{[1]} & & Z
\end{array}$$

where the arrow marked $[1]$ isn't a morphism as shown but rather the morphism $h\colon Z \to X[1]$. The value of drawing it this way is that it can be rotated:

$$\begin{array}{ccccc}
 & & Z & & \\
 & \overset{g}{\nearrow} & & \overset{h}{\searrow} & \\
Y & & \xleftarrow[-f]{[1]} & & X[1]
\end{array}
\qquad \text{or} \qquad
\begin{array}{ccccc}
 & & X & & \\
 & \overset{-h[-1]}{\nearrow} & & \overset{f}{\searrow} & \\
Z[-1] & & \xleftarrow[g]{[1]} & & Y
\end{array}$$

The homotopy category hA, with shift functor $X \mapsto X[1]$ and distinguished triangles as above, is in fact a ***triangulated category***. The proof of this claim is Theorem 1.1.2.14 in [168]. However, working with a stable ∞-category is always preferable to — and usually easier than — working with a triangulated category.

Universal property of Sp

The ∞-category **Sp** admits a universal property as an object of the ∞-category $\mathbf{Pr}^L_{st}$ of presentable stable ∞-categories and colimit-preserving functors. Precisely, **Sp** is the free presentable stable ∞-category on one generator; that is, for any presentable stable ∞-category E, evaluation at S^0 defines an equivalence $\mathrm{Fun}^L(\mathbf{Sp}, E) \simeq E$, where Fun^L is the category of colimit-preserving functors.

Though we won't go into detail about symmetric monoidal structures on ∞-categories, it is useful to note that $\mathbf{Pr}^L_{st}$ has such a structure: for any pair of presentable stable ∞-categories C and D, there exists a presentable stable ∞-category $C \otimes D$ such that $\mathrm{Fun}^L(C \otimes D, A)$ is equivalent to the ∞-category of functors $C \times D \to A$ that preserve colimits separately in each variable. In this symmetric monoidal structure, the unit is **Sp**.

Since the ∞-category **Sp** is the unit for the symmetric monoidal structure $\mathbf{Pr}^L_{st}$ that we discussed above, it follows that **Sp** admits a unique symmetric monoidal structure $\mathbf{Sp} \times \mathbf{Sp} \to \mathbf{Sp}$ that preserves colimits separately in each variable. This gives a pleasant universal characterisation of the smash product.

One consequence of the universal property of the ∞-category of spectra is the following omnibus comparison result to the models of spectra considered in Chapter 3.

Theorem 4.4.9. *The underlying ∞-categories of the categories of orthogonal spectra, symmetric spectra, classical prespectra, and EKMM spectra with the stable equivalences are all equivalent to the ∞-category of spectra.*

Additivity

One point that we will address carefully is the presence of *direct sums* and the *additivity* of a stable ∞-category.

Definition 4.4.10. If A is an ∞-category, we say that A *admits direct sums* if the following conditions obtain.

1. The ∞-category A admits finite products and finite coproducts.
2. The natural morphism from the initial object to the terminal object is an equivalence, so that there is a zero object in A.
3. For any objects $X, Y \in A$, the map

$$I = \begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix} : X \sqcup Y \to X \times Y$$

is an equivalence.

In this case, we write $X \oplus Y$ for the identified product and coproduct.

If A admits direct sums, then the homotopy category hA acquires an enrichment in the category of commutative monoids: for any morphisms $f, g\colon X \to Y$, one defines

$$f + g = (id \;\; id)\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} id \\ id \end{pmatrix}\colon X \to X \oplus X \to Y \oplus Y \to Y$$

One says moreover that A is *additive* if $\mathrm{Map}_{hA}(X, Y)$ is an abelian group.

The homotopy category hA of a stable ∞-category A is automatically enriched in abelian groups, thanks to the natural equivalence $\mathrm{Map}_A(S, T) \simeq \Omega^2 \mathrm{Map}_A(S[-2], T)$. But in fact even more is true:

Proposition 4.4.11. *Any stable ∞-category A is additive.*

Proof. A contains a zero object, and it admits finite products and finite coproducts. To see that these coincide, we claim that $id \times 0\colon X \to X \times Y$ and $0 \times id\colon Y \to X \times Y$ together exhibit $X \times Y$ as the coproduct $X \sqcup Y$. For any object Z, the induced map

$$\mathrm{Map}_A(X \times Y, Z) \to \mathrm{Map}_A(X, Z) \times \mathrm{Map}_A(Y, Z)$$

admits a homotopy inverse given by the formula $(f, g) \mapsto (f \times 0) + (0 \times g)$ (using the enrichment of hA in abelian groups). Finally, the Eckmann–Hilton argument shows that the commutative monoid enrichment of hA arising from the presence of direct sums coincides with the abelian group enrichment of hA arising from the stability of A. □

Notation 4.4.12. If A is a stable ∞-category and if $X, Y \in A$, we obtain abelian groups

$$\mathrm{Ext}^n_A(X, Y) = \mathrm{Mor}_{hA}(X[-n], Y) \simeq \mathrm{Mor}_{hA}(X, Y[n]).$$

When $n \leq 0$, we have

$$\mathrm{Ext}^n_A(X, Y) \cong \pi_{-n} \mathrm{Map}_A(X, Y).$$

These abelian groups are the homotopy groups of *mapping spectra* associated to objects $X, Y \in A$. In fact, in a precise sense the category of stable ∞-categories is equivalent to the category of spectral categories (where equivalences are the "Morita equivalences" of spectral categories, defined in terms of equivalences on associated module categories). See for example [46, 4.23] for a discussion of this.

Loopspace and suspension

The argument of 4.1.3 works in general here, and it implies that, in order to verify stability, it is enough to check that Σ and Ω are inverse:

Theorem 4.4.13. *Let A be an ∞-category with a zero object, all finite limits, and all finite colimits. If the functors Σ and Ω on A are inverse, then A is stable.*

Example 4.4.14. If A is stable, then a *stable subcategory* is a full subcategory that is stable under equivalences, contains the zero object, and is stable under finite limits and colimits.

Exercise 4.4.15. Let C be an ∞-category with a zero object and all finite limits. Show that the limit $\mathbf{Sp}^{seq}(C)$ in $\mathbf{Cat}_\infty$ of the sequence

$$\cdots \xrightarrow{\Omega} C \xrightarrow{\Omega} C \xrightarrow{\Omega} C$$

is stable. (Hint: the tricky point is to confirm that $\mathbf{Sp}^{seq}(C)$ admits all finite colimits.)

The “cartesian section” point of view on $\mathbf{Sp}^{seq}(C)$ is that its objects are sequences $\{X_n\}_{n\geq 0}$ of objects of C along with sequences of equivalences $\{X_n \to \Omega X_{n+1}\}_{n\geq 0}$. This goes some way to explaining the notation. The equivalence with “true” spectra will be addressed in the next section.

4.5 Generalisations

One may ask what happens when one has only part of the axioms of a stable ∞-category. These ∞-categories often appear as subcategories of stable ∞-categories, but they also arise directly from applications.

Prestable ∞-categories

For example, the subcategory of *connective spectra* — those whose homotopy is confined to nonnegative degrees — is only closed under suspension, but not loopspace. More generally, we have the following subcategories of spectra:

Example 4.5.1. For any integer k, write $\mathbf{Sp}_{\geq k} \subset \mathbf{Sp}$ for the full subcategory spanned by the *k-connective spectra*, i.e., those spectra E such that $E_n = 0$ for $n < k$. Dually, write $\mathbf{Sp}_{\leq k} \subset \mathbf{Sp}$ for the full subcategory spanned by the *k-coconnective spectra*, i.e., those spectra E such that $E_n = 0$ for $n > k$.

The objects of the ∞-category $\mathbf{Sp}_{\geq k}$ are called the *k-connective spectra*, and the objects of $\mathbf{Sp}_{\leq k}$ are called the *k-truncated spectra*.

We will study systems of subcategories like this in detail in Section 4.7. Here, we are more interested in the intrinsic properties of the ∞-category $\mathbf{Sp}_{\geq k}$. Right away, we notice that the suspension of a k-connective spectrum remains k-connective, but the loopspace of a k-connective spectrum is in general no longer k-connective. Consequently, we are interested in situations in which we have “half” of our stability conditions:

Definition 4.5.2. An ∞-category A is said to be ***prestable*** if and only if the following conditions obtain.

1. The ∞-category A admits a zero object.
2. The ∞-category A has all finite colimits.

3. Every morphism $f: X \to \Sigma Y$ of A admits a kernel $i: F \to X$, and the square

$$\begin{array}{ccc} F & \xrightarrow{i} & X \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & \Sigma Y \end{array}$$

exhibits f as the cokernel of i.

4.5.3. By the same argument as 4.1.3, we see that an ∞-category that contains a zero object and all finite limits and colimits is prestable if and only if the suspension Σ is fully faithful.

4.5.4. As in Proposition 4.4.11, a prestable ∞-category A is automatically additive. To see this, note that we have the abelian group enrichment, thanks to the equivalence $\mathrm{Map}_A(S,T) \simeq \Omega^2 \mathrm{Map}_A(S,T[2])$. The rest of the argument is as in 4.4.11.

Example 4.5.5. For any $k \in \mathbb{Z}$, the ∞-category $\mathbf{Sp}_{\geq k}$ of k-connective spectra is prestable.

Derived ∞-categories

The triangulated derived category $D(R)$ of the category of R-modules has some disadvantages:

- The formation of cones is not functorial; they are generally not unique, but rather they are unique up to a noncanonical isomorphism in the derived category. This is because diagrams in $D(R)$ commute up to homotopy, but the data of such a homotopy is not part of the data of such a diagram.
- In a similar vein, there is not a good theory of sheaves valued in $D(R)$. For instance, if $\{U, V, W\}$ is an open cover of a topological space X, and if F is a sheaf on X valued in $D(R)$, then the sheaf condition ensures that global sections can be recovered from local sections that agree up to homotopy on double overlaps, but this is true even without any compatibility for these homotopies on the triple overlap.

Consequently, it is often more convenient to work with the derived ∞-category of R. Here is the construction:

Construction 4.5.6. Let E be an abelian category, which we shall regard as an ∞-category. Assume that E has enough projective objects. Write $E_{proj} \subseteq E$ for the full subcategory spanned by the projective objects.

We will construct the *nonnegative derived ∞-category of E*. It's actually convenient to start by defining the nonnegative derived ∞-category of $\mathrm{Ind}(E)$.

We write $D_{\geq 0}(\mathrm{Ind}(E)) \subseteq \mathrm{Fun}(E_{proj}^{op}, \mathcal{S})$ for the full subcategory spanned by those functors $E_{proj}^{op} \to \mathcal{S}_*$ that carry finite direct sums to products.

One has the Yoneda embedding $j: E_{proj} \hookrightarrow D_{\geq 0}(\mathrm{Ind}(E))$, which can be thought of *either* as freely generating $D_{\geq 0}(\mathrm{Ind}(E))$ under sifted colimits (that is, filtered colimits and geometric realisations) *or* as generating $D_{\geq 0}(\mathrm{Ind}(E))$ under *all* colimits, subject to the condition that j preserve finite coproducts. That is:

1. For any ∞-category C that admits all sifted colimits, the functor

$$j^*\colon \mathrm{Fun}(D_{\geq 0}(\mathrm{Ind}(E)), C) \to \mathrm{Fun}(E_{proj}, C)$$

restricts to an equivalence between the full subcategory of $\mathrm{Fun}(D_{\geq 0}(\mathrm{Ind}(E)), C)$ spanned by those functors $D_{\geq 0}(\mathrm{Ind}(E)) \to C$ that preserve sifted colimits and $\mathrm{Fun}(E_{proj}, C)$.
2. If E admits ***all*** colimits, then j^* restricts to an equivalence between the full subcategory of $\mathrm{Fun}(D_{\geq 0}(\mathrm{Ind}(E)), C)$ spanned by those functors $D_{\geq 0}(\mathrm{Ind}(E)) \to C$ that preserve all colimits and the full subcategory of $\mathrm{Fun}(E_{proj}, C)$ spanned by those functors $A \to E$ that preserve finite coproducts.

The ∞-category $D_{\geq 0}(\mathrm{Ind}(E))$ is called the ***nonnegative derived ∞-category*** of $\mathrm{Ind}(E)$.

We write $D_{\geq 0}(E) \subseteq D_{\geq 0}(\mathrm{Ind}(E))$ for the smallest full subcategory that contains E_{proj} and is closed under geometric realisations. Thus $D_{\geq 0}(E)$ is obtained from A by freely adding geometric realisations; that is, for any ∞-category E that admits all geometric realisations, the functor

$$j^*\colon \mathrm{Fun}(D_{\geq 0}(E), E) \to \mathrm{Fun}(E_{proj}, E)$$

restricts to an equivalence between the full subcategory of $\mathrm{Fun}(D_{\geq 0}(\mathrm{Ind}(E)), E)$ spanned by those functors $D_{\geq 0}(E) \to E$ that preserve geometric realisations and $\mathrm{Fun}(E_{proj}, E)$.

The ∞-category $D_{\geq 0}(E)$ is called the ***nonnegative derived ∞-category*** of E. A functor $F\colon D_{\geq 0}(E) \to E$ that preserves geometric realisations will be said to be the ***left derived functor*** of j^*F.

4.5.7. There is no ambiguity in our notation. If E is an abelian category with enough projectives and $E' = \mathrm{Ind}(E)$, then E' also has enough projectives, and our definition of $D_{\geq 0}(E')$ agrees with our definition of $D_{\geq 0}(\mathrm{Ind}(E))$: each freely adds sifted colimits to E_{proj}.

Example 4.5.8. For any abelian category E with enough projectives, the ∞-categories $D_{\geq 0}(\mathrm{Ind}(E))$ and $D_{\geq 0}(E)$ are prestable. Indeed, the second universal property makes it clear that $D_{\geq 0}(E)$ admits direct sums. To prove that the suspension on $D_{\geq 0}(\mathrm{Ind}(E))$ is fully faithful, let $C\colon E_{proj}^{op} \to \mathcal{S}_*$ be an object; then since products and geometric realisations are computed objectwise in $D_{\geq 0}(\mathrm{Ind}(E))$, we may write ΣC as the functor that carries an object $X \in A$ to the geometric realisation of the bar construction

$$B_*(0, C(X), 0)\colon n \mapsto C(X)^n.$$

This simplicial space is a grouplike Segal space, and so we have an equivalence $C(X) \simeq \Omega|B_*(0, C(X), 0)| \simeq \Omega\Sigma C(X)$.

Construction 4.5.9. Let E be an abelian category with enough projectives, and let $C\colon E_{proj}^{op} \to \mathcal{S}_*$ be an object of $D_{\geq 0}(E)$. Then we obtain, for any integer $n \geq 0$, a functor

$$H_n(C) = \pi_n C\colon E_{proj}^{op} \to \mathbf{Set}_*$$

that carries direct sums to products. In other words, $H_n(C) \in \mathrm{Ind}(E)$.

Let us quickly check that $H_n(C)$ actually lies in E. For any object $M \in E$, one has $H_0(j(M)) = M$, and for $n \geq 1$, $H_n(j(M)) = 0$. Furthermore, if C_* is a simplicial object of $D_{\geq 0}(E)$ with the property that $H_n(C_k) \in E$ for every $k, n \geq 0$, then the obvious spectral sequence argument ensures that for every $n \geq 0$, one has $H_n|C_*| \in A$.

We have thus defined the ***homology functors*** $H_n \colon D_{\geq 0}(E) \to E$.

If E is an abelian category with enough projective objects, the homotopy category $hD_{\geq 0}(E)$ can be shown to be the derived category of nonnegatively graded complexes in E_{proj}. Under this equivalence, the homology functors above agree with the classically defined functors, and the left derived functor of a functor $E \to E'$ in our sense coincides with the left derived functor in the classical sense.

Definition 4.5.10. For any abelian category E with enough projectives, we write $\mathrm{Fun}^{\oplus}(E_{proj}^{op}, \mathbf{Sp})$ for the (stable) ∞-category of functors $E_{proj}^{op} \to \mathbf{Sp}$ that preserve direct sums. We then define $D^-(E)$ as the smallest stable full subcategory of $\mathrm{Fun}^{\oplus}(E_{proj}^{op}, \mathbf{Sp})$ that contains the essential image of $\Sigma^\infty j \colon E_{proj} \to \mathrm{Fun}^{\oplus}(E_{proj}^{op}, \mathbf{Sp})$ and is closed under geometric realisations. This is the ***right bounded derived*** ∞***-category*** of E.

Exercise 4.5.11. Verify that the functor $\Sigma^\infty \colon D_{\geq 0}(E) \to D^-(E)$ is fully faithful.

4.5.12. Let E be an abelian category with enough projectives. If $C \colon E_{proj}^{op} \to \mathbf{Sp}$ is an object of $D_{\geq 0}(E)$, then as in 4.5.9 we obtain, for any integer $n \in \mathbb{Z}$, a functor

$$H_n(C) = \pi_n C \colon E_{proj}^{op} \to \mathbf{Set}_*$$

that carries direct sums to products, so that $H_n(C) \in \mathrm{Ind}(E)$, and once again it turns out that $H_n(C)$ lies in E itself.

We have thus defined the ***homology functors*** $H_n \colon D^-(E) \to A$.

Construction 4.5.13. If E is an abelian category with enough ***injective*** objects, then we can define

$$D_{\leq 0}(E) = D_{\geq 0}(E^{op})^{op} \qquad \text{and} \qquad D^+(E) = D^-(E^{op})^{op};$$

we call $D^+(E)$ the ***left bounded derived*** ∞***-category*** of E. We can also define the ***cohomology functors*** $H^{-n} = H_n$.

4.5.14. An even more dramatic generalisation of the stable ∞-categories is the notion of an ***exact*** ∞***-category.*** These were introduced in [24] as a simultaneous generalisation of the exact categories of Quillen and stable ∞-categories. Exact ∞-categories are a natural setting for algebraic K-theory and Quillen's Q construction.

4.6 Stabilisation

In this section, we give a machine for printing examples of stable ∞-categories. This machine is really nothing more than a formal extension of our definition of spectra.

Definition 4.6.1. Let C and D be ∞-categories. Assume that C admits all finite colimits. We say that a functor $F\colon C \to D$ is *reduced* if it carries the initial object of C to a terminal object of D, and we say that F is *excisive* if it carries any pushout square in C to a pullback square in D.

We write $\mathrm{Fun}_*(C,D) \subseteq \mathrm{Fun}(C,D)$ for the full subcategory spanned by the reduced functors; $\mathrm{Exc}(C,D) \subseteq \mathrm{Fun}(C,D)$ for the full subcategory spanned by the excisive functors; and $\mathrm{Exc}_*(C,D) \subseteq \mathrm{Fun}(C,D)$ for the full subcategory spanned by the reduced excisive functors.

4.6.2. Let C, D, and F be as above. If C is stable, then F is reduced excisive if and only if F is left exact. If D is stable, then F is reduced excisive if and only if F is right exact.

If D admits all finite limits, then the argument of 4.1.3 applies again to ensure that F is excisive if and only if, for any object $X \in C$, the natural map $FX \to \Omega F \Sigma X$ is an equivalence.

Exercise 4.6.3. Check that, if C is an ∞-category C with all finite colimits and D is an ∞-category D with all finite limits, the ∞-category $\mathrm{Exc}_*(C,D)$ is stable.

Definition 4.6.4. For any ∞-category D with all finite limits, a ***spectrum in*** D is a reduced excisive functor $\mathcal{S}_*^{fin} \to D$. We write $\mathbf{Sp}(D) = \mathrm{Exc}_*(\mathcal{S}_*^{fin}, D)$, and we call this ∞-category the ***stabilisation*** of D.

Evaluation at S^0 defines a functor $\Omega^\infty\colon \mathbf{Sp}(D) \to D$.

Example 4.6.5. Of course $\mathbf{Sp} \simeq \mathbf{Sp}(\mathcal{S})$.

We haven't got much of an excuse for the notation Ω^∞ at the moment, but we will explain it soon.

Universal property of stabilisation

Exercise 4.6.6. For any ∞-category D with all finite limits, show that $\Omega^\infty\colon \mathbf{Sp}(D) \to D$ is an equivalence if and only if D is stable.

4.6.7. Let C be an ∞-category with all finite colimits, and let D be an ∞-category with all finite limits. Then a reduced excisive functor $C \to \mathbf{Sp}(D)$ is the same thing as a functor $C \times \mathcal{S}_*^{fin} \to D$ that is reduced and excisive separately in each variable. This, in turn, is the same thing as a spectrum in the ∞-category $\mathrm{Exc}_*(C,D)$.

Proposition 4.6.8. *Let C be an ∞-category with all finite colimits, and let D be an ∞-category with all finite limits. The functor $\Omega^\infty\colon \mathbf{Sp}(D) \to D$ induces an equivalence*

$$\mathrm{Exc}_*(C, \mathbf{Sp}(D)) \simeq \mathrm{Exc}_*(C,D).$$

Proof. The induced functor $\mathbf{Sp}(\mathrm{Exc}_*(C,D)) \simeq \mathrm{Exc}_*(C,\mathbf{Sp}(D)) \to \mathrm{Exc}_*(C,D)$ is Ω^∞, which is an equivalence since $\mathrm{Exc}_*(C,D)$ is stable. □

This result reveals a universal property of the stabilisation: if we look at the subcategory $\mathbf{Cat}_\infty^{lex}$ whose objects are ∞-categories with all finite limits and whose morphisms are left exact functors, then the ∞-category $\mathbf{Stab}_\infty$ is a full subcategory of $\mathbf{Cat}_\infty^{lex}$. Now the previous proposition reveals that the stabilisation is in fact the right adjoint to the inclusion $\mathbf{Stab}_\infty \hookrightarrow \mathbf{Cat}_\infty^{lex}$. In other words, $\mathbf{Stab}_\infty$ is a colocalisation of $\mathbf{Cat}_\infty^{lex}$.

Spectra and sequential spectra

Here is another perspective, which explains the notation Ω^∞ and refers to the construction of $\mathbf{Sp}^{seq}$ of 4.4.15:

Proposition 4.6.9. *Let D be an ∞-category with a zero object and all finite limits. Then the functor*

$$\mathbf{Sp}^{seq}(D) \to D$$

given informally by $\{X_n\}_{n\geq 0} \mapsto X_0$ exhibits $\mathbf{Sp}^{seq}(D)$ as the stabilisation of D:

$$\mathbf{Sp}(D) \simeq \mathbf{Sp}^{seq}(D).$$

Proof. One knows from (4.4.15) that $\mathbf{Sp}^{seq}(D)$ is stable. Therefore it suffices to prove that for any stable ∞-category A, the induced functor

$$\mathrm{Exc}_*(A, \mathbf{Sp}^{seq}(D)) \to \mathrm{Exc}_*(A, D)$$

is an equivalence. But this functor is the limit of the sequence

$$\cdots \xrightarrow{\Omega} \mathrm{Exc}_*(A, D) \xrightarrow{\Omega} \mathrm{Exc}_*(A, D) \xrightarrow{\Omega} \mathrm{Exc}_*(A, D),$$

which is a diagram of equivalences over a weakly contractible ∞-category. □

Complete derived ∞-categories

We can apply the stabilisation process to the nonnegative derived ∞-category:

Definition 4.6.10. Let E be an abelian category with enough projective objects. We write $D^{-,\wedge}(E)$ for the stabilisation $\mathbf{Sp}(D_{\geq 0}(E))$. This is the *right complete derived ∞-category* of E.

Dually, if E is an abelian category with enough injective objects, we write $D^{+,\wedge}(E)$ for the stabilisation $\mathbf{Sp}(D_{\geq 0}(E^{op}))^{op}$. This is the *left complete derived ∞-category* of E.

In the next section, we will be able to characterise these ∞-categories in an intrinsic manner.

4.7 t-structures

The most basic examples of triangulated categories possess additional structure given by shift and truncation functors. For example, for the derived category of a commutative ring, there are inverse auto-equivalences given by shifting complexes up and down, and it is often useful to study about truncated complexes that live entirely in positive or negative degrees. This structure is axiomatized in terms of additional data referred to as a t-structure, which specifies positive and negative subcategories whose intersection is an abelian category known as the heart of the t-structure. There is a natural generalisation of this theory to the setting of stable ∞-categories.

Definition 4.7.1. Let A be a stable ∞-category, and let $A_{\geq 0}, A_{\leq 0} \subseteq A$ be a pair of full subcategories. We may shift these subcategories about:

$$A_{\geq n} = (A_{\geq 0})[n] \quad \text{and} \quad A_{\leq n} = (A_{\leq 0})[n].$$

We say that the pair $(A_{\geq 0}, A_{\leq 0})$ constitute a *t-structure* on A if it enjoys the following properties.

1. If $X \in A_{\geq 0}$ and $Y \in A_{\leq -1}$, then the space $\mathrm{Map}_A(X, Y)$ is contractible.
2. The subcategory $A_{\geq 0}$ is closed under positive shifts, and the subcategory $A_{\leq 0}$ is closed under negative shifts. So $A_{\geq 1} \subseteq A_{\geq 0}$, and, dually, $A_{\leq -1} \subseteq A_{\leq 0}$ as well.
3. For every $X \in A$, there is a distinguished triangle

$$\tau_{\geq 0}X \to X \to \tau_{\leq -1}X \to (\tau_{\geq 0}X)[1],$$

where $\tau_{\geq 0}X \in A_{\geq 0}$ and $(\tau_{\leq -1}X)[1] \in A_{\leq 0}$.

4.7.2. For any object X, a distinguished triangle $\tau_{\geq 0}X \to X \to \tau_{\leq -1}X \to (\tau_{\geq 0}X)[1]$ exhibits $\tau_{\leq -1}X \in A_{\leq -1}$ as a $(A_{\leq -1})$-localisation of X. Consequently, $\tau_{\leq -1}$ organises itself into a left adjoint to the inclusion $A_{\leq -1} \hookrightarrow A$. One may shift to find that for any $n \in \mathbb{Z}$, the functor $\tau_{\leq n}$ defined by

$$\tau_{\leq n}X = (\tau_{\leq -1}(X[-n]))[n]$$

exhibits the subcategory $A_{\leq n} \subseteq A$ as a localisation.

Dually, the distinguished triangle $\tau_{\geq 0}X \to X \to \tau_{\leq -1}X \to (\tau_{\geq 0}X)[1]$ exhibits $\tau_{\geq 0}X \in A_{\geq 0}$ as a $(A_{\geq 0})$-colocalisation of X, and for any $n \in \mathbb{Z}$, the functor $\tau_{\geq n}$ defined by

$$\tau_{\geq n}X = (\tau_{\geq 0}(X[n]))[-n]$$

exhibits the subcategory $A_{\geq n} \subseteq A$ as a colocalisation.

Example 4.7.3. We have already encountered the t-structure on the ∞-category **Sp** of spectra. The spectra that lie in $\mathbf{Sp}_{\geq 0}$ are called *connective*.

Example 4.7.4. Let E be an abelian category with enough projectives. We have also encountered the t-structure on right bounded derived ∞-category. Then $D_{\leq 0}(E)$, regarded as a full subcategory of $D^{-}(E)$, is a t-structure.

This notion is compatible with the classical notion; if A is a stable ∞-category with a t-structure in the sense above, then the homotopy category of A is a triangulated category with a t-structure.

Warning 4.7.5. Here we are following the homotopy theory convention of homological indexing. This is mostly for the sake of compatibility with Lurie's text.

However, one may expect to encounter cohomological indexing in the literature. These *should* be written with superscripts rather than subscripts:

$$A^{\leq n} = A_{\geq -n}, \qquad A^{\geq n} = A_{\leq -n}, \qquad \tau^{\leq n} = \tau_{\geq -n}, \qquad \tau^{\geq n} = \tau_{\leq -n}.$$

Unfortunately, even when cohomological indexing is being employed, the truncation functors are sometimes written with subscripts (notably, in [34]), so one must remain vigilant.

We emphasise that the meaning of the shift functor $X \mapsto X[1]$ is *always* suspension. So one has the formulas

$$A^{\leq n} = (A^{\leq 0})[-n] \qquad \text{and} \qquad A^{\geq n} = (A^{\geq 0})[-n].$$

4.7.6. A t-structure on a stable ∞-category A is uniquely specified by giving, for some $n \in \mathbb{Z}$, any one of the following pieces of data:

1. the full subcategory $A_{\geq n} \subseteq A$;
2. the full subcategory $A_{\leq n} \subseteq A$;
3. the functor $\tau_{\geq n} \colon A \to A$; or
4. the functor $\tau_{\leq n} \colon A \to A$.

4.7.7. For any $n \in \mathbb{Z}$, the ∞-category $A_{\geq n}$ is an exact ∞-category in which every morphism is ingressive, and $A_{\leq n}$ is an exact ∞-category in which every morphism is egressive.

4.7.8. For integers $a \leq b$, one may define $A_{[a,b]} = A_{\geq a} \cap A_{\leq b}$. The restriction of $\tau_{\leq b}$ to $A_{\geq a}$ is a left adjoint $A_{\geq a} \to A_{[a,b]}$, and the restriction of $\tau_{\geq a}$ to $A_{\leq b}$ is a left adjoint $A_{\leq b} \to A_{[a,b]}$. A simple "five lemma" argument furnishes us with a natural equivalence

$$\tau_{\leq b}\tau_{\geq a} \simeq \tau_{\geq a}\tau_{\leq b} \colon A \to A_{[a,b]},$$

and we shall write $\tau_{[a,b]}$ for this functor.

The ∞-category $A_{[a,b]}$ is an exact ∞-category in which the ingressive morphisms are those morphisms that are ingressive in $A_{\geq a}$, and the egressive morphisms are those morphisms that are egressive in $A_{\leq b}$.

As a special case, we write $A^{\heartsuit} = A_{[0,0]}$; this is called the ***heart*** of the t-structure. Note that the shift functor restricts to a *specified* equivalence $A^{\heartsuit} \simeq A_{[n,n]}$ for any $n \in \mathbb{Z}$; we now define the ***homological functors*** attached to the t-structure:

$$\pi_n = \tau_{[n,n]} \colon A \to A_{[n,n]} \simeq A^{\heartsuit}.$$

We have chosen this notation again for the sake of compatibility with Lurie. Other authors may write H_n for this functor, and those who use cohomological indexing are liable to write $H^n = \tau^{[n,n]}$.

Proposition 4.7.9. *Let A be a stable ∞-category endowed with a t-structure. Then the heart $A^{\heartsuit}$ is (equivalent to the ∞-category corresponding to) an ordinary abelian category.*

Proof. If $X, Y \in A^{\heartsuit}$, then for any $n \geq 1$, one has

$$\pi_n \operatorname{Map}_A(X, Y) \cong \operatorname{Ext}_A^{-n}(X, Y) \cong 0,$$

whence $A^{\heartsuit}$ is (equivalent to) a 1-category. We have already seen that $A^{\heartsuit}$ is an exact ∞-category, whence it is an exact category in Quillen's sense. To show that it is abelian, one just has to note that the ingressives are precisely the monomorphisms, and the egressives are precisely the epimorphisms. □

Example 4.7.10. The heart $\mathbf{Sp}^{\heartsuit}$ is the category **Ab** of abelian groups. The homological functors attached to this t-structure are precisely the usual stable homotopy group functors π_n.

Example 4.7.11. Let E be an abelian category with enough projectives. Then the heart $D^-(E)^{\heartsuit}$ is again E. The homological functors attached to this t-structure are precisely the homology functors H_n.

Boundedness and completeness

The previous examples show that stable ∞-categories with t-structures are not determined by their hearts. There is, however, a special class of stable ∞-categories with t-structures that ***are*** determined by their hearts. These are the ***derived*** ∞-categories of abelian categories. To describe them, we must discuss some different kinds of t-structures.

Definition 4.7.12. Let A be a stable ∞-category equipped with a t-structure. Define

$$A^- = \bigcup_{m \in \mathbb{Z}} A_{\geq m}, \qquad A^+ = \bigcup_{n \in \mathbb{Z}} A_{\leq n}, \qquad A^b = A^+ \cap A^- = \bigcup_{m,n \in \mathbb{Z}} A_{[m,n]}.$$

We call

1. the objects of A^- ***bounded below***,
2. the objects of A^+ ***bounded above***, and
3. the objects of A^b ***bounded***.

We say that the t-structure is

4. ***right bounded*** if $A = A^-$,
5. ***left bounded*** if $A = A^+$, and
6. ***bounded*** if $A = A^b$.

Example 4.7.13. The t-structure on $\mathbf{Sp}^{fin}$ is right bounded.

Definition 4.7.14. Let A be a stable ∞-category equipped with a t-structure. We define $A^{\wedge,R} \subseteq \operatorname{Fun}(\mathbb{Z}^{op}, A)$ as the full subcategory spanned by those sequences X such that $X(m) \in A_{\geq m}$ for any $m \in \mathbb{Z}$, and that the induced morphism $X(n) \to \tau_{\geq n} X(m)$ is an equivalence for any $m \leq n$. Dually, we define $A^{\wedge,L} \subseteq \operatorname{Fun}(\mathbb{Z}^{op}, A)$ as the full subcategory spanned by those objects X such that $X(m) \in A_{\leq m}$ for any $m \in \mathbb{Z}$, and that the induced morphism $\tau_{\leq m} X(n) \to X(m)$ is an equivalence for any $m \leq n$.

We call

1. $A^{\wedge,R}$ the *right completion* of A with respect to its t-structure, and
2. $A^{\wedge,L}$ the *left completion* of A with respect to its t-structure.

We say that the t-structure is

3. *right complete* if the natural map $A \to A^{\wedge,R}$ is an equivalence,
4. *left complete* if the natural map $A \to A^{\wedge,L}$ is an equivalence, and
5. *complete* if it is both left and right complete.

4.7.15. If A is a stable ∞-category equipped with a t-structure, the right completion of A^- coincides with the right completion of A itself, and the bounded below objects of the right completion $A^{\wedge,R}$ coincide with the bounded below objects of A itself. It follows that there is an equivalence between the ∞-category of right bounded t-structures and that of right complete t-structures.

Example 4.7.16. Let E be an abelian category. If E has enough projectives, then the t-structure on $D^-(E)$ is right bounded (whence the notation!) and left complete. Dually, if E has enough injectives, then the t-structure on $D^+(E)$ is left bounded and right complete.

In the same vein, if E has enough projectives, then $D^{-,\wedge}(E)$ is complete, and if E has enough injectives, then $D^{+,\wedge}(E)$ is complete. In fact, $D^{-,\wedge}(E)$ is the right completion of $D^-(E)$, and $D^{+,\wedge}(E)$ is the left completion of $D^+(E)$.

It is *a priori* difficult to determine whether a t-structure is right or left complete. Fortunately, there is a reasonable criterion for this.

Definition 4.7.17. Let A be a stable ∞-category equipped with a t-structure. We define

$$A_{-\infty} = \bigcap_{n\in\mathbb{Z}} A_{\leq n} \qquad \text{and} \qquad A_{+\infty} = \bigcap_{n\in\mathbb{Z}} A_{\geq n}.$$

We say that the t-structure is

1. *right separated* if $A_{-\infty} = 0$,
2. *left separated* if $A_{+\infty} = 0$, and
3. *separated* if it is both left and right separated.

Proposition 4.7.18 ([168, Proposition 1.2.1.19]). *Let A be a stable ∞-category with countable coproducts. Let τ be a t-structure on A with the property that $A_{\geq 0}$ is stable under countable coproducts. Then τ is right complete if and only if it is right separated.*

Exercise 4.7.19. Use this criterion to check that the t-structure on **Sp** is complete.

Derived ∞-categories

Roughly speaking, the constructions $E \mapsto D^{-}(E)$ and $E \mapsto D^{+}(E)$ are left adjoint to the construction $A \mapsto A^{\heartsuit}$. To make this precise, we must specify which ∞-category of stable ∞-categories we will to use:

Definition 4.7.20. Let A and B be stable ∞-categories equipped with t-structures. An exact functor $f\colon A \to B$ is said to be ***right t-exact*** if $f(A_{\geq 0}) \subseteq B_{\geq 0}$. Dually, an exact functor $f\colon A \to B$ is said to be ***left t-exact*** if $f(A_{\leq 0}) \subseteq B_{\leq 0}$. An exact functor $f\colon A \to B$ is said to be ***t-exact*** if it is both left and right t-exact.

Let us say that a right t-exact functor $A \to B$ is ***left derived*** if it carries the projective objects of $A^{\heartsuit}$ into $B^{\heartsuit}$. We write $\mathrm{Fun}^{lder}(A,B) \subseteq \mathrm{Fun}(A,B)$ for the full subcategory spanned by the left derived right t-exact functors $A \to B$. Dually, let us say that a left t-exact functor $A \to B$ is ***right derived*** if it carries the injective objects of $A^{\heartsuit}$ into $B^{\heartsuit}$. We write $\mathrm{Fun}^{rder}(A,B) \subseteq \mathrm{Fun}(A,B)$ for the full subcategory spanned by the right derived right t-exact functors $A \to B$.

Theorem 4.7.21. *Let E be an abelian category with enough projectives, and let B be a stable ∞-category equipped with a left complete t-structure. The construction $F \mapsto \tau_{\leq 0}F|_E$ is an equivalence of ∞-categories*

$$\mathrm{Fun}^{lder}(D^{-}(E),B) \to \mathrm{Fun}^{rex}(E,B^{\heartsuit}),$$

where $\mathrm{Fun}^{rex}(E,B^{\heartsuit}) \subseteq \mathrm{Fun}(E,B^{\heartsuit})$ is the full subcategory spanned by the right exact functors $E \to B^{\heartsuit}$.

Dually, let E be an abelian category with enough injectives, and let B be a stable ∞-category equipped with a right complete t-structure. The construction $G \mapsto \tau_{\geq 0}G|_E$ is an equivalence of ∞-categories

$$\mathrm{Fun}^{rder}(D^{+}(E),B) \to \mathrm{Fun}^{lex}(E,B^{\heartsuit}),$$

where $\mathrm{Fun}^{lex}(E,B^{\heartsuit}) \subseteq \mathrm{Fun}(E,B^{\heartsuit})$ is the full subcategory spanned by the left exact functors $E \to B^{\heartsuit}$.

4.7.22. If E has enough projectives and B is a stable ∞-category equipped with a left complete t-structure, then we call $F\colon D^{-}(E) \to B$ the *left derived functor* of $f = \tau_{\leq 0}F|_A$, and we write $\mathbb{L}f = F$.

Dually, if E has enough injectives and B is a stable ∞-category equipped with a right complete t-structure, then we call $G\colon D^{+}(E) \to B$ the *right derived functor* of $g = \tau_{\geq 0}G|_E$, and we write $\mathbb{R}g = G$.

Example 4.7.23. Since $\mathbf{Sp}^{\heartsuit} \simeq \mathbf{Ab}$, we obtain a t-exact functor $H\colon D^{-}(\mathbf{Ab}) \to \mathbf{Sp}$, which carries a chain complex C to the *generalised Eilenberg–Mac Lane spectrum* HC.

This result also allows us to recognise derived ∞-categories.

Corollary 4.7.24. *Let A be a stable ∞-category equipped with a left complete t-structure. Assume that $A^{\heartsuit}$ has enough projectives. The unique t-exact functor $K\colon D^{-}(A^{\heartsuit}) \to A$ is fully faithful if and only if, for any projective object $M \in A^{\heartsuit}$ and any object $N \in A^{\heartsuit}$, the groups $\mathrm{Ext}^n(M,N) = 0$ for any $n \geq 1$. In this case, the essential image of K is $A^{-} \subseteq A$.*

4.7.25. As a final comment, we observe that if E is a Grothendieck abelian category, then there is also an ***unbounded derived*** ∞***-category*** $D(E)$ equipped with a t-structure. Since E has enough injectives, we obtain a stable ∞-category $D^+(E)$. The previous corollary ensures that the unique t-exact functor $D^+(E) \to D(E)$ is fully faithful, and it identifies $D^+(E)$ with the bounded above objects of $D(E)$.

One might be therefore tempted to believe that $D(E)$ coincides with the left complete derived ∞-category $D^{+,\wedge}(E)$. We emphasise, however, that this is not generally true: there are abelian categories E, such as the category of representations of $\mathbb{G}_a$ over a field of positive characteristic, for which $D(E)$ is not left complete.

If, however, countable products in E are exact, the criterion of 4.7.18 works to ensure that $D(E)$ is left complete. Then we can identify $D(E)$ with $D^{+,\wedge}(E)$, and so 4.7.21 furnishes us with a universal characterisation of $D(E)$ in this case. The author does not know a universal characterisation of $D(E)$ for a general Grothendieck abelian category.