3 Stable categories and spectra via model categories

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3.1 Introduction

The first popular model category of spectra was due to Bousfield–Friedlander [56], and for many years it was the only one in common use (a previous model due to K. Brown [61] never really caught on). But this category does not admit a suitable smash product on the model category level. Following an early but limited attempt by Robinson [248], in the late 1990s several new model categories of spectra appeared that fixed this problem. These days a working topologist should know a little about each of these models, and about their various advantages and disadvantages.

Here is a list of the main players:

(1) Bousfield–Friedlander spectra
(2) Symmetric spectra
(3) Orthogonal spectra
(4) EKMM spectra
(5) Γ-spaces (which only model connective spectra)
(6) $\mathcal{W}$-spaces (generalizing "functors with smash product")

While it would be nice to pick out one model and say *this* is the one everyone should learn, life is not that simple. An algebraic topologist is likely to encounter each of the above models at some point, and some will have advantages over others depending on the context. For example, at this point there is a developing consensus that orthogonal spectra work best for equivariant homotopy theory; but some constructions — like Waldhausen K-theory — naturally produce a symmetric spectrum, not an orthogonal one. Functors with smash product (FSPs) have largely disappeared from the stage, being eclipsed by (2) and (3), but they are still worth a passing familiarity. In this survey we concentrate on (1)–(4), with (5) and (6) only making a quick appearance at the end.

To describe the organization of this survey, it is helpful to use an analogy from daily life: the automobile. For most of us, an automobile is a box with wheels that has certain behaviors when we turn the steering wheel or step on the pedals. That very primitive level of understanding is sufficient for most day-to-day functioning, and it is rare that any of us have to actually look under the hood. To some extent, the same holds true of spectra. Much of daily life can be covered just by knowing that there exists a model category of spectra with a smash product satisfying a small list of basic properties. This kind of superficial knowledge is fine for driving around town, but unlike the automobile analogy my experience has been that nearly every trip on the homotopy-theory highway requires one or two stops to mess around with the engine. It bothers me that this is so, and I usually find myself cursing at the injustice when I have to do it, but this seems to be the nature of the subject.

To continue beating our analogy to death, when one *is* messing around under the hood there is simply no substitute for the technical manuals. For spectra these are [94], [133], [178], [267], and [132]. The present survey cannot replace them. Instead, we concentrate on two aims. The first is to give a kind of "driver's manual" to the world of stable model categories, monoidal model categories, and general properties that are satisfied by all the commonly used model categories of spectra. This takes roughly the first half of the chapter. The second goal is to give enough of a technical introduction to the different categories that readers can confidently go open up the manuals and feel that they have a fighting chance.

Before moving on let's state the definitions of the basic objects:

1. A **classical spectrum** is a collection of pointed spaces X_n for $n \geq 0$ together with structure maps $\sigma_n \colon S^1 \wedge X_n \to X_{n+1}$. The notion of a spectrum originated with Lima [158], but the first model structure was developed by Bousfield–Friedlander. The phrase "Bousfield–Friedlander spectra" sometimes gets used for these objects, even though the definition of the objects themselves came much earlier. They are also sometimes called "prespectra", mainly in the work of Peter May and his collaborators. A **suspension spectrum** is a spectrum where the structure maps are all identity maps, and an Ω-spectrum (read "omega spectrum") is one where the adjoints $X_n \to \Omega X_{n+1}$ of the structure maps σ_n are weak equivalences.
2. A **symmetric spectrum** is a classical spectrum where each X_n comes equipped with an action of the symmetric group Σ_n, and where each of the iterated structure maps
$$\sigma^p \colon (S^1)^{\wedge(p)} \wedge X_q \to X_{p+q}$$
is $\Sigma_p \times \Sigma_q$-equivariant. Here σ^p is actually a composite of associativity maps with p different applications of σ, the $\Sigma_p \times \Sigma_q$-action on the domain is the evident one, and the action on the target comes from the embedding of groups $\Sigma_p \times \Sigma_q \hookrightarrow \Sigma_{p+q}$ where the image consists of permutations that permute the first p elements and last q elements without mixing the two blocks.
3. An **orthogonal spectrum** is an assignment that sends each finite-dimensional real inner product space V to a pointed space X_V equipped with an action of the orthogonal group $O(V)$, together with structure maps $\sigma_{V,W} \colon S^V \wedge X_W \to X_{V \oplus W}$ that are $O(V) \times O(W)$-equivariant (with S^V the one-point compactification of V). In addition, to any isometry $V \to W$ is assigned (continuously) a homeomorphism $X_V \to X_W$, and these must be compatible with all the previous structure. Finally,

the structure maps must satisfy some evident unital and associativity conditions. (If we drop the orthogonal group actions then the assignment $V \mapsto X_V$ together with the structure maps is often called a **coordinate-free spectrum**).

4. The definition of an **EKMM spectrum** cannot be given in a few lines, but the following words at least give a rough idea. An EKMM spectrum is a coordinate-free Ω-spectrum where the adjoints of the structure maps are all homeomorphisms, together with an action of a certain linear isometries monad on this spectrum, and satisfying an extra "S-unital" condition.
5. For each $n \geq 0$ write $\mathbf{n}^+ = \{0, 1, \ldots, n\}$ for the pointed set with 0 as basepoint. Let $\mathcal{F}$ be the category whose objects are all the $\mathbf{n}^+$ and whose morphisms are the based maps. A **$\boldsymbol{\Gamma}$-space** is simply a functor $\mathcal{F} \to \mathcal{T}op_*$.
6. Let $\mathcal{W}$ be the category of pointed spaces homeomorphic to finite CW-complexes. Regard this as a category enriched over topological spaces. A **$\boldsymbol{\mathcal{W}}$-space** is just an enriched functor $\Phi \colon \mathcal{W} \to \mathcal{T}op_*$. Note that for every X and Y there is a natural map $X \to \mathcal{T}op_*(Y, X \wedge Y)$ (adjoint to the identity); composing with the map $\mathcal{T}op_*(Y, X \wedge Y) \to \mathcal{T}op_*(\Phi(Y), \Phi(X \wedge Y))$ and taking the adjoint therefore gives a family of natural structure maps

$$X \wedge \Phi(Y) \to \Phi(X \wedge Y).$$

These maps are broad generalizations of the structure maps for classical spectra — for example, we could get a classical spectrum by setting $\Phi_n = \Phi(S^n)$ and letting $X = S^1$, or more generally by fixing Y and setting $\Phi_n^Y = \Phi(S^n \wedge Y)$. The notion of $\mathcal{W}$-space is roughly equivalent to that of "simplicial functor", and the objects classically called "functors with smash product" are the monoids in this category.

Remark 3.1.1. What we here call "EKMM spectra" were called "S-modules" when first introduced, and are often still called that. Unfortunately, both symmetric spectra and orthogonal spectra are also S-modules, just in different contexts. So the phrase "S-module" is now very ambiguous, whereas "EKMM spectrum" cannot be confused with anything else.

From a historical perspective, the objects in (1) and (5) date to the 1960s and 1970s and vastly predate all of the others in the list. The objects in (2), (3), (4), and (6) all appeared in the 1990s, and their importance is that they admit a symmetric monoidal smash product on the model category level (sometimes colloquially referred to as the "point-set level"), rather than just on the associated homotopy category — see Section 3.1.3 below for more discussion of this. (The objects in (6) actually first appeared in the 1970s, but didn't enter the limelight until the 1990s with the other models).

Having such a point-set level smash product quickly led to a flurry of advances, and nowadays this is a standard part of any algebraic topologist's toolkit. But because there are four models and not just one, learning to use the toolkit also means learning what the different models do best, and how to navigate between them. The different models come with their own advantages and disadvantages, or pros and cons. These terms don't feel quite right, though, because the pros and cons are so closely linked.

If something good only happens because of something bad, is the “bad” thing really all that bad? Rather than delve into this philosophical quagmire, we take the elementary-school approach in the table below (focusing only on the three most commonly used models):

	Things that make us happy	Things that make us sad
EKMM spectra	All objects are fibrant. Weak equivalences are easy. Plays well with the linear isometries operad.	The unit is not cofibrant. Definition of the category is quite hard, with several layers of machinery.
Symmetric spectra	Easy definition of the objects. The unit is cofibrant.	Weak equivalences are hard to understand. Need fibrant replacement, and this can destroy other structure. One can make a theory of genuine G-spectra, but it feels a bit unnatural.
Orthogonal spectra	Works well for G-spectra. Unit is cofibrant. Weak equivalences are easy. Objects are not as easy as symmetric spectra, but not hard.	Need fibrant replacement.

By “weak equivalences are easy” we mean that they coincide with the π_*-isomorphisms on the underlying classical spectrum. The issue of whether every object is fibrant has a surprisingly large simplifying effect on how one ends up handling certain monoidal phenomena — we discuss this more in Section 3.3.2.

For the rest of this introduction I am going to do something a bit unusual. Mathematical narratives tend to have two sides: one consists of the definitions and theorems, and the other is the *story* behind those definitions and theorems (sometimes called *motivation*). The latter might try to answer why a certain definition is the right one, or why a certain theorem should be expected. It is an odd phenomenon that these two sides of mathematical narration sometimes end up getting in the way of each other.

To help try to combat this, for the rest of this introduction I am going to give a series of mathematical vignettes that attempt to highlight various important issues or ideas behind the “story” of spectra. These come in no particular order, and are also by no means exhaustive. The hope is that a reader can get some basic picture from the vignettes right away, even if they don’t make complete sense on first reading. Be assured that we will return to each of these ideas in more formal ways later in the text.

3.1.1 Why use model categories?

Let me begin by painting a picture. Somewhere up in the heavens is a wondrous paradise where lives the homotopy theory of spectra. You are welcome to think of this realm as an infinity-category if you like, but I will intentionally keep things more vague. Regardless, it is a magical shangri-la where the theories of associative and commutative ring spectra, their modules, equivariant analogs, and so forth all work out easily and naturally. The gods who walk that land are happy and content, and can do many fine things.

Most of us mortals cannot inhabit this kingdom directly, and so instead we gain limited access by choosing a *model*. As with all attempts at creating paradise down on earth, this doesn't entirely succeed. These models are not canonical, different models come with different pros and cons, and no one model seems to be completely satisfactory for everything. But such is the price we pay for our mortality. Dan Kan used to compare choosing a model to choosing coordinates on a manifold, and Jeff Smith once remarked that model categories give a way of bringing infinity-categorical phenomena down into the realm of 1-categories. These are good ways of thinking about the situation.

As one reaches for more and more sophisticated structures, any fixed model seems to inevitably run its course. Early models of spectra adequately capture the homotopy category but fail to admit a point-set-level smash product. Other models capture the smash product but fail to give an adequate theory of commutative ring spectra, or of equivariant spectra. Recent work [221] suggests that none of the existing models can handle coalgebra spectra correctly. The homotopy theorists' version of Murphy's Law is that after choosing any particular model for spectra, a topologist will eventually want to do something where the model seems to get in the way and make things harder than they should be.

This picture so far gives a somewhat skewed view, because the heavenly paradise is not always one's main goal. Down here on earth we have concrete objects like manifolds, chain complexes, and differential graded algebras, and often at the end of the day we are trying to prove theorems about these concrete things. The more one ascends into the heavens, the more blurred these objects become in their very existence. It is not always clear what infinity-categorical theorems are actually saying about our concrete objects, and this is another place where model categories turn out to be helpful. In addition to giving us a view into heavenly realms, model categories are also a tool for taking theorems from those realms and applying them down here on earth.

3.1.2 Where do models come from?

There is no one answer to this question, but the following schema covers very many cases. Recall that for any two objects X and Y in a "homotopy theory" there is a homotopy mapping space $\mathrm{hMap}(X, Y)$, well-defined up to weak homotopy equivalence. If X and X' are related in some homotopy-theoretic sense, then there will be some corresponding relation between $\mathrm{hMap}(X,Y)$ and $\mathrm{hMap}(X',Y)$. The simplest example is that if there is a map $X \to X'$ then there should be an associated $\mathrm{hMap}(X',Y) \to \mathrm{hMap}(X,Y)$.

If $\mathcal{C}$ is a collection of "test objects" in our homotopy theory, we can attempt to understand an object Y by remembering the collection of all function spaces $\mathrm{hMap}(U,Y)$ for $U \in \mathcal{C}$. That is, we understand Y by remembering how all of our test objects map into it. That's the basic idea. If there are some relations between our test objects, we should remember the corresponding relations between our mapping spaces. In this way we are attempting to model our homotopy theory as certain functions $\mathcal{C} \to \mathcal{T}op$. Often $\mathcal{C}$ will be a category, and so we actually look at functors $\mathcal{C}^{op} \to \mathcal{T}op$.

For example, the homotopy theory of spectra should have objects S^{-n} for $n \geq 0$, together with equivalences $\Sigma(S^{-n}) \simeq S^{-(n-1)}$. If we take these as our test objects, then a spectrum Y will be modeled by the collection of spaces $Y_n = \mathrm{hMap}(S^{-n},Y)$ together with the relations $\Omega Y_n \simeq Y_{n-1}$. In this way we arrive at the classical definition of an Ω-spectrum.

Instead of starting with the objects S^{-n} we could just start with S^{-1} together with the spectra $I_n = (S^{-1})^{\wedge(n)}$. The symmetric group Σ_n acts on I_n, and so there will be an induced action on the function complexes $\mathrm{Map}(I_n,Y)$. This perspective leads directly to the notion of a symmetric spectrum.

Likewise, the fact that the orthogonal group $O(n)$ acts on S^n might lead one to believe that it should also act on S^{-n}, in which case there would be an induced action of $O(n)$ on $Y_n = \mathrm{Map}(S^{-n},Y)$. Thus one is led to the notion of an orthogonal spectrum.

3.1.3 The smash product

Let's go back to the most basic model of a spectrum: a collection of pointed spaces X_n for $n \geq 0$ with structure maps $\sigma_n\colon S^1 \wedge X_n \to X_{n+1}$. Given spectra X and Y, how could we make a spectrum that deserves to be called $X \wedge Y$? In level 0 there is only one thing that makes sense, which is $X_0 \wedge Y_0$. We will need a structure map $\Sigma(X_0 \wedge Y_0) \to (X \wedge Y)_1$, and there are two obvious choices: we could use σ_X to get into $X_1 \wedge Y_0$, or we could use σ_Y to get into $X_0 \wedge Y_1$. There is no reason for choosing one over the other, so let's randomly choose $(X \wedge Y)_1 = X_0 \wedge Y_1$. Similar reasoning leads to choices for $(X \wedge Y)_n$ for each n, and it's not hard to believe that we will be fine as long as we don't keep making the same choice over and over again: that is, we should make sure to use each of σ_X and σ_Y infinitely many times. These considerations do indeed produce a spectrum $X \wedge Y$, but because of all the choices it is far from canonical. In fact we have an uncountable collection of models for $X \wedge Y$. In the old days these were called handcrafted smash products. One can prove that they all are homotopy equivalent, thereby giving a well-defined smash product on the homotopy category, but clearly this is not a very good state of affairs. Still, this at least shows immediately that there is some kind of smash product around.

Rather than constructing $X \wedge Y$ by making these arbitrary choices, another approach is to build all the choices into the spectrum from the beginning. All the modern incarnations of the smash product involve some form of this, but let us start by exploring the most naive. We still take $(X \wedge Y)_0 = X_0 \wedge Y_0$, but now for $(X \wedge Y)_1$ we might first make the guess $(X_0 \wedge Y_1) \vee (X_1 \wedge Y_0)$. The suspension operators σ_X and σ_Y then take us into opposite wedge summands, which is no good, so we fix this by identifying them in an appropriate way:

$$(X \wedge Y)_1 = \text{pushout of } [\, X_0 \wedge Y_1 \longleftarrow S^1 \wedge (X_0 \wedge Y_0) \longrightarrow X_1 \wedge Y_0 \,],$$

where the maps are the evident ones coming from σ_Y and σ_X. Note that the left-pointing map must involve the twist map, used to commute the S^1 and the X_0. We leave the reader to derive the definition for $(X \wedge Y)_n$ for $n \geq 2$, along the same lines.

This definition does not give us what we want, but it is informative to see why. The first problem one encounters is that the sphere spectrum S is not a unit (recall that S is the suspension spectrum of S^0). To see this, let us compute $S \wedge S$. One readily checks that $(S \wedge S)_0 = S^0$ and $(S \wedge S)_1 = S^1$, but $(S \wedge S)_2$ is the colimit of the diagram

$$\begin{array}{ccccccc}
(S^0 \wedge S^2) & & & (S^1 \wedge S^1) & & & (S^2 \wedge S^0) \\
& \nwarrow^{\gamma} & \nearrow^{\gamma} & & \nwarrow^{-\gamma} & \nearrow^{\gamma} & \\
& S^1 \wedge (S^0 \wedge S^1) & & & & S^1 \wedge (S^1 \wedge S^0) & \\
& & \nwarrow_{id\wedge\gamma} & & \nearrow_{id\wedge\gamma} & & \\
& & & S^1 \wedge S^1 \wedge (S^0 \wedge S^0) & & &
\end{array}$$

Replacing each parenthesized $(S^i \wedge S^j)$ in the diagram with $(X_i \wedge Y_j)$ gives the diagram for $(X \wedge Y)_2$ and helps one understand the various maps. Each map in the diagram uses associativity, twist, and the structure maps from S in the evident way — for example, the left map in the bottom row commutes the second S^1 past the S^0 and then uses the structure map on the rightmost two terms. Upon analyzing these maps, one finds that they are all canonical identifications (labeled γ in the diagram), except for one: this last map involves the twist map on S^1 and so ends up being $-\gamma$. Consequently, the colimit of this diagram is the coequalizer of $(id, -id)\colon S^2 \rightrightarrows S^2$, which is $\mathbb{R}P^2$. So we see that $S \wedge S \neq S$.

Exercise 3.1.2. For an arbitrary spectrum Y, convince yourself that under the above definition $(S \wedge Y)_2$ is the colimit of the following diagram:

$$\begin{array}{ccccc}
S^1 \wedge S^1 \wedge Y_0 & \xrightarrow{id\wedge\sigma} & S^1 \wedge Y_1 & \xrightarrow{\sigma} & Y_2 \\
{\scriptstyle t\wedge id}\big\downarrow & \nearrow_{id\wedge\sigma} & & & \\
S^1 \wedge S^1 \wedge Y_0 & & & &
\end{array}$$

Working through the simple example preceding Exercise 3.1.2 already suggests the key for fixing the situation. The problem is that we are not keeping track of the "twists" that occur when we apply our structure maps, so we need to build in some machinery for doing so. This is what symmetric spectra do, by building in symmetric groups. In symmetric spectra, $(X \wedge Y)_2$ is made from $X_0 \wedge Y_2$, $X_2 \wedge Y_0$, and *two* copies of $X_1 \wedge Y_1$ (indexed by the elements of the symmetric group Σ_2), and then one quotients by the same kind of relations we saw above. This fixes the problem. See Section 3.7.2 to find this worked out in detail.

Orthogonal spectra solve the problem in an even more elegant way (though secretly it is really the same way). Here spectra are indexed on the category of finite-dimensional inner product spaces, and the direct sum operation on this category already has twist maps built into it. If X is an orthogonal spectrum then $X_{V\oplus W}$ and $X_{W\oplus V}$ are different objects, though the twist $t\colon V\oplus W\to W\oplus V$ gives a homeomorphism between them. The moral here is that indexing things on inner product spaces forces one to keep track of the relevant twists in the very notation.

There is another way to see that symmetric groups should come into the picture. Let us imagine that we have a homotopy theory of spectra (off in some shangri-la) and we are attempting to model spectra X by the collection of mapping spaces $X_n = \mathrm{Map}(I^{\wedge(n)}, X)$ where I is a model for S^{-1}. We need to ask ourselves: if we have all the $\{X_n\}$ and all the $\{Y_n\}$, what is the best we can do in terms of approximating the spaces $\{(X\wedge Y)_n\}$? Clearly if $p+q=n$ we will have maps

$$\mathrm{Map}(I^{\wedge(p)}, X)\wedge \mathrm{Map}(I^{\wedge(q)}, Y)\to \mathrm{Map}(I^{\wedge(p+q)}, X\wedge Y) = \mathrm{Map}(I^{\wedge(n)}, X\wedge Y) \quad (3.1.4)$$

induced by the shangri-la smash product. However, this kind of process only gives maps $I^{\wedge(n)}\to X\wedge Y$ which send the first set of "coordinates" into X and the second set into Y. Not all maps will look this way! Indeed, the action of Σ_n on $I^{\wedge(n)}$ induces an action on $\mathrm{Map}(I^{\wedge(n)}, X\wedge Y)$ and lets us scramble the "coordinates" any way we want. This suggests, though, that if we use the maps in (3.1.4) ***together with*** a superimposed symmetric group action, then we might get a sensible approximation to $\mathrm{Map}(I^{\wedge(n)}, X\wedge Y)$. This leads us to write down the space

$$\Big[\bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p\times\Sigma_q} (X_p\wedge Y_q)\Big]\Big/\sim$$

as a model for $\mathrm{Map}(I^{\wedge(n)}, X\wedge Y)$, where the equivalence relation just comes from thinking about the evident ways that the maps (3.1.4) interact with symmetric group actions and the structure maps. We have just invented the smash product for symmetric spectra!

3.1.5 Coordinate-free spectra

The world of classical spectra provides inverses (under the smash product) for the standard spheres S^n. If V is a finite-dimensional real vector space then its one-point compactification S^V is isomorphic to $S^{\dim V}$, and so of course S^V has an inverse in this world as well. But this inverse is not canonical, because the isomorphism $V\cong \mathbb{R}^{\dim V}$ is not canonical. This might seem like a small point, but in some constructions (like Pontryagin–Thom) it is very convenient to have a canonical inverse for S^V.

A larger issue arises in the setting of G-equivariant homotopy theory. Here one has different spheres S^V for each finite-dimensional G-representation V, so to introduce inverses for these it is not enough to just work with the standard spheres S^n. Thus, for various reasons we are led to the need for a notion of "coordinate-free" spectra.

The first idea of what a coordinate-free spectrum should be is an assignment $V \mapsto X_V$ that sends every finite-dimensional vector space to a pointed space. For $V \subseteq W$ there should be structure maps $S^{??} \wedge X_V \to X_W$, but already one runs into trouble as far as what sphere to put in the domain. This sphere should be related to the complement of V in W, but there is no canonical such complement. To get around this, we assume that the vector spaces have inner products on them so that we can take orthogonal complements. If $W - V$ denotes the orthogonal complement of V in W, then our structure map should have the form $S^{W-V} \wedge X_V \to X_W$.

Finally, since the collection of ***all*** finite-dimensional inner product spaces is not a set, we prefer to set things up so that there is an intrinsic bound to where these live — an underlying "universe". To be precise, define a **May universe** to be a real inner product space of countably infinite dimension. Any universe $\mathcal{U}$ is isometric to $\mathbb{R}^\infty$ with the dot product, but not canonically. Then a **coordinate-free spectrum on** $\mathcal{U}$ is defined to be an assignment $V \mapsto X_V$ for finite-dimensional $V \subseteq \mathcal{U}$, together with maps $S^{W-V} \wedge X_V \to X_W$ for every pair $V \subseteq W \subseteq \mathcal{U}$. These must satisfy some evident unital and associativity conditions.

Example 3.1.3. The definitions of some familiar classical spectra immediately generalize to give coordinate-free spectra:

(a) The sphere spectrum is $V \mapsto S^V$.
(b) If A is an abelian group, the Eilenberg–MacLane spectrum HA is the spectrum $V \mapsto C(S^V;A)$ where for any pointed space X the space $C(X;A)$ is the Dold–Thom space of finite configurations of points on X labeled by elements of A.
(c) The real cobordism spectrum MO is $V \mapsto \mathrm{Th}(EO(V) \times_{O(V)} V \to BO(V))$, where $O(V)$ is the group of isometries of V (with its natural topology) and $\mathrm{Th}(E \to B)$ is the Thom space. This is also commonly written in the form $V \mapsto EO(V)_+ \wedge_{O(V)} S^V$.

For orthogonal spectra, it is important that we are able to form the direct sum of our inner product spaces. That is to say, if X is an orthogonal spectrum we need $X_{V \oplus W}$ to make sense when X_V and X_W do. For this reason we cannot restrict ourselves to subspaces of a universe $\mathcal{U}$ anymore. To avoid set-theoretical issues we must either fix a small skeletal subcategory of the category of finite-dimensional inner product spaces, or else fix some Grothendieck universe at the very beginning. See Remark 3.5.4 for more details.

3.1.6 Rings, modules, and algebras

Let $(\mathcal{C}, \otimes, S)$ be a symmetric monoidal category. A monoid in $\mathcal{C}$ is an object R together with a unit map $S \to R$ and a product $R \otimes R \to R$ satisfying the evident associativity and unital actions. A monoid in $(\mathcal{A}b, \otimes, \mathbb{Z})$ is just a ring, and for this reason we will sometimes call monoids in other symmetric monoidal categories "rings" as well.

If R is a ring in $\mathcal{C}$ then one likewise has notions of left and right R-modules, and if R is a commutative ring then one can talk about R-algebras. The definitions are all the obvious ones.

In the 1970s, after Boardman had constructed the symmetric monoidal structure on Ho($\mathcal{S}pectra$), one could apply the above ideas and talk about ring- and module-spectra. Nowadays these would probably be called “homotopy ring spectra”, or “naive ring spectra”, to differentiate them from more rigid notions. Suppose that R is one of these homotopy ring spectra and that $f\colon M \to N$ is a map of left R-modules. One would like for the homotopy cofiber Cf to be again a left R-module in a canonical way, but this doesn’t work out. Try it: there is a diagram in the homotopy category that looks like

$$\begin{array}{ccccc} R\wedge M & \longrightarrow & R\wedge N & \longrightarrow & R\wedge Cf \\ \downarrow & & \downarrow & & \\ M & \longrightarrow & N & \longrightarrow & Cf \end{array}$$

and both rows are homotopy cofiber sequences, so there does indeed exist an extension $\mu\colon R\wedge Cf \to Cf$ (apply $[-,Cf]$ to the top cofiber sequence and use the resulting long exact sequence). However, the homotopy class of μ is not unique and moreover one cannot prove that μ satisfies the necessary associativity condition.

So this is a deficiency. Using the naive definitions of rings and modules in Ho($\mathcal{S}pectra$) does not lead to a situation where we can do homotopy theory for R-modules. The problem is the usual one: the homotopy category itself is not robust enough for most purposes. The above problem with cofibers is coming from the fact that the homotopy category doesn’t have colimits.

This was one of the motivations for desiring a symmetric monoidal smash product on the model category level. Assuming that one has a model category $\mathcal{S}pectra$ with a smash product that commutes with colimits in either variable, it follows at once that colimits of left R-modules are again left R-modules in a canonical way. One would hope that the adjoint functors

$$R\wedge(-)\colon \mathcal{S}pectra \rightleftarrows R\text{–Mod}\colon U$$

would lift the model category structure on $\mathcal{S}pectra$ to a corresponding model structure on the category of left R-modules. Similarly, if R is a commutative ring spectrum then one might hope for a model category structure on R-algebras, and also one on commutative R-algebras.

In short, the hope would be that the model structure on $\mathcal{S}pectra$ could be passed to various categories of algebraic structures on spectra. This basically works out, but it doesn’t work out for free. One approach was developed in [94] for topological model categories where all objects are fibrant, which reduced things down to their so-called “Cofibration Hypothesis”. For more general model categories another approach was developed by Schwede–Shipley [267], who identified the need for a separate axiom they called the “Monoid Axiom”. The Monoid Axiom is one of those things that is safely left under the hood on regular days, but that one needs to be prepared to play with when the car breaks down.

We discuss the Monoid Axiom and its applications to model categories of modules and algebras in Section 3.3.2.

3.1.7 The Lewis enigma

In 1991, before the advent of the modern categories of spectra, Lewis discovered an argument showing that some of the expected properties of such categories were mutually inconsistent [156]. It is worth understanding this argument not only to see how the modern categories of spectra interface with it, but also because this same argument explains some of the complications in various theories of commutative ring spectra.

Let $\mathcal{S}$ be a category with the following properties:

(A1) There exists a symmetric monoidal functor $\wedge\colon \mathcal{S}\times\mathcal{S}\to\mathcal{S}$.
(A2) There exists an adjoint pair $\Sigma^\infty\colon \mathcal{T}op_* \rightleftarrows \mathcal{S}\colon \Omega^\infty$.
(A3) There is a natural transformation

$$\eta_{X,Y}\colon \Sigma^\infty(X\wedge Y)\to \Sigma^\infty X\wedge\Sigma^\infty Y$$

that is compatible with the associativity and commutativity isomorphisms for $(\mathcal{T}op_*,\wedge)$ and $(\mathcal{S},\wedge)$.
(A4) $\Sigma^\infty S^0$ is the unit for $\wedge$, and η is compatible with the unital isomorphism.
(A5) There is a natural weak equivalence $\Omega^\infty\Sigma^\infty X\simeq QX$, where as usual one defines $QX = \mathrm{hocolim}_n\,\Omega^n\Sigma^n X$.

Putting $X=\Omega^\infty E$ and $Y=\Omega^\infty F$ into (A3) and using the counit of the adjunction gives a natural transformation $\epsilon_{E,F}\colon \Omega^\infty E\wedge\Omega^\infty F\to\Omega^\infty(E\wedge F)$, and this will also be compatible with the associativity and commutativity isomorphisms.

Given such a category, set $S=\Sigma^\infty S^0$. The unit isomorphism $S\wedge S\to S$ makes S into a commutative ring spectrum. Then $\epsilon\colon \Omega^\infty S\wedge\Omega^\infty S\to\Omega^\infty S$ makes $\Omega^\infty S$ into a commutative monoid. So its identity component is a generalized Eilenberg–MacLane space. But this contradicts (A5), which says $\Omega^\infty S=\Omega^\infty\Sigma^\infty S^0\simeq QS^0$. So the conclusion is that (A1)–(A5) are mutually incompatible.

Symmetric and orthogonal spectra satisfy (A1)–(A4), but get around the problem via the failure of (A5). Here $\Sigma^\infty S^0$ is not fibrant and so $\Omega^\infty\Sigma^\infty S^0$ has the "wrong" homotopy type; said differently, (A5) must be modified to say that $\Omega^\infty\mathcal{F}\Sigma^\infty X\simeq QX$, where $\mathcal{F}$ is a fibrant replacement functor.

The EKMM setup gets around this problem by having two sets of adjoint functors, called here $(\Sigma^\infty_S,\Omega^\infty_S)$ and $(\Sigma^\infty,\Omega^\infty)$ (see Section 3.9 for more details). There is a natural transformation $\Sigma^\infty_S\to\Sigma^\infty$ that is a weak equivalence on cofibrant pointed spaces, and there is its adjoint $\Omega^\infty\to\Omega^\infty_S$. The pair $(\Sigma^\infty_S,\Omega^\infty_S)$ is the one with homotopical meaning (it turns out to be a Quillen pair, with the right model category structures), whereas $(\Sigma^\infty,\Omega^\infty)$ is the one with the good monoidal properties. So Σ^∞ satisfies (A3) and (A4), but $\Omega^\infty\Sigma^\infty$ does not satisfy (A5); whereas $\Omega^\infty_S\Sigma^\infty_S$ satisfies (A5), but Σ^∞_S does not satisfy (A3) and (A4).

Returning to the simpler setting of symmetric spectra, replacing (A5) with its derived version is not the end of the story. Even with this modified (A5), Lewis's argument shows that if R is a fibrant spectrum with a commutative and associative product then $\Omega^\infty R$ (which is already appropriately derived) must be a generalized

Eilenberg–MacLane space. This is obviously a matter of concern, since we would like spectra such as S, K, MO, and MU to have models which are commutative ring spectra on the nose. That is not prohibited, but such models cannot *also* be fibrant in the usual model structure for symmetric (or orthogonal) spectra. The standard way for dealing with this is to use a different model structure called the **positive model category structure**. We will discuss this briefly in Section 3.10.5.

3.1.8 Organization of the chapter

We assume a basic familiarity with model categories, as provided by sources like [91], [124], [130], and [229]. See also Chapter 2 of this volume. Specifically, we assume the reader is familiar with the model category axioms, cylinder and path objects, the homotopy category, Quillen functors, derived functors, the small object argument, simplicial model categories, and the notion of cofibrant-generation.

We occasionally assume the reader has a passing acquaintance with the classical aspects of spectra and their connection to (co)homology theories, as represented for example in any of [1], [2, Part III], and [288].

We also assume the reader has a basic knowledge of closed symmetric monoidal categories; MacLane's book [174] is a good source. Finally, we use enriched categories to a certain extent. Not much more is needed than the basic definition and the notion of enriched functor, which are essentially obvious; but consult [148] for any needed background here.

With homotopy-theoretic machinery, there is the usual issue of whether to take as foundation simplicial sets or topological spaces. For the most part we have tried to present results in a way that applies to either situation, but this is not always convenient. To avoid having to constantly work in two situations at once, we choose topological spaces as our main framework. The reader who prefers to work simplicially should be able to make the necessary modifications to the exposition with little trouble.

3.1.9 Notation and terminology

When $\mathcal{C}$ is a category we write $\mathcal{C}(X,Y)$ for $\mathrm{Hom}_{\mathcal{C}}(X,Y)$. If $\mathcal{C}$ is a category enriched over some symmetric monoidal category $\mathcal{V}$, we write $\underline{\mathcal{C}}(X,Y)$ for the corresponding $\mathcal{V}$-mapping object. We write $\mathcal{T}op_*$ for the category of pointed topological spaces. We fix $S^1 = I/\partial I$ and define $S^n = S^1 \wedge (S^1 \wedge (S^1 \wedge (\cdots \wedge S^1)))$.

3.1.10 Acknowledgments

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3.2 Stable model categories

A model category is called stable when the suspension functor is a self-equivalence on the homotopy category. The homotopy categories of stable model categories enjoy several nice properties: they are additive, triangulated, and the notions of homotopy cofiber and fiber sequences are the same. These simply stated facts take a nontrivial amount of effort to set up and prove carefully. Most of Chapters 6 and 7 of [130] are devoted to this. We aim to give a quick tour for those who are new to this machinery, partly because the depth of the results in [130] make them a bit of a maze. We hope the treatment here can serve as a guide through that material.

A category $\mathcal{M}$ is called **pointed** if it has an initial object, a terminal object, and the two are isomorphic. Quillen [229, Chapter I.2] showed that if $\mathcal{M}$ is a pointed model category then the homotopy category $\mathrm{Ho}(\mathcal{M})$ comes equipped with a special pair of adjoint functors

$$\Sigma\colon \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{M})\colon \Omega,$$

called **suspension** and **loop** functors. If X is a cofibrant object, factor $X \to *$ as $X \rightarrowtail CX \overset{\sim}{\twoheadrightarrow} *$. Then ΣX can be defined to be the pushout of $* \leftarrow X \to CX$. Likewise, if Z is a fibrant object then factor $* \to Z$ as $* \overset{\sim}{\rightarrowtail} PZ \twoheadrightarrow Z$ and define ΩZ as the pullback of $* \to Z \leftarrow PZ$. It is easy to show that these homotopy types do not depend on the choice of CX or PZ, and moreover that these definitions extend to give the desired functors. (Note that "C" and "P" stand for "cone" and "path object").

Let X be cofibrant and consider the diagram

$$\begin{array}{ccccc} CX & \leftarrowtail & X & \rightarrowtail & CX \\ {\scriptstyle\simeq}\downarrow & & \| & & \| \\ * & \longleftarrow & X & \rightarrowtail & CX \end{array}$$

Taking pushouts gives a map $CX \amalg_X CX \to \Sigma X$, and the model category axioms force this to be a weak equivalence (see [232, Corollary to Theorem B]). But collapsing X gives $CX \amalg_X CX \to \Sigma X \vee \Sigma X$, and so we have constructed a map $\Sigma X \to \Sigma X \vee \Sigma X$ in $\mathrm{Ho}(\mathcal{M})$. A little work shows that this makes ΣX into a cogroup object in $\mathrm{Ho}(\mathcal{M})$, and that $\Sigma^2 X$ is a cocommutative cogroup object. Similarly, when Y is fibrant, ΩY is a group object in $\mathrm{Ho}(\mathcal{M})$ and $\Omega^2 Y$ is a commutative group object. It follows that $[\Sigma^2 X, Z]$ and $[A, \Omega^2 Y]$ have natural structures of abelian groups, where from now on we will write $[-,-]$ for maps in $\mathrm{Ho}(\mathcal{M})$.

Definition 3.2.1. A pointed model category $\mathcal{M}$ is called **stable** if the suspension functor $\Sigma\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$ is an equivalence of categories.

The (Σ, Ω) adjunction shows that it is equivalent to require that Ω be an equivalence. Moreover, when $\mathcal{M}$ is stable the functors Σ and Ω will be inverses. The following is an easy exercise:

Proposition 3.2.2. *Let $\mathcal{M}$ be a pointed model category. The following conditions are equivalent:*

(a) $\mathcal{M}$ *is stable.*

(b) *For all objects* X *and* Y *the maps* $\Sigma\Omega X \to X$ *and* $Y \to \Omega\Sigma Y$ *are isomorphisms in* $\mathrm{Ho}(\mathcal{M})$.

If $\mathcal{M}$ is a stable model category then every object in $\mathrm{Ho}(\mathcal{M})$ is a double suspension (and a double loop space), and so the hom sets are all abelian groups and composition is additive in both variables. The homotopy category inherits coproducts and products from $\mathcal{M}$, so $\mathrm{Ho}(\mathcal{M})$ is additive. In particular, it follows formally that the canonical map $i\colon A \vee B \to A \times B$ is an isomorphism in $\mathrm{Ho}(\mathcal{M})$. We recall the brief proof: If $j_A\colon A \to A \vee B$ and $\pi_A\colon A \times B \to A$ are the canonical inclusions and projections, then $j_A\pi_A + j_B\pi_B$ is a two-sided inverse to i.

When $\mathcal{M}$ is a pointed model category Quillen also showed that $\mathrm{Ho}(\mathcal{M})$ comes equipped with special "triangles" called homotopy fiber and cofiber sequences. An **Ω-triangle** is a diagram $\Omega C \to A \to B \to C$ in $\mathrm{Ho}(\mathcal{M})$ such that the composition of any two maps is zero, and a **Σ-triangle** is a diagram $A \to B \to C \to \Sigma A$ with the same property. A map of Ω-triangles is a commutative diagram

$$\begin{array}{ccccccc} \Omega C & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\ {\scriptstyle \Omega h}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ \Omega C' & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

and an isomorphism of Ω-triangles is a map where all the vertical maps are isomorphisms. We use similar notions for maps and isomorphisms of Σ-triangles.

Exercise 3.2.3. Check that changing the signs of two maps in an Ω-triangle (or Σ-triangle) produces an isomorphic triangle.

If $p\colon X \twoheadrightarrow Y$ is a fibration between fibrant objects, there exists a lifting in the square

$$\begin{array}{ccc} * & \longrightarrow & X \\ {\scriptstyle \simeq}\downarrow & \overset{\lambda}{\nearrow} & \downarrow \\ PY & \longrightarrow & Y \end{array}$$

and therefore an induced map $\Omega Y \to F$, where F is the fiber of $X \twoheadrightarrow Y$. We leave it as an exercise to check that a different choice for λ gives the same map $\Omega Y \to F$ in $\mathrm{Ho}(\mathcal{M})$. The Ω-triangle $\Omega Y \to F \to X \to Y$ is called the **homotopy fiber sequence** corresponding to p. More generally:

Definition 3.2.4. An Ω-triangle is called a **homotopy fiber sequence** if it is isomorphic to the homotopy fiber sequence corresponding to some fibration between fibrant objects $p\colon X \to Y$.

Remark 3.2.5. It is a common abuse of terminology to say things like "$F \to X \to Y$ is a homotopy fiber sequence", leaving the map $\Omega Y \to F$ implicit.

We leave the reader to write down the dual notion of a homotopy cofiber sequence, which yields a special class of Σ-triangles.

Remark 3.2.6. In addition to the map $\Omega F \to Y$ we constructed above, one can show that there is a map $\gamma\colon \Omega F \times Y \to Y$ giving an action of ΩF on Y in $\mathrm{Ho}(\mathcal{M})$. Our map $\Omega F \to Y$ is the restriction of γ along $\Omega F \times * \to \Omega F \times Y$. The notion of "homotopy fiber sequence" should really include this map γ as part of the data. But when $\mathcal{M}$ is stable $\Omega F \vee Y \to \Omega F \times Y$ is an equivalence, and the restriction of γ to the Y summand is just the identity. So in this case there is no more information in γ than in our map $\Omega F \to Y$. We refer to [130, Chapter 6.3] or [229, Chapter I.3] for careful studies of homotopy fiber and cofiber sequences in the unstable setting.

From now on assume that $\mathcal{M}$ is stable. The first result about homotopy cofiber and fiber sequences is the following:

Proposition 3.2.7. *Let $\mathcal{M}$ be a stable model category and let T be any object.*

(a) *For any homotopy fiber sequence $\Omega Y \to F \to X \to Y$, the induced sequence of abelian groups*

$$[T, \Omega Y] \to [T, F] \to [T, X] \to [T, Y]$$

is exact at the two middle spots.

(b) *For any homotopy cofiber sequence $A \to B \to C \to \Sigma A$, the induced sequence of abelian groups*

$$[\Sigma A, T] \to [C, T] \to [B, T] \to [A, T]$$

is exact at the two middle spots.

If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a homotopy cofiber sequence, we get associated maps

$$\Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \cong X \quad \text{and} \quad Y \xrightarrow{g} Z \cong \Sigma\Omega Z,$$

where the two isomorphisms are the unit and counit of the $\Sigma - \Omega$ adjunction. One might expect the evident sequence $\Omega Z \to X \to Y \to \Sigma\Omega Z$ made from these maps to be a homotopy cofiber sequence, but this is not correct — there is a sign issue. To get a homotopy cofiber sequence one must negate one of the maps.

The following proposition gives several results of this form. Rather than give names to all the maps, we adopt the convention that a minus sign by itself means "take the negative of the evident map one would get by using Σ, Ω, and the adjunctions".

Proposition 3.2.8. *Let $\mathcal{M}$ be a stable model category.*

(a) *Given a diagram in* $\mathrm{Ho}(\mathcal{M})$ *of the form*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow & & \downarrow & & & & \downarrow{\scriptstyle -} \\ \Omega Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

in which the top row is a homotopy cofiber sequence and the bottom row is a homotopy fiber sequence, there is a map $C \to Y$ making the diagram commute.

(b) *Given a diagram in* $\mathrm{Ho}(\mathcal{M})$ *of the form*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow & & & & \downarrow & & \downarrow{\scriptstyle -} \\ \Omega Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

in which the top row is a homotopy cofiber sequence and the bottom row is a homotopy fiber sequence, there is a map $B \to X$ *making the diagram commute.*

(c) *Given a diagram in* $\mathrm{Ho}(\mathcal{M})$ *of the form*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow & & \downarrow & & & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma A' \end{array}$$

in which both rows are homotopy cofiber sequences, there is a map $C \to C'$ *making the diagram commute. The dual statement for homotopy fiber sequences holds as well.*

(d) *If any of the following* Σ*-triangles are homotopy cofiber sequences, then so are the others:*

(i) $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$, (ii) $Y \longrightarrow Z \longrightarrow \Sigma X \xrightarrow{-} \Sigma Y$

(iii) $\Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma Z \xrightarrow{-} \Sigma^2 X$, (iv) $\Omega Z \xrightarrow{-} X \longrightarrow Y \longrightarrow \Sigma\Omega Z$.

(e) *If any of the following* Ω*-triangles are homotopy fiber sequences, then so are the others:*

(i) $\Omega Z \longrightarrow X \to Y \longrightarrow Z$, (ii) $\Omega Y \xrightarrow{-} \Omega Z \longrightarrow X \longrightarrow Y$,

(iii) $\Omega^2 Z \xrightarrow{-} \Omega X \longrightarrow \Omega Y \longrightarrow \Omega Z$, (iv) $\Omega\Sigma X \xrightarrow{-} Y \longrightarrow Z \longrightarrow \Sigma X$.

Reading this extensive list of results is a bit tedious, but having it around is very useful. It captures several of the main points from [130, Chapter 6]. A good (but challenging) exercise is to try to prove all of these facts from first principles yourself. If you get stuck, parts (a) and (b) are the content of [130, Proposition 6.3.7], and (c) is [130, Proposition 6.3.5]. The equivalence of (i) and (ii) in parts (d,e) is covered in [130, Proposition 6.3.4], and the equivalence with (iii) comes from repeatedly applying (i) $\Longleftrightarrow$ (ii) and using Exercise 3.2.3. Finally, the equivalence with (iv) is an easy exercise using the other parts.

Remark 3.2.9. Although it is necessary to get the signs right in cofiber or fiber sequences, in practice one almost always passes at some point to a long exact sequence of homotopy classes. In these long exact sequences, one can indiscriminately alter the signs on the maps without changing exactness. This is why one can sometimes get away with a cavalier attitude about some of these sign issues.

Part (c) of the following result is a lynchpin of the theory of stable model categories. It is often phrased colloquially as saying that in a stable model category the classes of homotopy fiber sequences and homotopy cofiber sequences are the same. We include the proof here because of the key nature of the result, and because it takes a bit of work to extract it from [130].

Proposition 3.2.10. *Let* $\mathcal{M}$ *be a stable model category.*

(a) *If* $X \to Y \to Z \to \Sigma X$ *is a homotopy cofiber sequence and* T *is any object, then*

$$[T,X] \to [T,Y] \to [T,Z] \to [T,\Sigma X]$$

is exact in the middle two spots.

(b) *More generally, given a homotopy cofiber sequence* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ *and an object* T*,*

$$\cdots \to [T,\Omega Y] \to [T,\Omega Z] \to [T,X] \to [T,Y] \to [T,Z] \to [T,\Sigma X] \to \cdots$$

is a long exact sequence, where each map is the obvious one obtained by applying Σ *and* Ω *to* f*,* g*, or* h *and (if necessary) using the unit and counit of the adjunction.*

(c) *The triangle* $\Omega Z \longrightarrow X \longrightarrow Y \longrightarrow Z$ *is a homotopy fiber sequence if and only if* $\Omega Z \longrightarrow X \longrightarrow Y \xrightarrow{-} \Sigma\Omega Z$ *is a homotopy cofiber sequence, or equivalently if and only if* $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ *is a homotopy cofiber sequence.*

Proof. Denote the maps by $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$. For (a), suppose $u\colon T \to Y$ is such that $gu = *$ (we work always in the homotopy category). Rotate the cofiber sequence and construct the following diagram:

$$\begin{array}{ccccccc}
Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y \\
\uparrow{\scriptstyle u} & & \uparrow & & & & \uparrow{\scriptstyle \Sigma u} \\
T & \longrightarrow & * & \longrightarrow & \Sigma T & \xrightarrow{id} & \Sigma T
\end{array}$$

Both rows are homotopy cofiber sequences, so by Proposition 3.2.8(c) there is a fill-in $v\colon \Sigma T \to \Sigma X$. But $\Sigma\colon [T,X] \to [\Sigma T,\Sigma X]$ is an isomorphism, so let $\bar{v}$ be a preimage of v. Then $f \circ \bar{v} = -u$, so $-\bar{v}$ is the desired lift of u in our sequence. Exactness at $[T,Z]$ can be proven by rotating the homotopy cofiber sequence and then applying what we just proved.

Part (b) is a direct consequence of (a) and stability. We can iteratively rotate the homotopy cofiber sequence to get the Puppe sequence

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \xrightarrow{-} \Sigma Y \xrightarrow{-} \Sigma Z \xrightarrow{-} \Sigma^2 X \longrightarrow \cdots$$

(where each four terms are a homotopy cofiber sequence), and then apply $[T,-]$. But we can also apply $[\Sigma T,-]$ and then use both adjunction and stability to rewrite this as

$$[T,\Omega X] \to [T,\Omega Y] \to [T,\Omega Z] \to [T,X] \to \cdots$$

Similarly, we repeatedly extend the long exact sequence to the left by applying $[\Sigma^N T,-]$ to our Puppe sequence. The signs can be neglected because leaving them off does not change exactness.

For (c) we just prove one direction as the other is similar. Assume given that $\Omega Z \longrightarrow X \longrightarrow Y \xrightarrow{-} \Sigma\Omega Z$ is a homotopy cofiber sequence. Let $\Omega Z \to F \to Y \to Z$ be a homotopy fiber sequence and consider the diagram

$$\begin{array}{ccccccc}
\Omega Z & \longrightarrow & X & \longrightarrow & Y & \xrightarrow{-} & \Sigma\Omega Z \\
\downarrow{\scriptstyle id} & & & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle -} \\
\Omega Z & \longrightarrow & F & \longrightarrow & Y & \longrightarrow & Z
\end{array}$$

By Proposition 3.2.8(b) there is a fill-in $u\colon X \to F$. Now let T be any object and consider the diagram below:

$$\begin{array}{ccccccccc}
[T,\Omega Y] & \longrightarrow & [T,\Omega Z] & \longrightarrow & [T,X] & \longrightarrow & [T,Y] & \xrightarrow{-} & [T,\Sigma\Omega Z] \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle u_*} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle -}\,{\scriptstyle\cong} \\
[T,\Omega Y] & \longrightarrow & [T,\Omega Z] & \longrightarrow & [T,F] & \longrightarrow & [T,Y] & \longrightarrow & [T,Z]
\end{array}$$

Here we have mostly just applied $[T,-]$ to our diagram in Ho($\mathcal{M}$), but we have used (b) to extend the top sequence to the left by one term. The top row is exact by (b), and the bottom row is exact by Proposition 3.2.7(a). The Five Lemma then implies that u_* is an isomorphism. Since this holds for all T we conclude that u itself was an isomorphism.

Finally, consider the commutative diagram

$$\begin{array}{ccccccc}
\Omega Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\
\Omega Z & \longrightarrow & F & \longrightarrow & Y & \longrightarrow & Z
\end{array}$$

The bottom row was a homotopy fiber sequence by construction, and u is an isomorphism, so the top row is a homotopy fiber sequence as well.

For the last statement in (c), use Proposition 3.2.8(d). □

We refer the reader to [297, Chapter 10.2] for the axioms of a triangulated category. The culmination of the above line of work is the following:

Proposition 3.2.11. *Let $\mathcal{M}$ be a stable model category. Then the suspension functor and the class of homotopy cofiber sequences make* Ho($\mathcal{M}$) *into a triangulated category.*

Proof. Axiom TR1 is routine, and TR2 is Proposition 3.2.8(d). Axiom TR3 is Proposition 3.2.8(c). So the only part that requires additional work is TR4, the Octahedral Axiom. The main point of this final axiom is to relate the homotopy cofiber sequence for a composition fg to the homotopy cofiber sequences for f and g. The reader can find a proof of this axiom (in the unstable version) in [130, Proposition 6.3.6]. □

3.3 Monoidal machinery

This section concerns categorical (and model categorical) material that is not specific to the theory of spectra, mostly centering around monoidal structures. We survey some basic facts about monoidal categories and monoidal model categories, as well as invertible objects.

3.3.1 Sufficiently combinatorial model categories

A common issue in model categories is that one wants to take a model structure on a given category $\mathcal{M}$ and produce an associated model structure on a related category

$\mathcal{M}'$. The first example is where $\mathcal{M}'$ is diagrams (of a fixed shape) inside of $\mathcal{M}$, but we will see others as well. There are almost no general theorems along these lines; in most cases some extra structure is required on $\mathcal{M}$ or $\mathcal{M}'$ or both. These structures typically take the form of sets of generating maps where the domains and codomains satisfy certain smallness properties — whatever one needs to run the small object argument.

The first notion of this type is that of a ***cofibrantly generated*** model category; see [124]. This notion works well for some purposes, but is too weak for others. Later notions are that of a ***cellular model category*** (also in [124]), and Jeff Smith's notion of a ***combinatorial model category***. A combinatorial model category is one that is cofibrantly generated and where the underlying category is locally presentable; see [35] and [77] for written accounts. The combinatorial setting is especially appealing, because here all objects are small (with respect to large enough cardinals) and this property passes to most associated categories.

Most model categories built in some way starting from $s\mathcal{S}et$ or $\mathcal{T}op$ are cofibrantly generated, and the ones built from $s\mathcal{S}et$ are almost all combinatorial. Jeff Smith observed that one can make combinatorial forms of $\mathcal{T}op$-based model categories by replacing $\mathcal{T}op$ with the category of Δ-generated spaces.

In this chapter we will sometimes want to phrase results in a way that applies both to categories of spectra based on simplicial sets and those made from topological spaces. The safe thing is to always assume the categories in question are combinatorial, but this does not apply to the category of compactly generated spaces used in [94]. To cut the Gordian knot, we will use the phrase **sufficiently combinatorial** as an intentionally imprecise stand-in for "assume enough hypotheses so that the smallness conditions necessary for the arguments actually work".

3.3.2 Monoids and models

Let $(\mathcal{M}, \otimes, I)$ be a monoidal category (I is the unit). Recall that a monoid in this category is an object R together with unit map $I \to R$ and multiplication $R \otimes R \to R$ satisfying the evident axioms. The monoids in $(\mathcal{A}b, \otimes, \mathbb{Z})$ are usually called rings, and in stable homotopy contexts the monoids are often called rings as well. For this reason we will use the word "ring" as a synonym for "monoid", although the latter is really the correct term.

If R is a ring in $\mathcal{M}$, a left R-module is an object X together with a map $R \otimes X \to X$ satisfying the evident axioms. One similarly defines right-modules and bimodules. By convention, we mean "left R-module" whenever we say "R-module" without qualification. Recall that if M is a right R-module and N is a left R-module then one defines $M \otimes_R N$ to be the coequalizer (if it exists) of the two action maps $M \otimes R \otimes N \rightrightarrows M \otimes N$.

When $\mathcal{M}$ is a symmetric monoidal category we can talk about commutative rings in $\mathcal{M}$, and for such rings there is an evident way of turning any left module into a right module, and vice versa. If R is a commutative ring then we define an ***R*-algebra** to be a ring map $f: R \to W$ such that R is central in W, meaning that the diagram

$$\begin{array}{ccccc} R\otimes W & \xrightarrow{f\otimes id} & W\otimes W & & \\ {\scriptstyle t}\big\downarrow & & & \overset{\mu}{\searrow} & \\ & & & & W \\ & & & \underset{\mu}{\nearrow} & \\ W\otimes R & \xrightarrow{id\otimes f} & W\otimes W & & \end{array}$$

is commutative. Observe that if $\mathcal{M}$ has coproducts and the tensor distributes over them, then we have the expected "tensor algebra" functor $T: R\text{–Mod} \to R\text{–Alg}$ given by $T(V) = R \amalg V \amalg (V \otimes_R V) \amalg \cdots$ with the evident multiplication. This gives an adjoint pair $T: R\text{–Mod} \rightleftarrows R\text{–Alg}: U$, where U is the forgetful functor.

We will be interested in the question of when certain structures on $\mathcal{M}$ pass to the category of R-modules. For example, if $\mathcal{M}$ is complete then so is R–Mod. To see this, let $\{M_\alpha\}$ be a diagram of R-modules and write $\lim_\alpha M_\alpha$ for the limit in $\mathcal{M}$. The canonical map $R\otimes(\lim_\alpha M_\alpha) \to \lim_\alpha(R\otimes M_\alpha)$ makes $\lim_\alpha M_\alpha$ into an R-module, and one readily checks that this has the properties of the limit in the category R–Mod. To say the same thing in fancier language, the forgetful functor $U: R\text{–Mod} \to \mathcal{M}$ is right adjoint to the free R-module functor $X \mapsto R\otimes X$ and therefore preserves all limits.

The situation for colimits is a little more challenging. Here the canonical map $\operatorname{colim}_\alpha(R\otimes M_\alpha) \to R\otimes \operatorname{colim}_\alpha M_\alpha$ goes in the wrong direction, and so does not give an R-module structure on $\operatorname{colim}_\alpha M_\alpha$. However, in many cases the functor $R\otimes(-)$ is a left adjoint and hence preserves colimits; so in these cases the above map *is* an isomorphism and everything works as before.

A symmetric monoidal category $(\mathcal{M}, \otimes, I)$ is called **closed** if there exists a cotensor (or "internal hom") functor $\mathcal{F}: \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ together with natural adjunctions

$$\mathcal{M}(A\otimes B, C) \cong \mathcal{M}(A, \mathcal{F}(B,C)).$$

Note that this implies that $(-)\otimes(-)$ commutes with colimits in both variables.

Proposition 3.3.1. *Suppose* $(\mathcal{M}, \otimes, I, \mathcal{F})$ *is a closed symmetric monoidal category. Then both* R-Mod *and* R-Alg *are complete and cocomplete.*

Proof. We have already discussed the situation for R–Mod. For R–Alg, limits are created by the forgetful functor U in the adjoint pair $T: R\text{–Mod} \rightleftarrows R\text{–Alg}: U$. Colimits in R–Alg are more complicated, but by [53, Proposition 4.3.6] the category is cocomplete provided that the tensor functor $T(-)$ preserves filtered colimits. The latter condition is immediate from the fact that $\otimes$ preserves colimits in each variable. □

See Section 5.6 in Chapter 5 of this volume for a more detailed discussion of limits and colimits in categories of operadic algebras.

We will next discuss the issue of compatibility between a monoidal structure and a model structure.

Definition 3.3.2. A **monoidal model category** is a model category $\mathcal{M}$ equipped with a monoidal structure $(\otimes, I)$ satisfying the following two axioms:

(1) [Pushout-Product Axiom] For any two cofibrations $f: A \rightarrowtail B$ and $j: K \rightarrowtail L$ in $\mathcal{M}$, the induced map

$$f \square j: (A \otimes L) \amalg_{A \otimes K} (B \otimes K) \longrightarrow B \otimes L$$

is a cofibration. Moreover, $f \square j$ is a weak equivalence if either f or j is a trivial cofibration.

(2) [Unit Axiom] There exists a cofibrant replacement $QI \xrightarrow{\sim} I$ having the property that for all cofibrant X the map $QI \otimes X \to I \otimes X$ is a weak equivalence.

The notion of monoidal model category was introduced in [130]. The Pushout-Product Axiom is analogous to one common form of Quillen's SM7 axiom for simplicial model categories; it is the standard axiom for compatibility of a tensor with the model structure. In the presence of the Pushout-Product Axiom, the Unit Axiom is equivalent to requiring that *every* cofibrant replacement $QI \xrightarrow{\sim} I$ has the stated property. This axiom is automatically satisfied if the unit I is itself cofibrant.

It is an easy exercise to verify that in a monoidal model category the derived functor of $\otimes$ descends to give a monoidal structure on the homotopy category.

By a **closed symmetric monoidal model category** we simply mean a monoidal model category where the underlying monoidal category is symmetric and closed. It is an easy exercise in adjoint functors to check the following:

Proposition 3.3.3. *Let $\mathcal{M}$ be a closed symmetric monoidal model category. If $f: A \rightarrowtail B$ and $g: X \twoheadrightarrow Y$ are maps in $\mathcal{M}$ then the induced map*

$$\mathcal{F}(B, X) \to \mathcal{F}(A, X) \times_{\mathcal{F}(A,Y)} \mathcal{F}(B, Y)$$

is a fibration, and moreover it is a weak equivalence if either f or g is so.

We next consider when a model category structure on $\mathcal{M}$ induces an associated model structure for R–Mod and for R–Alg. Suppose given a model category $\mathcal{M}$ together with an adjoint pair $L: \mathcal{M} \rightleftarrows \mathcal{N}: U$. In good cases one can put a model category structure on $\mathcal{N}$ where a map f is a weak equivalence (respectively, fibration) if and only if Uf is a weak equivalence (respectively, fibration). The cofibrations are forced to be the maps with the left lifting property with respect to the trivial fibrations, but often this is about all one can say about them. When such a model structure on $\mathcal{N}$ exists, one refers to it as the model structure **created by** the right adjoint U.

The main result on such structures is Kan's Recognition Theorem [124, Theorem 11.3.2], which says that U creates a model structure on $\mathcal{N}$ if the following conditions are satisfied:

(1) $\mathcal{M}$ is cofibrantly generated.

(2) The images under L of the generating cofibrations and trivial cofibrations permit the small object argument.

(3) If J denotes the set of generating trivial cofibrations for $\mathcal{M}$, then U takes all maps in $\widehat{LJ}$ to weak equivalences, where $\widehat{LJ}$ is the class of maps obtained from $L(J)$ by taking cobase changes and transfinite compositions.

Conditions (1) and (2) are technical conditions that are always satisfied in the cases of interest; we will bundle them into the "sufficiently combinatorial" adjective. Condition (3) is where the real content is.

Let $\mathcal{M}$ be a monoidal model category and let R be a monoid in $\mathcal{M}$. Then we have adjoint functors

$$\mathcal{M} \underset{U}{\overset{F_R}{\rightleftarrows}} R\text{–Mod}$$

where U is the forgetful functor and $F_R(X) = R \otimes X$. If we are lucky, then U will create a model category structure on R–Mod. Here are some general conditions where this happens:

Proposition 3.3.4. *Let $\mathcal{M}$ be a sufficiently combinatorial monoidal model category.*

(a) *If R is cofibrant in $\mathcal{M}$, then R-Mod has the model structure created by U.*
(b) *Start with the collection of maps $f \otimes id_R \colon R \otimes A \to R \otimes B$, where $f \colon A \stackrel{\sim}{\rightarrowtail} B$ is a trivial cofibration. Let $\mathcal{S}$ be the collection of maps obtained from the original collection using cobase change and transfinite composition. If every element of $\mathcal{S}$ is a weak equivalence, then R-Mod has the model structure created by U.*

Proof. In (b), the stated hypothesis exactly verifies condition (3) from Kan's Recognition Theorem. For (a), the point is that when R is cofibrant the functor $R \otimes (-)$ preserves trivial cofibrations by the Pushout-Product Axiom. Since trivial cofibrations are closed under cobase change and transfinite composition, the condition from (b) is automatically satisfied. □

Now assume that $\mathcal{M}$ is a closed symmetric monoidal model category. This allows us to talk about *commutative* monoids in $\mathcal{M}$. Let R be a commutative monoid and let M and N be R-modules (we will identify left and right R-modules, as usual). Define

$$M \otimes_R N = \operatorname{coeq}(M \otimes R \otimes N \rightrightarrows M \otimes N),$$

where the two maps in the coequalizer come from the R-module structures on M and N. Then $\otimes_R$ is a symmetric monoidal product on R–Mod with unit R. Likewise, define

$$\mathcal{F}_R(M,N) = \operatorname{eq}(\mathcal{F}(M,N) \rightrightarrows \mathcal{F}(R \otimes M, N)),$$

where the two maps in the equalizer are the adjoints to the two evident maps $F(M,N) \otimes R \otimes M \to N$ (twist-evaluate-multiply and multiply-evaluate). It follows by quite general considerations that these definitions give a closed symmetric monoidal structure on R–Mod with unit R. We can hope that this makes R–Mod into a closed symmetric monoidal model category.

Finally, let us turn to algebras. If R is a commutative monoid in $\mathcal{M}$ then we have the adjoint functors $T_R \colon R\text{–Mod} \rightleftarrows R\text{–Alg} \colon U$. We can again hope that U creates a model structure on R–Alg.

We now bundle all of these "hopes" into the following definition:

Definition 3.3.5. Let $\mathcal{M}$ be a closed symmetric monoidal model category. We say that $\mathcal{M}$ satisfies the **Algebraic Creation Property** if

(1) for every monoid R in $\mathcal{M}$, the forgetful functor R–Mod $\to \mathcal{M}$ creates a model structure on $\mathcal{M}$;
(2) when R is a commutative monoid, $\otimes_R$ and $\mathcal{F}_R(-,-)$ make R–Mod into a closed symmetric monoidal model category; and
(3) when R is a commutative monoid, the forgetful functor R–Alg $\to R$–Mod creates a model structure on R–Alg.

There are essentially two separate circumstances where the Algebraic Creation Property is known to hold. The first is when all objects of $\mathcal{M}$ are fibrant, and a few other conditions are satisfied — this kind of case was treated in [94, Chapter VII], though some of the ideas go back as far as [229]. When it is not true that all objects of $\mathcal{M}$ are fibrant, the situation is more delicate; it was first analyzed in [267]. The following proposition, though somewhat awkward, brings together these different threads.

Proposition 3.3.6. *Let $(\mathcal{M},\otimes,I)$ be a symmetric monoidal model category that is sufficiently combinatorial and consider the following hypotheses:*

(1) *For some cofibrant replacement $QI \xrightarrow{\sim} I$ and any object X, the map $QI\otimes X \to I\otimes X$ is a weak equivalence.*
(2) *All objects of $\mathcal{M}$ are fibrant, and $\mathcal{M}$ is a simplicial or topological model category.*
(3) [The Monoid Axiom] *For any trivial cofibration $A \rightarrowtail B$ and any object X, the map $A\otimes X \to B\otimes X$ is a weak equivalence. Additionally, all maps obtained from the class*

$$\{A\otimes X \to B\otimes X \mid A \to B \text{ is a trivial cofibration and } X \text{ is any object}\}$$

by cobase change and transfinite composition are also weak equivalences.

Assume that (1) *holds and that* **either** (2) *or* (3) *holds. Then $\mathcal{M}$ satisfies the Algebraic Creation Property.*

Remark 3.3.7. Condition (1) is automatic if the unit is cofibrant. In general condition (1) seems much too strong, but it is not clear how to weaken it. Condition (3) was isolated by Schwede–Shipley [267] and christened by them.

Proof of Proposition 3.3.6. Condition (2) implies that the appropriate model structures are created on R–Mod and R–Alg; this is by [267, Lemma 2.3(2)] and the fact that the simplicial (or topological) structure on $\mathcal{M}$ gives canonical path objects on both R–Mod and R–Alg. See also [267, Remark 4.5].

Condition (3) also implies that the appropriate model structures are created on R–Mod and R–Alg. For R–Mod this is automatic, because the condition of Proposition 3.3.4(b) is a special case of (3). For R–Alg this is a little more difficult, but was worked out in [267, Theorem 4.1(3)].

It remains to prove that R–Mod is a monoidal model category. For the Pushout–Product Axiom, as in [267, Theorem 4.1(2)] it suffices to check this on the generating cofibrations and trivial cofibrations of R–Mod. But these are of the form $id_R \otimes f$, where f is a generating cofibration or trivial cofibration of $\mathcal{M}$, and the pushout-product is readily analyzed. The necessary condition follows at once from the Pushout–Product Axiom on $\mathcal{M}$.

The trouble arises with the Unit Axiom for R–Mod. This was not dealt with in [267]. Let $QI \to I$ be a cofibrant replacement in $\mathcal{M}$. Hypothesis (1) implies that $R \otimes QI \to R \otimes I = R$ is a weak equivalence, and of course $R \otimes QI$ is cofibrant in R–Mod. So we must check that for every cofibrant R-module M, the map $(R \otimes QI) \otimes_R M \to R \otimes_R M$ is a weak equivalence. This is just the map $QI \otimes M \to M$, and so hypothesis (1) completes the verification. □

If we have model categories on R–Mod and R–Alg, we should of course be concerned with the extent to which they depend on the homotopy type of R. If $R \to T$ is a map of monoids then there is an adjoint pair

$$T \otimes_R (-) \colon R\text{–Mod} \rightleftarrows T\text{–Mod} \colon V, \tag{3.3.3}$$

where here the right adjoint V is restriction of scalars, and this will be a Quillen pair if the categories have the model structures created by U (because V will preserve both fibrations and trivial fibrations).

Similarly, if $R \to T$ is a map of commutative monoids then $T \otimes_R (-)$ takes R-algebras to T-algebras and we have a similar Quillen pair

$$T \otimes_R (-) \colon R\text{–Alg} \rightleftarrows T\text{–Alg} \colon V. \tag{3.3.4}$$

In both cases, if $R \to T$ is a weak equivalence one would hope that the above adjoint pairs are Quillen equivalences. Unfortunately, this does not work out for free and is not known without various unsatisfying extra hypotheses. To sweep some of these under the rug, we make the following definition:

Definition 3.3.8. Let $\mathcal{M}$ be a symmetric monoidal model category that satisfies the Algebraic Creation Property. Then $\mathcal{M}$ satisfies the **Algebraic Invariance Property** if for every weak equivalence of monoids $R \to T$ the Quillen pair of (3.3.3) is a Quillen equivalence, and if for every weak equivalence of commutative monoids $R \to T$ the pair (3.3.4) is a Quillen equivalence.

The following result is basically Theorems 4.3 and 4.4 of [267]. It follows readily from Quillen's criterion for checking that an adjoint pair is a Quillen equivalence. The proof is an easy exercise.

Proposition 3.3.9. *Let $\mathcal{M}$ be a symmetric monoidal model category satisfying the Algebraic Creation Property. Suppose further that*

(1) *for every monoid R and every cofibrant R-module M, the functor $(-) \otimes_R M$ preserves all weak equivalences, and*
(2) *every cofibration $R \to T$ in R-Alg is a cofibration in R-Mod as well.*

Then $\mathcal{M}$ satisfies the Algebraic Invariance Property.

The conditions in this proposition seem like a lot to check, and in some sense they are. But they have been verified for all the modern model categories of spectra. Condition (1) turns out to be surprisingly important, and deserves its own name:

Definition 3.3.10. Let $\mathcal{M}$ be a symmetric monoidal model category satisfying the Algebraic Creation Property. Say that $\mathcal{M}$ satisfies the **Strong Flatness Property** if for every monoid R in $\mathcal{M}$ and every cofibrant R-module M, the functor $(-) \otimes_R M$ preserves all weak equivalences of right R-modules.

While this property seems somewhat unnatural from the perspective of model category theory, it nevertheless is a crucial element of all the modern model categories of spectra. It automatically implies condition (1) of Proposition 3.3.6, using the Unit Axiom. One of the lessons of this whole section is that when it comes to model structures on categories of modules and algebras in a monoidal model category, none of the existing theory works out quite as naturally as one would like.

Remark 3.3.11. Lewis and Mandell in [157] have some interesting things to say about the Algebraic Invariance Property. Define an object C of $\mathcal{M}$ to be **semi-cofibrant** if $\mathcal{F}(C,-)$ preserves fibrations and trivial fibrations; by adjointness this is equivalent to saying that $C \otimes (-)$ preserves cofibrations and trivial cofibrations. Every cofibrant object is semi-cofibrant, but the converse does not necessarily hold. Lewis–Mandell prove that if one has a weak equivalence of monoids $R \to T$, where R and T are semi-cofibrant, then the Quillen pair of (3.3.3) is a Quillen equivalence. The same paper has many other interesting results about the homotopy theory of module categories.

Remark 3.3.12. If T is a monad on $\mathcal{M}$, one can consider the category of T-algebras $\mathcal{M}[T]$ and again ask whether the forgetful functor $U\colon \mathcal{M}[T] \to \mathcal{M}$ creates a model structure on $\mathcal{M}[T]$. This question generalizes the specific cases of R–Mod and R–Alg we have considered in this section. While we will not address the general version here, we refer the reader to [94, Chapter VII.4] for techniques that apply to the case where $\mathcal{M}$ is a topological model category where all objects are fibrant. The task of creating the model structures is essentially reduced to verifying two criteria, embodied in the so-called "Cofibration Hypothesis" [94, Remark IV.4.12].

See also Section 5.8 in Chapter 5 of this volume for a detailed discussion of model structures on operadic algebras more generally.

3.3.5 Invertible objects

If one had to describe the idea of spectra in a single sentence, one approach would be to say that it is a modification of $\mathcal{T}op_*$ that makes the spheres invertible in the homotopy category. So it is good to know a little about the general theory of invertible objects.

Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. An object X in $\mathcal{C}$ is **invertible** if the functor $X \otimes (-)\colon \mathcal{C} \to \mathcal{C}$ is an equivalence of categories. This is equivalent to saying that there exists an object Y and an isomorphism $\alpha\colon I \xrightarrow{\cong} Y \otimes X$, and here we say that the pair (Y, α) is an inverse for X. Note that α is not unique, since given one choice one can make others by precomposing with automorphisms of I. Likewise, Y is unique up to isomorphism but not up to *unique* isomorphism. However, given an inverse (Y, α_Y) and another inverse (Z, α_Z) it is easy to check that there is a unique map $f\colon Y \to Z$ making the diagram

$$\begin{array}{ccc} I & \xrightarrow{\alpha_Y} & Y \otimes X \\ & \searrow^{\alpha_Z} & \downarrow{\scriptstyle f\otimes id} \\ & & Z \otimes X \end{array}$$

commute, and moreover f is an isomorphism.

Note that the tensor product of invertible objects is again invertible.

In a symmetric monoidal category, the endomorphisms of the unit always form a *commutative* monoid: this is an easy exercise using that if f and g are any two maps then $f \otimes g = (f \otimes id)(id \otimes g) = (id \otimes g)(f \otimes id)$. Given any object X in $\mathcal{C}$, there is a map of monoids $\Gamma_X\colon \operatorname{End}(I) \to \operatorname{End}(X)$ that sends $f\colon I \to I$ to the composite

$$X \xrightarrow{\cong} I \otimes X \xrightarrow{f\otimes id} I \otimes X \xrightarrow{\cong} X.$$

When X is invertible, the map Γ_X is an isomorphism. So in particular, the endomorphisms of an invertible object are always commutative. One checks that if (Y, α) is an inverse to X and $f\colon X \to X$ then $\Gamma_X^{-1}(f)$ is the composite

$$I \xrightarrow{\alpha} Y \otimes X \xrightarrow{id\otimes f} Y \otimes X \xrightarrow{\alpha^{-1}} I.$$

Now let X be any object in $\mathcal{C}$. For $n \geq 0$ set $X^{\otimes n} = X \otimes (X \otimes (X \otimes \cdots \otimes X))$. Let $\sigma \in \Sigma_n$ and consider natural transformations

$$X_1 \otimes (X_2 \otimes (X_3 \otimes \cdots \otimes X_n)) \longrightarrow X_{\sigma^{-1}(1)} \otimes (X_{\sigma^{-1}(2)} \otimes (X_{\sigma^{-1}(3)} \otimes \cdots \otimes X_{\sigma^{-1}(n)})),$$

where the domain and codomain are considered as functors $\mathcal{C}^{\times n} \to \mathcal{C}$. MacLane's Coherence Theorem for symmetric monoidal categories says that all natural transformations of this form, made from composites of associativity and commutativity isomorphisms, are identical; see [174, Theorem XI.1.1]. So we have a canonical such transformation. Evaluating at the case where all X_i equal X gives a map $\sigma_*\colon X^{\otimes n} \to X^{\otimes n}$, and one readily checks that this gives a group homomorphism $\Sigma_n \to \operatorname{Aut}(X^{\otimes n})$. If X is invertible then so is $X^{\otimes n}$, which means $\operatorname{Aut}(X^{\otimes n})$ is abelian and therefore this map factors through the abelianization of Σ_n (which is $\mathbb{Z}/2$). In particular, every commutator in Σ_n acts as the identity on $X^{\otimes n}$. The first interesting case is $n = 3$, where the commutator subgroup is generated by the cyclic permutation (123). Moreover, via block sum of permutations and conjugation this case generates the relations for all higher n as well.

Proposition 3.3.13 (The cyclic permutation condition). *If X is an invertible object in a symmetric monoidal category then the composite*

$$X\otimes(X\otimes X) \xrightarrow{id\otimes t} X\otimes(X\otimes X) \xrightarrow{a} (X\otimes X)\otimes X \xrightarrow{t\otimes id} (X\otimes X)\otimes X \xrightarrow{a} X\otimes(X\otimes X)$$

must equal the identity, where all maps labeled a and t are associativity and commutativity isomorphisms, respectively.

The cyclic permutation condition seems to have first been identified by Voevodsky, when attempting to construct symmetric spectra in motivic homotopy theory. See the discussion preceding Theorem 4.3 in [294].

Invertible objects are, in particular, examples of ***dualizable*** objects. Self-maps of dualizable objects have a *trace*. We will not recount the general theory here, but just give a very streamlined version suitable for our present context. For the general theory, see [155, Section III.1] or the survey in [76].

Assume X is invertible and (Y,α) is a chosen inverse. Then there is a unique map $\hat{\alpha}\colon X\otimes Y\to I$ with the property that the composite

$$X \xrightarrow{\cong} X\otimes I \xrightarrow{id\otimes\alpha} X\otimes(Y\otimes X) \xrightarrow{a} (X\otimes Y)\otimes X \xrightarrow{\hat{\alpha}\otimes id} I\otimes X \xrightarrow{\cong} X$$

equals the identity. If $f\colon X\to X$ then the **trace** of f is the element $\mathrm{tr}(f)\in\mathrm{End}(I)$ defined by the composite

$$I \xrightarrow{\alpha} Y\otimes X \xrightarrow{id\otimes f} Y\otimes X \xrightarrow{t} X\otimes Y \xrightarrow{\hat{\alpha}} I.$$

Given $f\colon X\to X$ we now have two ways to extract an element of $\mathrm{End}(I)$: via $\Gamma_X^{-1}(f)$ and via $\mathrm{tr}(f)$. These don't always give the same element! The following results explain the relation between them. They certainly must be classical, but see [76] for a written account:

Proposition 3.3.14. *Let X be an invertible object in a symmetric monoidal category, and let $\tau_X=\mathrm{tr}(id_X)\in\mathrm{End}(I)$.*

(a) $\tau_X=\Gamma_{X\otimes X}^{-1}(t_X)=\mathrm{tr}(t_X)$ *where $t_X\colon X\otimes X\to X\otimes X$ is the twist.*
(b) $\tau_X^2=id$.
(c) *For any $f\colon X\to X$, $\Gamma_X^{-1}(f)=\tau_X\cdot\mathrm{tr}(f)$.*
(d) *If Y is another invertible object then $\tau_{X\otimes Y}=\tau_X\tau_Y$.*

The elements τ_X should be thought of as "generalized signs". They appear as control factors in commutation formulas involving X, in the same way that ± 1 terms appear in the standard formulas of topology.

Example 3.3.15. Fix a field k and consider the category of $\mathbb{Z}$-graded vector spaces, equipped with the graded tensor product, standard associativity isomorphism, and the twist isomorphism that incorporates the Koszul sign rule. Write $k[n]$ for the graded vector space consisting of a single k in degree n and zero in all other degrees. We identify k with $\mathrm{End}(k[0])$ by letting $x\in k$ correspond to multiplication by x.

The object $k[1]$ is invertible. For an inverse we may choose $k[-1]$ and the map $\alpha\colon k[0]\to k[-1]\otimes k[1]$ sending 1 to $1\otimes 1$. The map $\hat{\alpha}\colon k[1]\otimes k[-1]\to k[0]$ then sends $1\otimes 1$ to 1. If $x\in k$ and $\rho_x\colon k[1]\to k[1]$ is multiplication by x, we leave it as an exercise to check that $\Gamma_X^{-1}(\rho_x)=x$ and $\mathrm{tr}(\rho_x)=-x$. In particular, $\tau_{k[1]}=-1$ here.

3.4 Spectra for Sulu and Chekov

For many applications one needs a model category of spectra but doesn't care much about the inner workings, other than a few basic properties. In the words of one eloquent topologist, "Sometimes one just needs to drive the Enterprise, not necessarily be Mr. Scott." The goal of this section is to supply a list of properties that are shared by most of the existing models, and to give some standard examples of how they can be used. These examples were all originally worked out in [94].

In this section we assume the existence of a pointed category $\mathcal{S}pectra$ equipped with a closed symmetric monoidal smash product $\wedge$ with unit S and cotensor $\mathcal{F}(-,-)$. Additionally, we suppose given adjoint functors $\Sigma^\infty\colon \mathcal{T}op_* \rightleftarrows \mathcal{S}pectra\colon \Omega^\infty$ and a stable model category structure on $\mathcal{S}pectra$. We assume the following properties:

1. $\Sigma^\infty\colon \mathcal{T}op_* \rightleftarrows \mathcal{S}pectra\colon \Omega^\infty$ is a Quillen pair.
2. The smash product makes $\mathcal{S}pectra$ into a monoidal model category. So we have (a) the pushout-product axiom: given cofibrations $f\colon A \rightarrowtail B$ and $g\colon C \rightarrowtail D$, the induced map
$$f \square g\colon (A \wedge D) \amalg_{A\wedge C} (B \wedge C) \to B \wedge D$$
is a cofibration, and additionally it is a weak equivalence if either f or g is so. And (b): for every cofibrant object X and every cofibrant replacement $QS \xrightarrow{\sim} S$, the induced map $QS \wedge X \to S \wedge X$ is a weak equivalence.
3. There exists a weak equivalence $\epsilon\colon \Sigma^\infty S^0 \to S$ and a natural transformation
$$\eta\colon \Sigma^\infty(X \wedge Y) \to \Sigma^\infty X \wedge \Sigma^\infty Y$$
that is oplax monoidal: this says that the evident associativity and unital squares commute. Additionally, η is a weak equivalence when X and Y are cofibrant.
4. $(\mathcal{S}pectra, \wedge)$ satisfies the Algebraic Creation and Invariance Properties (see Definitions 3.3.5 and 3.3.8).
5. $(\mathcal{S}pectra, \wedge)$ satisfies the Strong Flatness Condition of Definition 3.3.10. In particular, for any cofibrant spectrum A and any weak equivalence of spectra $X \to Y$, the induced map $A \wedge X \to A \wedge Y$ is a weak equivalence.
6. There is an equivalence of triangulated categories between $\mathrm{Ho}(\mathcal{S}pectra)$ and the homotopy category of Bousfield–Friedlander spectra that carries the spectra $\Sigma^\infty(S^n)$ to the standard n-sphere.
7. For any directed system $X_0 \to X_1 \to X_2 \to \cdots$ in $\mathcal{S}pectra$ and any $n \geq 0$, the canonical map
$$\mathrm{colim}_k [\Sigma^\infty(S^n), X_k] \to [\Sigma^\infty(S^n), \mathrm{hocolim}_k X_k]$$
is an isomorphism, and similarly sequences indexed by other transfinite ordinals.

All these properties are satisfied by the categories of symmetric spectra, orthogonal spectra, and $\mathcal{W}$-spaces (all to be defined in subsequent sections). Actually (7) is a consequence of (6) (using the smallness of spheres in $\mathcal{T}op$), but is included separately here for emphasis. Note also that Γ-spaces are eliminated from the discussion because they are not a stable model category, but except for this and the related property (6) all the other properties are satisfied.

Remark 3.4.1. EKMM spectra are a special case as they do NOT satisfy property (3), although they satisfy all of the others. Instead, in EKMM spectra there are two pairs of adjoint functors called $(\Sigma^\infty_S, \Omega^\infty_S)$ and $(\Sigma^\infty, \Omega^\infty)$ together with natural maps $\Sigma^\infty_S X \to \Sigma^\infty X$ which are weak equivalences whenever X is cofibrant as a pointed space. The pair $(\Sigma^\infty_S, \Omega^\infty_S)$ satisfies (1), and the pair $(\Sigma^\infty, \Omega^\infty)$ satisfies (3). But if we use the pair $(\Sigma^\infty_S, \Omega^\infty_S)$ for (1)–(7) then we can replace (3) above with (3′) stating that there is a contractible space of choices for an η, giving an oplax symmetric monoidal map in the homotopy category. Keeping this small variation in mind, all of the arguments in the remainder of this section apply to EKMM spectra as well. (It is unfortunate that the EKMM $(\Sigma^\infty, \Omega^\infty)$ notation conflicts with what we use above, but we will just live with this).

3.4.1 Homotopy groups of spectra

Write $S^0 = \Sigma^\infty(S^0)$ and $S^1 = \Sigma^\infty(S^1)$. For $p > 1$ define the stable sphere S^p recursively by $S^p = S^1 \wedge S^{p-1}$, so that

$$S^p = S^1 \wedge (S^1 \wedge (S^1 \wedge \cdots))).$$

Note that S^1 is cofibrant by property (1), and then S^p is cofibrant by the Pushout–Product Axiom. Also we see using property (3) that there is a canonical weak equivalence $\eta\colon \Sigma^\infty(S^p) \to S^p$. Some authors prefer to adopt $\Sigma^\infty(S^p)$ as the ***definition*** of the stable sphere, but η shows that for homotopical purposes this is equivalent to our approach.

Since Σ is an autoequivalence of the homotopy category, there exists a desuspension of S^0. Let S^{-1} be any chosen cofibrant spectrum for which there exists an isomorphism $\alpha\colon S \to S^{-1} \wedge S^1$ in Ho($\mathcal{S}pectra$). For $p \geq 1$ inductively define $S^{-p} = S^{-1} \wedge S^{1-p}$. Let $\hat{\alpha}\colon S^1 \wedge S^{-1} \to S$ be the dual map to α in Ho($\mathcal{S}pectra$) as defined after Proposition 3.3.13.

Under these definitions, there are canonical isomorphisms in Ho($\mathcal{S}pectra$) of the form

$$\gamma\colon S^k \wedge S^l \to S^{k+l}$$

for any $k, l \in \mathbb{Z}$. If $k, l > 0$ then we define γ as a composite of associativity isomorphisms, and MacLane's Coherence Theorem for monoidal categories says that all choices for such associativity isomorphisms lead to the same map γ. Similar remarks apply when $k, l < 0$. When $k = 0$ we use

$$S^0 \wedge S^l \xrightarrow{\epsilon \wedge id} S \wedge S^l \cong S^l,$$

which uses property (3) and also property (2) to know that the first map is an isomorphism. Likewise for $l = 0$. When $k < 0$ and $l > 0$ we use associativity isomorphisms together with repeated applications of the map α^{-1} and the unit map. Again, one can prove that the exact choice of maps here does not affect the final composite. Finally, when $k > 0$ and $l < 0$ we do the same thing but using $\hat{\alpha}$ instead of α.

It is a theorem that these specified isomorphisms are compatible, in the sense that the evident pentagon containing $S^k \wedge (S^l \wedge S^n)$ and S^{k+l+n} is commutative in the homotopy category. More generally, any two composites derived from these canonical maps (but having the same domain and range) are identical (again, in the homotopy category). See [76] for a complete discussion.

Here is why this tedious discussion is actually important. For any spectrum X we write $\pi_p(X)$ for $\mathrm{Ho}(\mathcal{S}pectra)(S^p, X)$. If X is a ring spectrum and $f\colon S^p \to X$ and $g\colon S^q \to X$ we may form the composite

$$S^{p+q} \xrightarrow{\gamma} S^p \wedge S^q \xrightarrow{f\wedge g} X \wedge X \xrightarrow{\mu} X,$$

and this determines a pairing $\pi_p(X) \otimes \pi_q(X) \to \pi_{p+q}(X)$. Also, the composite map $S^0 \xrightarrow{\epsilon} S \to X$ determines a special element $1 \in \pi_0(X)$.

Lemma 3.4.2. *When X is a ring spectrum, $\pi_*(X)$ is a ring. If M is a left X-module then $\pi_*(M)$ is a left $\pi_*(X)$-module.*

Proof. Left to the reader as an exercise, but note that the properties of the canonical maps γ are important here. See [76] for details and generalizations. □

3.4.2 Homotopy groups of tensors and cotensors

Let R be a commutative ring spectrum and let M and N be R-modules. We will construct a spectral sequence of the form

$$\mathrm{Tor}^{\pi_* R}_{p,q}(\pi_* M, \pi_* N) \Rightarrow \pi_{p+q}(M \wedge^{\mathbb{L}}_R N),$$

where $\wedge^{\mathbb{L}}_R$ denotes the derived version of $\wedge_R$. When $M = R \wedge X$ and $N = R \wedge Y$ this gives the Künneth spectral sequence $\mathrm{Tor}^{\pi_* R}(R_*(X), R_*(Y)) \Rightarrow R_*(X \wedge Y)$.

The following argument can be made almost entirely in the homotopy category $\mathrm{Ho}(R\text{–Mod})$, using only the triangulated structure. However, the model structure on R–Mod is key to setting up this homotopy category to begin with. The model structure also plays a small role in the following lemma:

Lemma 3.4.3. *Let R be a commutative ring spectrum and let M be an R-module. Then there exists an R-module F of the form $F = \bigvee_i R \wedge S^{n_i}$ together with a map $F \to M$ in* $\mathrm{Ho}(R\text{-Mod})$ *that is surjective on homotopy groups.*

Proof. Let $M \to M^{fib}$ be a fibrant replacement in R–Mod. Choose a set of $\pi_* R$-module generators $\alpha_i \in \pi_*(M)$, together with representative maps $\alpha_i\colon S^{n_i} \to M^{fib}$ in $\mathcal{S}pectra$. We then get R-module maps $R \wedge S^{n_i} \to M^{fib}$ using the adjoint pair $\mathcal{S}pectra \rightleftarrows R$–Mod. Let $F = \bigvee_i R \wedge S^{n_i}$ and let $\alpha\colon F \to M^{fib}$ be the evident map.

Since α is a map of R-modules, $\pi_*\alpha$ is a map of $\pi_* R$-modules. So to see that $\pi_*\alpha$ is surjective we only need argue that each α_i is in the image. This follows from the commutative diagram

$$\begin{array}{ccccc}
R \wedge S^{n_i} & \xrightarrow{id \wedge \alpha_i} & R \wedge M^{fib} & \xrightarrow{\mu} & M^{fib} \\
\uparrow & & \uparrow & \nearrow\!\!\!\!= & \\
S \wedge S^{n_i} & \xrightarrow{id \wedge \alpha_i} & S \wedge M^{fib} & & \\
\cong\uparrow & & \cong\uparrow & & \\
S^{n_i} & \xrightarrow{\alpha_i} & M^{fib} & &
\end{array}$$

□

Let R be a commutative ring spectrum and let M be an R-module. The following argument takes place entirely in the category $\mathrm{Ho}(R\text{–Mod})$. Set $X_0 = M$. Using Lemma 3.4.3 choose an R-module $F_0 = \bigvee_i R \wedge S^{n_i}$ and a map $F_0 \to X_0$ that is a surjection on $\pi_*(-)$. Let $X_1 \to F_0 \to X_0$ be a homotopy fiber sequence in $\mathrm{Ho}(R\text{–Mod})$ (see the discussion of fiber and cofiber sequences in Section 3.2, and in particular Remark 3.2.5).

Repeat this process inductively to likewise construct homotopy fiber sequences $X_n \to F_{n-1} \to X_{n-1}$ where F_{n-1} is a wedge of suspensions of R and $F_{n-1} \to X_{n-1}$ is surjective on homotopy groups. One way to present all this information is through the diagram

$$\begin{array}{ccccccccc}
\cdots & \dashrightarrow & F_2 & & \dashrightarrow & F_1 & \dashrightarrow & F_0 & \\
 & & & \searrow\!\!\!\searrow & \nearrow & & \searrow\!\!\!\searrow \quad \nearrow & & \searrow\!\!\!\searrow \\
 & & & X_2 & & & X_1 & & M
\end{array}$$

where double-headed arrows represent maps that induce surjections on homotopy groups and tailed arrows represent maps that induce injections on homotopy groups. Observe that the induced sequence $\pi_*(F_\bullet)$ is a free π_*R-resolution of π_*M. (There are some subtleties in justifying this last claim, which for the moment we leave for the reader to try to uncover. But see Section 3.4.4 below.)

Our diagram can also be restructured as a diagram of homotopy fiber sequences. We rotate the fiber sequence $X_n \to F_{n-1} \to X_{n-1}$ to become $X_{n-1} \to \Sigma X_n \to \Sigma F_{n-1}$ and suspend $n-1$ times to get

$$\begin{array}{ccccccc}
 & & \Sigma F_0 & & \Sigma^2 F_1 & & \\
 & & \uparrow & & \uparrow & & \\
M = X_0 & \longrightarrow & \Sigma X_1 & \longrightarrow & \Sigma^2 X_2 & \longrightarrow & \cdots
\end{array}$$

where every "layer" is a homotopy fiber sequence (note that we are being cavalier about signs, but that will be okay for our application). Now apply the derived functor $(-)\wedge_R^{\mathbb{L}} N$. This is still taking place entirely within $\mathrm{Ho}(R\text{–Mod})$, but observe that we know this derived functor exists because of model category machinery. For convenience we will drop the derived "$\mathbb{L}$" in all smash products and write our new tower of homotopy fiber sequences as

$$\begin{array}{ccccccc} & & & & \Sigma F_0 \wedge_R N & & \Sigma^2 F_1 \wedge_R N \\ & & & & \uparrow & & \uparrow \\ M \wedge_R N & = & X_0 \wedge_R N & \longrightarrow & \Sigma X_1 \wedge_R N & \longrightarrow & \Sigma^2 X_2 \wedge_R N \longrightarrow \cdots \end{array}$$

Every layer of this tower induces a long exact sequence in homotopy groups, because homotopy fiber sequences of R-modules are also homotopy fiber sequences of spectra (the forgetful functor from R-modules to spectra is a right adjoint and preserves all weak equivalences, so is its own right derived functor). These long exact sequences braid together to give a spectral sequence in the usual way, taking the form

$$E^1_{a,b} = \pi_a(\Sigma^{b+1} F_b \wedge_R N) \Rightarrow \pi_{a-1}(M \wedge_R N), \qquad d^r \colon E^r_{a,b} \to E^r_{a-1,b-r}$$

(and recall once more that all smash products are derived).

Finally, observe that $F_b \wedge_R N = \bigvee_i (R \wedge S^{n_i}) \wedge_R N = \bigvee_i S^{n_i} \wedge N$, and so $\pi_*(F_b \wedge_R N)$ is a direct sum of shifted copies of $\pi_*(N)$. Said in the most canonical way possible, for any R-module W we have a natural map

$$\pi_*(W) \otimes_{\pi_*(R)} \pi_*(N) \to \pi_*(W \wedge_R N)$$

and when W is $R \wedge S^n$ or a wedge of such things this map is an isomorphism. This identifies the E_1-term of our spectral sequence as $\pi_*(F_\bullet) \otimes_{\pi_*(R)} \pi_*(N)$, and a little thought shows the d^1 maps are the boundary maps in this complex. So the E_2-term is $\mathrm{Tor}^{\pi_* R}(\pi_* M, \pi_* N)$, as desired. Specifically, $E^2_{a,b} = \mathrm{Tor}^{\pi_* R}_{b,a-b-1}(\pi_* M, \pi_* N)$ and this converges to $\pi_{a-1}(M \wedge_R N)$. Recoordinatizing the spectral sequence by setting $b = p$ and $a - b - 1 = q$ yields the following:

Theorem 3.4.4. *Let R be a commutative ring spectrum and let M and N be R-modules. Then there is a spectral sequence*

$$E^2_{p,q} = \mathrm{Tor}^{\pi_* R}_{p,q}(\pi_* M, \pi_* N) \Rightarrow \pi_{p+q}(M \wedge^{\mathbb{L}}_R N)$$

with differentials of the form $d^r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$.

The construction of a spectral sequence for $\pi_* \mathcal{F}_R(M, N)$ is entirely similar. Start with the same tower of homotopy fiber sequences and apply $\mathcal{F}_R(-, N)$. The key part of the calculation is that

$$\mathcal{F}_R(R \wedge S^n, N) \simeq \mathcal{F}(S^n, N) \simeq \Sigma^{-n} N,$$

and so $\pi_*(\mathcal{F}_R(F_q, N)) \cong \mathrm{Hom}_{\pi_* R}(\pi_* F_q, \pi_* N)$. We leave the reader to work out the details for the following:

Theorem 3.4.5. *Let R be a commutative ring spectrum and let M and N be R-modules. Write $\mathbb{R}\mathcal{F}(M, N)$ for the derived cotensor. Then there is a spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{\pi_* R}(\pi_* M, \pi_* N) \Rightarrow \pi_{-(p+q)} \mathbb{R}\mathcal{F}_R(M, N)$$

with differentials of the form $d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$.

For more about the above two spectral sequences, see [94, Chapter IV.4].

3.4.3 Constructing Morava K-theory

For each prime p the n-th Morava K-theory spectrum is a certain ring spectrum $K(n)$ having the property that $\pi_* K(n) = \mathbb{Z}/p[v_n^{\pm 1}]$, where $|v_n| = 2(p^n - 1)$. In addition to those properties it can be characterized by the existence of a map $MU \to K(n)$ having a prescribed behavior on homotopy groups (where MU is the usual complex cobordism spectrum). As a demonstration of the model-category-theoretic tools we have been describing, we show how they lead to a construction of the spectrum $K(n)$ starting with MU.

We start with the assumption that there is a commutative ring spectrum MU in our category $\mathcal{S}pectra$ and a ring isomorphism $\pi_*(MU) \cong \mathbb{Z}[x_1, x_2, \ldots]$ with $|x_i| = 2i$ for all i. Let $MU \to X$ be a fibrant replacement in the category of MU-modules, and recall that this implies X is fibrant in $\mathcal{S}pectra$.

Fix a prime p. Since $\pi_0(MU) = \mathbb{Z}$ and X is fibrant, there exists a map $S^0 \to X$ that represents the element $p \in \pi_0(MU)$. Consider the composite $MU \wedge S^0 \to MU \wedge X \xrightarrow{\mu} X$, and let MU_1 be the homotopy cofiber in the category MU–Mod. This is also a homotopy cofiber in $\mathcal{S}pectra$: the forgetful functor from MU-modules to spectra is its own right derived functor and therefore preserves homotopy fiber sequences, and homotopy cofiber and fiber sequences are the same by Proposition 3.2.10(c). The long exact sequence on homotopy groups immediately shows that $\pi_*(MU_1) = \mathbb{Z}/p[x_1, x_2, \ldots]$. (Note: There is a subtlety here! The reader may try to uncover it, or see the end of Section 3.4.4.)

Now let $MU_1 \to X_1$ be a fibrant replacement of MU-modules, and choose a map $S^2 \to X_1$ that represents x_1. Let MU_2 be the homotopy cofiber in MU–Mod of the composite $MU \wedge S^2 \to MU \wedge X_1 \to X_1$, and verify that $\pi_*(MU_2) = \mathbb{Z}/p[x_2, x_3, \ldots]$.

The only thing we are ever using is that we are quotienting by an element x_i which is a nonzerodivisor on homotopy groups, so we can continue to do this for whichever x_i we choose. Fix an n and successively kill off all the x_i except for $x_{p^n - 1}$. For convenience set $r = p^n - 1$. This produces a sequence in Ho(MU–Mod) of the form

$$MU = MU_0 \to MU_1 \to MU_2 \to \cdots \to MU_{r-1} \to MU_{r+1} \to \cdots$$

Lift this to a directed system in MU–Mod, and let Z be the homotopy colimit in MU–Mod. Then Z sits in a homotopy cofiber sequence $\bigvee_n MU_n \to \bigvee_n MU_n \to Z$, where the first map is the difference between the identity and the shift map. This is also a homotopy fiber sequence, by Proposition 3.2.10(c), and that property is preserved after applying the forgetful functor to $\mathcal{S}pectra$. So Z is the homotopy colimit of the MU_n in $\mathcal{S}pectra$, not just in MU–Mod. We then know by property (7) that $\pi_*(Z) = \operatorname{colim}_n \pi_*(MU_n)$, and so $\pi_*(Z) \cong \mathbb{Z}/p[x_r]$.

Now consider the composite map $Z \wedge S^{2r} \longrightarrow Z \wedge MU \xrightarrow{t} MU \wedge Z \xrightarrow{\mu} Z$. This is a map of left MU-modules, using that MU is commutative. Applying $(-) \wedge S^{-2r}$ gives a map of MU-modules $Z \to Z \wedge S^{-2r}$. On homotopy groups this is multiplication by x_r. Consider the sequence in Ho(MU–Mod)

$$Z \to Z \wedge S^{-2r} \to Z \wedge S^{-2r} \wedge S^{-2r} \to \cdots$$

then lift it to MU–Mod, and let W be the homotopy colimit. It follows again from property (7) that $\pi_*(W) = \mathbb{Z}/p[x_r^{\pm 1}]$.

In this way we have constructed an MU-module spectrum W whose homotopy groups make it look like W is the n-th Morava K-theory spectrum. The construction has also produced a map $MU \to W$ which does the right thing on homotopy groups, so W really *is* Morava K-theory.

Note that we have not constructed W as a ring spectrum, only as an MU-module spectrum. In Chapters V.3 and V.4 of [94] (see especially Theorem V.4.1) it is explained how to construct a product $W \wedge W \to W$ making W into a homotopy ring spectrum, but this is much weaker than what is desired. To construct W as a ring spectrum one seems to need the full force of A_∞-obstruction theory, which we will not recount here.

Remark 3.4.6 (historical note). All of the arguments in this section first appeared in [94]. They needed very little of the inner workings of EKMM-spectra, however, and as we have seen here they work in any of the modern model categories of spectra.

3.4.4 Loose ends

In the course of the argument from Section 3.4.3 we had a homotopy cofiber sequence $MU \wedge S^0 \to MU \to MU_1$ and wanted to compute the homotopy groups of MU_1 using the long exact sequence. This required us to know $\pi_*(MU \wedge S^0)$ — but how exactly do we know these groups? Recall that $S^0 \to S$ is a cofibrant replacement, and so it is tempting to use property (2) to say that $MU \wedge S^0 \to MU \wedge S = MU$ is a weak equivalence. But that works only if MU is cofibrant as a spectrum, which we have not assumed!

To try to get around this issue, let $\widetilde{MU} \xrightarrow{\sim} MU$ be a cofibrant replacement in $\mathcal{S}pectra$. We certainly know $\widetilde{MU} \wedge S^0 \simeq \widetilde{MU} \simeq MU$ by property (2), so we know the homotopy groups of $\widetilde{MU} \wedge S^0$. We could go back to the beginning and try to do the entire construction with $\widetilde{MU}$ replacing MU, except we do not know that $\widetilde{MU}$ is a ring spectrum. The lifting diagram

$$\begin{array}{ccccc} & & & & \widetilde{MU} \\ & & & \nearrow & \downarrow \simeq \\ \widetilde{MU} \wedge \widetilde{MU} & \longrightarrow & MU \wedge MU & \longrightarrow & MU \end{array}$$

produces a multiplication, but in general it will only be associative up to homotopy. If $\widetilde{MU}$ is only a homotopy ring spectrum we do not have a good homotopy theory of $\widetilde{MU}$-modules, so we are again defeated.

What saves us here is the amazing property (5). Since S^0 is cofibrant this property guarantees that $\widetilde{MU} \wedge S^0 \to MU \wedge S^0$ is still an equivalence, and so we have $MU \wedge S^0 \simeq \widetilde{MU} \wedge S^0 \simeq \widetilde{MU} \simeq MU$. This analysis is actually needed at each stage of the construction, since at the n-th stage we need to know the homotopy groups of $MU \wedge S^{2n}$ and it is only property (5) that allows these to be identified with the homotopy groups of $MU \wedge^{\mathbb{L}} S^{2n}$ (which we know are just a shifted version of the homotopy groups of MU).

A similar subtle issue came up in §3.4.2. There we had a spectrum $X = \bigvee_\alpha (R \wedge S^{n_\alpha})$ and wanted to conclude that $\pi_*(X) \cong \bigoplus_\alpha \pi_{*-n_\alpha}(R)$. Given cofibrant spectra E_α, general model category considerations show that $\bigvee_\alpha E_\alpha$ is the homotopy colimit of a directed system $E_{\alpha_1} \to E_{\alpha_1} \vee E_{\alpha_2} \to \cdots$ (possibly indexed by an ordinal larger than ω). So property (7) implies that $\pi_*(\bigvee_\alpha E_\alpha) \cong \bigoplus_\alpha \pi_*(E_\alpha)$. The spectra $R \wedge S^{n_\alpha}$ need not be cofibrant, but if $\tilde{R} \to R$ is a cofibrant replacement in *Spectra* then we can write

$$X = \bigvee_\alpha (R \wedge S^{n_\alpha}) \cong R \wedge \Big(\bigvee_\alpha S^{n_\alpha}\Big) \simeq \tilde{R} \wedge \Big(\bigvee_\alpha S^{n_\alpha}\Big) \cong \bigvee_\alpha (\tilde{R} \wedge S^{n_\alpha})$$

where we have used property (5) for the weak equivalence in the middle. Since the spectra $\tilde{R} \wedge S^{n_\alpha}$ are cofibrant, we can use the previously mentioned result to see that $\pi_*(X)$ is as desired.

Though not necessarily the most important applications of property (5), these are good examples of how that property can come to the rescue at key moments.

3.5 Diagram categories and spectra

With the exception of the EKMM model, all of the common model categories of spectra are built on the foundation of diagram categories. It is perhaps not immediately apparent from the classical definition, but a spectrum is a kind of diagram. The goal of this section is to survey the general theory of model structures on diagram categories, and then to explain how spectra can be regarded as diagrams. This whole "diagrammatic" perspective is one of the main points of [178].

3.5.1 Model category structures on diagram categories

Let $\mathcal{M}$ be a category and let I be a small category. We write $\mathcal{M}^I$ for the category whose objects are the functors $X\colon I \to \mathcal{M}$ and whose morphisms are natural transformations. Such functors are also called ***I*-diagrams** in $\mathcal{M}$. When $\mathcal{M}$ has a notion of weak equivalence, $\mathcal{M}^I$ can be equipped with the **objectwise weak equivalences**, namely the maps $X \to Y$ such that $X_i \to Y_i$ is a weak equivalence for every object i in I. These are sometimes called **levelwise weak equivalences** as well.

If $\mathcal{M}$ has a model structure then one might expect there to be an associated model structure on $\mathcal{M}^I$ built around the objectwise weak equivalences, but unfortunately this doesn't seem to work out unless one assumes some extra hypotheses on $\mathcal{M}$.

Theorem 3.5.1. *Let $\mathcal{M}$ be a model category and let I be a small category.*

(a) *If $\mathcal{M}$ is cofibrantly generated then there is a model category structure on $\mathcal{M}^I$ in which a map $f: X \to Y$ is a weak equivalence (resp., fibration) if and only if $f_i : X_i \to Y_i$ is a weak equivalence (resp., fibration) for all objects i in I. This is called the **projective model structure** on $\mathcal{M}^I$. The cofibrations are forced to be those maps satisfying the left lifting property with respect to the trivial fibrations.*

(b) *If $\mathcal{M}$ is combinatorial (cofibrantly generated and locally presentable) then there is a model category structure on $\mathcal{M}^I$ in which a map $f: X \to Y$ is a weak equivalence (resp.,* *cofibration) if and only if $f_i\colon X_i \to Y_i$ is a weak equivalence (resp., cofibration) for all objects i in I. This is called the **injective model structure** on $\mathcal{M}^I$. The fibrations are forced to be those maps satisfying the right lifting property with respect to the trivial cofibrations.*

Both parts (a) and (b) were proven by Heller [117, Theorem II.4.5] in the case $\mathcal{M} = s\mathcal{S}et$, with (b) also following from work of Jardine in this case [135]. For part (a) in the above generality, see [124, Theorem 11.6.1]. Part (b) in the above generality is due to Jeff Smith; it follows from [35, Theorem 1.7 and Propositions 1.15, 1.18], using the forgetful functor $\mathcal{M}^I \to \prod_{i\in I}\mathcal{M}$ as the "detection functor" for Beke's Proposition 1.18.

Let us say a little about how Theorem 3.5.1 is proven, since the main idea is easy and also useful in a variety of situations. For each i in I there are adjoint functors

$$F_i\colon \mathcal{M} \rightleftarrows \mathcal{M}^I\colon \mathrm{Ev}_i,$$

where the right adjoint Ev_i is the "evaluation at i" functor. The diagram F_iX is the free diagram generated by starting with an X at spot i. One readily checks that for each X in $\mathcal{M}$ and j in I,

$$(F_iX)(j) = \coprod_{I(i,j)} X.$$

That is, $(F_iX)(j)$ is a coproduct of copies of X indexed by $I(i,j)$. For T a set it is convenient to write $T \odot X$ for the coproduct of copies of X generated by T, so that $(F_iX)(j) = I(i,j) \odot X$.

Start with sets $\{f_\alpha\colon A_\alpha \rightarrowtail B_\alpha\}$ and $\{\tilde{f}_\alpha\colon \tilde{A}_\alpha \stackrel{\sim}{\rightarrowtail} \tilde{B}_\alpha\}$ of generating cofibrations and trivial cofibrations for $\mathcal{M}$. The collections $\mathcal{I} = \{F_i(f_\alpha)\}_{i,\alpha}$ and $\mathcal{J} = \{F_i(\tilde{f}_\alpha)\}_{i,\alpha}$ are potential sets of generating cofibrations and trivial cofibrations for $\mathcal{M}^I$: the maps with the right lifting property with respect to $\mathcal{I}$ and $\mathcal{J}$ are clearly the objectwise trivial fibrations and the objectwise fibrations, respectively. The only thing nontrivial in setting up the projective model category structure is the factorization axiom, and this can be proven by the small object argument — it works in $\mathcal{M}^I$ as long as it worked in $\mathcal{M}$, which is the cofibrant-generation assumption. This proves (a).

Another way of describing the proof of (a) is to package all the pairs (F_i, Ev_i) into a single adjoint pair

$$F\colon \prod_{i\in I}\mathcal{M} \rightleftarrows \mathcal{M}^I\colon \mathrm{Ev}.$$

Kan's Recognition Theorem [124, Theorem 11.3.2] immediately yields that the right adjoint Ev creates the projective model structure on $\mathcal{M}^I$.

The proof of (b) works a little differently; it is a direct descendant of the classical proof that categories of sheaves have enough injectives. Here one fixes a large cardinal λ (depending on I and $\mathcal{M}$) and looks at a skeletal set of all objectwise cofibrations (or objectwise trivial cofibrations) where the domain and codomain are both λ-small. The λ-small conditions guarantee that the isomorphism classes of such things actually form a set and not a proper class. By choosing λ large enough, one can show that these give generating cofibrations and trivial cofibrations for the desired injective model structure.

Remark 3.5.2. The cofibrations in the projective model structure on $\mathcal{M}^I$ are often called “projective cofibrations”. For general I they are hard to identify explicitly, but for some special classes of indexing categories I this can be done. One such class consists of the “upwards-directed Reedy categories”, that is, categories whose objects can be assigned a degree in $\mathbb{N}$ in such a way that all non-identity maps raise degree. Maps of diagrams over such categories can be built inductively, degree by degree, and this is what makes it easy to identify the projective cofibrations. See Corollary 3.5.8 below for an example, or [75, Section 14] for a detailed discussion.

Remark 3.5.3 (Comparing diagram categories). Suppose $f : I \to J$ is a functor between small categories. Then there is an induced “restriction” map $f_*: \mathcal{M}^J \to \mathcal{M}^I$, obtained by precomposition with f. The functor f_* has a left adjoint f^* given by left Kan extension, and the pair (f^*, f_*) is a Quillen pair between the projective model structures (since f_* clearly preserves objectwise fibrations and trivial fibrations). We will often make use of this Quillen pair.

We will not have need of the following, but note that f_* also has a right adjoint $f_!$ given by right Kan extension, and the pair $(f_*, f_!)$ is a Quillen pair when $\mathcal{M}^I$ and $\mathcal{M}^J$ are given the injective model structures.

Remark 3.5.4. We have assumed I is a small category, otherwise we run into set-theoretic difficulties in constructing $\mathcal{M}^I$. However, in applications one often wants to apply these ideas to non-small categories as well. One typical approach is to fix a Grothendieck universe and to redefine “small” to mean “small with respect to the universe”. Then one can still construct $\mathcal{M}^I$ for non-small I, but at the expense of passing to a larger universe.

If $I_0 \hookrightarrow I$ is a small skeletal subcategory, the adjoint functors from Remark 3.5.3 give an equivalence between $\mathcal{M}^I$ and $\mathcal{M}^{I_0}$. So one could instead just use $\mathcal{M}^{I_0}$ as a substitute for $\mathcal{M}^I$ and thereby ***avoid*** passing to the larger universe.

In practice a combination of these two ideas is often used, mostly without explanation. When I has a small skeletal subcategory one *can* stay on firm ground by using $\mathcal{M}^{I_0}$, and common practice is to regard this as allowing one to use $\mathcal{M}^I$ with impunity.

3.5.2 Enriched diagrams

If I is a category enriched over $s\mathcal{S}et$ and $\mathcal{M}$ is a simplicial model category, then one can look at enriched diagrams $X : I \to \mathcal{M}$. These are collections of objects X_i for $i \in I$ together with maps of simplicial sets $I(i, j) \to \underline{\mathcal{M}}(X_i, X_j)$ that satisfy the evident unital and associativity axioms. Here we will write $\mathcal{M}^I$ for the category of enriched diagrams, with the comment that in practice this abuse of notation never leads to any confusion. The analog of Theorem 3.5.1 still holds for enriched diagrams, and the proof is the same. The only modification is to realize that here one has $(F_i X)(j) = I(i, j) \otimes X$, where the simplicial tensor now replaces the previous $\odot$ symbol.

Similar results hold when $\mathcal{M}$ is a model category enriched over $\mathcal{T}op$ (satisfying the analog of SM7) and I is a $\mathcal{T}op$-enriched category, or the same with $\mathcal{T}op$ replaced by $\mathcal{T}op_*$. This will be the case most relevant to spectra.

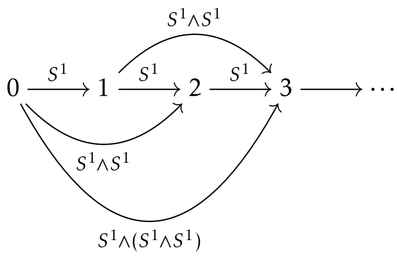
3.5.3 Spectra and diagram categories

Classically, a spectrum is a sequence of pointed spaces X_n together with maps $\Sigma X_n \to X_{n+1}$. Such an object does not manifestly suggest a diagram, but it turns out that spectra are ***precisely*** certain enriched diagrams. The key here is to realize that a map $\Sigma X_n \to X_{n+1}$ corresponds under the usual adjunctions to a pointed map $S^1 \to \underline{\mathcal{T}op}_*(X_n, X_{n+1})$ (where $\underline{\mathcal{T}op}_*(A, B)$ denotes the ***space*** of maps from A to B).

Define a $\mathcal{T}op_*$-enriched category Θ where the objects are non-negative integers n, and where

$$\Theta(k,n) = \begin{cases} * & \text{if } k > n, \\ (S^1)^{\wedge(n-k)} & \text{if } k \le n. \end{cases}$$

The pairings $\Theta(l,n) \wedge \Theta(k,l) \to \Theta(k,n)$ are the canonical maps obtained from the associativity isomorphisms for the smash product in $\mathcal{T}op_*$, and the identity maps in $\Theta(n,n)$ are given by the non-basepoint in S^0. It is routine to check that this really is a $\mathcal{T}op_*$-enriched category. Here is a depiction of the first few levels of Θ:



At this point it is an exercise to check that a classical spectrum is the same as an enriched diagram $\Theta \to \mathcal{T}op_*$.

3.5.4 The level model structure on classical spectra

We are going to construct this model category in two ways: by brute force (as is normally done) and then by the diagrammatic perspective. The two ways are really the same, but it is informative to see that firsthand.

So for the moment let us pause and start from scratch. A spectrum X is a sequence of pointed spaces X_n for $n \ge 0$ together with structure maps $\sigma_n \colon \Sigma X_n \to X_{n+1}$. A map of spectra $f \colon X \to Y$ is a collection of pointed maps $f_n \colon X_n \to Y_n$ such that the diagrams

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\sigma_X} & X_{n+1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma Y_n & \xrightarrow{\sigma_Y} & Y_{n+1} \end{array}$$

all commute. Let $\mathrm{Sp}^{\mathbb{N}}$ denote the resulting category.

Let $\mathrm{Ev}_n \colon \mathrm{Sp}^{\mathbb{N}} \to \mathcal{T}op_*$ be the functor $X \mapsto X_n$. This has a left adjoint which takes a pointed space W, puts it in level n, and generates a spectrum from that information in the freest way possible. Specifically, one readily checks that

$$(F_n W)_k = \begin{cases} * & \text{if } k < n, \\ \Sigma^{k-n} W & \text{if } k \geq n, \end{cases}$$

with the evident structure maps.

Exercise 3.5.5. Check that Ev_n also has a right adjoint $I_n\colon \mathcal{T}op_* \to \mathrm{Sp}^{\mathbb{N}}$, that sends a pointed space W to the spectrum with

$$(I_n W)_k = \begin{cases} \Omega^{n-k} W & \text{if } k \leq n, \\ * & \text{if } k > n, \end{cases}$$

again with the evident structure maps.

As another exercise with these adjoints, observe that there are canonical maps $F_{n+1}(\Sigma W) \to F_n W$ and $I_n W \to I_{n-1}(\Omega W)$. The first is an isomorphism in degrees larger than n, and the second is an isomorphism in degrees lower than n.

Theorem 3.5.6. *There exists a model category structure on* $\mathrm{Sp}^{\mathbb{N}}$ *in which a map* $f\colon X \to Y$ *is a weak equivalence (resp., fibration) if and only if* $f_n\colon X_n \to Y_n$ *is a weak equivalence (resp., fibration) for all* n*. This is called the* ***projective, level model structure*** *on* $\mathrm{Sp}^{\mathbb{N}}$.

Additionally, the adjoint pairs

$$\mathcal{T}op_* \underset{\mathrm{Ev}_n}{\overset{F_n}{\rightleftarrows}} \mathrm{Sp}^{\mathbb{N}} \qquad \textit{and} \qquad \mathrm{Sp}^{\mathbb{N}} \underset{I_n}{\overset{\mathrm{Ev}_n}{\rightleftarrows}} \mathcal{T}op_*$$

are Quillen pairs (with the left adjoint always drawn on top, going left to right).

Proof. We explain the proof in two ways. The first is to take the generating cofibrations and trivial cofibrations in $\mathcal{T}op_*$ and apply all the functors F_n to them, thereby getting generating sets for $\mathrm{Sp}^{\mathbb{N}}$. The model structure then basically constructs itself, using the small object argument. The second way, which says the same thing, is to use the observation that $\mathrm{Sp}^{\mathbb{N}}$ is secretly the category $\mathcal{T}op_*^{\Theta}$ and then use Theorem 3.5.1(a).

For the statements about Quillen pairs, the right adjoints Ev_n and I_n clearly preserve fibrations and trivial fibrations. □

Remark 3.5.7. Using Theorem 3.5.1(b) there is also a "level, injective" model category structure on $\mathrm{Sp}^{\mathbb{N}}$, which is sometimes useful. However, the model structures derived from the projective one end up having better properties when we get to symmetric and orthogonal spectra. See Remark 3.7.8(2).

The category Θ acts like an upwards-directed Reedy category, in the sense that all the interesting maps raise degree. As in Remark 3.5.2, this is a case where we can explicitly identify the projective cofibrations:

Corollary 3.5.8. *A map of spectra* $f\colon X \to Y$ *is a cofibration in the projective, level model structure if and only if the evident maps*

$$X_n \amalg_{\Sigma X_{n-1}} \Sigma Y_{n-1} \longrightarrow Y_n$$

are cofibrations for all n*, where by convention we set* $X_{-1} = Y_{-1} = *$.

Sketch of proof. Let $W \xrightarrow{\sim} Z$ be a levelwise trivial fibration of spectra, and suppose given a square

$$\begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

We will attempt to produce a lifting $Y \to W$ by constructing it inductively on the levels. At level 0 we have the diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & W_0 \\ \downarrow & & \downarrow \simeq \\ Y_0 & \longrightarrow & Z_0 \end{array}$$

and so get a lifting if $X_0 \to Y_0$ is a cofibration. At level 1 we have a similar diagram, but we can't just take any lifting — because we need the map $Y_1 \to W_1$ to be compatible with the already chosen $Y_0 \to W_0$. This compatibility is encoded by the diagram

$$\begin{array}{ccc} X_1 \amalg_{\Sigma X_0} (\Sigma Y_0) & \longrightarrow & W_1 \\ \downarrow & & \downarrow \simeq \\ Y_1 & \longrightarrow & Z_1 \end{array}$$

and we will get a lift provided the vertical map on the left is a cofibration. Continuing inductively in the evident manner, one sees that a map satisfying the conditions started in the corollary is a cofibration in the projective level model structure.

For the converse, assume $X \to Y$ is a projective cofibration and suppose given a lifting diagram

$$\begin{array}{ccc} X_n \amalg_{\Sigma X_{n-1}} \Sigma Y_{n-1} & \longrightarrow & W \\ \downarrow & & \downarrow \simeq \\ Y_n & \longrightarrow & Z \end{array}$$

This adjoints over to a diagram

$$\begin{array}{ccc} X & \longrightarrow & I_n W \\ \downarrow & & \downarrow \\ Y & \longrightarrow & I_n Z \times_{I_{n-1}(\Omega Z)} I_{n-1}(\Omega W) \end{array}$$

and the right vertical map is a levelwise trivial fibration by inspection, so there is a lifting. Now adjoint back. □

Remark 3.5.9. The level model structure is a rather formal thing, not capturing any kind of stabilization phenomenon. It treats spectra as mere diagrams, and not as true stable objects. For example, a spectrum X and its truncation $\{*, X_1, X_2, \ldots\}$ should represent the same "stable object", but the level model structure sees them as different. The suspension functor on $\mathrm{Sp}^{\mathbb{N}}$ just applies suspension objectwise, and clearly this is not an equivalence on the homotopy category level — so we do not have a stable model category. In Section 3.6 we will see how to impose relations into the level model structure that incorporate stability.

3.5.5 The level model structure on coordinate-free spectra

This is an easy modification of what we have already done. Fix a May universe $\mathcal{U}$, as in Section 3.1.5. For $V \subseteq W \subseteq \mathcal{U}$, write $W - V$ for the orthogonal complement of V in W. Define a **coordinate-free spectrum** to be an assignment $V \mapsto X_V$ for $V \subseteq \mathcal{U}$ a finite-dimensional subspace, together with maps $S^{W-V} \wedge X_V \to X_W$ for every pair $V \subseteq W$, subject to the evident unital and associativity conditions. Write $\mathrm{Sp}^{\mathcal{U}}$ for the evident category of coordinate-free spectra on $\mathcal{U}$.

Define a $\mathcal{T}op_*$-enriched category $\Theta_{\mathcal{U}}$ whose objects are the finite-dimensional subspaces of $\mathcal{U}$. Let the morphisms be given by

$$\Theta_{\mathcal{U}}(V,W) = \begin{cases} S^{W-V} & \text{if } V \subseteq W, \\ * & \text{otherwise.} \end{cases}$$

For $V \subseteq W \subseteq Z$, the evident isomorphism $S^{Z-W} \wedge S^{W-V} \to S^{Z-V}$ gives a composition map for Θ that is readily checked to be unital and associative. Observe that an enriched diagram $\Theta_{\mathcal{U}} \to \mathcal{T}op_*$ is the same as a coordinate-free spectrum defined on $\mathcal{U}$.

The projective model structure on the diagram category $\mathcal{T}op_*^{\Theta_{\mathcal{U}}}$ is called the projective, level model structure on $\mathrm{Sp}^{\mathcal{U}}$.

To compare this construction to classical spectra, pick an orthonormal basis $e_1, e_2, \dots$ for $\mathcal{U}$ and let $\mathbb{R}^n$ be the span of the first n basis elements. Consider the particular linear map $\mathbb{R} \to \mathbb{R}^{n+1} - \mathbb{R}^n$ sending 1 to e_{n+1}, so that compactifying gives us a preferred homeomorphism $S^1 \cong S^{(\mathbb{R}^{n+1}-\mathbb{R}^n)}$. If X is a coordinate-free spectrum then the assignment $[n] \mapsto X_{\mathbb{R}^n}$ gives a classical spectrum. Let $U\colon \mathrm{Sp}^{\mathcal{U}} \to \mathrm{Sp}^{\mathbb{N}}$ denote this forgetful functor. From the diagrammatic viewpoint we have described an embedding $j\colon \Theta \hookrightarrow \Theta_{\mathcal{U}}$ and U is just restriction along this embedding. Category theory automatically tells us that U has a left adjoint G: it sends a spectrum $X\colon \Theta \to \mathcal{T}op_*$ to its left Kan extension along j. Note that $(GX)_V$ is an appropriate (enriched) colimit over the category of all $\mathbb{R}^n$ contained in V. One easy but important fact is that the map $X_n \to (UGX)_n$ is an isomorphism, for all n.

It is formal that the pair $G\colon \mathrm{Sp}^{\mathbb{N}} \rightleftarrows \mathrm{Sp}^{\mathcal{U}} \colon U$ is a Quillen pair, since U preserves fibrations and trivial fibrations. It is of course not a Quillen equivalence, because we are using the levelwise model structures. This will change when we pass to the stable model structures in the next section.

Remark 3.5.10 (Change of universe). Suppose that $\mathcal{U}$ and $\mathcal{U}'$ are two May universes, and $f\colon \mathcal{U} \to \mathcal{U}'$ is an isometry (which will necessarily be injective, but possibly not surjective). Then there is an enriched functor $\Theta_{\mathcal{U}} \to \Theta_{\mathcal{U}'}$ that on objects behaves as $V \mapsto f(V)$ and on maps as $S^{W-V} \mapsto S^{f(W)-f(V)}$ (induced by f). We therefore get a restriction functor $f_*\colon \mathcal{T}op_*^{\Theta'_{\mathcal{U}}} \to \mathcal{T}op_*^{\Theta_{\mathcal{U}}}$ and its left adjoint f^* as in Remark 3.5.3.

Again, these are not Quillen equivalences — but their analogs will become Quillen equivalences after stabilization.

3.6 Localization and the stable model structures on spectra

In this section we will see how to modify the level model structure on spectra in a way that captures true stable phenomena. This uses a technique that is now called Bousfield localization, although it of course did not have this name when it first appeared back in [56]. Here we review the relevant model category theoretic techniques and then we repeat the work of [56] to obtain the stable structure on spectra. This works in both the classical and coordinate-free contexts. See also Chapter 7 of this volume for more on Bousfield localization.

3.6.1 Homotopy mapping spaces

Let $\mathcal{M}$ be a model category. To any two objects X and Y in $\mathcal{M}$ one can associate a homotopy mapping space $\underline{\mathcal{M}}(X,Y)$, also sometimes called a homotopy function complex. This is a simplicial set, well defined up to weak homotopy equivalence, which only depends on the weak homotopy types of X and Y. Given maps $X \to X'$ and $Y \to Y'$ one can construct models for these function complexes that come with maps $\underline{\mathcal{M}}(X',Y) \to \underline{\mathcal{M}}(X,Y)$ and $\underline{\mathcal{M}}(X,Y) \to \underline{\mathcal{M}}(X,Y')$.

Here are four standard ways to construct models of $\underline{\mathcal{M}}(X,Y)$:

(1) Replace X by a cosimplicial resolution QX^*, choose a fibrant replacement $Y \to RY$, and use the simplicial set $\mathcal{M}(QX^*,RY)$ obtained by applying $\mathcal{M}(-,RY)$ to QX^*.
(2) Choose a cofibrant replacement $QX \to X$, a simplicial resolution $Y \to RY_*$, and use the simplicial set $\mathcal{M}(QX,RY_*)$.
(3) Use nerves of categories of zig-zags from X to Y to form the so-called ***hammock localization space*** $L_H\mathcal{M}(X,Y)$.
(4) When $\mathcal{M}$ is a simplicial model category, choose a cofibrant replacement $QX \to X$ and a fibrant replacement $Y \to RY$ and use the simplicial mapping space from QX to RY.

See [124] and [130] for more on (1) and (2), and [88] or Chapter 2 of this volume for (3). But all the model categories considered in this chapter are simplicial, so feel free to focus on (4). Depending on the context one might also write $\mathrm{Map}(X,Y)$ or $\mathrm{hMap}(X,Y)$ as a synonym for $\underline{\mathcal{M}}(X,Y)$.

3.6.2 Localization of model categories

Given a model category $\mathcal{M}$ and a collection of maps T in $\mathcal{M}$, one sometimes wants to construct a new model category structure that is obtained from $\mathcal{M}$ by adjoining the maps in T to the already existing weak equivalences. This will likely force even more maps to be weak equivalences (at the very least one has to close up the set under two-out-of-three), and at least one of the notions of cofibration/fibration will have to change as well. The main technique for accomplishing this is called *Bousfield localization.*

Definition 3.6.1. Let $\mathcal{M}$ be a model category and let T be a set of maps in $\mathcal{M}$.

(a) An object X in $\mathcal{M}$ is ***T*-local** if, for all $f\colon A \to B$ in T, the induced map $\underline{\mathcal{M}}(B,X) \to \underline{\mathcal{M}}(A,X)$ is a weak equivalence.
(b) A map $f\colon A \to B$ is a ***T*-local equivalence** if, for all T-local objects X, the induced map $\underline{\mathcal{M}}(B,X) \to \underline{\mathcal{M}}(A,X)$ is a weak equivalence.

Briefly, an object X is T-local if it sees all the maps in T as weak equivalences, where "see" amounts to looking at things from the perspective of $\underline{\mathcal{M}}(-,X)$. Likewise, the T-local equivalences are the maps that are seen as weak equivalences by all the T-local objects. So the T-local equivalences include all of T, but will usually include other maps as well.

The following result is due to Hirschhorn [124] in the cellular case, and to Jeff Smith in the combinatorial case (see [35] for a written account):

Theorem 3.6.2. *Let $\mathcal{M}$ be a cellular or combinatorial model category, and let T be a set of maps in $\mathcal{M}$. Then there exists a new model structure $T^{-1}\mathcal{M}$ on the same underlying category as $\mathcal{M}$ such that*

(i) *the cofibrations in $T^{-1}\mathcal{M}$ are the same as the cofibrations in $\mathcal{M}$,*
(ii) *the weak equivalences in $T^{-1}\mathcal{M}$ are the T-local equivalences,*
(iii) *the fibrations are the maps with the right lifting property with respect to cofibrations that are T-local equivalences.*

Moreover, an object X is fibrant in $T^{-1}\mathcal{M}$ if and only if X is fibrant in $\mathcal{M}$ and X is T-local. Finally, if X and Y are T-local then a map $f\colon X \to Y$ is a weak equivalence in $T^{-1}\mathcal{M}$ if and only if it is a weak equivalence in $\mathcal{M}$.

When it exists, the model category $T^{-1}\mathcal{M}$ is called the **left Bousfield localization** of $\mathcal{M}$ at the set T. A fibrant replacement functor in $T^{-1}\mathcal{M}$ is called a ***T*-localization** functor.

Remark 3.6.3. It is useful to know a bit about how Theorem 3.6.2 is proven and about the construction of the localization functor. For each map in T choose a model $f\colon A \rightarrowtail B$ that is a cofibration. For each simplicial horn $j\colon \Lambda^{n,k} \hookrightarrow \Delta^n$ consider the box product $j \,\square\, f$, which is the map

$$j \,\square\, f\colon (\Lambda^{n,k} \otimes B) \amalg_{(\Lambda^{n,k}\otimes A)} (\Delta^n \otimes A) \longrightarrow \Delta^n \otimes B.$$

Here the tensor refers to the simplicial tensor if $\mathcal{M}$ is a simplicial model category, or more generally it refers to a version built using cosimplicial frames (see [124] for details). Formally add these maps $j \,\square\, f$ (for every j and f) to the set of generating trivial cofibrations of $\mathcal{M}$, and then repeat the small object argument using this new set to get the required factorization. In particular, the localization functor is obtained as a transfinite composition of cobase changes of the generating trivial cofibrations in $\mathcal{M}$ together with the maps $j \square f$.

3.6.3 Bousfield–Friedlander spectra

If X is a spectrum and $n \geq 0$, define the n-truncation $\tau_{\geq n}X$ to be the spectrum $\{*,*,\ldots,*,X_n,X_{n+1},\ldots\}$. There is a natural map $\tau_{\geq n}X \to X$. Our basic goal will be to localize the level, projective model structure on spectra at the class $\{\tau_{\geq n}X \to X \mid n, X\}$. However, this class is not a set and so the first task is to pare it down somewhat. To this end, define

$$\mathcal{T} = \{\tau_{\geq(n+1)}F_n(S^k) \to F_n(S^k) \mid n,k \geq 0\}.$$

Observe that $\tau_{\geq(n+1)}F_n(X)$ is canonically isomorphic to $F_{n+1}(\Sigma X)$, so we can also describe the set as

$$\mathcal{T} = \{F_{n+1}(S^{k+1}) \to F_n(S^k) \mid n,k \geq 0\},$$

where the map in question is adjoint to the identity $S^{k+1} \to \mathrm{Ev}_{n+1}(F_nS^k)$.

Definition 3.6.4. The **stable projective model structure** on $\mathrm{Sp}^{\mathbb{N}}$ is the localization of the level projective model structure at the set $\mathcal{T}$.

Let us analyze the $\mathcal{T}$-local objects. Here the relevant observation is that

$$\underline{\mathrm{Sp}^{\mathbb{N}}}\big(F_n(S^k),X\big) \simeq \underline{\mathcal{T}op_*}(S^k,X_n)$$

by adjoint functors. If f denotes our map $F_{n+1}(S^{k+1}) \to F_n(S^k)$ then on mapping spaces this is

$$\begin{array}{ccccc} \underline{\mathrm{Sp}^{\mathbb{N}}}\big(F_nS^k,X\big) & \longrightarrow & \underline{\mathrm{Sp}^{\mathbb{N}}}\big(F_{n+1}S^{k+1},X\big) & & \\ \simeq\downarrow & & \simeq\downarrow & & \\ \underline{\mathcal{T}op_*}(S^k,X_n) & \longrightarrow & \underline{\mathcal{T}op_*}(S^{k+1},X_{n+1}) & = & \underline{\mathcal{T}op_*}(S^k,\Omega X_{n+1}) \end{array}$$

and one checks that the lower horizontal composite is induced by the structure map $X_n \to \Omega X_{n+1}$. So a spectrum X is $\mathcal{T}$-local precisely when it is an Ω-spectrum.

Remark 3.6.5. We only needed $k=0$ to make this argument. So the maps in $\mathcal{T}$ where $k>0$ represent redundant information, and we could throw them out of $\mathcal{T}$ and still get the same localization.

For the following result, recall that if X is a spectrum and $n \in \mathbb{Z}$ then

$$\pi_n(X) = \mathrm{colim}_k\, \pi_{n+k}(X_k)$$

where the maps in the colimit system are induced by the structure maps of X.

Proposition 3.6.6. *In the stable projective model structure on* $\mathrm{Sp}^{\mathbb{N}}$,

(a) *the fibrant objects are the levelwise fibrant* Ω*-spectra, and*

(b) *a map $f\colon X \to Y$ is a weak equivalence if and only if it induces isomorphisms $\pi_n(X) \to \pi_n(Y)$ for all $n \in \mathbb{Z}$.*

Note that the levelwise fibrant condition is vacuous if we are defining spectra in terms of topological spaces, but not if we are doing so in terms of simplicial sets.

Proof. We have already proven (a). For (b), first note that for a map of Ω-spectra the notions of level weak equivalence, π_*-isomorphism, and stable equivalence all coincide: level equivalence = stable equivalence by the last line of Theorem 3.6.2, and level equivalence = π_*-isomorphism by inspection.

Next consider the map $f_{n,k}\colon F_{n+1}(S^{k+1}) \to F_n(S^k)$. This is an isomorphism in levels $n+1$ and higher, so this same property passes to any cobase change. Hence any cobase change of an $f_{n,k}$ is a π_*-isomorphism. Similarly, for any set of horns $j\colon \Lambda^{p,r} \hookrightarrow \Delta^p$ the box product $j \,\square\, f_{n,k}$ is also an isomorphism in levels $n+1$ and higher. It follows that any map obtained from these box products by cobase changes and transfinite compositions is a π_*-isomorphism. In particular, the fibrant replacement functors $X \to RX$ in the stable projective structure are made this way (see Remark 3.6.3) and are therefore π_*-isomorphisms.

Finally, suppose given a map of spectra $g\colon X \to Y$ and consider the square

$$\begin{array}{ccc} X & \longrightarrow & RX \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Rg} \\ Y & \longrightarrow & RY \end{array}$$

The horizontal maps are both stable equivalences and π_*-isomorphisms. So g is a stable equivalence (resp., π_*-isomorphism) if and only if Rg is so. But RX and RY are Ω-spectra, so the conditions of Rg being a stable equivalence or π_*-isomorphism are equivalent; hence, the same must hold for g. □

In general, it can be very hard to give a nice description for the fibrations in a Bousfield localization. In the present case one can actually do it, though. Note that since there are more trivial cofibrations in $\mathcal{T}^{-1}\mathcal{M}$ than in $\mathcal{M}$, there will be fewer fibrations.

Proposition 3.6.7. *For a spectrum X, let $QX = \operatorname{hocolim}_n \Omega^n X_n$. Then a map of spectra $X \to Y$ is a fibration in the projective stable structure on $\mathrm{Sp}^{\mathbb{N}}$ if and only if it is a levelwise fibration and for every $n \geq 0$ the square*

$$\begin{array}{ccc} X_n & \longrightarrow & QX \\ \downarrow & & \downarrow \\ Y_n & \longrightarrow & QY \end{array}$$

is homotopy Cartesian.

Proof. See [56]. In that paper the projective stable category is not constructed by Bousfield localization, but directly by brute force. The cofibrations and weak equivalences, however, match the ones in our structure, and fibrations are always determined by the trivial cofibrations, so the two structures are in fact the same. □

3.6.4 The coordinate-free setting

Recall the coordinate-free setting of Section 3.5.5. Here we localize at the maps $F_W(S^{W-V} \wedge S^k) \to F_V(S^k)$ for all k and all pairs $V \subseteq W \subseteq \mathcal{U}$. The functor G from Section 3.5.5 sends the maps in $\mathcal{T}$ to these kinds of maps, so by general localization theory the adjoint pair (G, U) descends to give Quillen functors between the resulting stable model categories:

$$G\colon \mathrm{Sp}^{\mathbb{N}}_{stable} \rightleftarrows \mathrm{Sp}^{\mathcal{U}}_{stable}\colon U.$$

By the same arguments that we have seen for $\mathrm{Sp}^{\mathbb{N}}$, the stable equivalences in $\mathrm{Sp}^{\mathcal{U}}$ are all π_*-isomorphisms. Since $X \to UGX$ is a levelwise isomorphism (see Section 3.5.5), it follows at once that the above pair is a Quillen equivalence.

We leave the reader to think about change of universe in this setting, building off of Remark 3.5.10.

3.7 Symmetric spectra

The definitions and basic results about symmetric spectra are very elegant and beautiful. Understanding the details of what is going on beneath the surface is a different matter. Our approach here will be to quickly survey the basic theory from [133] and then go back and work on some motivation afterwards.

Definition 3.7.1. A **symmetric sequence** in a category $\mathcal{C}$ is a collection of objects X_n together with group homomorphisms $\Sigma_n \to \mathrm{Aut}(X_n)$, for each $n \geq 0$.

It will be convenient to have a more diagrammatic way of phrasing this definition. Let ΣI be the subcategory of $\mathcal{S}et$ consisting of the objects $\underline{n} = \{1, 2, \ldots, n\}$ for $n \geq 0$ (with $\underline{0} = \emptyset$) together with all automorphisms. A symmetric sequence in $\mathcal{C}$ is simply a functor $X\colon \Sigma I \to \mathcal{C}$. As usual, we write $\mathcal{C}^{\Sigma I}$ for the category of all such functors.

Now assume that $(\mathcal{C}, \otimes, I, \mathcal{F})$ is closed symmetric monoidal and also cocomplete. Given symmetric sequences X and Y, define a new symmetric sequence $X \otimes Y$ by

$$(X \otimes Y)_n = \coprod_{p+q=n} (\Sigma_n)_+ \odot_{\Sigma_p \times \Sigma_q} (X_p \otimes X_q).$$

To explain the $\odot$ notation, regard any group G as a groupoid with one object and G as its endomorphism group. If $H \leq G$ and W is an object with a left H-action, then $G \odot_H W$ is the left Kan extension in the diagram

$$\begin{array}{ccc} H & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \\ G & & \end{array}$$

Equivalently, one can write

$$G \odot_H W = \mathrm{coeq}\Big(\coprod_{G\times H} W \rightrightarrows \coprod_G W \Big),$$

where the top map sends the copy of W indexed by (g,h) to the copy of W indexed by g via left multiplication by h, and the bottom map sends the copy of W indexed by (g,h) to the copy of W indexed by gh via the identity. The action of G on $\coprod_G W$ by permutation of the factors descends to give a left action of G on $G \odot_H W$.

There is a self evident, though tedious to write down, associativity isomorphism $X\otimes(Y\otimes Z) \cong (X\otimes Y)\otimes Z$. Define the twist isomorphism $\tau_{X,Y}\colon X\otimes Y \to Y\otimes X$ on level n to be the coproduct of maps $\Sigma_n \odot_{\Sigma_a\times\Sigma_b} (X_a \otimes Y_b) \to \Sigma_n \odot_{\Sigma_b\times\Sigma_a} (Y_b\otimes X_a)$ (where $a+b=n$) sending $[\alpha, X_a\otimes Y_b]$ to $[\alpha\rho_{b,a}, Y_b\otimes X_a]$ via the twist map from $\mathcal{C}$, where $\rho_{b,a}\in\Sigma_n$ is the evident (b,a)-shuffle. It is a good exercise to check that without $\rho_{b,a}$ in the formula this is not a well-defined map, as it does not exhibit the required $\Sigma_a\times\Sigma_b$-equivariance; indeed, check that one needs to include a permutation ρ having the property that $(\beta_a|\gamma_b)\circ\rho = \rho\circ(\gamma_b|\beta_a)$ for every $\beta_a\in\Sigma_a$, $\gamma_b\in\Sigma_b$. The only permutation that does the job is $\rho=\rho_{b,a}$. (For a general schema that helps one quickly determine the correct permutation to use in situations like this, see Remark 3.7.9).

When $\mathcal{C}$ is complete one can also define a cotensor $X,Y\mapsto \mathcal{F}(X,Y)$ for symmetric sequences. Before giving the definition, let us record the basic property it should have:

Lemma 3.7.2. *Let X, Y, and Z be symmetric sequences in $\mathcal{C}$. There are natural bijections between the following three sets:*

(1) $\mathcal{C}^{\Sigma I}(X\otimes Y, Z)$;
(2) *collections of* $\Sigma_p\times\Sigma_q$*-equivariant maps* $X_p\otimes Y_q \to Z_{p+q}$ *for all* $p,q\geq 0$;
(3) $\mathcal{C}^{\Sigma I}(X, \mathcal{F}(Y,Z))$.

Parts (2) and (3) of the lemma lead one directly to the definition of the cotensor. For X and Y in $\mathcal{C}^{\Sigma I}$ define $\mathcal{F}(X,Y)$ by

$$\mathcal{F}(X,Y)_n = \prod_q \mathcal{F}(X_q, Y_{n+q})^{\Sigma_q},$$

where the Σ_q action is as follows. If $\alpha\in\Sigma_q$ then we have maps $\alpha_X\colon X_q\to X_q$ and $(id_n|\alpha)_Y\colon Y_{n+q}\to Y_{n+q}$, where $(id_n|\alpha)\in\Sigma_{n+q}$ is the map that permutes the last q elements according to α. Then α acts on $\mathcal{F}(X_q,Y_{n+q})$ via the composite

$$\mathcal{F}(X_q,Y_{n+q}) \xrightarrow{((id_n|\alpha)_Y)_*} \mathcal{F}(X_q,Y_{n+q}) \xrightarrow{(\alpha_X^{-1})^*} \mathcal{F}(X_q,Y_{n+q}).$$

This gives an action of Σ_q, and $\mathcal{F}(X_q,Y_{n+q})^{\Sigma_q}$ is the fixed object (the limit of the corresponding functor $\Sigma_q\to\mathcal{C}$). The action of Σ_n on Y_{n+q} coming from permutation of the first block of n elements descends to an action of Σ_n on $\mathcal{F}(X_q,Y_{n+q})^{\Sigma_q}$.

The following is a routine exercise:

Proposition 3.7.3. *With the above associativity and twist isomorphisms, the tensor product on $\mathcal{C}^{\Sigma I}$ is closed symmetric monoidal with unit $\mathbb{I}=\{I,\emptyset,\emptyset,\ldots\}$ and cotensor $\mathcal{F}(-,-)$.*

Now fix any object X in $\mathcal{C}$. Recall from Section 3.3.5 that $X^{\otimes n}$ is defined inductively by $X^{\otimes n} = X \otimes X^{\otimes(n-1)}$, and that there is a natural left action of Σ_n on $X^{\otimes n}$. Define $\mathbb{X}$ to be the symmetric sequence $\mathbb{X}_n = X^{\otimes n}$, and let $\mathbb{I} \to \mathbb{X}$ be the unique map that is the identity in level 0.

The associativity maps give natural isomorphisms $\mu_{a,b} \colon \mathbb{X}_a \otimes \mathbb{X}_b \to \mathbb{X}_{a+b}$. We use these to define a pairing $\mathbb{X} \otimes \mathbb{X} \to \mathbb{X}$ that on level n is the coproduct of maps

$$\Sigma_n \odot_{\Sigma_a \times \Sigma_b} (X^{\otimes a} \otimes X^{\otimes b}) \to X^{\otimes(a+b)},$$

which on the summand $[\alpha, X^{\otimes a} \otimes X^{\otimes b}]$ equals $\alpha \circ \mu_{a,b}$. One readily checks that this is well-defined and makes $\mathbb{X}$ into a commutative monoid. The category of left $\mathbb{X}$-modules then inherits a closed symmetric monoidal structure as in Section 3.3.2, where for example the tensor is $(-) \otimes_{\mathbb{X}} (-)$.

Definition 3.7.4. A **symmetric X-spectrum** is a left $\mathbb{X}$-module.

Unwinding the definitions, a left $\mathbb{X}$-module M is a sequence of objects M_n in $\mathcal{C}$ together with an action of Σ_n on M_n and structure maps

$$\alpha_{p,q} \colon X^{\otimes p} \otimes M_q \to M_{p+q}$$

that are $\Sigma_p \times \Sigma_q$-equivariant. The unital condition says that $\alpha_{0,q}$ is the identity, and associativity says that for $p = a + b$ one has $\alpha_{p,q} = \alpha_{a,b+q} \circ (id \otimes \alpha_{b,q})$; that is, the diagram

$$\begin{array}{ccccc}
X^{\otimes a} \otimes (X^{\otimes b} \otimes M_q) & \xrightarrow{id \otimes \alpha_{b,q}} & X^{\otimes a} \otimes M_{b+q} & \xrightarrow{\alpha_{a,b+q}} & M_{a+b+q} \\
\downarrow \cong & & & \nearrow \alpha_{p,q} & \\
(X^{\otimes a} \otimes X^{\otimes b}) \otimes M_q & \xrightarrow{\cong} & X^{\otimes(a+b)} \otimes M_q & &
\end{array}$$

is commutative. This implies that the maps $\alpha_{p,q}$ with $p > 1$ can be built up from the $\alpha_{1,*}$ maps.

So at the end of the day, a symmetric $\mathbb{X}$-spectrum is a collection of objects M_n in $\mathcal{C}$ equipped with a left Σ_n-action and structure maps $\alpha \colon X \otimes M_n \to M_{n+1}$ having the property that the iterated structure maps

$$X^{\otimes p} \otimes M_n \to M_{n+p}$$

are $\Sigma_p \times \Sigma_n$-equivariant, for all $n, p \geq 0$. Here "iterated structure map" means an evident composition of associativity maps with the structure maps α.

3.7.1 The model category of symmetric spectra

We now specialize to the case where $\mathcal{C}$ is $\mathcal{T}op_*$ and $X = S^1$. The spectrum $\mathbb{X} = \{S^0, S^1, S^2, \ldots\}$ is called the sphere spectrum and denoted simply by S. So symmetric spectra are just left S-modules. Write Sp^Σ for the category of symmetric spectra.

The evaluation map $\mathrm{Ev}_n \colon \mathrm{Sp}^\Sigma \to \mathcal{T}op_*$ has a left adjoint F_n given by

$$(F_n X)_k = \begin{cases} * & \text{if } k < n, \\ \Sigma_k \odot_{\Sigma_{k-n}} (S^{k-n} \wedge X) & \text{if } k \geq n, \end{cases}$$

where in the second line Σ_{k-n} sits in Σ_k as permutations of the front $(k-n)$-block. Note that there are canonical maps

$$F_{n+1}(S^1 \wedge X) \to F_n(X)$$

that are equal to the identity in level $n+1$. (Warning: Unlike the case of Bousfield–Friedlander spectra, these maps are not isomorphisms in degrees larger than $n+1$. See the discussion below for an example.)

Proposition 3.7.5. *There is a model category structure on* Sp^Σ *where the weak equivalences and fibrations are objectwise. This is called the* ***level, projective model structure****.*

Proof. One can do this directly using the functors F_n and Kan's Recognition Theorem, just as we did for Bousfield–Friedlander spectra. Alternatively, one can realize that symmetric spectra are just certain enriched functors and use Theorem 3.5.1(a). See Section 3.7.4 below for more on this perspective. □

Definition 3.7.6. The **projective stable model structure** on Sp^Σ is the left Bousfield localization of the projective level model category structure at the set of maps $\{F_{n+1}(S^1 \wedge S^k) \to F_n(S^k) \mid n,k \geq 0\}$.

Say that a symmetric spectrum is an Ω-spectrum if its underlying classical spectrum is an Ω-spectrum. Here is the main foundational result about symmetric spectra, pulling together various statements from [133]:

Theorem 3.7.7.

(a) *The projective stable model structure on* Sp^Σ *is a stable, closed symmetric monoidal model category satisfying the Monoid Axiom as well as the Algebraic Creation and Invariance Properties.*
(b) *The fibrant objects are the objectwise fibrant* Ω*-spectra.*
(c) *The forgetful functor* $U\colon \mathrm{Sp}^\Sigma \to \mathrm{Sp}^\mathbb{N}$ *has a left adjoint* G*, and the adjoint pair* $G\colon \mathrm{Sp}^\mathbb{N} \rightleftarrows \mathrm{Sp}^\Sigma\colon U$ *is a Quillen equivalence between the projective stable model structures.*

Remarks 3.7.8.

(1) Part (b) is automatic from the way we choose the maps to localize, just as for Bousfield–Friedlander spectra.
(2) In (a) it suffices to verify the Pushout-Product Axiom for box products of generating cofibrations and trivial cofibrations. This is where it is finally important that we started with the projective level structure and not the injective level structure. In the former, the generating maps are well understood and it is easy to analyze their box products. In the latter, there are far too many cofibrations and in fact the Pushout-Product Axiom does not hold.
(3) The Quillen equivalence in part (c) is not unexpected, but it is not as easy as one might think. The left adjoint just comes as in Remark 3.5.3, and the fact that it is a Quillen pair is easy. But the equivalence part takes a bit of work. See Section 3.10.3 for further discussion.

(4) The precise references for the different parts of Theorem 3.7.7 are these: monoidal model category, [133, 5.3.8]; monoid axiom, [133, 5.4.1]; Algebraic Creation Property, [133, 5.4.2 and 5.4.3]; Algebraic Invariance Property, [133, 5.4.5]; Strong Flatness Property, [133, 5.4.4]; Quillen equivalence with $\mathrm{Sp}^{\mathbb{N}}$, [133, 4.2.5].

The derived functors of the Quillen equivalence from Theorem 3.7.7(c) give an equivalence of categories

$$\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}) \underset{\mathbb{R}U}{\overset{\mathbb{L}G}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Sp}^{\Sigma}).$$

A common misconception is to confuse $\mathbb{R}U$ and U. That is, if E is a symmetric spectrum, it is tempting to believe that the image of E in $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}})$ is represented by the underlying classical spectrum UE. This is false in general — an example is $E = F_1(S^1)$, discussed below. Two other related issues are these:

(1) The functor U does *not* preserve all stable weak equivalences.
(2) If X is a symmetric spectrum then define

$$\pi_n^{\text{naive}}(X) = \pi_n(UX) = \operatorname{colim}_k \pi_{n+k}(X_k).$$

It is *not* true that all stable weak equivalences induce isomorphisms on $\pi_*^{\text{naive}}(-)$. In particular, the groups $\pi_*^{\text{naive}}(X)$ are not guaranteed to be the “correct” homotopy groups unless X is fibrant.

One source of confusion here is that $\pi_*^{\text{naive}}(X)$ sometimes *are* the correct homotopy groups even when X is not fibrant. The paper [261] gives a detailed discussion of which spectra X are well-behaved in this regard.

The following example from [133, Example 3.1.10] demonstrates (1) and (2) above. It is worth examining in some detail. Consider the canonical map $f\colon F_1(S^1) \to F_0(S^0)$ that is the identity in level 1. This is one of the maps we localized to form the stable model structure, so it is a stable weak equivalence by definition. Note that $F_0(S^0)$ is just the sphere spectrum S. For X any pointed space, $(F_1X)_n = \Sigma_n \odot_{\Sigma_{n-1}} ((S^1)^{\wedge(n-1)} \wedge X)$ for $n \geq 1$; in particular, $(F_1S^1)_n = \Sigma_n \odot_{\Sigma_{n-1}} S^n$. As a space, this is a wedge of n copies of S^n, and the copies may be regarded as indexed by the set of permutations $T_n = \{Id, (1n), (2n), \ldots, (n{-}1, n)\}$ (these are coset representatives for Σ_n/Σ_{n-1}). Our map f takes the form

$$\begin{array}{ccccc}
S^0 & S^1 & S^2 & S^3 & \cdots \\
\uparrow & \uparrow{=} & \uparrow & \uparrow & \\
* & S^1 & \bigvee_{T_2}(S^1 \wedge S^1) & \bigvee_{T_3}(S^1 \wedge S^1 \wedge S^1) & \cdots
\end{array}$$

where in each level the component indexed by $\alpha \in T_n$ is mapped into S^n via the canonical identification followed by α.

Of course we know $\pi_0^{\text{naive}}(S) = \mathbb{Z}$. The colimit system for $\pi_0^{\text{naive}}(F_1 S^1)$ is

$$0 \to \mathbb{Z} \hookrightarrow \mathbb{Z}^2 \hookrightarrow \mathbb{Z}^3 \hookrightarrow \mathbb{Z}^4 \hookrightarrow \cdots,$$

where in each case the group includes into the next as a direct summand. So $\pi_0^{\text{naive}}(F_1 S^1)$ is an infinite direct sum of copies of $\mathbb{Z}$. In particular, we see that Uf is not a stable equivalence and (equivalently) that f does not induce isomorphisms on $\pi_*^{\text{naive}}(-)$. Note that $\pi_*^{\text{naive}}(-)$ gives the "correct" answer for S, but not for $F_1 S^1$.

3.7.2 Understanding the smash product

Let us open up the definition of the smash product and look inside. If X and Y are symmetric spectra (left S-modules) recall that $X \wedge Y$ (also known as $X \wedge_S Y$) is the coequalizer of $X \otimes S \otimes Y \rightrightarrows X \otimes Y$. Note that here X is being implicitly converted from a left S-module into a right S-module via the twist map. Looking level by level, we find that $(X \wedge Y)_n$ is the coequalizer of

$$\bigvee_{a+b+c=n} \Sigma_n \odot_{\Sigma_a \times \Sigma_b \times \Sigma_c} (X_a \wedge (S^1)^{\otimes b} \wedge Y_c) \rightrightarrows \bigvee_{p+q=n} \Sigma_n \odot_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q).$$

This looks scary, but we can tame things a bit by adopting a more algebraic notation, which we now pause to explain.

If $a + b + c + d + e = n$ write $\rho_{a[b]c[d]e}$ for the permutation in Σ_n that interchanges the b-block and the d-block and otherwise maintains the internal order within all 5 blocks. When a or c or e is zero we will drop them from the notation. Also, if $\alpha \in \Sigma_p$ and $\beta \in \Sigma_q$ write $\alpha|\beta \in \Sigma_{p+q}$ for the permutation that is α on the front p-block and β on the back q-block.

Let us denote elements of symmetric groups by Greek letters, elements of $(S^1)^{\wedge n}$ by capital Roman letters, and elements of X_* and Y_* by lowercase Roman letters. In addition, we write subscripts x_n to denote elements of degree n, e.g., $x_n \in X_n$. Denote the iterated structure map $(S^1)^{\wedge n} \wedge X_p \to X_{p+n}$ by $(A_n, x_p) \mapsto A_n x_p$, and the Σ_n action on X_n by $(\alpha_n, x_n) \mapsto \alpha_n x_n$. Observe that the equivariance of the structure map is the relation

$$(\alpha_n A_n)(\beta_p x_p) = (\alpha_n|\beta_p)(A_n x_p). \tag{3.7.3}$$

We claim that the spaces $(X \wedge Y)_n$ consist of all elements $\alpha_n[x_p \wedge y_q]$ for $p + q = n$ subject to the following relations:

(1) $(\alpha_n(\beta_p|\gamma_q))[x_p \wedge y_q] = \alpha_n[\beta_p x_p \wedge \gamma_q y_q]$ for $p + q = n$.

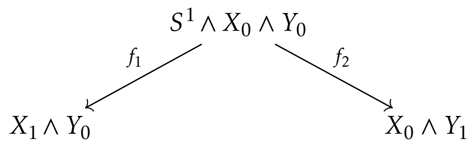
(2) $A_k[x_r \wedge y_s] = A_k x_r \wedge y_s = \rho_{[r][k]s}[x_r \wedge A_k y_s]$.

(3) $(\alpha_k A_k)(\gamma_{r+s}[x_r \wedge y_s]) = (\alpha_k|\gamma_{r+s})(A_k x_r \wedge y_s) = (\alpha_k|\gamma_{r+s})\rho_{[r][k]s}[x_r \wedge A_k y_s]$

Relation (2) is a special case of (3); we have listed it separately because it is easier to absorb in this simpler form. Also, relation (3) is really just relation (2) plus equivariance.

Remark 3.7.9. There is a procedure for determining the permutations ρ appearing in formulas like the ones above. For an equation of the form "ρ(formula P) = formula Q", regard each subscript u in P as a block of u symbols. Then ρ is the permutation that rearranges the blocks as listed in P into the order listed in Q. For example, in equation (2) consecutive blocks of length r, k, and s must be reordered by bringing the k-block in front of the r-block.

As an example of how to use the above notation, we work out $X \wedge Y$ in the first three levels. Level 0 is easy, as there are no relations: $(X \wedge Y)_0 = X_0 \wedge Y_0$. Level 1 has $(X \wedge Y)_1 = [(X_0 \wedge Y_1) \wedge (X_1 \wedge Y_0)]/\sim$, where the relation is $A_1(x_0 \wedge y_0) = (A_1 x_0) \wedge y_0 = x_0 \wedge (A_1 y_0)$. If desired we can translate this back into categorical language and say that $(X \wedge Y)_1$ is the pushout of the diagram



with $f_1(A_1, x_0, y_0) = A_1 x_0 \wedge y_0$ and $f_2(A_1, x_0, y_0) = x_0 \wedge A_1 y_0$.

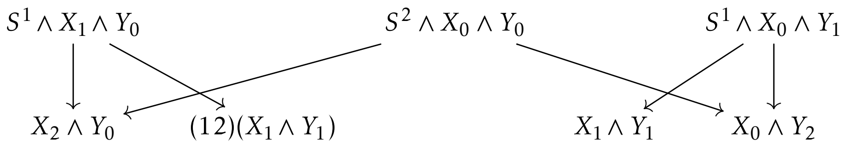
In general, for $(X \wedge Y)_n$ one writes down a big wedge of $X_p \wedge Y_q$ (with extra symmetric groups out front) and then quotients by relations coming from structure maps out of lower levels. So for $n = 2$ we start with

$$(X_2 \wedge Y_0) \vee (X_1 \wedge Y_1) \vee (12)(X_1 \wedge Y_1) \vee (X_0 \wedge Y_2),$$

where (12) is the generator of Σ_2 and appears here as a bookkeeping factor. The relations are

$$(A_2 x_0) \wedge y_0 = x_0 \wedge A_2 y_0, \quad A_1 x_0 \wedge y_1 = x_0 \wedge A_1 y_1, \quad A_1 x_1 \wedge y_0 = \rho_{[1],[1]} x_1 \wedge A_1 y_0.$$

Translating again to categorical language, $(X \wedge Y)_2$ is the colimit of a diagram



where the maps are easily written down from the algebraic relations. As an exercise, check that when $Y = S$ this colimit gives exactly X_2. Note that this fixes the problem we saw in our naive attempt back in Section 3.1.3, where the factors $X_1 \wedge Y_1$ and $(12)(X_1 \wedge Y_1)$ were compressed into a single term.

This discussion also leads to the following useful fact:

Proposition 3.7.10. *Let X, Y, and Z be symmetric spectra. To give a map of symmetric spectra $X \wedge Y \to Z$ is equivalent to giving maps $X_p \wedge Y_q \to Z_{p+q}$ for all $p, q \geq 0$ that are $\Sigma_p \times \Sigma_q$-equivariant and satisfy the identities*

$$A_k(x_p \cdot y_q) = A_k x_p \cdot y_q = \rho_{[p],[k],q}(x_p \cdot A_k y_q).$$

A pairing $X \wedge X \to Z$ is commutative if it also satisfies the identity

$$x_p \cdot x'_q = \rho_{[q],[p]}(x'_q \cdot x_p).$$

Proof. For the first claim, note that relation (3) is a consequence of the listed relations and the equivariance of the structure maps in Z. The second claim is routine. □

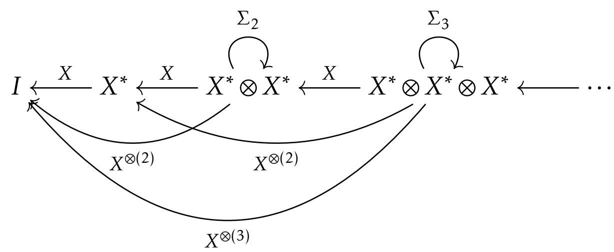
This would be a good moment to see some examples of symmetric ring spectra, but most of the standard examples are also examples of orthogonal ring spectra and it is clearer to discuss them in that context. The curious reader might wish to look ahead at Section 3.8.8.

3.7.4 Symmetric spectra and diagram categories

Let $\mathcal{C}$ be a closed symmetric monoidal category and let X be an invertible object in $\mathcal{C}$. Let X^* and $\alpha\colon I \xrightarrow{\cong} X^* \otimes X$ be a choice for inverse, and recall the dual map $\hat{\alpha}\colon X \otimes X^* \to I$ from Section 3.3.5. The adjoint of $\hat{\alpha}$ is a map $X \to \mathcal{F}(X^*, I)$, and more generally we get canonical maps

$$X^{\otimes(k)} \to \mathcal{F}\Big((X^*)^{\otimes(n+k)}, (X^*)^{\otimes(n)}\Big) \tag{3.7.5}$$

adjoint to the map $X^{\otimes(k)} \otimes (X^*)^{\otimes(n+k)} \to (X^*)^{\otimes(n)}$ that reverses the order of the tensor factors in $X^{\otimes(k)}$ and then uses $\hat{\alpha}$ repeatedly to eliminate adjacent factors of X and X^* (note that there are various associativity isomorphisms as well, but we are ignoring them). This leads to the following picture of elements in $\mathcal{C}$ and canonical "maps" between them, where an arrow from A to B labeled Z means a map $Z \to \mathcal{F}(A, B)$ and Σ_n acts on the ***right*** of $(X^*)^{\otimes(n)}$ by permutation of the factors:



(3.7.6)

Remark 3.7.11. There are canonical isomorphisms $(X^*)^{\otimes(k)} \to \mathcal{F}(X^{\otimes(k)}, I)$ induced by the tensoring operation $\mathcal{F}(A,B) \otimes \mathcal{F}(C,D) \to \mathcal{F}(A \otimes C, B \otimes D)$, and the above descriptions might make more sense if one uses these isomorphisms to replace every appearance of the domain by the codomain. The usual left action of Σ_n on $X^{\otimes(n)}$ (see Section 3.3.5) gives a right action on $\mathcal{F}(X^{\otimes(n)}, I)$, and the maps in (3.7.5) were set up so that the adjoints generalize the evaluations $X^{\otimes(k)} \otimes \mathcal{F}(X^{\otimes(k)}, I) \to I$.

To capture the picture in (3.7.6) more formally, define a category Σ_X^{op} enriched over $\mathcal{C}$ as follows (apologies for the mysterious "op" but it will become clear in a moment). Σ_X^{op} has one object $[n]$ for every $n \geq 0$, and

$$\Sigma_X^{op}([n],[k]) = \begin{cases} \emptyset & \text{if } k > n, \\ X^{\otimes(n-k)} \odot_{\Sigma_{n-k}} \Sigma_n & \text{if } k \leq n. \end{cases}$$

In the last line, Σ_{n-k} sits in Σ_n as permutations of the first $n-k$ elements, and the notation means the evident analog of $\Sigma_n \odot_{\Sigma_{n-k}} X^{\otimes(n-k)}$ obtained by reversing left and right. To define this as a category we need to explain how to compose maps, and we will do this using algebraic notation as in the last section. If maps from $[n]$ to $[k]$ are written $B_1 \ldots B_{n-k}\beta_n$ then the rule is

$$(B_1 \ldots B_{k-l}\beta_k)(C_1 \ldots C_{n-k}\gamma_n) = C_1 \ldots C_{n-k}B_1 \ldots B_{k-l}(id_{n-k}|\beta_k)\gamma_n \tag{3.7.7}$$

(the switching of the B's and C's seems annoying but works itself out when we move from Σ_X^{op} to Σ_X). This rule comes from reading off how compositions work in (3.7.6). For example, pretend X is a one-dimensional vector space and $\hat{\alpha}$ is evaluation. The left-hand side of (3.7.7) takes a tensor product of functionals $\phi_1 \otimes \cdots \otimes \phi_n$ on X, permutes them into the new tensor $\phi_{\gamma(1)} \otimes \cdots \otimes \phi_{\gamma(n)}$, evaluates the first $n-k$ of these on the C's to get $[\phi_{\gamma(1)}(C_1)\phi_{\gamma(2)}(C_2)\cdots] \cdot \phi_{\gamma(n-k+1)} \otimes \cdots \otimes \phi_{\gamma(n)}$, permutes the remaining functionals according to β, and then evaluates the first $k-l$ of these at the B's. One readily verifies that the right-hand side of (3.7.7) does the same thing.

So we have a category Σ_X^{op} and (3.7.6) amounts to the observation that our choice of $(X^*, \hat{\alpha})$ determines a canonical (enriched) functor $\Sigma_X^{op} \to \mathcal{C}$ sending $[n]$ to $(X^*)^{\otimes(n)}$. This in turn means that if Z is any object in $\mathcal{C}$ then we get an (enriched) functor $\Sigma_X \to \mathcal{C}$ by $[n] \mapsto \mathcal{F}((X^*)^{\otimes(n)}, Z)$.

A brief amount of thought reveals that enriched functors $\Sigma_X \to \mathcal{C}$ are precisely symmetric X-spectra. Note that in Σ_X rule (3.7.7) becomes instead

$$\gamma_n^{-1}C_1 \cdots C_{n-k} \circ \beta_k^{-1}B_1 \cdots B_{k-l} = \gamma_n^{-1}(id_{n-k}|\beta_k^{-1})C_1 \cdots C_{n-1}B_1 \cdots B_{k-l}$$

which could be made prettier by removing all of the inverses.

To paraphrase this discussion, the category Σ_X^{op} in some sense encodes the universal structure an inverse of X would have in $\mathcal{C}$. Symmetric X-spectra arise by "remembering" how all the inverses of X map into some given object. This is how one could re-invent the notion of symmetric spectra, if one were trapped on a desert island and forgot how it all worked.

Let us push these ideas a little further. The subcategory of $\mathcal{C}$ pictured in (3.7.6) is symmetric monoidal, and this structure can be lifted back to Σ_X^{op}. Define the tensor by $[k] \otimes [l] = [k+l]$, let the associativity isomorphism be the identity, and let the symmetry isomorphism $t\colon [k] \otimes [l] \to [l] \otimes [k]$ be the permutation $\rho_{[k],[l]}$. We also have to define the tensor product of maps, and this is done using the formula

$$A_1 \ldots A_k\alpha_s \otimes B_1 \ldots B_l\beta_t = A_1 \ldots A_kB_1 \ldots B_l\rho_{k,[s-k],[l],t-l}(\alpha_s|\beta_t). \tag{3.7.8}$$

This formula is again easily derived by thinking about vector spaces and functionals. The left-hand side is the operation that takes functions $\phi_1, \ldots, \phi_s, \mu_1, \ldots, \mu_t$, permutes the first set according to α and the second set according to β, then successively evaluates the first part of each set at the A's and B's in order (with the first A getting plugged into the first ϕ, and so forth). The right-hand side also does the α and β scrambling but then moves the first group of μ's in front of the last group of ϕ's, before plugging in the A's and B's. These are clearly the same operation.

It is a good exercise to check that Σ_X^{op}, thus defined, is indeed symmetric monoidal.

The symmetric monoidal structure on Σ_X^{op} yields a corresponding structure on Σ_X, and then this passes to a symmetric monoidal structure on the functor category $F(\Sigma_X, \mathcal{C})$ through a process called ***Day convolution***. Briefly, given two functors $Y, Z\colon \Sigma_X \to \mathcal{C}$ one forms the diagram

$$\begin{array}{ccccc} \Sigma_X \times \Sigma_X & \xrightarrow{Y\times Z} & \mathcal{C}\times\mathcal{C} & \xrightarrow{\otimes} & \mathcal{C} \\ {\scriptstyle\otimes}\downarrow & & & \nearrow{\scriptstyle Y\otimes Z} & \\ \Sigma_X & & & & \end{array} \tag{3.7.9}$$

and $Y\otimes Z$ is the (enriched) left Kan extension. The fact that the tensors on Σ_X and $\mathcal{C}$ are both symmetric monoidal yields that the tensor product of functors is symmetric monoidal as well.

To summarize this discussion, we could have defined symmetric spectra as follows:

Definition 3.7.12 (Symmetric spectra, approach #2). Let Σ denote the category Σ_{S^1}, as defined above. This is a category enriched over $\mathcal{T}op_*$. A symmetric spectrum is simply an enriched functor $\Sigma \to \mathcal{T}op$.

This approach provides a useful perspective on the difference between classical spectra and symmetric spectra. Classical spectra are diagrams indexed by the evident subcategory $\mathbb{N}S^1$ of Σ_{S^1}. The monoidal structure on Σ_{S^1} does not descend to this subcategory: to define the tensor product of two maps one needs the ρ-permutations as in (3.7.8), and these are not available in $\mathbb{N}S^1$. This seems to be the core reason that classical spectra do not have a smash product at the model category level.

3.8 Orthogonal spectra

The development of orthogonal spectra proceeds along lines very similar to what we did for symmetric spectra, and so we will be able to cover it fairly quickly. We describe the two (equivalent) approaches, one going through S-modules and the other via enriched diagrams. In each case there are some annoying technicalities to be dealt with at the beginning, but after that everything works much as for symmetric spectra. Certain formulas that were a little complicated in symmetric spectra — because they required an introduction of a permutation — have an easier counterpart in the orthogonal case, because the machinery in some sense keeps track of the permutation for us. The theory of orthogonal spectra was developed in [178].

Very briefly, an orthogonal spectrum assigns to each finite-dimensional inner product space V a pointed space X_V, and to every linear isometric inclusion $f\colon V \hookrightarrow W$ a natural structure map $\sigma_f\colon S^{W-f(V)}\wedge X_V \to X_W$, where $W - f(V)$ is the orthogonal complement of $f(V)$ in W. The extra complication is that these structure maps must be continuous in f in an appropriate sense. Some other things are as expected: if f is an isomorphism then by naturality the structure map will be an isomorphism $X_V \xrightarrow{\cong} X_W$, in particular showing that the orthogonal group $O(V)$ of self-isometries will act on each X_V.

Why bother with orthogonal spectra? There are at least three reasons. As mentioned, the theory works out a bit more naturally, with simpler formulas. Secondly, orthogonal spectra adapt easily to the setting of equivariant spectra (see [177] or Appendix A of [120]). Finally, unlike symmetric spectra, orthogonal spectra have the nice property that the weak equivalences are just the maps inducing isomorphisms on stable homotopy groups.

In this section we will in fact discuss four types of spectra, interrelated thus:

$$\begin{array}{ccc} \text{1: symmetric spectra} & \rightsquigarrow & \text{2: generalized symmetric spectra} \\ \Big| & & \Big| \\ \text{3: coordinatized orthogonal spectra} & \rightsquigarrow & \text{4: orthogonal spectra} \end{array} \tag{3.8.1}$$

(Types 2 and 3 on the anti-diagonal seem to lack standard names; these are our own.) Our development will proceed in the order $1 \longrightarrow 2 \longrightarrow 4 \longrightarrow 3$, although other orders of navigation are also possible.

3.8.2 Prelude: generalized symmetric spectra

The generalized symmetric spectra we are about to introduce do not typically get much airtime, as there is little payoff for the extra work and they are not truly "coordinate-free". But they are a useful prelude to orthogonal spectra, and only a slight modification of the symmetric spectra story we saw in Section 3.7. They come up, for example, in Remark 2.1.5 of [133].

For any finite set T consider the real vector space $\mathbb{R}\langle T\rangle$ with basis T, together with its one-point compactification $S^T = S^{\mathbb{R}\langle T\rangle}$. Let $\Sigma(T)$ denote the group of permutations of T; it acts naturally on S^T. Write $\mathbf{n}$ for the set $\{1, 2, \dots, n\}$, so that $\Sigma_n = \Sigma(\mathbf{n})$.

A generalized symmetric spectrum should be, in part, a functor $T \mapsto X_T$ defined on the category of finite sets with isomorphisms, taking values in the category of pointed spaces. Functoriality will give each X_T a $\Sigma(T)$-action. In addition, the spectrum should assign to every subset inclusion $T \subseteq U$ a structure map

$$\sigma_{T,U} \colon S^{U-T} \wedge X_T \to X_U$$

that is $\Sigma(U-T) \times \Sigma(T)$-equivariant, with the assignment being compatible with the isomorphisms $X_J \cong X_{J'}$ for $J \cong J'$. By restricting to the special sets $\mathbf{n}$ and inclusions $\mathbf{n} \hookrightarrow \mathbf{k}$ for $n \le k$, we get a (classical) symmetric spectrum $\tilde{X}$. If $|T| = n$ then every bijection $T \to \mathbf{n}$ induces a homeomorphism $X_T \to X_{\mathbf{n}}$, and one can check that there is really no more information in X than in $\tilde{X}$. But what we have accomplished here is to produce a notion of symmetric spectrum that avoids any dependence on the particular choice of finite sets $\mathbf{n}$, which after all are a bit unnatural.

Remark 3.8.1. In fact the above yields structure maps for *any* inclusion $f \colon T \hookrightarrow U$, of the form

$$\sigma_f \colon S^{U-f(T)} \wedge X_T \to X_U,$$

via composition:

$$S^{U-f(T)} \wedge X_T \xrightarrow{id \wedge X_f} S^{U-f(T)} \wedge X_{f(T)} \xrightarrow{\sigma_{f(T),U}} X_U.$$

Just as for symmetric spectra, we can follow two approaches for setting up the generalized version. Let $\mathbf{\Sigma I}$ denote the category of finite sets and isomorphisms.

Approach #1: Define a $\mathbf{\Sigma I}$-sequence to be a functor $\mathbf{\Sigma I} \to \mathcal{T}op_*$. Define the tensor product of $\mathbf{\Sigma I}$-sequences X and Y by

$$(X \otimes Y)_U = \bigvee_{T \subseteq U} X_T \wedge Y_{U-T}. \tag{3.8.3}$$

For the $\Sigma(U)$-action, an $\alpha \in \Sigma(U)$ maps the summand $X_T \wedge Y_{U-T}$ to $X_{\alpha(T)} \wedge Y_{\alpha(U-T)}$ via $X_{\alpha|_T} \wedge X_{\alpha|_{U-T}}$. The twist map in the symmetric monoidal structure carries the summand $X_T \wedge Y_{U-T}$ (indexed by $T \subseteq U$) to $Y_{U-T} \wedge X_T$ (indexed by $U - T \subseteq U$) via the usual twist map from $\mathcal{T}op_*$.

The sphere spectrum S is the $\mathbf{\Sigma I}$-sequence $T \mapsto S^T$, which can be checked to be a commutative monoid. We define a generalized symmetric spectrum to be an S-module.

Unfortunately, because $\mathbf{\Sigma I}$ is not a small category we cannot form the category of $\mathbf{\Sigma I}$-sequences without running into set-theoretic issues. See Remark 3.5.4 for the common ways to get around this: for example, we can choose a skeletal subcategory $\mathbf{\Sigma I}_{skel} \hookrightarrow \mathbf{\Sigma I}$ together with a retraction r, and then transplant all the definitions for $\mathbf{\Sigma I}$-sequences to $\mathbf{\Sigma I}_{skel}$-sequences. One choice for skeletal subcategory is precisely the category ΣI from Section 3.7, leading to the previous (ungeneralized) notion of symmetric spectra.

The monoidal product on $\mathbf{\Sigma I}$-sequences is another example of Day convolution (see (3.7.9)): the category $\mathbf{\Sigma I}$ has the symmetric monoidal structure $\amalg$ given by disjoint union, and $X \otimes Y$ is the left Kan extension in the diagram

$$\begin{array}{ccc} \mathbf{\Sigma I} \times \mathbf{\Sigma I} & \xrightarrow{X \wedge Y} & \mathcal{T}op_* \\ {\scriptstyle \amalg}\downarrow & \nearrow_{X \otimes Y} & \\ \mathbf{\Sigma I} & & \end{array}$$

The most natural formula for this left Kan extension is

$$(X \otimes Y)(U) = \operatorname{colim}_{[A \amalg B \to U]} X_A \wedge Y_B,$$

where the indexing category consists of triples $(A, B, f\colon A \amalg B \to U)$ for f a map in $\mathbf{\Sigma I}$ and therefore an isomorphism. The maps between triples are the evident ones. This indexing category is not small, but again it has a small skeleton and so the colimit still exists. By associating the triple (A, B, f) with the image $f(A) \subseteq U$, one readily identifies the above colimit with the expression in (3.8.3).

Approach #2: For finite sets A and B define a category $[A, B]$ whose objects are sets C such that $A \subseteq C$ and $|C| = |B|$; morphisms $C \to C'$ are bijections $g\colon C \to C'$ which are the identity on A. Next define a category $\mathbf{\Sigma}$ enriched over $\mathcal{T}op_*$ whose objects are the finite sets and where the morphisms are given by

$$\mathbf{\Sigma}(A,B) = \operatorname{colim}_{[A,B]}\Big[\operatorname{Isom}(C,B)_+ \wedge S^{C-A}\Big]$$

(and $\operatorname{Isom}(C,B)$ is the set of bijections from C to B). The category $[A,B]$ indexing the colimit consists only of isomorphisms, and so the colimit can be identified with the co-invariants of the group of automorphisms acting on any spot of the diagram. In particular, for any subset $A \subseteq C$ such that $|C| = |B|$ one has

$$\mathbf{\Sigma}(A,B) \cong \operatorname{Isom}(C,B)_+ \wedge_{\Sigma(C-A)} S^{C-A}.$$

We can also regard $\mathbf{\Sigma}(A,B)$ as the subset of $\operatorname{Hom}(A,B)_+ \wedge S^B$ consisting of all pairs (f,x) where f is an injection and $x \in S^{B-f(A)}$; it is easy to check that the above colimit maps to this space in the evident way. If we do this, the composition is easy to describe: $\mathbf{\Sigma}(B,C) \times \mathbf{\Sigma}(A,B) \longrightarrow \mathbf{\Sigma}(A,C)$ is the map

$$((g,y),\ (f,x)) \mapsto (gf, y \wedge g(x)).$$

In this approach, a generalized symmetric spectrum is simply an enriched functor $\mathbf{\Sigma} \to \mathcal{T}op_*$. Just as in Approach #1, one runs into the difficulty that $\mathbf{\Sigma}$ is not a small category — and one way of dealing with this is to replace $\mathbf{\Sigma}$ with a skeletal subcategory, such as the category Σ from Definition 3.7.12.

3.8.4 Orthogonal spectra

Generalized symmetric spectra were built around the vector spaces $\mathbb{R}\langle A\rangle$, where A ranged over all finite sets. So these are vector spaces with a choice of basis, and one is naturally led to wonder about a basis-free approach. That is essentially what orthogonal spectra are. The role of the symmetric groups $\Sigma(A)$ is instead played by orthogonal groups $O(V)$.

Let $\mathbb{O}I$ be the category of finite-dimensional real inner product spaces, with linear isometric isomorphisms for the maps. This category only has maps from V to W when $\dim V = \dim W$, and all such maps are isomorphisms. We regard $\mathbb{O}I$ as being enriched over $\mathcal{T}op$, with $\mathbb{O}I(V,W)$ having the usual subspace topology induced by the compact-open topology on the space of all continuous maps W^V. For $W \in \operatorname{ob}\mathbb{O}I$ define $O(W) = \mathbb{O}I(W,W)$ to be the space of isometries from W to itself. If $V \subseteq W$ write $W - V$ for the orthogonal complement of V in W. Then we have a canonical inclusion $O(V) \hookrightarrow O(W)$: isometries of V extend to W by having them act as the identity on $W - V$. We will write $\operatorname{Isom}(U,V)$ for space of linear isometric inclusions from U into V, so when $\dim U = \dim V$ we have $\operatorname{Isom}(U,V) = \mathbb{O}I(U,V)$.

Approach #1: An **$\mathbb{O}I$-sequence** is simply an enriched functor $\mathbb{O}I \to \mathcal{T}op_*$. The symmetric monoidal structure $\oplus$ on $\mathbb{O}I$ induces a symmetric monoidal structure on $\mathbb{O}I$-sequences by Day convolution. Specifically, if X and Y are $\mathbb{O}I$-sequences then

$X \otimes Y$ is the (enriched) left Kan extension

$$\begin{array}{ccccc} \mathbb{O}I \times \mathbb{O}I & \xrightarrow{X \times Y} & \mathcal{T}op_* \times \mathcal{T}op_* & \xrightarrow{\wedge} & \mathcal{T}op_* \\ \otimes \big\downarrow & & & \nearrow_{X \otimes Y} & \\ \mathbb{O}I & & & & \end{array}$$

and can be given by the (enriched) colimit formula

$$(X \otimes Y)_W = \mathrm{colim}_{A \oplus B \to W}(X_A \wedge Y_B). \tag{3.8.5}$$

Here the indexing category has objects consisting of tuples $(A, B, f : A \oplus B \to W)$, where f is a map in $\mathbb{O}I$, and the evident morphisms (once again this is not a small category, but has a small skeleton). The enriched colimit is the coequalizer in $\mathcal{T}op$ of the two evident arrows

$$\begin{array}{c} \coprod_{A,B,A',B'} \mathrm{Isom}(A, A') \times \mathrm{Isom}(B, B') \times \mathrm{Isom}(A \oplus B, W) \times (X_A \wedge Y_B) \\ \downarrow\downarrow \\ \coprod_{A,B} \mathrm{Isom}(A \oplus B, W) \times (X_A \wedge Y_B) \end{array}$$

and so in particular the topology on $(X \otimes Y)_W$ comes from the topology on both $\mathrm{Isom}(A \oplus B, W)$ and on $X_A \wedge Y_B$. As a set (ignoring the topology) we can write

$$(X \otimes Y)_W = \bigvee_{V \subseteq W} X_Y \wedge Y_{W-V}. \tag{3.8.6}$$

by associating to every isometric isomorphism $f : A \oplus B \to W$ the subspace $f(A) \subseteq W$ (but this precisely ignores the topology on $\mathrm{Isom}(A \oplus B, W)$). Note that in this picture an isometry $h \colon W \to W'$ acts on this wedge by sending the summand $X_V \wedge Y_{W-V}$ to $X_{h(V)} \wedge Y_{h(W-V)}$ using the maps $X(h|_V)$ and $Y(h|_{W-V})$. The description in (3.8.5) readily gives the continuity of the maps

$$\mathbb{O}I(W, W') \times (X \otimes Y)_W \to (X \otimes Y)_{W'}.$$

The indexing category for the colimit in (3.8.5) has the property that all maps are isomorphisms; it follows formally that the colimit can be identified with the wedge of the co-invariants of the groups of automorphisms corresponding to every connected component of the category. So if we choose one $V_p \subseteq W$ of dimension p for every $0 \le p \le \dim W$ then we can write

$$(X \otimes Y)_W \cong \bigvee_p O(W)_+ \wedge_{O(V_p) \times O(W-V_p)} [X_{V_p} \wedge Y_{W-V_p}]. \tag{3.8.7}$$

This is correct as topological spaces but is non-canonical because of the choices of V_p. The bijection from (3.8.7) to (3.8.6) sends a tuple $(\alpha, x \wedge y \in X_{V_p} \wedge Y_{W-V_p})$ to $\alpha_*(x) \wedge \alpha_*(y) \in X_{\alpha(V_p)} \wedge Y_{\alpha(W-V_p)}$.

This tensor gives a closed symmetric monoidal product on the category of $\mathbb{O}I$-sequences, where the symmetry isomorphism $t \colon X \otimes Y \to Y \otimes X$ sends $x \wedge y \in X_A \wedge Y_B$ to $y \wedge x \in Y_B \wedge X_A$, using the description of (3.8.5).

Let S denote the $\mathbb{O}I$-sequence defined by $V \mapsto S^V$. It is easy to check that the maps $S^V \wedge S^W \to S^{V\oplus W}$ make S into a commutative monoid in the category of $\mathbb{O}I$-sequences. Define an **orthogonal spectrum** to be a left S-module. If X and Y are orthogonal spectra then their smash product is $X \wedge Y = X \otimes_S Y$.

We will write $\mathrm{Sp}^{\mathbb{O}}$ for the category of orthogonal spectra.

Remark 3.8.2. In wanting to consider all enriched functors $\mathbb{O}I \to \mathcal{T}op_*$ as a category, we run into the usual problem that $\mathbb{O}I$ is not small. To circumvent this using a small skeletal subcategory, as in Remark 3.5.4, we can take for such a subcategory the Euclidean spaces $(\mathbb{R}^n, \cdot)$ with standard dot product, for each $n \geq 0$. This leads to a spectrum being an assignment $n \mapsto X_n$, where X_n is a pointed space with an $O(n)$-action, together with structure maps $S^1 \wedge X_n \to X_{n+1}$ such that the iterated maps $S^p \wedge X_n \to X_{n+p}$ are $O(p) \times O(n)$-equivariant. Such an object could be called a "coordinatized orthogonal spectrum", and completes our tour of the square (3.8.1).

Approach #2: Here we define a $\mathcal{T}op_*$-enriched category $\mathbb{O}$ having the same objects as $\mathbb{O}I$ and where $\mathbb{O}(V,W)$ is supposed to parameterize the various suspension maps from X_V to X_W in a spectrum X. Recall that for every isometry $f\colon V \to W$ (which will necessarily be injective) we are supposed to have a suspension map $\sigma_f\colon S^{W-f(V)} \wedge X_V \to X_W$. The tricky part here is that there is not one single sphere involved in these maps: the sphere varies continuously with f. So to this end, let $\mathrm{Isom}(V,W)$ be the space of isometries from V into W and let $W-V \to \mathrm{Isom}(V,W)$ denote the bundle whose fiber over $f\colon V \to W$ is $W - f(V)$. Define

$$\mathbb{O}(V,W) = \mathrm{Th}(W - V \to \mathrm{Isom}(V,W)),$$

the Thom space of the bundle $W - V$. Note that if $|V| > |W|$ then $\mathrm{Isom}(V,W)$ is empty and this Thom space is a single point.

A point in $\mathbb{O}(V,W)$ can be represented by a pair (f,x) consisting of an isometry $f\colon V \to W$ and $x \in S^{W-f(V)}$. Using this notation, if $(g,y) \in \mathbb{O}(W,Z)$ then composition in $\mathbb{O}$ is given by the formula

$$(g,y) \circ (f,x) = (gf, g(x) + y),$$

the sum-of-vectors map $(g(W) - gf(V)) \times (Z - g(W)) \to Z - gf(V)$ being extended to the one-point compactifications in the usual way.

We can make the following identifications:

$$\mathbb{O}(V,W) = \begin{cases} O(W)_+ \wedge_{O(W-V)} S^{W-V} & \text{if } V \subseteq W, \\ \mathrm{Isom}(V,W) & \text{if } \dim V = \dim W, \\ \mathrm{Isom}(U,W)_+ \wedge_{O(U-V)} S^{U-V} & \text{if } \dim V \leq \dim W \text{ and } V \subseteq U \cong W, \\ * & \text{if } \dim W < \dim V. \end{cases}$$

The first two lines are actually special cases of the third, but are included separately for pedagogical purposes. For the third line use the map $\mathrm{Isom}(U,W)_+ \wedge_{O(U-V)} S^{U-V} \to \mathrm{Th}(W-V)$ given by $(h,x) \mapsto (h|_V, h(x))$.

The point to remember in the above descriptions is that when $\dim V = \dim W$ we have exactly $\mathrm{Isom}(V,W)$ as the space of maps from V to W. When $V \subseteq W$ we put an S^{W-V} into the space of maps from V to W, and then allow post-compositions with our $O(W)$ maps from W to itself — this accounts for the $O(W)_+ \wedge_{O(W-V)} S^{W-V}$ term. When V and W are incomparable we choose $U \supseteq V$ such that $\dim U = \dim W$ and allow compositions between our S^{U-V} maps from V to U and our $\mathrm{Isom}(U, W)$ maps from U to W, accounting for the $\mathrm{Isom}(U, W)_+ \wedge_{O(U-V)} S^{U-V}$ term.

In this approach an **orthogonal spectrum** is simply an enriched functor $\mathbb{O} \to \mathcal{T}op_*$. Unraveling this definition, an orthogonal spectrum X consists of

- a functor $X\colon \mathbb{O}I \to \mathcal{T}op_*$, and
- for every pair $V \subseteq W$ a structure map

$$\sigma_{V,W}\colon S^{W-V} \wedge X_V \to X_W$$

  that is $O(W-V) \times O(V)$-equivariant.

These structure maps must satisfy unital and associativity conditions that are easy to work out.

We leave the reader to justify the following analog of Proposition 3.7.10. Note that the isometry ρ that appears here is naturally forced upon us, since the second equality does not even make sense without it. In this sense the situation is a bit simpler than for symmetric spectra.

Proposition 3.8.3. *Let X, Y, and Z be orthogonal spectra. Giving a pairing $X \wedge Y \to Z$ is equivalent to giving a collection of maps $X_V \wedge Y_W \to Z_{V \oplus W}$ that are $O(V) \times O(W)$-equivariant and satisfy the identities*

$$A_U(x_V y_W) = (A_U x_V) y_W = \rho(x_V \cdot (A_U y_W)),$$

where ρ is the evident isometry $V \oplus (U \oplus W) \to (U \oplus V) \oplus W$ that is natural in the three variables. (Here we are using the algebraic notation from (3.7.2), adapted in the obvious way to the present context.) A pairing $X \wedge X \to Z$ is commutative if it also satisfies the identities $x_V \cdot y_W = \rho(y_W \cdot x_V)$, where ρ is the twist isometry $W \oplus V \to V \oplus W$.

3.8.8 Examples

We now give several standard examples of orthogonal and symmetric ring spectra.

(a) Let R be a ring and let HR be the spectrum $V \mapsto R\langle S^V \rangle$, where the latter is the free R-module on the set S^V with an appropriate topology (and where the basepoint is equal to zero). It is convenient to think of points in $R\langle S^V \rangle$ as finite configurations on S^V with labels in R, written formally as $\sum_i r_i x_i$ with $r_i \in R$, $x_i \in S^V$. The maps $S^W \wedge R\langle S^V \rangle \to R\langle S^{W \oplus V} \rangle$ send $(x, \sum r_i y_i) \to \sum r_i (x \wedge y_i)$. The product maps $R\langle S^V \rangle \wedge R\langle S^W \rangle \to R\langle S^{V \oplus W} \rangle$ send $(\sum r_i x_i, \sum s_j y_j) \to \sum_{i,j} r_i s_j [x_i \wedge y_j]$, and the unit maps $S^W \to R\langle S^W \rangle$ send $x \mapsto 1 \cdot x$.

(b) Let MO be the spectrum $V \mapsto MO_V = EO(V)_+ \wedge_{O(V)} S^V$. Here we take EG to be the geometric realization of the standard simplicial space $[n] \mapsto G^{n+1}$ with projections as face maps. Note that this comes with canonical maps $EH \to EG$ for $H \to G$ and

$EG_1 \times EG_2 \xrightarrow{\cong} E(G_1 \times G_2)$, and that G acts on EG from both the left and the right via its diagonal action on the G^{n+1} terms. The $O(V)$ action on MO_V comes from the left action on $EO(V)$.

The maps $S^W \wedge MO_V \to MO_{W\oplus V}$ are $(x,(\alpha,y)) \mapsto (\alpha, x\wedge y)$, where by abuse we write α for both an element of $EO(V)$ and its image in $EO(W\oplus V)$. It is informative to check the $O(W)\times O(V)$-equivariance. The $O(V)$-equivariance is clear, but the $O(W)$-equivariance looks wrong at first. One must use that $O(W)$ and $O(V)$ commute inside of $O(W\oplus V)$!

The pairings $MO_V \wedge MO_W \to MO_{V\oplus W}$ are the evident ones: $(\alpha,x)\wedge(\beta,y)\mapsto (\alpha\beta, x\wedge y)$, where $\alpha\beta$ refers to the pairing $EO(V)\times EO(W)\to EO(V\oplus W)$. The unit maps $S^V \to MO_V$ send x to (Id_V, x). We leave the reader to check the necessary relations to see that this is indeed a commutative ring spectrum.

(c) Constructing MU as an orthogonal ring spectrum is a little tricky. One can mimic our construction of MO using complexifications and unitary groups and write $MU(V) = EU(V_{\mathbb{C}})_+ \wedge_{U(V_{\mathbb{C}})} S^{V_{\mathbb{C}}}$, where $V_{\mathbb{C}}$ is the complexification of V, but then one only gets suspension operators by $S^{W_{\mathbb{C}}-V_{\mathbb{C}}}$ when one wants S^{W-V}. So this doesn't quite work. To explain the fix, if W is a Hermitian inner product space define

$$MU_W^{Herm} = EU(W)_+ \wedge_{U(W)} S^W.$$

This has a left $U(W)$-action coming from the left action on $EU(W)$. This construction satisfies all the analogous properties to (b) above, but only for Hermitian spaces. For a real inner product space V define $MU_V = \mathrm{Map}(S^{iV}, MU_{V_{\mathbb{C}}}^{Herm})$, where iV is the imaginary part of $V_{\mathbb{C}}$. Note that $O(V)$ acts on S^{iV} in the evident way, on $MU_{V_{\mathbb{C}}}^{Herm}$ through the map $O(V)\to U(V_{\mathbb{C}})$, and then on the mapping space via conjugation.

It is an easy exercise to check that one gets natural maps $S^V \wedge MU_W \to MU_{V\oplus W}$ making MU into an orthogonal Ω-spectrum. Moreover, smashing of maps gives the pairings

$$\begin{array}{ccc}
MU_V \wedge MU_W & = & \mathrm{Map}(S^{iV}, MU_{V_{\mathbb{C}}}^{Herm}) \wedge \mathrm{Map}(S^{iW}, MU_{W_{\mathbb{C}}}^{Herm}) \\
 & & \downarrow (f,g)\mapsto f\wedge g \\
 & & \mathrm{Map}(S^{iV\oplus iW}, MU_{V_{\mathbb{C}}}^{Herm} \wedge MU_{W_{\mathbb{C}}}^{Herm}) \\
 & & \downarrow \\
 & & \mathrm{Map}(S^{iV\oplus iW}, MU_{(V\oplus W)_{\mathbb{C}}}^{Herm}) = MU_{V\oplus W}
\end{array}$$

which make MU into an orthogonal commutative ring spectrum.

(d) Real K-theory was written down as a *symmetric* commutative ring spectrum by Joachim [136]. It is not completely obvious how to do this, but Joachim found a way using spaces of Fredholm operators. The Σ_n-actions come from the action on a tensor product of Hilbert spaces $\mathcal{H}^{\otimes n}$. This construction can be adapted to complex K-theory using techniques similar to those in (c), but it does not immediately yield an orthogonal spectrum.

(e) (Waldhausen K-theory). Let $\mathcal{C}$ be an exact category in the sense of [224] (or alternatively, a category with cofibrations and weak equivalences in the sense of Waldhausen). Waldhausen's $S_\bullet$-construction produces a spectrum $K(\mathcal{C})$ called the **Waldhausen K-theory spectrum** of $\mathcal{C}$. Geisser and Hesselholt observed in [102, Section 6] that if one sets things up carefully then this construction actually produces a symmetric spectrum, and that if $\mathcal{C}$ has a well-behaved tensor product then $K(\mathcal{C})$ is in fact a symmetric ring spectrum. While it would take us too far afield to give a rigorous development of these ideas, by doing a bit of handwaving we can nevertheless give the general idea. In this example we work entirely simplicially, mostly just to avoid the excess step of needing to apply geometric realization constantly.

The $S_\bullet$-construction applied to $\mathcal{C}$ gives a simplicial set $[n] \mapsto S_n\mathcal{C}$, where an element of $S_n\mathcal{C}$ is, roughly speaking, a filtered object $A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n$ in $\mathcal{C}$ together with a particular choice for every quotient A_i/A_j with $j \leq i$. We will refer to this as a "filtered object with quotient data". For $i \geq 1$ the face map d_i omits A_i from the filtration, whereas d_0 sends the filtered object to $A_2/A_1 \hookrightarrow A_3/A_1 \hookrightarrow \cdots \hookrightarrow A_n/A_1$. Note that $S_0\mathcal{C} = *$ by convention, and $S_1\mathcal{C}$ is the set of objects in $\mathcal{C}$.

Define $K(\mathcal{C})_0 = *$ and $K(\mathcal{C})_1 = S_\bullet\mathcal{C}$. We will extend this to a generalized symmetric spectrum (as discussed in Section 3.8.2) by defining $K(\mathcal{C})_Q$ for every finite set Q. To do this we need the notion of a Q-simplicial set. Recall that Δ denotes the simplicial indexing category, and define Δ^Q to be the product category $\prod_Q \Delta$ — a product of copies of Δ indexed by the set Q. An object in Δ^Q is a Q-tuple $\underline{n} = (n_q)_{q\in Q}$, or equivalently a function $Q \to \mathbb{N}$. We define a Q-simplicial set to be a functor $(\Delta^Q)^{op} \to \mathcal{S}et$. If $|Q| = k$, a Q-simplicial set is the same as a k-fold multi-simplicial set, but we think of the different simplicial directions as being indexed by Q.

If X is a Q-simplicial set, define $\operatorname{diag}(X)$ to be the simplicial set $[n] \mapsto X_{(n,n,\ldots,n)}$, where the subscript indicates the constant Q-tuple whose value is n. We will also need the notion of skeleton: if $T \subseteq Q$ and $r \geq 0$, define the (T, r)-skeleton of X to be the Q-simplicial set given by

$$(\mathrm{sk}_{(T,r)} X)_{(\underline{n})} = X_{(\underline{n}')}, \quad \text{where} \quad n'_q = \begin{cases} n_q & \text{if } q \notin T, \\ \min\{n_q, r\} & \text{if } q \in T. \end{cases}$$

Despite the cumbersome definition, this just says that whenever $q \in T$ we replace the simplicial q-direction of X by its usual r-skeleton.

Let S^Q be the smash product of copies of $S^1 = \Delta^1/\partial\Delta^1$ indexed by the set Q. In simplicial degree k the set $(S^Q)_k$ consists of $k+1$ elements, which correspond to the basepoint together with the k possible degeneracies of the 1-simplex $[01]$.

The following strange result turns out to be the key to producing our desired symmetric spectrum.

Proposition 3.8.4. *Let Q and Q' be finite sets, and let X be a $Q \sqcup Q'$-simplicial set. Assume that $\mathrm{sk}_{(Q',0)} X = *$. Then there is a natural map of simplicial sets*

$$S^{Q'} \wedge \operatorname{diag}(\mathrm{sk}_{(Q',1)} X) \longrightarrow \operatorname{diag}(X).$$

Proof. This is a combinatorial exercise left to the reader. The main point is that the non-basepoint elements of $(S^{Q'})_k$ can be thought of as exactly corresponding to the k different ways of applying degeneracies in the Q'-directions to move from simplicial degree 1 up to simplicial degree k. The desired map is defined to consist exactly of these degeneracy maps. □

With these tools in hand, we return to Waldhausen K-theory. Recall that every $[n]$ in Δ may be regarded as a category, in which there is a unique map from i to j whenever $i < j$. Filtered objects of length n in $\mathcal{C}$ may be identified with functors $[n] \to \mathcal{C}$ that send 0 to the zero object of $\mathcal{C}$. Likewise, we associate the tuple $\underline{n} = (n_q)_{q\in Q}$ to the product category $[\underline{n}] = \prod_{q\in Q}[n_q]$, and define an $\underline{n}$-filtered object to be a functor $[\underline{n}] \to \mathcal{C}$ which sends every tuple containing 0 to the zero object. For example, a $(1,1)$-filtered object is the same as an object of $\mathcal{C}$, and a $(2,3)$-filtered object is a diagram of the form

$$\begin{array}{ccccc} X_{11} & \longrightarrow & X_{12} & \longrightarrow & X_{13} \\ \downarrow & & \downarrow & & \downarrow \\ X_{21} & \longrightarrow & X_{22} & \longrightarrow & X_{23} \end{array}$$

For each finite set Q, define $S^Q_\bullet\mathcal{C}$ to be the Q-simplicial set which in multidegree $(\underline{n})$ consists of all $\underline{n}$-filtered objects of $\mathcal{C}$ satisfying certain cofibration conditions together with particular choices for various quotient objects (again, we are being intentionally vague and only giving the basic idea). Define $K(\mathcal{C})_Q = \operatorname{diag}(S^Q_\bullet\mathcal{C})$. Note that $\Sigma(Q)$ acts naturally on this construction, by permutation of the factors.

Observe that $\operatorname{sk}_{(Q',1)}(S^{Q\amalg Q'}_\bullet\mathcal{C}) = S^Q_\bullet\mathcal{C}$. So Proposition 3.8.4 gives maps

$$S^{Q'} \wedge K(\mathcal{C})_Q \to K(\mathcal{C})_{Q\amalg Q'}$$

which are readily checked to be $\Sigma(Q')\times\Sigma(Q)$-equivariant. Thus, we have a generalized symmetric spectrum. Note that there does not seem to be any obvious approach for producing an orthogonal spectrum here.

If in addition $\mathcal{C}$ has a well-behaved tensor product — one that preserves cofibrations and exactness — then we can take an $(\underline{n}_q)_{q\in Q}$-filtered object X and an $(\underline{k}_s)_{s\in Q'}$-filtered object Y and tensor them together to get a $(\underline{n} \amalg \underline{k})_{Q\amalg Q'}$-filtered object $X \otimes Y$. This yields maps

$$K(\mathcal{C})_Q \wedge K(\mathcal{C})_{Q'} \to K(\mathcal{C})_{Q\amalg Q'}$$

making $K(\mathcal{C})$ into a symmetric ring spectrum.

We again refer to [102, Section 6.1] for a detailed treatment of this material.

3.8.9 Model structures for orthogonal spectra

We now turn to the development of the commonly used model category structures for orthogonal spectra. By now the following series of results will be very familiar.

Proposition 3.8.5. *There exists a model category structure on* $\mathrm{Sp}^{\mathcal{O}}$ *where the weak equivalences and fibrations are levelwise. This is called the* ***level, projective model structure****.*

Proof. Direct application of Theorem 3.5.1(a) in the setting of enriched diagrams. □

The evaluation functors $\mathrm{Ev}_V\colon \mathrm{Sp}^{\mathbb{O}} \to \mathcal{T}op_*$ have left adjoints F_V given by

$$(F_V X)_W = \mathrm{Th}\Big(W - V \to \mathrm{Isom}(V, W)\Big) \wedge X$$

$$\cong \begin{cases} O(W)_+ \wedge_{O(W-V)} (S^{W-V} \wedge X) & \text{if } V \subseteq W, \\ \mathbb{O}I(U, W)_+ \wedge_{O(U-V)} (S^{U-V} \wedge X) & \text{if } V \subseteq U \text{ and } \dim U = \dim W, \\ * & \text{if } \dim W < \dim V. \end{cases}$$

If $V \subseteq W$ there is a canonical map $F_W(S^{W-V} \wedge X) \to F_V(X)$.

Definition 3.8.6. The **stable projective model structure** on $\mathrm{Sp}^{\mathbb{O}}$ is the Bousfield localization of the level projective model category structure at the set of maps

$$\Big\{F_W(S^{W-V} \wedge S^0) \to F_V(S^0) \,\Big|\, V \subseteq W\Big\}.$$

There is a simple comparison map between orthogonal spectra and symmetric spectra. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$, so that we have the usual inclusion $\mathbb{R}^n \subseteq \mathbb{R}^{n+1}$. The choice of vector e_{n+1} gives a map $\mathbb{R} \to \mathbb{R}^{n+1} - \mathbb{R}^n$ (sending 1 to e_{n+1}) and therefore an induced homeomorphism $S^1 \to S^{(\mathbb{R}^{n+1} - \mathbb{R}^n)}$. Permutation of basis elements gives a group map $\Sigma_n \to O(\mathbb{R}^n)$.

There is a forgetful functor $U\colon \mathrm{Sp}^{\mathbb{O}} \to \mathrm{Sp}^{\Sigma}$ that sends an orthogonal spectrum X to the symmetric spectrum $[n] \mapsto X_{\mathbb{R}^n}$, where the Σ_n-action on $X_{\mathbb{R}^n}$ comes from restricting the $O(\mathbb{R}^n)$-action and the structure maps come from those in X via the identification $S^1 \cong S^{(\mathbb{R}^{n+1} - \mathbb{R}^n)}$.

The following results are all proven in [178]:

Proposition 3.8.7.

(a) *The stable projective structure on* $\mathrm{Sp}^{\mathbb{O}}$ *is a stable, closed symmetric monoidal model category satisfying the Monoid Axiom, the Algebraic Creation and Invariance Properties and the Strong Flatness Property.*
(b) *The fibrant objects in* $\mathrm{Sp}^{\mathbb{O}}$ *are the levelwise fibrant* Ω*-spectra, meaning orthogonal spectra for which the adjoints to the structure maps* $X_V \to \Omega^{W-V} X_W$ *are all weak equivalences for* $V \subseteq W$.
(c) *The forgetful functor* $U\colon \mathrm{Sp}^{\mathbb{O}} \to \mathrm{Sp}^{\Sigma}$ *has a left adjoint* G *and the pair* (G, U) *is a Quillen equivalence.*
(d) *A map* $f\colon X \to Y$ *in* $\mathrm{Sp}^{\mathbb{O}}$ *is a stable weak equivalence if and only if* Uf *is a weak equivalence in* $\mathrm{Sp}^{\mathbb{N}}$ *(slightly abusing our use of* U *here).*

Proof. The precise references for the different parts are: model structure, [178, 9.2]; monoidal properties, [178, 12.1 (take $R = S$)]; Algebraic Creation Property, [178, 12.1(i)]; Algebraic Invariance, [178, 12.1vi,vii]; Strong Flatness, [178, 12.3, 12.7]; Quillen Equivalence, [178, 10.4]; U detects stable weak equivalences, [178, 8.7]. □

Statement (d) is something of a surprise, as this is not true when orthogonal spectra are replaced with symmetric spectra. The topology of the orthogonal groups turns out to be what makes this work, as we now explain. If X is an orthogonal spectrum define $\pi_k(X) = \operatorname{colim}_n \pi_{n+k}(X_{\mathbb{R}^n})$. These are precisely the homotopy groups of the underlying Bousfield–Friedlander spectrum. One might think to include other X_V in the colimit system, but there is no point as $X_{\mathbb{R}^n} \cong X_V$ when $\dim V = n$. Part (d) of Proposition 3.8.7 is equivalent to the statement that the stable equivalences of orthogonal spectra are just the π_*-isomorphisms.

The key to understanding this is to look at the map $F_{n+1}(S^1 \wedge A) \to F_n(A)$, where we now write F_n as short for $F_{\mathbb{R}^n}$. We claim this is a π_*-isomorphism (the analog was false for symmetric spectra). In level $n+k$ this map is

$$O(n+k)_+ \wedge_{O(k-1)} (S^{k-1} \wedge S^1 \wedge A) \longrightarrow O(n+k)_+ \wedge_{O(k)} (S^k \wedge A).$$

The A comes out on both sides as a smash factor, so we might as well throw it away. Also, we won't change the stable homotopy groups (except for a shift) if we smash both sides with S^n, and this gives

$$O(n+k)_+ \wedge_{O(k-1)} S^{n+k} \longrightarrow O(n+k)_+ \wedge_{O(k)} S^{n+k}.$$

Now, if X is a left G-space and $H \leq G$ then

$$G_+ \wedge_H X \cong G_+ \wedge_H (G_+ \wedge_G X) \cong (G_+ \wedge_H G_+) \wedge_G X \cong (G/H_+ \wedge G_+) \wedge_G X \cong G/H_+ \wedge X.$$

In our case $O(n+k)$ acts on S^{n+k}, so the map simplifies to

$$O(n+k)/O(k-1)_+ \wedge S^{n+k} \to O(n+k)/O(k)_+ \wedge S^{n+k}.$$

Since $O(k)/O(k-1) \cong S^{k-1}$, the map $O(n+k)/O(k-1) \to O(n+k)/O(k)$ is $(k-1)$-connected and so the smash with S^{n+k} is $(n+2k-1)$-connected. As this goes to infinity with k, we have our isomorphism on stable homotopy groups.

3.9 EKMM spectra

Unpacking the definitions of [94] takes time and energy. There are several layers to unravel, with quite a bit of intricate mathematics. Anything close to a complete account would involve reproducing a big chunk of the book [94]. Since our aim is only to survey this material, we will content ourselves with a very incomplete account, outlining the main steps but omitting the details behind them.

We first explain the basic idea. Start with the notion of a spectrum defined on a May universe $\mathcal{U}$. This is basically the idea of Bousfield–Friedlander spectra, but done in a coordinate-free way. If M and N are two such spectra, then the smash product $M \wedge N$ seems to be most naturally defined as a spectrum on the universe $\mathcal{U} \oplus \mathcal{U}$. To get a spectrum on $\mathcal{U}$ we can choose an isomorphism $\mathcal{U} \cong \mathcal{U} \oplus \mathcal{U}$, but this involves a choice. The space of all choices is contractible, so in some sense the choice doesn't matter. But if we want a smash product that is commutative and associative on the point-set level, we can't afford to make a single choice.

To get around this, one adopts a definition that builds all the choices in from the beginning. An EKMM-spectrum is (approximately) a coordinate-free spectrum that comes bundled together with its images under all possible changes of universe. The smash product of two such things gives a "bundle" (in a very non-technical sense) of spectra on $\mathcal{U} \oplus \mathcal{U}$, and then changing back to $\mathcal{U}$ in all possible ways just creates another bundle. No choices have been made, but at the expense of introducing extra complexity into the objects themselves.

It is informative to contrast symmetric (or orthogonal) spectra with EKMM-spectra. For the former, the category itself is fairly concrete and easy to understand. The complexities appear in the model structure, where the fibrant objects and weak equivalences are complicated. With EKMM-spectra all the complexity is built into the objects themselves. They are "flabby" enough to all be fibrant in the model structure, and the weak equivalences are quite simple to understand.

3.9.1 Outline for the EKMM approach

Fix a May universe $\mathcal{U}$, by which we mean a real inner product space isomorphic to $\mathbb{R}^\infty$ with the dot product. For subspaces $V \subseteq W \subseteq \mathcal{U}$ write $W - V$ for the orthogonal complement of V in W. Let S^V be the one-point compactification of V, and for X a pointed space write $\Omega^V X$ for the pointed function space $\mathcal{F}_*(S^V, X)$.

It is important to understand that the machinery we describe below was developed over a long time in the works of May and his collaborators. We note especially [155], [93], and [94], but there are plenty of precursors in [63] and [199] as well.

(1) A **prespectrum** is an assignment $V \mapsto E_V$ that sends finite-dimensional subspaces of $\mathcal{U}$ to pointed spaces, together with suspension maps $S^{W-V} \wedge E_V \to E_W$ for every pair $V \subseteq W$. These maps must satisfy an associativity condition and be the identity when $V = W$. Write $\mathcal{P}\mathcal{U}$ for the category of prespectra on $\mathcal{U}$, with the evident maps.

(2) A **spectrum** is a prespectrum where the adjoints $E_V \to \Omega^{W-V} E_W$ are homeomorphisms. Write $\mathcal{S}\mathcal{U}$ for the category of spectra on $\mathcal{U}$.

(3) There are adjoint functors $L\colon \mathcal{P}\mathcal{U} \rightleftarrows \mathcal{S}\mathcal{U}\colon i$ where the right adjoint i is the evident inclusion. The functor L is called "spectrification". (This functor is more mysterious than one might first guess, and having control over colimits in $\mathcal{S}\mathcal{U}$ is entirely dependent on having a good working knowledge of L, as provided by Lewis in [155, Appendix].)

(4) For universes $\mathcal{U}$, $\mathcal{U}'$ there is an external smash product $\wedge_{pre}\colon \mathcal{S}\mathcal{U} \times \mathcal{S}\mathcal{U}' \to \mathcal{P}(\mathcal{U} \oplus \mathcal{U}')$ defined as follows. For M and N in $\mathcal{S}\mathcal{U}$, define

$$(M \wedge_{pre} N)(V \oplus V') = M_V \wedge N_{V'}.$$

This only defines $M \wedge_{pre} N$ on subspaces of $\mathcal{U} \oplus \mathcal{U}$ of the form $V \oplus V'$, but these are cofinal amongst all subspaces; so extend $M \wedge_{pre} N$ to all subspaces in any reasonable way. For example, this can be done inductively on the dimension: given an arbitrary finite-dimensional subspace $W \subseteq \mathcal{U}$, choose V and V' with $W \subseteq V \oplus V'$ and define

$$(M \wedge_{pre} N)(W) = \Omega^{(V \oplus V')-W}(M_V \wedge N_{V'}).$$

Finally, define the external smash product $\wedge_{ext}\colon \mathcal{S}\mathcal{U} \times \mathcal{S}\mathcal{U}' \to \mathcal{S}(\mathcal{U} \oplus \mathcal{U}')$ by

$$M \wedge_{ext} N = L(M \wedge_{pre} N).$$

The choices involved in the definition of $M \wedge_{pre} N$ get ironed out by the spectrification functor L, and one can check that $M \wedge_{ext} N$ is well-defined.

(5) [Change of universe] By an isometry $f: \mathcal{U} \to \mathcal{U}'$ we mean a linear isometric embedding, not necessarily surjective. Given an isometry $f: \mathcal{U} \to \mathcal{U}'$ and a spectrum M on $\mathcal{U}'$, there is an induced spectrum f^*M given by $V \mapsto M_{f(V)}$. The functor $f^*: \mathcal{S}\mathcal{U}' \to \mathcal{S}\mathcal{U}$ has a left adjoint f_*, defined as follows. For $W \subseteq \mathcal{U}'$ write $W_f = W \cap \mathrm{im}(f)$. For a spectrum E defined on $\mathcal{U}$, define a prespectrum $f_*^{pre}E$ by

$$(f_*^{pre}E)(W) = S^{W-W_f} \wedge E_{f^{-1}(W_f)}.$$

We leave the reader the pleasant exercise of working out the structure maps. Then define $f_*M = L(f_*^{pre}M)$. See [155, II.1] for more details.

(6) Let $\mathcal{I}(\mathcal{U},\mathcal{U}')$ denote the space of linear isometries from $\mathcal{U}$ to $\mathcal{U}'$. This is a contractible space. One would therefore hope that if $f, g \in \mathcal{I}(\mathcal{U},\mathcal{U}')$ and E is a spectrum on $\mathcal{U}$ then f_*E and g_*E are weakly equivalent spectra on $\mathcal{U}'$. This is not known in general, but there is a special class of spectra for which it does hold. Define a spectrum E to be **Σ-cofibrant** if the structure maps $S^W \wedge E_V \to E_{V\oplus W}$ are all cofibrations, and define E to be **tame** if it is homotopy equivalent to a Σ-cofibrant spectrum. It is known that if E is tame then f_*E and g_*E are homotopy equivalent [94, I.2.5]. We will need to study all these different pushforwards at once.

(7) Given a space A, a map $\alpha: A \to \mathcal{I}(\mathcal{U},\mathcal{U}')$, and a spectrum E on $\mathcal{U}$, there is a construction $A \ltimes E$ which is a spectrum on $\mathcal{U}'$. It is called the “twisted half-smash product”. It depends on α, but this is omitted from the notation. Loosely speaking, $A \ltimes E$ contains all the ways of constructing a pushforward of E from $\mathcal{U}$ to $\mathcal{U}'$, as parameterized by the map α, all bundled together. When A is contractible and E is tame, this has the same homotopy type as the simple pushforwards f_*E.

(8) Write $\mathcal{L}(j) = \mathcal{I}(\mathcal{U}^j,\mathcal{U})$ where $\mathcal{U}^j$ is the direct sum of j copies of $\mathcal{U}$. The spaces $\mathcal{L}(j)$ together form an operad $\mathcal{L}$, called the **linear isometries operad**.

(9) Let $\mathbb{L}: \mathcal{S}\mathcal{U} \to \mathcal{S}\mathcal{U}$ denote the monad $\mathbb{L}(E) = \mathcal{L}(1) \ltimes E$. Then the composition map $\mathcal{L}(1) \times \mathcal{L}(1) \to \mathcal{L}(1)$ induces the natural transformation $\mu: \mathbb{L}\mathbb{L}E \to \mathbb{L}E$, and the identity element $id \in \mathcal{L}(1)$ induces the unit $\eta: E \to \mathbb{L}E$.

(10) An **$\mathbb{L}$-spectrum** is an $\mathbb{L}$-algebra: that is, an $\mathbb{L}$-spectrum is a spectrum X together with a map $\mathbb{L}X \to X$ making the usual diagrams commute.

(11) Given $\mathbb{L}$-spectra M and N, we define the smash product by

$$M \wedge_{\mathcal{L}} N = \mathcal{L}(2) \ltimes_{\mathcal{L}(1)\times\mathcal{L}(1)} (M \wedge_{ext} N).$$

Note that $M \wedge_{ext} N$ is a spectrum on $\mathcal{U}^2$. The object on the right in this definition is a coequalizer of certain evident maps coming from the $\mathbb{L}$-algebra structures on M and N and the operad maps in $\mathcal{L}$. The smash product $\wedge_{\mathcal{L}}$ turns out to be associative and symmetric (see [94, I.5]), but not unital.

(12) The sphere spectrum S is the spectrification of the prespectrum $V \mapsto S^V$. It turns out that S is an $\mathbb{L}$-algebra in a natural way, and that for any $\mathbb{L}$-spectrum M there is a natural map $\lambda_M: S \wedge_{\mathcal{L}} M \to M$. Define an **EKMM-spectrum** to be an $\mathbb{L}$-spectrum

M for which λ_M is an isomorphism. Denote the category of EKMM-spectra by EKMM_S. The spectrum S is itself an EKMM spectrum.

Remark. EKMM-spectra are called "S-modules" in [94]. While not a terrible name, it conflicts with the notions of S-modules that one has in other categories like symmetric spectra and orthogonal spectra. The name "EKMM-spectra" seems to lead to less confusion.

(13) The smash product of EKMM-spectra M and N is defined as $M \wedge_S N = M \wedge_{\mathcal{L}} N$. This gives a symmetric monoidal smash product on EKMM_S with unit S.

(14) Now suppress the universe and abbreviate $\mathcal{S}\mathcal{U}$ to just $\mathcal{S}$. There are adjunctions

$$\mathcal{S} \underset{\mathbb{U}}{\overset{\mathbb{L}(-)}{\rightleftarrows}} (\mathbb{L}-\mathcal{S}pectra) \underset{\mathcal{F}_{\mathcal{L}}(S,-)}{\overset{S\wedge_{\mathcal{L}}(-)}{\rightleftarrows}} \mathrm{EKMM}_S$$

where $\mathbb{U}$ is the forgetful functor and the left adjoints both point left to right.

(15) For each $V \subseteq \mathcal{U}$, the evaluation map $\mathrm{Ev}_V\colon \mathcal{S} \to \mathcal{T}op_*$ has a left adjoint, denoted F_V. We also write Σ^∞ for the functor F_0.

(16) For a map f in $\mathcal{S}$, say that f is a weak equivalence if f is a π_*-isomorphism on underlying spectra. Since the objects of $\mathcal{S}$ are all Ω-spectra, we can also characterize the weak equivalences as maps inducing objectwise weak equivalences in $\mathcal{T}op_*$ on application of Ev_V (for all V).

If $i\colon \mathrm{EKMM}_S \hookrightarrow \mathbb{L}-\mathcal{S}pectra$ denotes the inclusion then for any M in EKMM_S there is a canonical map $iM \to \mathcal{F}_{\mathcal{L}}(S,M)$ and this map is always a weak equivalence. So up to homotopy the functors i and $\mathcal{F}_{\mathcal{L}}(S,-)$ are really the same; as a consequence, a map in EKMM_S is a weak equivalence if and only if $\mathcal{F}_{\mathcal{L}}(S,-)$ is a weak equivalence.

Say that f is a fibration if it has the right lifting property with respect to all maps $F_n(I^k \times \{0\}) \to F_n(I^k \wedge I_+)$, for all n and k.

Then $\mathcal{S}$ has a model category structure with the weak equivalences and fibrations defined above, and the right adjoints $\mathbb{U}$ and $\mathcal{F}_{\mathcal{L}}(S,-)$ create induced model category structures on $\mathbb{L}-\mathcal{S}pectra$ and EKMM_S. Note that since all objects are fibrant in $\mathcal{T}op_*$, the same holds in each of the categories $\mathcal{S}$, $\mathbb{L}-\mathcal{S}pectra$, and EKMM_S.

Moreover, the two pairs of adjoint functors from (14) are both Quillen equivalences.

(17) For any pointed space X we define

$$\Sigma^\infty_S X = S \wedge_{\mathcal{L}} \mathbb{L}(\Sigma^\infty X).$$

This is just the composite of the left adjoints in the diagram

$$\mathcal{T}op_* \underset{\mathrm{Ev}_0}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathcal{S} \underset{\mathbb{U}}{\overset{\mathbb{L}(-)}{\rightleftarrows}} (\mathbb{L}-\mathcal{S}pectra) \underset{\mathcal{F}_{\mathcal{L}}(S,-)}{\overset{S\wedge_{\mathcal{L}}(-)}{\rightleftarrows}} \mathrm{EKMM}_S$$

and so in particular is a left Quillen functor. Write Ω^∞_S for the composition of the right adjoints in the above diagram. For $n \geq 0$ write

$$S^n_S = \Sigma^\infty_S(S^n) = S \wedge_{\mathcal{L}} (\mathbb{L}(\Sigma^\infty S^n)).$$

We regard S^n_S as a "stable n-sphere", and from this we can define the notion of CW-spectra for EKMM_S in the usual way. Such spectra will all be cofibrant.

(18) Now we come to a major point. We have the object $S = \Sigma^\infty S^0$, which is an EKMM-spectrum (see (12)) and the unit for the smash product. But we also have the stable 0-sphere $S^0_S = \Sigma^\infty_S S^0 = S \wedge_{\mathcal{L}} \mathbb{L}S$. The $\mathbb{L}$-algebra structure on S is a map $\mathbb{L}S \to S$, which induces the canonical map

$$S^0_S = S \wedge_{\mathcal{L}} \mathbb{L}S \to S \wedge_{\mathcal{L}} S = S.$$

This map is a weak equivalence, but it is *not* an isomorphism. In fact it turns out that S is not cofibrant in EKMM_S, and so S^0_S is a cofibrant replacement for S.

The fact that S is not cofibrant, and the distinction between S^0_S and S, is one of the major differences between EKMM-spectra and symmetric (or orthogonal) spectra.

(19) For any pointed space X, the spectrum $\Sigma^\infty X$ (from (15) above) turns out to be an $\mathbb{L}$-spectrum in a natural way and also an EKMM-spectrum. So we can think of Σ^∞ as a functor $\mathcal{T}op_* \to \mathrm{EKMM}_S$. It has a right adjoint Ω^∞. It is dangerous to confuse Σ^∞_S and Σ^∞. The first is a left Quillen functor, but the second is not. We have the comparison map

$$\Sigma^\infty_S X = S \wedge_{\mathcal{L}} \mathbb{L}(\Sigma^\infty X) \longrightarrow S \wedge_{\mathcal{L}} \Sigma^\infty X \cong \Sigma^\infty X,$$

with the middle map coming from the $\mathbb{L}$-structure on $\Sigma^\infty X$, and the last isomorphism being because $\Sigma^\infty X$ is an S-module. This comparison map is a weak equivalence whenever X is nondegenerately based (i.e., $* \to X$ is a cofibration).

The functor Σ^∞ has good monoidal properties, such as a natural isomorphism $\Sigma^\infty(X \wedge Y) \cong (\Sigma^\infty X) \wedge_S (\Sigma^\infty Y)$ compatible with associativity and commutativity isomorphisms.

The work in [94] shows the following:

Theorem 3.9.1. *The category* EKMM_S *is a stable, closed symmetric monoidal model category satisfying the Algebraic Creation and Invariance Properties as well as the Strong Flatness Property. As a model category it is Quillen equivalent to the stable projective model structure on* $\mathrm{Sp}^{\mathbb{N}}$.

Proof. We sketch a proof here, since there seems to be no simple reference where this can be just looked up. Let $\mathbb{F}_n : \mathcal{T}op_* \to \mathrm{EKMM}_S$ be the functor $\mathbb{F}_n(X) = S \wedge_{\mathcal{L}} \mathbb{L}F_n(X)$.

In [94] the closed symmetric monoidal structure is established, as well as the model structure. The latter comes with the set $\{\mathbb{F}_m(S^n) \to \mathbb{F}_m(D^{n+1}) \mid m, n \geq 0\}$ of generating cofibrations and the set $\{\mathbb{F}_m(D^n) \to \mathbb{F}_m(D^n \wedge I_+) \mid m, n \geq 0\}$ of generating trivial cofibrations (see [94, VII.5.6–5.8]).

To prove the Pushout-Product Axiom, it suffices to check it on generating cofibrations and trivial cofibrations. So we need to analyze the box product of $\mathbb{F}_m(f)$ and $\mathbb{F}_n(g)$ for $f : A \rightarrowtail B$ and $g : C \rightarrowtail D$ cofibrations in $\mathcal{T}op_*$. The key point is then that a choice of homeomorphism $\mathcal{U}^2 \cong \mathcal{U}$ induces a homeomorphism $\mathcal{L}(2) \cong \mathcal{L}(1)$ and thus an identification $\mathbb{F}_m(f) \square \mathbb{F}_n(g) \cong \mathbb{F}_{m+n}(f \square g)$; the Pushout-Product Axiom then follows. (See [45, 4.21] for a version of this argument in the context of spaces.)

There is a canonical map $\mathbb{L}S \to S$, and the induced map $\alpha\colon S \wedge_{\mathcal{L}} \mathbb{L}S \to S \wedge_{\mathcal{L}} S \cong S$ is a cofibrant-approximation in EKMM_S. Note that the domain is $\Sigma^\infty_S(S^0)$. We must show for any M in EKMM_S that $(S \wedge_{\mathcal{L}} \mathbb{L}S) \wedge_S M \to S \wedge_S M = M$ is a weak equivalence. Remembering that $\wedge_S = \wedge_{\mathcal{L}}$, consider the diagram

$$\begin{array}{ccccc} S \wedge_{\mathcal{L}} \mathbb{L}S \wedge_{\mathcal{L}} M & \xrightarrow{\mu_{\mathbb{L}S} \wedge id_M} & \mathbb{L}S \wedge_{\mathcal{L}} M & \xrightarrow{g} & M \\ \downarrow & & \downarrow & \nearrow_{\cong} & \\ S \wedge_{\mathcal{L}} S \wedge_{\mathcal{L}} M & \xrightarrow[\mu_S \wedge id_M]{\cong} & S \wedge_{\mathcal{L}} M & & \end{array}$$

The diagonal map is an isomorphism by the definition of EKMM_S. The map g is a weak equivalence by [94, I.6.2], and $\mu_{\mathbb{L}S} \wedge id_M$ is a weak equivalence by [94, I.8.5(iii)]. It follows that every map in the diagram is a weak equivalence, and this verifies the Unit Axiom in the definition of monoidal model category. It also verifies condition (1) in Proposition 3.3.6.

Condition (2) of Proposition 3.3.6 also holds, since EKMM_S is a topological model category where all objects are fibrant. So Proposition 3.3.6 yields the Algebraic Creation Property.

The Strong Flatness Property follows from [94, III.3.8] together with the fact that every cofibrant R-module is a retract of a cell-module. For the Algebraic Invariance Property we verify the conditions of Proposition 3.3.9: condition (1) is the Strong Flatness Property, and condition (2) is [94, VII.6.2].

For the Quillen equivalence between EKMM_S and $\mathrm{Sp}^{\mathbb{N}}$, it is easiest to go through $\mathrm{Sp}^{\mathbb{O}}$ or Sp^{Σ}. The Quillen equivalence with $\mathrm{Sp}^{\mathbb{O}}$ is in [177], and the equivalence with Sp^{Σ} is in [262]. □

3.10 Afterthoughts

One of the drawbacks of a survey like this is that there is never enough time or space to say everything that one would like. This final section will give a blitz treatment of various topics that are important and should not go unmentioned.

3.10.1 Functors with smash product

This was an early attempt at a strict model for ring spectra, due to Bökstedt and used by him in his work on topological Hochschild homology [52]. In modern times these have been eclipsed by ring objects in either symmetric or orthogonal spectra, but it is still good to know the basic idea.

Let $\mathcal{W}$ be the category of pointed spaces that are homeomorphic to a finite CW-complex, Regard $\mathcal{W}$ as a $\mathcal{T}op_*$-enriched category. A **$\mathcal{W}$-sequence** is an enriched functor $\mathcal{W} \to \mathcal{T}op_*$ (these are also called $\mathcal{W}$-spaces sometimes). Day convolution, as in (3.7.9), gives a symmetric monoidal product on $\mathcal{W}$-sequences.

There is a "sphere sequence" S given by the inclusion $\mathcal{W} \hookrightarrow \mathcal{T}op_*$, and this is a commutative monoid. We define a **$\mathcal{W}$-spectrum** to be a left S-module. Unraveling this, a $\mathcal{W}$-spectrum is an enriched functor $\Phi\colon \mathcal{W} \to \mathcal{T}op_*$ together with structure maps $X \wedge \Phi(Y) \to \Phi(X \wedge Y)$ satisfying unital and associativity conditions. However, these extra structure maps do not provide new information — they are an automatic consequence of being an enriched functor, as was explained back in Section 3.1. So in this case $\mathcal{W}$-sequences and $\mathcal{W}$-spectra are the same thing.

There is a functor $\mathbb{O}I \to \mathcal{W}$ given by $V \mapsto S^V$, and restriction along this functor takes $\mathcal{W}$-spectra to orthogonal spectra. One can restrict further along the composite $\Sigma I \to \mathbb{O}I \to \mathcal{W}$ to get a symmetric spectrum.

The model category story works out in the same way as for orthogonal spectra. See [178].

A "functor with smash product" (FSP) is a monoid in the category of $\mathcal{W}$-spectra. This amounts to an enriched functor $\Phi\colon \mathcal{W} \to \mathcal{T}op_*$ equipped with maps $X \to \Phi(X)$ and $\Phi(X) \wedge \Phi(Y) \to \Phi(X \wedge Y)$ satisfying various properties that are not hard to work out.

Remark 3.10.1. We saw in Section 3.1.2 that the notion of a classical spectrum comes from the idea of "remembering" the mapping spaces $E_n = \mathrm{Map}(S^{-n}, E)$ for a fantasy stable object E. In a similar vein, a pointed finite CW-complex X should give rise to a stable object $\Sigma^\infty X$, which should have a Spanier–Whitehead dual $(\Sigma^\infty X)^*$. The idea of $\mathcal{W}$-spectra is that they "remember" the mapping spaces $E(X) = \mathrm{Map}((\Sigma^\infty X)^*, E)$.

We remark that the notion of $\mathcal{W}$-sequence is essentially equivalent (homotopically speaking) to the notion of a simplicial functor from $s\mathcal{S}et$ to $s\mathcal{S}et$. The connection between these kinds of functors and spectra was initially raised by Anderson [3]. Lydakis [171] first produced (in the simplicial setting) a model category structure as well as the symmetric monoidal product, showed the Quillen equivalence with Bousfield–Friedlander spectra, and identified the ring objects with Bökstedt's FSPs.

3.10.2 Γ-spaces

Let Γ^{op} be the category of finite based sets $\mathbf{n}_+ = \{0, 1, \ldots, n\}$ (based at 0) and based maps. A functor $\Gamma^{op} \to \mathcal{T}op_*$ is called a **Γ-space**. The smash product of based sets induces a symmetric monoidal product on Γ^{op}: specifically, we identify $\mathbf{m}_+ \wedge \mathbf{n}_+$ with $(\mathbf{m} \cdot \mathbf{n})_+$ using the lexicographic ordering. Day convolution then gives a monoidal structure on the category of Γ-spaces.

Γ-spaces were introduced by Segal [268], who showed that the homotopy category is equivalent to the full subcategory of the stable homotopy category consisting of the connective spectra. The first model category structure on Γ-spaces goes back to Bousfield–Friedlander [56] (note that no such model category could be stable, given that the suspension functor on the homotopy category is not an equivalence). Lydakis [172] introduced the symmetric monoidal product on Γ-spaces and showed that it models the smash product of spectra, and [264] produced a model category structure on the ring objects. See also the discussion in [178].

The idea behind Γ-spaces comes from considerations similar to those made in

Remark 3.10.1. In any homotopy theory of spectra we would have objects $\Sigma^\infty T$ for every pointed set T (this will just be a wedge of copies of the sphere spectrum S, indexed by the non-basepoints in T). Therefore we would also have Spanier–Whitehead duals $(\Sigma^\infty T)^*$. The assignment $T \mapsto (\Sigma^\infty T)^*$ would be a contravariant functor defined on Γ^{op}, and for a stable object E the assignment $T \mapsto \mathrm{Map}((\Sigma^\infty T)^*, E)$ would therefore be a Γ-space.

If $T = [n]$ then $\Sigma^\infty T = \bigvee_{i=1}^n S$, and so $(\Sigma^\infty T)^*$ can be identified with the product $\prod_{i=1}^n S$ (using that $S^* = S$). So another way to say the above is that a Γ-space comes from remembering what a spectrum looks like through the eyes of the finite products $*$, S, $S \times S$, $S \times S \times S$, and so forth. That is to say, if E is a spectrum we remember $[n] \mapsto E_n = \mathrm{Map}(S^{\times n}, E)$. As finite products are weakly equivalent to finite wedges in spectra, it's clear that this data can only remember the connective part of a spectrum.

In fact, since $\prod_{i=1}^n S \simeq \bigvee_{i=1}^n S$ we would additionally have the relations

$$E_n = \mathrm{Map}\Big(\prod_{i=1}^n S, E\Big) \simeq \mathrm{Map}\Big(\bigvee_{i=1}^n S, E\Big) \simeq \prod_{i=1}^n \mathrm{Map}(S, E) = \prod_{i=1}^n E_1.$$

This suggests that what we really care about are Γ-spaces X such that a canonical map $X_n \to \prod_{i=1}^n X_1$ is an equivalence (and when $n = 0$ this should be interpreted as $X_0 \simeq *$). These were called "special" Γ-spaces in [56]. This turns out to equip $\pi_0(X_1)$ with the structure of an abelian monoid via the multiplication

$$\pi_0(X_1) \times \pi_0(X_1) \xleftarrow{\cong} \pi_0(X_2) \xrightarrow{\mu} \pi_0(X_1),$$

where μ is induced by the map $[2]_+ \to [1]_+$ sending $1, 2 \mapsto 1$. But if $X_1 = \mathrm{Map}(S, E)$ then we should have $X_1 \simeq \Omega^2 \mathrm{Map}(S^{-2}, E)$, which means $\pi_0(X_1)$ would actually be an abelian group. Adding on this condition yields what [56] called "very special" Γ-spaces. The pleasant surprise is that there are no further "relations" that one has to keep track of here: that is, the model category structure on Γ-spaces is set up so that the fibrant objects are precisely these very special Γ-spaces, and this is enough to get the Quillen equivalence with connective spectra. See also [79, Example 5.7] for another perspective on these "relations".

The inclusion of categories $\Gamma^{op} \hookrightarrow \mathcal{W}$, regarding every pointed set as a discrete topological space, yields comparison functors between $\mathcal{W}$-spaces and Γ-spaces in the usual way. See Remark 3.5.3.

Segal introduced Γ-spaces in [268] because they were a natural receptor for a certain version of algebraic K-theory. We outline this briefly. Let $\mathcal{C}$ be a category with finite coproducts. For a finite set T write $\mathcal{P}(T)$ for the category whose elements are the subsets of T and whose maps are subset inclusions. Let $\mathcal{C}(T)$ be the category whose objects are functors $F\colon \mathcal{P}(T) \to \mathcal{C}$ having the property that whenever $A_1, \ldots, A_n \subseteq T$ are disjoint the set of maps $\{F(A_i) \to F(\cup_i A_i)\}$ induces an isomorphism

$$\coprod_i F(A_i) \xrightarrow{\cong} F\Big(\bigcup_i A_i\Big).$$

When $n = 0$ this property implies that $F(\emptyset)$ is an initial object in $\mathcal{C}$.

If T is a pointed set, let $(K\mathcal{C})(T) = B\mathcal{C}(T - *)$, where $B(-)$ denotes the usual classifying space of a small category (that is, the geometric realization of the nerve).

If $f: T \to U$ is a map of pointed sets, there is an induced map $\mathcal{P}(U-*) \to \mathcal{P}(T-*)$ sending $A \mapsto f^{-1}(A) \cap (T-*)$, and this in turn induces a functor $\mathcal{C}(T-*) \to \mathcal{C}(U-*)$. So $K\mathcal{C}$ is a Γ-space. (The basepoint is playing the role of a "sink" here, in the sense that pointed maps $f: T \to U$ are the same as pairs $(A \subseteq T, A \to U)$, where in the correspondence one has $T - A = f^{-1}(*)$. The reader is advised to work out the maps in $K\mathcal{C}$ where T and U are $\{0,1\}$ and $\{0,1,2\}$ — in either order — to get a feeling for what is happening here.)

Note that an object in $\mathcal{C}(T)$ can be thought of as a T-indexed collection of objects in $\mathcal{C}$ together with consistent choices of coproducts for all subsets of T. Compare the description of Waldhausen K-theory from Section 3.8.8.

3.10.3 Spectra in other settings

Let $\mathcal{M}$ be a symmetric monoidal model category and let K be a cofibrant object. Just as spectra stabilize $\mathcal{T}op_*$ under the operation of smashing with S^1, one might want to stabilize $\mathcal{M}$ under the operation of tensoring with K. Under mild "sufficiently combinatorial" hypotheses on $\mathcal{M}$, this works out just fine. Hovey [132] showed that one can form both Bousfield–Friedlander and symmetric spectra in this generalized setting, and all the basic model structures work out just as expected.

Standard applications include stabilizing the model category of G-spaces along a representation sphere S^V, or stabilizing a model category of motivic spaces along the motivic sphere $S^{2,1}$.

Hovey in fact showed that the Bousfield–Friedlander construction is really about inverting a *functor* $G: \mathcal{M} \to \mathcal{M}$, whereas (as discussed in Section 3.7.4) the symmetric spectrum construction is about making an object invertible in the symmetric monoidal sense. This difference has consequences for the comparison of the two constructions $\mathrm{Sp}^{\mathbb{N},\wedge K}$ and $\mathrm{Sp}^{\Sigma,K}$. In the latter, the suspension spectrum of K is an invertible object and so must satisfy the cyclic permutation condition (3.3.13). In the former, where we are only inverting the functor $(-) \wedge K$ and don't necessarily have a monoidal product around anymore, there is no guarantee that this holds. So there is no reason to suspect a Quillen equivalence here: in general, $\mathrm{Sp}^{\Sigma,K}$ has more "relations" than $\mathrm{Sp}^{\mathbb{N},\wedge K}$. Hovey [132] has some results showing that in the presence of the cyclic permutation condition these two constructions *are* Quillen equivalent, but he also observes that the results are perhaps not as general as one would like.

A version of $\mathcal{W}$-spaces (or simplicial functors) for model categories satisfying certain technical hypotheses has also been developed, by Dundas–Röndigs–Østvaer [84].

3.10.4 G-spectra

Let G be a compact Lie group, but feel free to think only of a finite group if desired. There should of course be a model category of genuine G-spectra, where one stabilizes with respect to all finite-dimensional representation spheres. The associated homotopy category was first developed in [155], and is nicely summarized in [190].

To construct an appropriate model category via symmetric spectra, one could pick representatives $V_1, V_2, \ldots, V_n$ for all finite-dimensional irreducible G-representations and set $V = V_1 \oplus \cdots \oplus V_n$. Performing the symmetric spectra construction on G-spaces using the object S^V makes a perfectly good model category of genuine G-spectra. Although this is fine for some purposes, it is a little unnatural. The fact that all finite-dimensional G-representations aren't inherently built into the machinery can make some things more trouble than they should be.

The construction of orthogonal spectra works right out of the box for G-spaces, requiring only the obvious modifications. See [177] or [120, Appendix A] for details. Currently this is the preferred setting for G-equivariant spectra.

The equivariant version of EKMM spectra is developed in [177]. One starts with a G-universe $\mathcal{U}$ that is "complete" in the sense that it contains infinitely many copies of every irreducible representation. One of the surprises is that there are ***two*** naturally arising model category structures on G-equivariant EKMM-spectra, both having the same notion of stable weak equivalence. One has cofibrations built from cellular inclusions based on cells of the form $F_n(G/H_+ \wedge S^k)$ for $n, k \geq 0$, and the other has cofibrations built from cells of the form $F_V(G/H_+ \wedge S^k)$ with $k \geq 0$ and V a G-representation. These model structures are Quillen equivalent, but different. We refer to [177, Chapter IV.2] for details.

When G is finite, versions of equivariant symmetric spectra have been produced by Mandell [183] and Hausmann [115]. Ostermayr [218] developed a model structure for equivariant Γ-spaces. A model category structure for an equivariant version of $\mathcal{W}$-spaces is developed in [84] (see also [43]).

3.10.5 Model categories for commutative algebras

Let $(\mathcal{S}pectra, \wedge, S)$ be a closed symmetric monoidal model category of spectra that satisfies the Algebraic Creation Property. Let R be a commutative ring spectrum, and write R–ComAlg for the category of commutative R-algebras. The forgetful functor $U\colon R\text{–ComAlg} \to R\text{–Mod}$ has a left adjoint Sym given by the symmetric algebra functor

$$\mathrm{Sym}(M) = R \vee M \vee (M \wedge_R M)/\Sigma_2 \vee (M \wedge_R M \wedge_R M)/\Sigma_3 \vee \cdots .$$

We can ask if the forgetful functor creates a model structure on R–ComAlg.

In EKMM_S, this works with no trouble — in part because all objects are fibrant. See [94, VII.4.7–4.10]. In contrast, for symmetric and orthogonal spectra there is a difficulty and such a model structure ***cannot*** exist in general. For example, it cannot exist when $R = S$: as we saw in Section 3.1.7, there cannot exist a commutative ring spectrum that is weakly equivalent to S and whose underlying spectrum is fibrant.

One solution to this problem is via the ***positive model structure*** on symmetric (or orthogonal) spectra, suggested originally by Jeff Smith. Basically, go back and mimic the development of the level and stable structures but remove all references to what happens in level 0. Change the levelwise weak equivalences to maps that are weak equivalences in levels greater than zero, and so forth. The fibrant objects in the positive stable model structure are then spectra X with the property that $X_n \to \Omega X_{n+1}$ is a weak equivalence for all $n \geq 1$ (these are called "positive Ω-spectra"). This model structure is Quillen equivalent to the one we already had, and it is also monoidal and satisfies all the nice properties we are used to.

The adjoint to the Σ^∞ functor is Ev_0 as always, but note that Ev_0 no longer has the behavior of Ω^∞ for fibrant objects. So there is no problem with having a model for S that is a commutative ring spectrum and is fibrant in the positive model structure.

The positive model structures on symmetric and orthogonal spectra are developed in [178], which also shows that if one uses these structures the forgetful functor does create a model structure on R–ComAlg for any commutative ring spectrum R.

For more work related to these issues, including yet another model structure on symmetric spectra, see [274].

As another application, the positive model structure on Sp^Σ is used in [262] to get a monoidal Quillen equivalence between Sp^Σ and EKMM_S.

Commutative ring spectra are discussed in more detail in Chapter 6 of this volume.

3.10.6 Stable categories and categories of modules

This is only a very brief remark, but if you want to better understand stable model categories in general and how they interact with the modern monoidal categories of spectra, go read [266]. That paper provides a basic technique that is pervasive in how we approach these categories.