

2 Homotopical categories: from model categories to $(\infty, 1)$ -categories

by Emily Riehl

2.1 The history of homotopical categories

A *homotopical category* is a category equipped with some collection of morphisms traditionally called “weak equivalences” that somewhat resemble isomorphisms but fail to be invertible in any reasonable sense, and might in fact not even be reversible: that is, the presence of a weak equivalence $X \xrightarrow{\sim} Y$ need not imply the presence of a weak equivalence $Y \xrightarrow{\sim} X$. Frequently, the weak equivalences are defined as the class of morphisms in a category \mathcal{K} that are “inverted by a functor” $F: \mathcal{K} \rightarrow \mathcal{L}$, in the sense of being precisely those morphisms in \mathcal{K} that are sent to isomorphisms in \mathcal{L} . For instance:

- Weak homotopy equivalences of spaces or spectra are those maps inverted by the homotopy group functors $\pi_*: \mathbf{Top} \rightarrow \mathbf{GrSet}$ or $\pi_*: \mathbf{Spectra} \rightarrow \mathbf{GrAb}$.
- Quasi-isomorphisms of chain complexes are those maps inverted by the homology functor $H_*: \mathbf{Ch} \rightarrow \mathbf{GrAb}$.
- Equivariant weak homotopy equivalences of G -spaces are those maps inverted by the homotopy functors on the fixed point subspaces for each compact subgroup of G .

The term used to describe the equivalence class represented by a topological space up to weak homotopy equivalence is a *homotopy type*. Since the weak homotopy equivalence relation is created by the functor π_* , a homotopy type can loosely be thought of as a collection of algebraic invariants of the space X , as encoded by the homotopy groups π_*X . Homotopy types live in a category called the *homotopy category of spaces*, which is related to the classical category of spaces as follows: a genuine continuous function $X \rightarrow Y$ certainly represents a map (graded homomorphism) between homotopy types. But a weak homotopy equivalence of spaces, defining an isomorphism of homotopy types, should now be regarded as formally invertible.

In their 1967 manuscript *Calculus of fractions and homotopy theory*, Gabriel and Zisman [100] formalized the construction of what they call the *category of fractions* associated to any class of morphisms in any category together with an associated localization functor $\pi: \mathcal{K} \rightarrow \mathcal{K}[\mathcal{W}^{-1}]$ that is universal among functors with domain \mathcal{K} that invert the class \mathcal{W} of weak equivalences. This construction and its universal

property are presented in §2.2. For instance, the homotopy category of spaces arises as the category of fractions associated to the weak homotopy equivalences of spaces.

There is another classical model of the homotopy category of spaces that defines an equivalent category. The objects in this category are the *CW-complexes*, spaces built by gluing disks along their boundary spheres, and the morphisms are now taken to be homotopy classes of maps. By construction the isomorphisms in this category are the homotopy equivalences of CW-complexes. Because any space is weak homotopy equivalent to a CW-complex and because Whitehead's theorem proves that the weak homotopy equivalences between CW-complexes are precisely the homotopy equivalences, it can be shown that this new homotopy category is equivalent to the Gabriel-Zisman category of fractions.

Quillen introduced a formal framework which draws attention to the essential features of these equivalent constructions. His axiomatization of an abstract “homotopy theory” was motivated by the following question: When does it make sense to invert a class of morphisms in a category and call the result a homotopy category, rather than simply a localization? In the introduction to his 1967 manuscript *Homotopical Algebra* [229], Quillen reports that Kan's theorem that the homotopy theory of simplicial groups is equivalent to the homotopy theory of connected pointed spaces [143] suggested to Quillen that simplicial objects over a suitable category A might form a homotopy theory analogous to classical homotopy theory in algebraic topology. In pursuing this analogy he observed that

there were a large number of arguments which were formally similar to well-known ones in algebraic topology, so it was decided to define the notion of a homotopy theory in sufficient generality to cover in a uniform way the different homotopy theories encountered. [229, pp. 1–2]

Quillen named these homotopy theories *model categories*, meaning “categories of models for a homotopy theory.” He entitled his explorations “homotopical algebra,” as they describe both a generalization of and a close analogy to homological algebra — in which the relationship between an abelian category and its derived category parallels the relationship between a model category and its homotopy category. We introduce Quillen's model categories and his construction of their homotopy categories as a category of “homotopy” classes of maps between sufficiently “fat” objects in §2.3. A theorem of Quillen proven as Theorem 2.3.29 below shows that the weak equivalences in any model category are precisely those morphisms inverted by the Gabriel-Zisman localization functor to the homotopy category. In particular, in the homotopical categories that we will most frequently encounter, the weak equivalences satisfy a number of closure properties, to be introduced in Definition 2.3.1.

To a large extent, homological algebra is motivated by the problem of constructing derived versions of functors between categories of chain complexes that fail to preserve weak equivalences. A similar question arises in Quillen's model categories. Because natural transformations can point either to or from a given functor, derived functors come with a “handedness”: either left or right. In §2.4, we introduce dual notions of left and right Quillen functors between model categories and construct their derived functors via a slightly unusual route that demands a stricter (but in our view improved)

definition of derived functors than the conventional one. In parallel, we study the additional properties borne by Quillen’s original model structure on simplicial sets, later axiomatized by Hovey [130] in the notion of a monoidal or enriched model category, which derives to define monoidal structures or enrichments on the homotopy category.

These considerations also permit us to describe when two “homotopy theories” are equivalent. For instance, the analogy between homological and homotopical algebra is solidified by a homotopical reinterpretation of the Dold–Kan theorem as an equivalence between the homotopy theory of simplicial objects of modules and chain complexes of modules presented in Theorem 2.4.33.

As an application of the theory of derived functors, in §2.5 we study homotopy limits and colimits, which correct for the defect that classically defined limit and colimit constructions frequently fail to be weak equivalence invariant. We begin by observing that the homotopy category admits few strict limits. It does admit weak ones, as we shall see in Theorem 2.5.3, but their construction requires higher homotopical information which will soon become a primary focus.

By convention, a full Quillen model structure can only be borne by a category possessing all limits and colimits, and hence the homotopy limits and homotopy colimits introduced in §2.5 are also guaranteed to exist. This supports the point of view that a model category is a presentation of a homotopy theory with all homotopy limits and homotopy colimits. In a series of papers from 1980 [89, 87, 88], Dwyer and Kan describe more general “homotopy theories” as *simplicial localizations* of categories with weak equivalences, which augment the Gabriel–Zisman category of fractions with homotopy types of the mapping spaces between any pair of objects. The *hammock localization* construction described in §2.6 is very intuitive, allowing us to re-conceptualize the construction of the category of fractions not by imposing relations in the same dimension, but by adding maps in the next dimension — “imposing homotopy relations” if you will.

The hammock localization defines a simplicially enriched category associated to any homotopical category. A simplicially enriched category is a non-prototypical exemplification of the notion of an $(\infty, 1)$ -category, that is, a category weakly enriched over ∞ -groupoids or homotopy types. Model categories also equip each pair of their objects with a well-defined homotopy type of maps, and hence also present $(\infty, 1)$ -categories. Before exploring $(\infty, 1)$ -categories in a systematic way, in §2.7 we introduce the most popular model, the *quasi-categories* first defined in 1973 by Boardman and Vogt [48] and further developed by Joyal [140, 141] and Lurie [169].

In §2.8 we turn our attention to other models of $(\infty, 1)$ -categories, studying six in total: quasi-categories, Segal categories, complete Segal spaces, naturally marked quasi-categories, simplicial categories, and relative categories. The last two models are strictly-defined objects, which are quite easy to define, but the model categories in which they live are poorly behaved. By contrast, the first four of these models live in model categories that have many pleasant properties, which are collected together in a new axiomatic notion of an ∞ -cosmos.

After introducing this abstract definition, we see in §2.9 how the ∞ -cosmos axiomatization allows us to develop the basic theory of these four models of $(\infty, 1)$ -categories

model-independently, that is, simultaneously and uniformly across these models. Specifically, we study adjunctions and equivalences between $(\infty, 1)$ -categories and limits and colimits in an $(\infty, 1)$ -category to provide points of comparison for the corresponding notions of Quillen adjunction, Quillen equivalence, and homotopy limits and colimits developed for model categories in §2.4 and §2.5. A brief epilogue, §2.10, contains a few closing thoughts and anticipates future chapters in this volume.

2.1.1 Acknowledgments

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2.2 Categories of fractions and localization

In one of the first textbook accounts of abstract homotopy theory [100], Gabriel and Zisman construct the universal category that inverts a collection of morphisms together with accompanying “calculus-of-fractions” techniques for calculating this categorical “localization.” Gabriel and Zisman prove that a class of morphisms in a category with finite colimits admits a “calculus of left fractions” if and only if the corresponding localization preserves them, which then implies that the category of fractions also admits finite colimits [100, §1.3]; dual results relate finite limits to their “calculus of right fractions.” For this reason, their calculi of fractions fail to exist in the examples of greatest interest to modern homotopy theorists, and so we will not introduce them here, focusing instead in §2.2.1 on the general construction of the category of fractions.

2.2.1 The Gabriel–Zisman category of fractions

For any class of morphisms \mathcal{W} in a category \mathcal{K} , the category of fractions $\mathcal{K}[\mathcal{W}^{-1}]$ is the universal category equipped with a functor $\iota: \mathcal{K} \rightarrow \mathcal{K}[\mathcal{W}^{-1}]$ that inverts \mathcal{W} , in the sense of sending each morphism to an isomorphism. Its objects are the same as the objects of \mathcal{K} and its morphisms are finite zigzags of morphisms in \mathcal{K} , with all “backwards” arrows finite composites of arrows belonging to \mathcal{W} , modulo a few relations which convert the canonical graph morphism $\iota: \mathcal{K} \rightarrow \mathcal{K}[\mathcal{W}^{-1}]$ into a functor and stipulate that the backwards copies of each arrow in \mathcal{W} define two-sided inverses to the morphisms in \mathcal{W} .

Definition 2.2.1 (category of fractions [100, 1.1]). For any class of morphisms \mathcal{W} in a category \mathcal{K} , the **category of fractions** $\mathcal{K}[\mathcal{W}^{-1}]$ is a quotient of the free category on the directed graph obtained by adding backwards copies of the morphisms in \mathcal{W} to the underlying graph of the category \mathcal{K} modulo certain relations:

- Adjacent arrows pointing forwards can be composed.
- Forward-pointing identities may be removed.
- Adjacent pairs of zigzags

$$x \xrightarrow{s} y \xleftarrow{s} x \quad \text{or} \quad y \xleftarrow{s} x \xrightarrow{s} y$$

indexed by any $s \in \mathcal{W}$ can be removed.¹

The image of the functor $\iota: \mathcal{K} \rightarrow \mathcal{K}[\mathcal{W}^{-1}]$ is comprised of those morphisms that can be represented by unary zigzags pointing forwards.

The following proposition expresses the 2-categorical universal property of the category of fractions construction in terms of categories $\text{Fun}(\mathcal{K}, \mathcal{M})$ of functors and natural transformations:

Proposition 2.2.2 (the universal property of localization [100, 1.2]). *For any category \mathcal{M} , restriction along ι defines a fully faithful embedding*

$$\begin{array}{ccc} \text{Fun}(\mathcal{K}[\mathcal{W}^{-1}], \mathcal{M}) & \xrightarrow{-\circ\iota} & \text{Fun}(\mathcal{K}, \mathcal{M}) \\ & \searrow \cong & \swarrow \cong \\ & \text{Fun}(\mathcal{K}, \mathcal{M}) & \\ & \mathcal{W} \mapsto \cong & \end{array}$$

defining an isomorphism

$$\text{Fun}(\mathcal{K}[\mathcal{W}^{-1}], \mathcal{M}) \cong \text{Fun}(\mathcal{K}, \mathcal{M})_{\mathcal{W} \mapsto \cong}$$

of categories onto its essential image, the full subcategory spanned by those functors that invert \mathcal{W} .

Proof. As in the analogous case of rings, the functor $\iota: \mathcal{K} \rightarrow \mathcal{K}[\mathcal{W}^{-1}]$ is an epimorphism and so any functor $F: \mathcal{K} \rightarrow \mathcal{M}$ admits at most one extension along ι . To show that any functor $F: \mathcal{K} \rightarrow \mathcal{M}$ that inverts \mathcal{W} does extend to $\mathcal{K}[\mathcal{W}^{-1}]$, we define a graph morphism from the graph described in Definition 2.2.1 to \mathcal{M} by sending the backwards copy of s to the isomorphism $(Fs)^{-1}$ and thus a functor from the free category generated by this graph to \mathcal{M} . Functoriality of F ensures that the enumerated relations are respected by this functor, which therefore defines an extension $\hat{F}: \mathcal{K}[\mathcal{W}^{-1}] \rightarrow \mathcal{M}$ as claimed.

The 2-dimensional aspect of this universal property follows from the 1-dimensional one by considering functors valued in arrow categories [146, §3]. □

Example 2.2.3 (groupoid reflection). When all the morphisms in \mathcal{K} are inverted, the universal property of Proposition 2.2.2 establishes an isomorphism $\text{Fun}(\mathcal{K}[\mathcal{K}^{-1}], \mathcal{M}) \cong \text{Fun}(\mathcal{K}, \text{core}\mathcal{M})$ between functors from the category of fractions of \mathcal{K} to functors valued in the **groupoid core**, which is the maximal subgroupoid contained in \mathcal{M} . In this

¹ It follows that adjacent arrows in \mathcal{W} pointing backwards can also be composed whenever their composite in \mathcal{K} also lies in \mathcal{W} .

way, the category of fractions construction specializes to define a left adjoint² to the inclusion of groupoids into categories:

$$\text{Cat} \begin{array}{c} \xrightarrow{\text{fractions}} \\ \xleftarrow{\perp} \\ \xrightarrow{\perp} \\ \xleftarrow{\text{core}} \end{array} \text{Gpd}$$

The universal property of Proposition 2.2.2 applies to the class of morphisms inverted by any functor admitting a fully faithful right adjoint [100, 1.3]. In this case, the category of fractions defines a *reflective subcategory* of \mathcal{K} , which admits a variety of useful characterizations, one being as the *local* objects orthogonal to the class of morphisms being inverted [238, 4.5.12, 4.5.vii, 5.3.3, 5.3.i]. For instance, if $R \rightarrow R[S^{-1}]$ is the localization of a commutative ring at a multiplicatively closed set, then the category of $R[S^{-1}]$ -modules defines a reflective subcategory of the category of R -modules [238, 4.5.14], and hence the extension of scalars functor $R[S^{-1}] \otimes_R -$ can be understood as a Gabriel–Zisman localization.

However, reflective subcategories inherit all limits and colimits present in the larger category [238, 4.5.15], which is not typical behavior for categories of fractions that are “homotopy categories” in a sense to be discussed in §5.8. With the question of when a category of fractions is a homotopy category in mind, we now turn our attention to Quillen’s homotopical algebra.

2.3 Model category presentations of homotopical categories

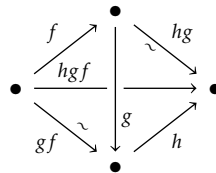
A question that motivated Quillen’s introduction of model categories [229] and also Dwyer, Kan, Hirschhorn, and Smith’s later generalization [92] is: When is a category of fractions a homotopy category? Certainly, the localization functor must invert some class of morphisms that are suitably thought of as “weak equivalences.” Perhaps these weak equivalences coincide with a more structured class of “homotopy equivalences” on a suitable subcategory of “fat” objects that spans each weak equivalence class — such as given in the classical case by Whitehead’s theorem that any weak homotopy equivalence between CW complexes admits a homotopy inverse — in such a way that the homotopy category is equivalent to the category of homotopy classes of maps in this full subcategory. Finally, one might ask that the homotopy category admit certain derived constructions, such as the loop and suspension functors definable on the homotopy category of based spaces. On account of this final desideratum, we will impose the blanket requirement that a category that bears a model structure must be complete and cocomplete.

² More precisely, this left adjoint takes values in a larger universe of groupoids, since the category of fractions $\mathcal{K}[\mathcal{K}^{-1}]$ associated to a locally small category \mathcal{K} need not be locally small. Toy examples illustrating this phenomenon are easy to describe. For instance, let \mathcal{K} be a category with a proper class of objects whose morphisms define a “double asterisk”: each non-identity morphism has a common domain object and for each other object there are precisely two non-identity morphisms with that codomain.

Consider a class of morphisms \mathcal{W} , denoted by “ $\xrightarrow{\sim}$ ”, in a category \mathcal{M} . Such morphisms might reasonably be referred to as “weak equivalences” if they somewhat resemble isomorphisms, aside from failing to be invertible in any reasonable sense. The meaning of “somewhat resembling isomorphisms” may be made precise via any of the following axioms, all of which are satisfied by the isomorphisms in any category.

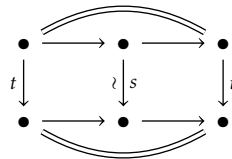
Definition 2.3.1. The following hypotheses are commonly applied to a class of “weak equivalences” \mathcal{W} in a category \mathcal{M} :

- The **two-of-three property**: for any composable pair of morphisms if any two of f , g , and gf is in \mathcal{W} then so is the third.
- The **two-of-six property**: for any composable triple of morphisms



if $gf, hg \in \mathcal{W}$ then $f, g, h, hgf \in \mathcal{W}$.

- The class \mathcal{W} is **closed under retracts** in the arrow category: given a commutative diagram



if s is in \mathcal{W} then so is its retract t .

- The class \mathcal{W} might define a **wide subcategory**, meaning that \mathcal{W} is closed under composition and contains all identity morphisms.
- More prosaically, it is reasonable to suppose that \mathcal{W} contains the isomorphisms.
- At a bare minimum, one might insist that \mathcal{W} contains all of the identities.

Lemma 2.3.2. *Let \mathcal{W} be the class of morphisms in \mathcal{M} inverted by a functor $F: \mathcal{M} \rightarrow \mathcal{K}$. Then \mathcal{W} satisfies each of the closure properties just enumerated.*

Proof. This follows immediately from the axioms of functoriality. □

In practice, most classes of weak equivalences arise as in Lemma 2.3.2. For instance, the quasi-isomorphisms are those chain maps inverted by the homology functor H_\bullet from chain complexes to graded modules, while the weak homotopy equivalences are those continuous functions inverted by the homotopy group functors π_\bullet . Rather than adopt a universal set of axioms that may or may not fit the specific situation at hand, we will use the term **homotopical category** to refer to any pair $(\mathcal{M}, \mathcal{W})$ comprised of a category and a class of morphisms and enumerate the specific properties we need for each result or construction. When the homotopical category $(\mathcal{M}, \mathcal{W})$ underlies a model category structure, to be defined, Theorem 2.3.29 below proves \mathcal{W} is precisely

the class of morphisms inverted by the Gabriel–Zisman localization functor and hence satisfies all of the enumerated closure properties.

The data of a *model structure* borne by a homotopical category is given by two additional classes of morphisms — the *cofibrations* \mathcal{C} denoted “ \twoheadrightarrow ”, and the *fibrations* \mathcal{F} denoted “ \twoheadrightarrow ” — satisfying axioms to be enumerated. In §2.3.1, we present a modern reformulation of Quillen’s axioms that more clearly highlights the central features of a model structure borne by a complete and cocomplete category. In §2.3.2, we discuss the delicate question of the functoriality of the factorizations in a model category with the aim of justifying our view that this condition is harmless to assume in practice.

In §2.3.3, we explain what it means for a parallel pair of morphisms in a model category to be *homotopic*; more precisely, we introduce distinct *left homotopy* and *right homotopy* relations that define a common equivalence relation when the domain is *cofibrant* and the codomain is *fibrant*. The homotopy relation is used in §2.3.4 to construct and compare three equivalent models for the homotopy category of a model category: the Gabriel–Zisman category of fractions $M[\mathcal{W}^{-1}]$ defined by formally inverting the weak equivalences, the category hM_{cf} of fibrant-cofibrant objects in M and homotopy classes of maps, and an intermediary HoM which has the objects of the former and hom-sets of the later, designed to facilitate the comparison. Finally, §2.3.5 presents a fundamental example: Quillen’s model structure on the category of simplicial sets.

2.3.1 Model category structures via weak factorization systems

When Quillen first introduces the definition of a model category in the introduction to “Chapter I. Axiomatic Homotopy Theory” [229], he highlights the factorization and lifting axioms as being the most important. These axioms are most clearly encapsulated in the categorical notion of a weak factorization system, a concept which was codified later.

Definition 2.3.3. A **weak factorization system** $(\mathcal{L}, \mathcal{R})$ on a category M is comprised of two classes of morphisms \mathcal{L} and \mathcal{R} such that:

- (i) Every morphism in M may be factored as a morphism in \mathcal{L} followed by a morphism in \mathcal{R} .

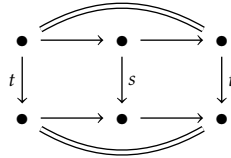
$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ & \searrow & \nearrow \\ & \bullet & \\ \mathcal{L} \ni \ell & & r \in \mathcal{R} \end{array}$$

- (ii) The maps in \mathcal{L} have the **left lifting property** with respect to each map in \mathcal{R} and the maps in \mathcal{R} have the **right lifting property** with respect to each map in \mathcal{L} : that is, any commutative square

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \mathcal{L} \ni \ell \downarrow & \dashrightarrow & \downarrow r \in \mathcal{R} \\ \bullet & \longrightarrow & \bullet \end{array}$$

admits a diagonal filler as indicated, making both triangles commute.

- (iii) The classes \mathcal{L} and \mathcal{R} are each closed under retracts in the arrow category: given a commutative diagram



if s is in that class then so is its retract t .

The following reformulation of Quillen’s definition [229, I.5.1] was given by Joyal and Tierney [142, 7.7], who prove that a homotopical category (M, \mathcal{W}) , with the weak equivalences satisfying the two-of-three property, admits a model structure just when there exist classes \mathcal{C} and \mathcal{F} that define a pair of weak factorization systems:

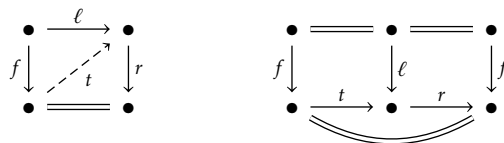
Definition 2.3.4 (model category). A **model structure** on a homotopical category (M, \mathcal{W}) consists of three classes of maps — the **weak equivalences** \mathcal{W} denoted “ $\overset{\sim}{\rightarrow}$ ”, which must satisfy the two-of-three property,³ the **cofibrations** \mathcal{C} denoted “ \rightarrow ”, and the **fibrations** \mathcal{F} denoted “ \twoheadrightarrow ” — so that $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ each define weak factorization systems on M .

Remark 2.3.5 (on self-duality). Definitions 2.3.3 and 2.3.4 are self-dual: if $(\mathcal{L}, \mathcal{R})$ defines a weak factorization system on M then $(\mathcal{R}, \mathcal{L})$ defines a weak factorization system on M^{op} . Thus the statements we prove about the left classes \mathcal{C} of cofibrations and $\mathcal{C} \cap \mathcal{W}$ of **trivial cofibrations** “ $\overset{\sim}{\rightarrow}$ ” will have dual statements involving the right classes \mathcal{F} of fibrations and $\mathcal{F} \cap \mathcal{W}$ of **trivial fibrations** “ $\overset{\sim}{\twoheadrightarrow}$ ”.

Axiom 3 of Definition 2.3.3 was missing from Quillen’s original definition of a model category; he referred to those model categories that have the retract closure property as “closed model categories.” The importance of this closure property is that it implies that the left class of a weak factorization system is comprised of all maps that have the left lifting property with respect to the right class and dually, that the right class is comprised of all of those maps that have the right lifting property with respect to the left class. These results follow as a direct corollary of the famous “retract argument”:

Lemma 2.3.6 (retract argument). *Suppose $f = r \circ \ell$ and f has the left lifting property with respect to its right factor r . Then f is a retract of its left factor ℓ .*

Proof. The solution to the lifting problem displayed on the left



defines the retract diagram on the right. □

³ The standard definition of a model category also requires the weak equivalences to be closed under retracts, but this is a consequence of the axioms given here [142, 7.8].

Corollary 2.3.7. *Either class of a weak factorization system determines the other: the left class consists of those morphisms that have the left lifting property with respect to the right class, and the right class consists of those morphisms that have the right lifting property with respect to the left class.*

Proof. Any map with the left lifting property with respect to the right class of a weak factorization system lifts against its right factor of the factorization guaranteed by axiom 1 of Definition 2.3.3 and so belongs to the left class by axiom 3. \square

It follows that the trivial cofibrations can be defined without reference to either the cofibrations or weak equivalences as those maps that have the left lifting property with respect to the fibrations, and dually the trivial fibrations are precisely those maps that have the right lifting property with respect to the cofibrations.

Exercise 2.3.8. Verify that a model structure on M , if it exists, is uniquely determined by any of the following data:

- (i) The cofibrations and weak equivalences.
- (ii) The fibrations and weak equivalences.
- (iii) The cofibrations and fibrations.

By a more delicate observation of Joyal [141, E.1.10] using terminology to be introduced in Definition 2.3.14, a model structure is also uniquely determined by

- (iv) The cofibrations and fibrant objects.
- (v) The fibrations and cofibrant objects.

As a further consequence of the characterizations of the cofibrations, trivial cofibrations, fibrations, and trivial fibrations by lifting properties, each class automatically enjoys certain closure properties.

Lemma 2.3.9. *Let \mathcal{L} be any class of maps characterized by a left lifting property with respect to a fixed class of maps \mathcal{R} . Then \mathcal{L} contains the isomorphisms and is closed under coproduct, pushout, retract, and (transfinite) composition.*

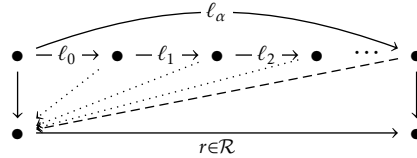
Proof. We prove the cases of pushout and transfinite composition to clarify the meaning of these terms, the other arguments being similar. Let k be a pushout of a morphism $\ell \in \mathcal{L}$ as in the left square below, and consider a lifting problem against a morphism $r \in \mathcal{R}$ as presented by the right square:

$$\begin{array}{ccccc}
 \bullet & \xrightarrow{a} & \bullet & \xrightarrow{u} & \bullet \\
 \mathcal{L} \ni \ell \downarrow & \nearrow s & \downarrow k & \nearrow t & \downarrow r \in \mathcal{R} \\
 \bullet & \xrightarrow{b} & \bullet & \xrightarrow{b} & \bullet
 \end{array}$$

Then there exists a lift s in the composite rectangle and this lift and u together define a cone under the pushout diagram, inducing the desired lift t .

Now let α denote any ordinal category. The **transfinite composite** of a diagram $\alpha \rightarrow M$ is the leg ℓ_α of the colimit cone from the initial object in this diagram to its colimit. To see that this morphism lies in \mathcal{L} under the hypothesis that the generating

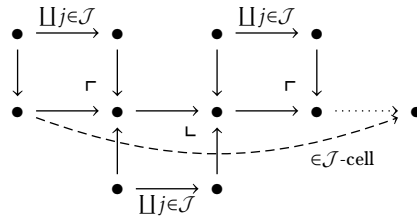
morphisms ℓ_i in the diagram do, it suffices to construct the solution to any lifting problem against a map $r \in \mathcal{R}$.



By the universal property of the colimit object, this dashed morphism exists once the commutative cone of dotted lifts do, and these may be constructed sequentially starting by lifting ℓ_0 against r . \square

Exercise 2.3.10. Verify that the class of morphisms \mathcal{L} characterized by the left lifting property against a fixed class of morphisms \mathcal{R} is closed under coproducts, closed under retracts, and contains the isomorphisms.

Definition 2.3.11. Let \mathcal{J} be any class of maps. A \mathcal{J} -**cell complex** is a map built as a transfinite composite of pushouts of coproducts of maps in \mathcal{J} , which may then be referred to in this context as the basic **cells**.



Lemma 2.3.9 implies that the left class of a weak factorization is closed under the formation of cell complexes.

Exercise 2.3.12. Explore the reason why the class of morphisms \mathcal{L} characterized by the left lifting property against a fixed class of morphisms \mathcal{R} may fail to be closed under coequalizers, formed in the arrow category.⁴

2.3.2 On functoriality of factorizations

The weak factorization systems that arise in practice, such as those that define the components of a model category, tend to admit *functorial* factorizations in the following sense.

Definition 2.3.13. A **functorial factorization** on a category M is given by a functor $M^2 \rightarrow M^3$ from the category of arrows in M to the category of composable pairs of arrows in M that defines a section to the composition functor $\circ: M^3 \rightarrow M^2$. The

⁴ Note, however, that if the maps in \mathcal{L} are equipped with specified solutions to every lifting problem posed by \mathcal{R} and if the squares in the coequalizer diagram commute with these specified lifts, then the coequalizer inherits canonically defined solutions to every lifting problem posed by \mathcal{R} and is consequently in the class \mathcal{L} .

action of this functor on objects in M^2 (which are arrows, displayed vertically) and morphisms in M^2 (which are commutative squares) is displayed below:

$$\begin{array}{ccc} X & \xrightarrow{u} & Z \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{v} & W \end{array} \mapsto \begin{array}{ccc} X & \xrightarrow{u} & Z \\ \left(\begin{array}{ccc} \downarrow Lf & & Lg \downarrow \\ Ef & \xrightarrow{E(u,v)} & Eg \\ \downarrow Rf & & Rg \downarrow \end{array} \right) & & \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{v} & W \end{array}$$

This data is equivalently presented by a pair of endofunctors $L, R: M^2 \rightrightarrows M^2$ satisfying compatibility conditions relative to the domain and codomain projections $\text{dom}, \text{cod}: M^2 \rightrightarrows M$, namely that

$$\text{dom}L = \text{dom}, \quad \text{cod}R = \text{cod}, \quad \text{and} \quad E := \text{cod}L = \text{dom}R$$

as functors $M^2 \rightarrow M$.

The functoriality of Definition 2.3.13 is with respect to (horizontal) composition of squares and is encapsulated most clearly by the functor E which carries a square (u, v) to the morphism $E(u, v)$ between the objects through which f and g factor. Even without assuming the existence of functorial factorizations, in any category with a weak factorization system $(\mathcal{L}, \mathcal{R})$, commutative squares may be factored into a square between morphisms in \mathcal{L} on top of a square between morphisms in \mathcal{R}

$$\begin{array}{ccc} X & \xrightarrow{u} & Z \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{v} & W \end{array} \mapsto \begin{array}{ccc} X & \xrightarrow{u} & Z \\ \left(\begin{array}{ccc} \downarrow \ell \in \mathcal{L} & & \mathcal{L} \ni \ell' \downarrow \\ E & \xrightarrow{\ell} & F \\ \downarrow r \in \mathcal{R} & & \mathcal{R} \ni r' \downarrow \end{array} \right) & & \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{v} & W \end{array}$$

with the dotted horizontal morphism defined by lifting ℓ against r' . These factorizations will not be strictly functorial because the solutions to the lifting problems postulated by axiom 2 of 2.3.3 are not unique. However, for either of the weak factorizations systems in a model category, any two solutions to a lifting problem are *homotopic* in a sense defined by Quillen, appearing below as Definition 2.3.19. As homotopic maps are identified in the homotopy category, this means that any model category has functorial factorizations up to homotopy, which suffices for most purposes.⁵ Despite the moral sufficiency of the standard axioms, for economy of language we henceforth tacitly assume that our model categories have functorial factorizations and take comfort in the fact that it seems to be exceedingly difficult to find model categories that fail to satisfy this condition.

⁵ While the derived functors constructed in Corollary 2.4.10 make use of explicit point-set level functorial factorizations, their total derived functors in the sense of Definition 2.4.4 are well-defined without strict functoriality.

2.3.3 The homotopy relation on arrows

Our aim now is to define Quillen’s homotopy relation, which will be used to construct a relatively concrete model hM_{cf} for the homotopy category of the model category M , which is equivalent to the Gabriel–Zisman category of fractions $M[\mathcal{W}^{-1}]$ but provides better control over the sets of morphisms between each pair of objects. Quillen’s key observation appears as Proposition 2.3.23, which shows that the weak equivalences between objects of M that are both *fibrant* and *cofibrant*, in a sense to be defined momentarily, are more structured, always admitting a homotopy inverse for a suitable notion of homotopy. The homotopy relation is respected by pre- and post-composition, which means that hM_{cf} may be defined simply to be the category of fibrant-cofibrant objects and homotopy classes of maps. In this section, we give all of these definitions. In §2.3.4, we construct the category hM_{cf} sketched above and prove its equivalence with the category of fractions $M[\mathcal{W}^{-1}]$.

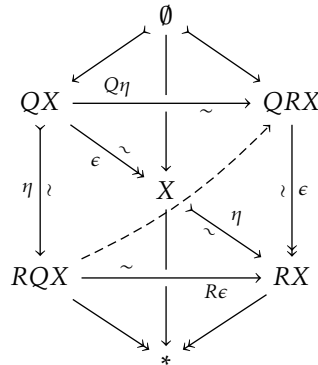
Definition 2.3.14. An object X in a model category M is **fibrant** when the unique map $X \rightarrow *$ to the terminal object is a fibration and **cofibrant** when the unique map $\emptyset \rightarrow X$ from the initial object is a cofibration.

Objects that are not fibrant or cofibrant can always be replaced by weakly equivalent objects that are, by factoring the maps to the terminal object or from the initial object, as appropriate.

Exercise 2.3.15 (fibrant and cofibrant replacement). Assuming the functorial factorizations of §2.3.2, define a **fibrant replacement** functor $R: M \rightarrow M$ and a **cofibrant replacement** functor $Q: M \rightarrow M$ equipped with natural weak equivalences

$$\eta: \text{id}_M \xrightarrow{\sim} R \quad \text{and} \quad \epsilon: Q \xrightarrow{\sim} \text{id}_M.$$

Applying both constructions, one obtains a **fibrant-cofibrant** replacement of any object X as either RQX or QRX . In the diagram



the middle square commutes because its two subdivided triangles do, by naturality of the maps η and ϵ of Exercise 2.3.15. This induces a direct comparison weak equivalence $RQX \xrightarrow{\sim} QRX$ by lifting η_{QX} against ϵ_{RX} .

Exercise 2.3.16. Show that any map in a model category may be replaced, up to a zigzag of weak equivalences, by one between fibrant-cofibrant objects that moreover may be taken to be either a fibration or a cofibration, as desired.⁶

The reason for our particular interest in the subcategory of fibrant-cofibrant objects in a model category is that between such objects the weak equivalences become more structured, coinciding with a class of “homotopy equivalences” in a sense we now define.

Definition 2.3.17. Let A be an object in a model category. A **cylinder object** for A is given by a factorization of the fold map

$$\begin{array}{ccc} A \amalg A & \xrightarrow{(1_A, 1_A)} & A \\ & \searrow (i_0, i_1) & \nearrow q \\ & & \text{cyl}(A) \end{array}$$

into a cofibration followed by a trivial fibration. Dually, a **path object** for A is given by any factorization of the diagonal map

$$\begin{array}{ccc} & \text{path}(A) & \\ & \nearrow j & \searrow (p_0, p_1) \\ A & \xrightarrow{(1_A, 1_A)} & A \times A \end{array}$$

into a trivial cofibration followed by a fibration.

Remark 2.3.18. For many purposes it suffices to drop the hypotheses that the maps in the cylinder and path object factorizations are cofibrations and fibrations, and retain only the hypothesis that the second and first factors, respectively, are weak equivalences. The standard terminology for the cylinder and path objects defined here adds the accolade “very good.” But since “very good” cylinder and path objects always exist, we eschew the usual convention and adopt these as the default notions.

Definition 2.3.19. Consider a parallel pair of maps $f, g: A \rightrightarrows B$ in a model category. A **left homotopy** H from f to g is given by a map from a cylinder object of A to B extending $(f, g): A \amalg A \rightarrow B$:

$$\begin{array}{ccccc} A & \xrightarrow{i_0} & \text{cyl}(A) & \xleftarrow{i_1} & A \\ & \searrow f & \downarrow H & \swarrow g & \\ & & B & & \end{array}$$

in which case one writes $f \sim_\ell g$ and says that f and g are **left homotopic**.

A **right homotopy** K from f to g is given by a map from A to a path object for B

⁶ Exercise 2.3.16 reveals that the notions of “cofibration” and “fibration” are not homotopically meaningful: up to isomorphism in $\mathbf{M}[\mathcal{W}^{-1}]$, any map in a model category can be taken to be either a fibration or a cofibration.

extending $(f, g): A \rightarrow B \times B$:

$$\begin{array}{ccccc}
 & & A & & \\
 & f \swarrow & \downarrow K & \searrow g & \\
 B & \xleftarrow{p_0} & \text{path}(B) & \xrightarrow{p_1} & B
 \end{array}$$

in which case one writes $f \sim_r g$ and says that f and g are **right homotopic**.

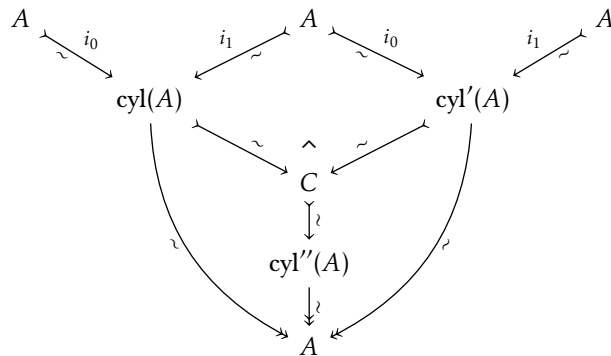
Exercise 2.3.20. Prove that the endpoint inclusions $i_0, i_1: A \rightrightarrows \text{cyl}(A)$ into a cylinder object are weak equivalences always and also cofibrations if A is cofibrant. Conclude that if $f \sim_\ell g$ then f is a weak equivalence if and only if g is. Dually, the projections $p_0, p_1: \text{path}(B) \rightrightarrows B$ are weak equivalences always and also fibrations if B is fibrant, and if $f \sim_r g$ then f is a weak equivalence if and only if g is.

A much more fine-grained analysis of the left and right homotopy relations is presented in the classic expository paper “Homotopy theories and model categories” of Dwyer and Spalinski [91]. Here we focus on the essential facts for understanding the homotopy relation on maps between cofibrant and fibrant objects.

Proposition 2.3.21. *If A is cofibrant and B is fibrant then left and right homotopy define equivalence relations on the set $\text{Hom}(A, B)$ of arrows and moreover these relations coincide.*

In light of Proposition 2.3.21, we say that maps $f, g: A \rightrightarrows B$ from a cofibrant object to a fibrant one are **homotopic** and write $f \sim g$ to mean that they are left or equivalently right homotopic.

Proof. The left homotopy relation is reflexive and symmetric without any cofibrancy or fibrancy hypotheses on the domains or codomains. To prove transitivity, consider a pair of left homotopies $H: \text{cyl}(A) \rightarrow B$ from f to g and $K: \text{cyl}'(A) \rightarrow B$ from g to h , possibly constructed using different cylinder objects for A . By cofibrancy of A and Exercise 2.3.20, a new cylinder object $\text{cyl}''(A)$ for A may be constructed by factoring the map from the following pushout C to A :



The homotopies H and K define a cone under the pushout diagram inducing a map $H \cup_A K: C \rightarrow B$. By fibrancy of B , this map may be extended along the trivial cofibration $C \xrightarrow{\sim} \text{cyl}''(A)$ to define a homotopy $\text{cyl}''(A) \rightarrow B$ from f to h . This proves that left homotopy is an equivalence relation.

Finally we argue that if $H: \text{cyl}(A) \rightarrow B$ defines a left homotopy from f to g then $f \sim_r g$. The desired right homotopy from f to g is constructed as the restriction of the displayed lift

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{\sim} & \text{path}(B) \\ \downarrow i_0 \wr & \nearrow & & & \downarrow (p_0, p_1) \\ A & \xrightarrow{\sim} & \text{cyl}(A) & \xrightarrow{(f, q, H)} & B \times B \end{array}$$

along the endpoint inclusion i_1 . The remaining assertions are dual to ones already proven. \square

Moreover, the homotopy relation is respected by pre- and post-composition:

Proposition 2.3.22. *Suppose $f, g: A \rightrightarrows B$ are left or right homotopic maps and consider any maps $h: A' \rightarrow A$ and $k: B \rightarrow B'$. Then $kfh, kgh: A' \rightrightarrows B'$ are again left or right homotopic, respectively.*

Proof. By lifting the endpoint inclusion $(i_0, i_1): A' \amalg A' \rightarrow \text{cyl}(A')$ against the projection $\text{cyl}(A) \xrightarrow{\sim} A$ — or by functoriality of the cylinder construction in the sense discussed in §2.3.2 — there is a map $\text{cyl}(h): \text{cyl}(A') \rightarrow \text{cyl}(A)$. Then, for any left homotopy $H: \text{cyl}(A) \rightarrow B$ from f to g , the horizontal composite defines a left homotopy $kfh \sim_\ell kgh$:

$$\begin{array}{ccccc} & & A' \amalg A' & \xrightarrow{h \sqcup h} & A \amalg A \\ & \swarrow & & \swarrow & \searrow (f, g) \\ \text{cyl}(A') & \xrightarrow{\text{cyl}(h)} & \text{cyl}(A) & \xrightarrow{H} & B \xrightarrow{k} B' \\ \downarrow \wr & & \downarrow \wr & & \\ A' & \xrightarrow{h} & A & & \end{array} \quad \square$$

Proposition 2.3.23. *Let $f: A \rightarrow B$ be a map between objects that are both fibrant and cofibrant. Then f is a weak equivalence if and only if it has a homotopy inverse.*

Proof. For both implications we make use of the fact that any map between fibrant-cofibrant objects may be factored as a trivial cofibration followed by a fibration through an object that is again fibrant-cofibrant:

$$\begin{array}{ccc} & & P \\ & \nearrow j & \searrow p \\ A & \xrightarrow{f} & B \end{array}$$

If f is a weak equivalence then p is a trivial fibration. We argue that any trivial fibration p between fibrant-cofibrant objects extends to a deformation retraction: admitting a right inverse that is also a left homotopy inverse. A dual argument proves that the trivial cofibration j admits a left inverse that is also a right homotopy inverse. These homotopy equivalences compose in the sense of Proposition 2.3.22 to define a homotopy inverse for f .

If p is a trivial fibration, then cofibrancy of B implies that it admits a right inverse i . The homotopy constructed in the lifting problem

$$\begin{array}{ccc}
 \emptyset & \longrightarrow & P \\
 \downarrow & \nearrow i & \downarrow p \\
 B & \xlongequal{\quad} & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 P \amalg P & \xrightarrow{(1_P, ip)} & P \\
 \downarrow & \nearrow \wr & \downarrow \wr p \\
 \text{cyl}(P) & \xrightarrow{\sim} P & \xrightarrow{p} B
 \end{array}$$

proves that $ip \sim 1_P$ as desired.

For the converse we suppose that f admits a homotopy inverse g . To prove that f is a weak equivalence it suffices to prove that p is a weak equivalence. A right inverse i to p may be found by lifting the endpoint of the homotopy $H: fg \sim 1_B$:

$$\begin{array}{ccccc}
 & B & \xrightarrow{g} & A & \xrightarrow{j} & P \\
 & \wr \downarrow i_0 & & & \nearrow \wr & \downarrow p \\
 B & \xrightarrow{i_1} & \text{cyl}(B) & \xrightarrow{H} & B & \\
 & \wr & & & &
 \end{array}$$

and then restricting this lift along i_1 . By construction this section i is homotopic to fg . The argument of the previous paragraph applies to the trivial cofibration j to prove that it has a left inverse and right homotopy inverse q . Composing the homotopies $1_P \sim jq$, $i \sim fg$, and $gf \sim 1_A$ we see that

$$ip \sim ipjq = ifq \sim jgfq \sim jq \sim 1_P$$

By Exercise 2.3.20 we conclude that ip is a weak equivalence. But by construction p is a retract of ip :

$$\begin{array}{ccccc}
 P & \xlongequal{\quad} & P & \xlongequal{\quad} & P \\
 p \downarrow & & \wr \downarrow ip & & \downarrow p \\
 B & \xrightarrow{i} & P & \xrightarrow{p} & B
 \end{array}$$

so it follows from the retract stability of the weak equivalences [142, 7.8] that p is a weak equivalence, as desired. \square

2.3.4 The homotopy category of a model category

In this section, we prove that the category of fractions $M[\mathcal{W}^{-1}]$, defined by formally inverting the weak equivalences, is equivalent to the category hM_{cf} of fibrant-cofibrant objects and homotopy classes of maps. Our proof appeals to the universal property of Proposition 2.2.2, which characterizes those categories that are *isomorphic* to the category of fractions. For categories to be isomorphic, they must have the same object sets, so we define a larger version of the homotopy category HoM , which has the same objects as $M[\mathcal{W}^{-1}]$ and is equivalent to its full subcategory hM_{cf} .

Definition 2.3.24. For any model category M , there is a category hM_{cf}

- whose objects are the fibrant-cofibrant objects in M and

- in which the set of morphisms from A to B is taken to be the set of homotopy classes of maps

$$[A, B] := \text{Hom}(A, B)_{/\sim}.$$

Proposition 2.3.22 ensures that composition in hM_{cf} is well-defined.

Definition 2.3.25. The **homotopy category** HoM of a model category M is defined by applying the (bijective-on-objects, fully faithful) factorization to the composite functor

$$\begin{array}{ccccc} \text{M} & \xrightarrow{RQ} & \text{M}_{\text{cf}} & \xrightarrow{\pi} & \text{hM}_{\text{cf}} \\ & \searrow \gamma & & \searrow \nu & \\ & \text{bij obj} & \text{HoM} & \xrightarrow{f+f} & \end{array} \quad (2.1)$$

That is, the objects in HoM are the objects in M and

$$\text{HoM}(A, B) := \text{M}(RQA, RQB)_{/\sim}.$$

Exercise 2.3.26 ($\text{HoM} \simeq \text{hM}_{\text{cf}}$).

- Verify that the category hM_{cf} defined by Definition 2.3.24 is equivalent to the full subcategory of HoM spanned by the fibrant-cofibrant objects of M .
- Show that every object in M is isomorphic in HoM to a fibrant-cofibrant object.
- Conclude that the categories HoM and hM_{cf} are equivalent.

Theorem 2.3.27 (Quillen). *For any model category M , the category of fractions $\text{M}[\mathcal{W}^{-1}]$ obtained by formally inverting the weak equivalences is isomorphic to the homotopy category HoM .*

Proof. We will prove that $\gamma: \text{M} \rightarrow \text{HoM}$ satisfies the universal property of Proposition 2.2.2 that characterizes the category of fractions $\text{M}[\mathcal{W}^{-1}]$. First we must verify that γ inverts the weak equivalences. The functor RQ carries weak equivalences in M to weak equivalences between fibrant-cofibrant objects. Proposition 2.3.23 then implies that these admit homotopy inverses and thus become isomorphisms in hM_{cf} . This proves that the composite horizontal functor of (2.1) inverts the weak equivalences. By fully-faithfulness of ν , the functor $\gamma: \text{M} \rightarrow \text{HoM}$ also inverts the weak equivalences.

It remains to verify that any functor $F: \text{M} \rightarrow \text{E}$ that inverts the weak equivalences factors uniquely through γ :

$$\begin{array}{ccc} \text{M} & \xrightarrow{F} & \text{E} \\ & \searrow \gamma & \searrow \bar{F} \\ & \text{HoM} & \end{array}$$

Since γ is identity-on-objects, we must define \bar{F} to agree with F on objects. Recall that the fibrant and cofibrant replacement functors come with natural weak equivalences $\epsilon_X: QX \xrightarrow{\sim} X$ and $\eta_X: X \xrightarrow{\sim} RX$. Because F inverts weak equivalences, these natural transformations define a natural isomorphism $\alpha: F \Rightarrow FRQ$ of functors from M to E . By the definition $\text{HoM}(X, Y) := \text{M}(RQX, RQY)_{/\sim}$, the morphisms from X to Y in HoM correspond to homotopy classes of morphisms from RQX to RQY in M . Choose

any representative $h: RQX \rightarrow RQY$ for the corresponding homotopy class of maps and define its image to be the composite

$$\bar{F}h: FX \xrightarrow{\alpha_X} FRQX \xrightarrow{Fh} FRQY \xrightarrow{\alpha_Y^{-1}} FY.$$

This is well-defined because if $h \sim h'$ then there exists a left homotopy such that $Hi_0 = h$ and $Hi_1 = h'$, where i_0 and i_1 are both sections to a common weak equivalence (the projection from the cylinder). Since F inverts weak equivalences, Fi_0 and Fi_1 are both right inverses to a common isomorphism, so it follows that $Fi_0 = Fi_1$ and hence $Fh = Fh'$.

Functoriality of \bar{F} follows immediately from naturality of α and functoriality of FRQ . To see that $\bar{F}\gamma = F$, recall that for any $f: X \rightarrow Y$ in M , $\gamma(f)$ is defined to be the map in $\text{HoM}(X, Y)$ represented by the homotopy class $RQf: RQX \rightarrow RQY$. By naturality of α , $\bar{F}\gamma(f) = Ff$, so that the triangle of functors commutes.

Finally, to verify that \bar{F} is unique observe that from the following commutative diagram in M any map $h \in \text{HoM}(X, Y)$, the leftmost vertical arrow, is isomorphic in HoM to a map in the image of γ , the vertical arrow on the right:

$$\begin{array}{ccccc} RQX & \xleftarrow{\epsilon_{RQX}} & QRQX & \xrightarrow{\eta_{QRQX}} & RQRQX \\ h \downarrow & & \downarrow Qh & & \downarrow RQh = \gamma(h) \\ RQY & \xleftarrow{\epsilon_{RQY}} & QRQY & \xrightarrow{\eta_{QRQY}} & RQRQY \end{array}$$

Since the image of \bar{F} on the right vertical morphism is uniquely determined and the top and bottom morphisms are isomorphisms, the image of \bar{F} on the left vertical morphism is also uniquely determined. \square

Remark 2.3.28. The universal property of hM_{cf} is slightly weaker than the universal property described in Proposition 2.2.2 for the category of fractions $M[\mathcal{W}^{-1}]$. For any category E , restriction along $\gamma: M \rightarrow \text{hM}_{\text{cf}}$ defines a fully faithful embedding $\text{Fun}(\text{hM}_{\text{cf}}, E) \hookrightarrow \text{Fun}(M, E)$ and equivalence onto the full subcategory of functors from M to E that carry weak equivalences to isomorphisms. The difference is that a given homotopical functor on M may not factor strictly through hM_{cf} but may only factor up to natural isomorphism. In practice, this presents no serious difficulty.

As a corollary, it is now easy to see that the only maps inverted by the localization functor are weak equivalences. By Lemma 2.3.2, this proves that the weak equivalences in a model category have all of the closure properties enumerated at the outset of this section.

Theorem 2.3.29 ([229, 5.1]). *A morphism in a model category M is inverted by the localization functor*

$$M \rightarrow M[\mathcal{W}^{-1}]$$

if and only if it is a weak equivalence.

Proof. Cofibrantly and then fibrantly replacing the map it suffices to consider a map between fibrant-cofibrant objects. By Theorem 2.3.27 we may prove this result for

$M_{\text{cf}} \rightarrow \text{h}M_{\text{cf}}$ instead. But now this is clear by construction: since morphisms in $\text{h}M_{\text{cf}}$ are homotopy classes of maps, the isomorphisms are the homotopy equivalences, which coincide exactly with the weak equivalences between fibrant-cofibrant objects by Proposition 2.3.23. \square

2.3.5 Quillen’s model structure on simplicial sets

We conclude this section with a prototypical example. Quillen’s original model structure is borne by the category of **simplicial sets**, presheaves on the category Δ of finite non-empty ordinals $[n] = \{0 < 1 < \dots < n\}$ and order-preserving maps. A **simplicial set** $X: \Delta^{\text{op}} \rightarrow \text{Set}$ is a graded set $\{X_n\}_{n \geq 0}$ — where elements of X_n are called “ n -simplices” — equipped with dimension-decreasing “face” maps $X_n \rightarrow X_m$ arising from monomorphisms $[m] \hookrightarrow [n] \in \Delta$ and dimension-increasing “degeneracy” maps $X_m \rightarrow X_n$ arising from epimorphisms $[n] \twoheadrightarrow [m] \in \Delta$. An n -simplex has $n + 1$ codimension-one faces, each of which avoids one of its $n + 1$ vertices.

There is a **geometric realization** functor $|-|: \text{sSet} \rightarrow \text{Top}$ that produces a topological space $|X|$ from a simplicial set X by gluing together topological n -simplices for each non-degenerate n -simplex along its lower-dimensional faces. The simplicial set represented by $[n]$ defines the standard n -simplex Δ^n . Its boundary $\partial\Delta^n$ is the union of its codimension-one faces, while a horn Λ_k^n is the further subspace formed by omitting the face opposite the vertex $k \in [n]$.

Theorem 2.3.30 (Quillen). *The category sSet admits a model structure whose*

- *weak equivalences are those maps $f: X \rightarrow Y$ that induce a weak homotopy equivalence $f: |X| \rightarrow |Y|$ on geometric realizations,*
- *cofibrations are monomorphisms, and*
- *fibrations are the **Kan fibration**, which are characterized by the left lifting property with respect to the set of all horn inclusions:*

$$\begin{array}{ccc}
 \Lambda_k^n & \longrightarrow & X \\
 \wr \downarrow & \nearrow & \downarrow \\
 \Delta^n & \longrightarrow & Y
 \end{array}$$

All objects are cofibrant. The fibrant objects are the **Kan complexes**, those simplicial sets in which all horns can be filled. The fibrant objects are those simplicial sets that most closely resemble topological spaces. In particular, two vertices in a Kan complex lie in the same path component if and only if they are connected by a single 1-simplex, with may be chosen to point in either direction. By Proposition 2.3.23 a weak equivalence between Kan complexes is a homotopy equivalence where the notion of homotopy is defined with respect to the interval Δ^1 using $\Delta^1 \times X$ as a cylinder object or X^{Δ^1} as a path object.

Quillen’s model category of simplicial sets is of interest because, on the one hand, the category of simplicial sets is very well behaved and, on the other hand, the geometric realization functor defines an “equivalence of homotopy theories”: in particular, the

homotopy category of simplicial sets gives another model for the homotopy category of spaces. To explain this, we turn our focus to derived functors and derived equivalences between model categories, the subject of §2.4.

2.4 Derived functors between model categories

Quillen’s model category axioms allow us to conjure a homotopy relation between parallel maps in any model category, whatever the objects of that category might be. For this reason, model categories are often regarded as “abstract homotopy theories.” We will now zoom out to consider functors comparing such homotopy theories.

More generally, we might consider functors between homotopical categories equipped with weak equivalences that at least satisfy the two-of-three property. A great deal of the subtlety in “category theory up to weak equivalence” has to do with the fact that functors between homotopical categories need not necessarily preserve weak equivalences. In the case where a functor fails to preserve weak equivalence the next best hope is that it admits a universal approximation by a functor that does, where the approximation is either “from the left” or “from the right.” Such approximations are referred to as *left* or *right derived functors*.

The universal properties of left or right derived functors exist at the level of homotopy categories though the derived functors of greatest utility, and the ones that are most easily constructed in practice, can be constructed at the “point-set level.” One of the selling points of Quillen’s theory of model categories is that they highlight classes of functors — the left or right Quillen functors — whose left or right derived functors can be constructed in a uniform way making the passage to total derived functors pseudofunctorial. However, it turns out a full model structure is not necessary for this construction; morally speaking, all that matters for the specification of derived functors is the weak equivalences.

In §2.4.1, we give a non-standard and in our view greatly improved presentation of the theory of derived functors guided by a recent axiomatization of Dwyer–Hirschhorn–Kan–Smith [92] paired with a result of Maltsiniotis [176]. The key point of difference is that we give a much stronger definition of what constitutes a derived functor than the usual one. In §2.4.2 we introduce left and right Quillen functors between model categories and show that such functors have a left or right derived functor satisfying this stronger property. Then, in §2.4.3, we see that the abstract theory of this stronger class of derived functors is considerably better than the theory of the weaker ones. A highlight is an efficient expression of the properties of composite or adjoint derived functors proven by Shulman [277] and reproduced as Theorem 2.4.15.

In §2.4.4, we extend the theory of derived functors to allow functors of two variables, with the aim of proving that the homotopy category of spaces is cartesian closed, inheriting an internal hom defined as the derived functor of the point-set level mapping spaces. Implicit in our approach to the proof of this statement is a result promised at the end of §2.3.5. In §2.4.5, we define a precise notion of equivalence between abstract homotopy theories encoded by model categories, which specializes to establish an equivalence between the homotopy theory of spaces and the homotopy theory of

simplicial sets. Finally, in §2.4.6 we briefly sketch the connection between *homotopical algebra* and *homological algebra* by considering suitable model structures appropriate for a theory of derived functors between chain complexes.

2.4.1 Derived functors and equivalence of homotopy theories

As a warning to the reader, this definition of a derived functor is stronger than the usual one in two ways:

- We explicitly require our derived functors to be defined “at the point-set level” rather than simply as functors between homotopy categories.
- We require the universal property of the corresponding “total derived functors” between homotopy categories to define *absolute* Kan extensions.

Before defining our derived functors we should explain the general meaning of absolute Kan extensions.

Definition 2.4.1. A **left Kan extension** of $F: C \rightarrow E$ along $K: C \rightarrow D$ is a functor $\text{Lan}_K F: D \rightarrow E$ together with a natural transformation $\eta: F \Rightarrow \text{Lan}_K F \cdot K$ such that for any other such pair $(G: D \rightarrow E, \gamma: F \Rightarrow GK)$, γ factors uniquely through η .⁷

$$\begin{array}{ccc} C & \xrightarrow{F} & E \\ & \searrow K & \nearrow \text{Lan}_K F \\ & D & \end{array} \quad \Downarrow \eta \quad \begin{array}{ccc} C & \xrightarrow{F} & E \\ & \searrow K & \nearrow G \\ & D & \end{array} = \begin{array}{ccc} C & \xrightarrow{F} & E \\ & \searrow K & \nearrow \text{Lan}_K F \\ & D & \end{array} \quad \Downarrow \eta \quad \begin{array}{ccc} C & \xrightarrow{F} & E \\ & \searrow K & \nearrow G \\ & D & \end{array} \quad \exists! \Downarrow$$

Dually, a **right Kan extension** of $F: C \rightarrow E$ along $K: C \rightarrow D$ is a functor $\text{Ran}_K F: D \rightarrow E$ together with a natural transformation $\epsilon: \text{Ran}_K F \cdot K \Rightarrow F$ such that for any $(G: D \rightarrow E, \delta: GK \Rightarrow F)$, δ factors uniquely through ϵ :

$$\begin{array}{ccc} C & \xrightarrow{F} & E \\ & \searrow K & \nearrow \text{Ran}_K F \\ & D & \end{array} \quad \Uparrow \epsilon \quad \begin{array}{ccc} C & \xrightarrow{F} & E \\ & \searrow K & \nearrow G \\ & D & \end{array} = \begin{array}{ccc} C & \xrightarrow{F} & E \\ & \searrow K & \nearrow \text{Ran}_K F \\ & D & \end{array} \quad \Uparrow \epsilon \quad \begin{array}{ccc} C & \xrightarrow{F} & E \\ & \searrow K & \nearrow G \\ & D & \end{array} \quad \exists! \Uparrow$$

A left or right Kan extension is **absolute** if for any functor $H: E \rightarrow F$, the whiskered composite $(H\text{Lan}_K F: D \rightarrow E, H\eta)$ or $(H\text{Ran}_K F: D \rightarrow E, H\epsilon)$ defines the left or right Kan extension of HF along K .

A functor between homotopical categories is a **homotopical functor** if it preserves the classes of weak equivalences, or carries the weak equivalences in the domain to isomorphisms in the codomain in the case where no class of weak equivalences is specified. Derived functors can be understood as universal homotopical approximations to a given functor in a sense we now define.

Definition 2.4.2 (derived functors). Let M and K be homotopical categories with weak equivalences satisfying the two-of-three property and with localization functors $\gamma: M \rightarrow \text{Ho}M$ and $\delta: K \rightarrow \text{Ho}K$.

⁷ Writing α for the natural transformation $\text{Lan}_K F \Rightarrow G$, the right-hand pasting diagrams express the equality $\gamma = \alpha K \cdot \eta$, i.e., that γ factors as $F \xrightarrow{\eta} \text{Lan}_K F \cdot K \xrightarrow{\alpha K} GK$.

- A **left derived functor** of $F: M \rightarrow K$ is a homotopical functor $\mathbb{L}F: M \rightarrow K$ equipped with a natural transformation $\lambda: \mathbb{L}F \Rightarrow F$ such that $\delta\mathbb{L}F$ and $\delta\lambda: \delta\mathbb{L}F \Rightarrow \delta F$ define an absolute *right* Kan extension of δF along γ :

$$\begin{array}{ccc} M & \begin{array}{c} \xrightarrow{F} \\ \uparrow \lambda \\ \xrightarrow{\mathbb{L}F} \end{array} & K \\ & \Leftrightarrow & \\ \begin{array}{ccc} M & \xrightarrow{F} & K \\ \gamma \downarrow & \uparrow \delta\lambda & \downarrow \delta \\ \text{Ho}M & \xrightarrow{\delta\mathbb{L}F} & \text{Ho}K \end{array} & & \end{array}$$

- A **right derived functor** of $F: M \rightarrow K$ is a homotopical functor $\mathbb{R}F: M \rightarrow K$ equipped with a natural transformation $\rho: F \Rightarrow \mathbb{R}F$ such that $\delta\mathbb{R}F$ and $\delta\rho: \delta F \Rightarrow \delta\mathbb{R}F$ define an absolute *left* Kan extension of δF along γ :

$$\begin{array}{ccc} M & \begin{array}{c} \xrightarrow{F} \\ \downarrow \rho \\ \xrightarrow{\mathbb{R}F} \end{array} & K \\ & \Leftrightarrow & \\ \begin{array}{ccc} M & \xrightarrow{F} & K \\ \gamma \downarrow & \downarrow \delta\rho & \downarrow \delta \\ \text{Ho}M & \xrightarrow{\delta\mathbb{R}F} & \text{Ho}K \end{array} & & \end{array}$$

Remark 2.4.3. Absolute Kan extensions are in particular “pointwise” Kan extensions, these being the left or right Kan extensions that are preserved by representable functors. The pointwise left or right Kan extensions are those definable as colimits or limits in the target category [238, §6.3], so it is somewhat surprising that these conditions are appropriate to require for functors valued in homotopy categories, which have few limits and colimits.⁸

As a consequence of Proposition 2.2.2, the homotopical functors

$$\delta\mathbb{L}F, \delta\mathbb{R}F: M \rightrightarrows \text{Ho}K$$

factor uniquely through γ and so may be equally regarded as functors

$$\delta\mathbb{L}F, \delta\mathbb{R}F: \text{Ho}M \rightrightarrows \text{Ho}K,$$

as appearing in the displayed diagrams of Definition 2.4.2.

Definition 2.4.4 (total derived functors). The **total left** or **total right derived functors** of F are the functors

$$\delta\mathbb{L}F, \delta\mathbb{R}F: \text{Ho}M \rightrightarrows \text{Ho}K,$$

defined as absolute Kan extensions in Definition 2.4.2 and henceforth denoted by

$$\mathbb{L}F, \mathbb{R}F: \text{Ho}M \rightrightarrows \text{Ho}K.$$

There is a common setting in which derived functors exist and admit a simple construction. Such categories have a subcategory of “good” objects on which the functor of interest becomes homotopical and a functorial reflection into this full subcategory. The details are encoded in the following axiomatization introduced in [92] and exposed in [276], though we diverge from their terminology to more thoroughly ground our intuition in the model categorical case.

⁸ With the exception of products and coproducts, the so-called “homotopy limits” and “homotopy colimits” introduced in §2.5 do not define limits and colimits in the homotopy category.

Definition 2.4.5. A **left deformation** on a homotopical category M consists of an endofunctor Q together with a natural weak equivalence $q: Q \xrightarrow{\cong} 1$.

The functor Q is necessarily homotopical. Let M_c be any full subcategory of M containing the image of Q . The inclusion $M_c \rightarrow M$ and the left deformation $Q: M \rightarrow M_c$ induce an equivalence between $\text{Ho}M$ and $\text{Ho}M_c$. As our notation suggests, any model category M admits a left deformation defined by cofibrant replacement. Accordingly, we refer to M_c as the subcategory of **cofibrant objects**, trusting the reader to understand that when we have not specified any model structures, Quillen's technical definition is not what we require.

Definition 2.4.6. A functor $F: M \rightarrow K$ between homotopical categories is **left deformable** if there exists a left deformation on M such that F is homotopical on an associated subcategory of cofibrant objects.

Our first main result proves that left deformations can be used to construct left derived functors. The basic framework of left deformations was set up in [92] while the fact that such derived functors are absolute Kan extensions was observed in [176].

Theorem 2.4.7 ([92, 41.2-5], [176]). *If $F: M \rightarrow K$ has a left deformation $q: Q \xrightarrow{\cong} 1$, then $\mathbb{L}F = FQ$ is a left derived functor of F .*

Proof. Write $\delta: K \rightarrow \text{Ho}K$ for the localization. To show that (FQ, Fq) is a point-set left derived functor, we must show that the functor δFQ and natural transformation $\delta Fq: \delta FQ \Rightarrow \delta F$ define a right Kan extension. The verification makes use of Proposition 2.2.2, which identifies the functor category $\text{Fun}(\text{Ho}M, \text{Ho}K)$ with the full subcategory of $\text{Fun}(M, \text{Ho}K)$ spanned by the homotopical functors. Suppose $G: M \rightarrow \text{Ho}K$ is homotopical and consider $\alpha: G \Rightarrow \delta F$. Because G is homotopical and $q: Q \Rightarrow 1_M$ is a natural weak equivalence, $Gq: GQ \Rightarrow G$ is a natural isomorphism. Using naturality of α , it follows that α factors through δFq as

$$G \xrightarrow{(Gq)^{-1}} GQ \xrightarrow{\alpha_Q} \delta FQ \xrightarrow{\delta Fq} \delta F.$$

To prove uniqueness, suppose α factors as

$$G \xrightarrow{\beta} \delta FQ \xrightarrow{\delta Fq} \delta F.$$

Naturality of β provides a commutative square of natural transformations:

$$\begin{array}{ccc} GQ & \xrightarrow{\beta_Q} & \delta FQ^2 \\ Gq \downarrow & & \downarrow \delta FQq \\ G & \xrightarrow{\beta} & \delta FQ \end{array}$$

Because q is a natural weak equivalence and the functors G and δFQ are homotopical, the vertical arrows are natural isomorphisms, so β is determined by β_Q . This restricted

natural transformation is uniquely determined: q_Q is a natural weak equivalence between objects in the image of Q . Since F is homotopical on this subcategory, this means that Fq_Q is a natural weak equivalence and thus δFq_Q is an isomorphism, so β_Q must equal the composite of the inverse of this natural isomorphism with α_Q .

Finally, to show that this right Kan extension is absolute, our task is to show that for any functor $H: \text{HoK} \rightarrow \text{E}$, the pair $(H\delta FQ, H\delta Fq)$ again defines a right Kan extension. Note that (Q, q) also defines a left deformation for $H\delta F$, simply because the functor $H: \text{HoK} \rightarrow \text{E}$ preserves isomorphisms. The argument just given now demonstrates that $(H\delta FQ, H\delta Fq)$ is a right Kan extension, as claimed. \square

2.4.2 Quillen functors

We'll now introduce important classes of functors between model categories that will admit derived functors.

Definition 2.4.8. A functor between model categories is

- **left Quillen** if it preserves cofibrations, trivial cofibrations, and cofibrant objects, and
- **right Quillen** if it preserves fibrations, trivial fibrations, and fibrant objects.

Most left Quillen functors are “cocontinuous,” preserving all colimits, while most right Quillen functors are “continuous,” preserving all limits; when this is the case there is no need to separately assume that cofibrant or fibrant objects are preserved. Importantly, cofibrant replacement defines a left deformation for any left Quillen functor, while fibrant replacement defines a right deformation for any right Quillen functor, as we now demonstrate:

Lemma 2.4.9 (Ken Brown’s lemma).

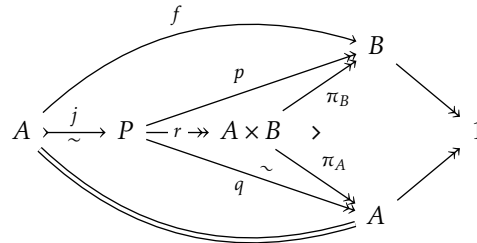
- (i) *Any map between fibrant objects in a model category can be factored as a right inverse to a trivial fibration followed by a fibration:*

$$\begin{array}{ccc}
 & P & \\
 \overset{q}{\curvearrowright} & \nearrow & \searrow p \\
 A & \xrightarrow{f} & B
 \end{array}
 \tag{2.2}$$

- (ii) *Let $F: \text{M} \rightarrow \text{K}$ be a functor from a model category to a category with a class of weak equivalences satisfying the two-of-three property. If F carries trivial fibrations in M to weak equivalences in K , then F carries all weak equivalences between fibrant objects in M to weak equivalences in K .*

Proof. For (i), given any map $f: A \rightarrow B$ factor its graph $(1_A, f): A \rightarrow A \times B$ as a

trivial cofibration j followed by a fibration r :



Since A and B are fibrant, the dual of Lemma 2.3.9 implies that the product projections are fibrations, and thus the composite maps p and q are fibrations. By the two-of-three property, q is also a weak equivalence.

To prove (ii) assume that $f : A \rightarrow B$ is a weak equivalence in M and construct the factorization (2.2). It follows from the two-of-three property that p is also a trivial fibration, so by hypothesis both Fp and Fq are weak equivalences in K . Since Fj is right inverse to Fq , it must also be a weak equivalence, and thus the closure of weak equivalences under composition implies that Ff is a weak equivalence as desired. \square

Specializing Theorem 2.4.7 we then have:

Corollary 2.4.10. *The left derived functor of any left Quillen functor F exists and is given by $\mathbb{L}F := FQ$, while the right derived functor of any right Quillen functor G exists and is given by $\mathbb{R}G := GR$, where Q and R denote any cofibrant and fibrant replacement functors, respectively.*

2.4.3 Derived composites and derived adjunctions

Left and right Quillen functors frequently occur in adjoint pairs, in which case the left adjoint is left Quillen if and only if the right adjoint is right Quillen:

Definition 2.4.11. Consider an adjunction between a pair of model categories.

$$M \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} N \tag{2.3}$$

Then the following are equivalent, defining a **Quillen adjunction**.

- (i) The left adjoint F is left Quillen.
- (ii) The right adjoint G is right Quillen.
- (iii) The left adjoint preserves cofibrations and the right adjoint preserves fibrations.
- (iv) The left adjoint preserves trivial cofibrations and the right adjoint preserves trivial fibrations.

Exercise 2.4.12. Justify the equivalence of the properties in the definition by proving:

- (i) In the presence of any adjunction (2.3) the lifting problem displayed below left in \mathcal{N} admits a solution if and only if the transposed lifting problem displayed below right admits a solution in \mathcal{M} .

$$\begin{array}{ccc}
 FA & \xrightarrow{f^\sharp} & X \\
 F\ell \downarrow & \dashrightarrow^{k^\sharp} & \downarrow r \\
 FB & \xrightarrow{g^\sharp} & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{f^b} & GX \\
 \ell \downarrow & \dashrightarrow^{k^b} & \downarrow Gr \\
 B & \xrightarrow{g^b} & GY
 \end{array}$$

- (ii) Conclude that if \mathcal{M} has a weak factorization system $(\mathcal{L}, \mathcal{R})$ and \mathcal{N} has a weak factorization system $(\mathcal{L}', \mathcal{R}')$ then F preserves the left classes if and only if G preserves the right classes.

Importantly, the total left and right derived functors of a Quillen pair form an adjunction between the appropriate homotopy categories.

Theorem 2.4.13 (Quillen [229, I.3]). *If*

$$\begin{array}{ccc}
 & \xrightarrow{F} & \\
 M & \perp & N \\
 & \xleftarrow{G} &
 \end{array}$$

is a Quillen adjunction, then the total derived functors form an adjunction

$$\begin{array}{ccc}
 & \xrightarrow{\mathbf{L}F} & \\
 \text{HoM} & \perp & \text{HoN} \\
 & \xleftarrow{\mathbf{R}G} &
 \end{array}$$

A particularly elegant proof of Theorem 2.4.13 is due to Maltsiniotis. Once the strategy is known, the details are elementary enough to be left as an exercise:

Exercise 2.4.14 ([176]). Use the fact that the total derived functors of a Quillen pair $F \dashv G$ define *absolute* Kan extensions to prove that $\mathbf{L}F \dashv \mathbf{R}G$. Conclude that Theorem 2.4.13 applies more generally to any pair of adjoint functors that are deformable in the sense of Definition 2.4.6 [92, 44.2].

A double categorical theorem of Shulman [277] consolidates into a single statement the adjointness of the total derived functors of a Quillen adjunction, the pseudo-functoriality of the construction of total derived functors of Quillen functors, and one further result about functors that are simultaneously left and right Quillen. A **double category** is a category internal to \mathbf{Cat} : it has a set of objects, a category of horizontal morphisms, a category of vertical morphisms, and a set of squares that are composable in both vertical and horizontal directions, defining the arrows in a pair of categories with the horizontal and vertical morphisms as objects, respectively [147].

For instance, \mathbf{Cat} is the double category of categories, functors, functors, and natural transformations inhabiting squares and pointing southwest. There is another double category \mathbf{Model} whose objects are model categories, whose vertical morphisms

are left Quillen functors, whose horizontal morphisms are right Quillen functors, and whose squares are natural transformations pointing southwest. The following theorem and a generalization, with deformable functors in place of Quillen functors [277, 8.10], is due to Shulman.

Theorem 2.4.15 ([277, 7.6]). *The map that sends a model category to its homotopy category and a left or right Quillen functor to its total left or right derived functor defines a double pseudofunctor $\mathbb{H}o: \mathbb{M}odel \rightarrow \mathbb{C}at$.*

The essential content of the pseudofunctoriality statement is that the composite of the left derived functors of a pair of left Quillen functors is coherently naturally weakly equivalent to the left derived functor of their composite. Explicitly, given a composable pair of left Quillen functors $M \xrightarrow{F} L \xrightarrow{G} K$, the map

$$\mathbb{L}G \circ \mathbb{L}F := GQ \circ FQ \xrightarrow{G\epsilon_{FQ}} GFQ =: \mathbb{L}(GF)$$

defines a comparison natural transformation. Since $Q: M \rightarrow M_c$ lands in the subcategory of cofibrant objects and F preserves cofibrant objects, $\epsilon_{FQ}: QFQ \Rightarrow FQ$ is a weak equivalence between cofibrant objects. Lemma 2.4.9(ii) then implies that $G\epsilon_{FQ}: GQFQ \rightarrow GFQ$ defines a natural weak equivalence $\mathbb{L}G \circ \mathbb{L}F \rightarrow \mathbb{L}(GF)$. Given a composable triple of left Quillen functors, there is a commutative square of natural weak equivalences $\mathbb{L}H \circ \mathbb{L}G \circ \mathbb{L}F \rightarrow \mathbb{L}(H \circ G \circ F)$. If we compose with the Gabriel–Zisman localizations to pass to homotopy categories and total left derived functors, these coherent natural weak equivalences become coherent natural isomorphisms, defining the claimed pseudofunctor.

Quillen adjunctions are encoded in the double category $\mathbb{M}odel$ as “conjoint” relationships between vertical and horizontal 1-cells; in this way Theorem 2.4.15 subsumes Theorem 2.4.13. Similarly, functors that are simultaneously left and right Quillen are presented as vertical and horizontal “companion” pairs. The double pseudofunctoriality of Theorem 2.4.15 contains a further result: if a functor is both left and right Quillen, then its total left and right derived functors are isomorphic.

2.4.4 Monoidal and enriched model categories

If M has a model structure and a monoidal structure it is natural to ask that these be compatible in some way, but what sort of compatibility should be required? In the most common examples, the monoidal product is *closed* — that is, the functors $A \otimes -$ and $- \otimes A$ admit right adjoints⁹ and consequently preserve colimits in each variable separately. This situation is summarized and generalized by the notion of a two-variable adjunction, which we introduce using notation that will suggest the most common examples.

Definition 2.4.16. A triple of bifunctors

$$K \times L \xrightarrow{\otimes} M, \quad K^{\text{op}} \times M \xrightarrow{\{, \}} L, \quad L^{\text{op}} \times M \xrightarrow{\text{Map}} K$$

⁹ Very frequently a monoidal structure is symmetric, in which case these functors are naturally isomorphic, and a single right adjoint suffices.

equipped with a natural isomorphism

$$M(K \otimes L, M) \cong L(L, \{K, M\}) \cong K(K, \text{Map}(L, M))$$

defines a **two-variable adjunction**.

Example 2.4.17. A symmetric monoidal category is **closed** just when its monoidal product $-\otimes -: V \times V \rightarrow V$ defines the left adjoint of a two-variable adjunction

$$V(A \otimes B, C) \cong V(B, \text{Map}(A, C)), V(A, \text{Map}(B, C)),$$

the right adjoint $\text{Map}: V^{\text{op}} \times V \rightarrow V$ defining an **internal hom**.

Example 2.4.18. A category M that is enriched over a monoidal category is **tensorred** and **cotensorred** just when the enriched hom functor $\text{Map}: M^{\text{op}} \times M \rightarrow V$ is one of the right adjoints of a two-variable adjunction

$$M(V \otimes M, N) \cong M(M, \{V, N\}) \cong V(V, \text{Map}(M, N)),$$

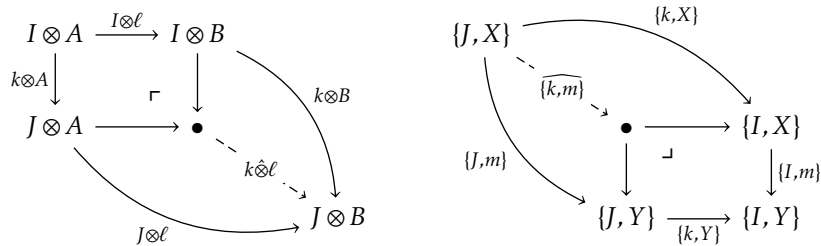
the other two adjoints defining the **tensor** $V \otimes M$ and **cotensor** $\{V, N\}$ of an object $V \in V$ with objects $M, N \in M$.¹⁰

A Quillen two-variable adjunction is a two-variable adjunction in which the left adjoint is a left Quillen bifunctor while the right adjoints are both right Quillen bifunctors, any one of these conditions implying the other two. To state these definitions, we must introduce the following construction. The “pushout-product” of a bifunctor $-\otimes -: K \times L \rightarrow M$ defines a bifunctor $-\hat{\otimes} -: K^2 \times L^2 \rightarrow M^2$ that we refer to as the “Leibniz tensor” (when the bifunctor \otimes is called a “tensor”). The “Leibniz cotensor” and “Leibniz hom”

$$\widehat{(-, -)}: (K^2)^{\text{op}} \times M^2 \rightarrow L^2 \quad \text{and} \quad \widehat{\text{Map}}(-, -): (L^2)^{\text{op}} \times M^2 \rightarrow K^2$$

are defined dually, using pullbacks in L and K respectively.

Definition 2.4.19 (the Leibniz construction). Given a bifunctor $-\otimes -: K \times L \rightarrow M$ valued in a category with pushouts, the **Leibniz tensor** of a map $k: I \rightarrow J$ in K and a map $\ell: A \rightarrow B$ in L is the map $k \hat{\otimes} \ell$ in M induced by the pushout diagram on the left:



¹⁰ As stated, this definition is a little too weak: one needs to ask in addition that (i) the tensors are associative relative to the monoidal product in V , (ii) dually that the cotensors are associative relative to the monoidal product in V , and (iii) that the two-variable adjunction is enriched in V . Any of these three conditions implies the other two.

Exercise 2.4.21 explains the equivalence between the following three definitions of a Quillen two-variable adjunction.

Definition 2.4.22. A two-variable adjunction

$$V \times M \xrightarrow{\otimes} N, \quad V^{\text{op}} \times N \xrightarrow{\{-,-\}} M, \quad M^{\text{op}} \times N \xrightarrow{\text{Map}} V$$

between model categories V , M , and N defines a **Quillen two-variable adjunction** if any, and hence all, of the following equivalent conditions are satisfied:

- (i) The functor $\hat{\otimes}: V^2 \times M^2 \rightarrow N^2$ carries any pair comprised of a cofibration in V and a cofibration in M to a cofibration in N , and this cofibration is a weak equivalence if either of the domain maps are.
- (ii) The functor $\widehat{\{-,-\}}: (V^2)^{\text{op}} \times N^2 \rightarrow M^2$ carries any pair comprised of a cofibration in V and a fibration in N to a fibration in M , and this fibration is a weak equivalence if either of the domain maps are.
- (iii) The functor $\widehat{\text{Map}}: (M^2)^{\text{op}} \times N^2 \rightarrow V^2$ carries any pair comprised of a cofibration in M and a fibration in N to a fibration in V , and this fibration is a weak equivalence if either of the domain maps are.

Exercise 2.4.23. Prove that if $-\otimes -: V \times M \rightarrow N$ is a left Quillen bifunctor and $V \in V$ is cofibrant then $V \otimes -: M \rightarrow N$ is a left Quillen functor.

Quillen's axiomatization of the additional properties enjoyed by his model structure on the category of simplicial sets has been generalized by Hovey [130, §4.2].

Definition 2.4.24. A **(closed symmetric) monoidal model category** is a (closed symmetric) monoidal category (V, \otimes, I) with a model structure so that the monoidal product and hom define a Quillen two-variable adjunction and furthermore so that the maps

$$QI \otimes v \rightarrow I \otimes v \cong v \quad \text{and} \quad v \otimes QI \rightarrow v \otimes I \cong v \quad (2.4)$$

are weak equivalences if v is cofibrant.¹¹

Definition 2.4.25. If V is a monoidal model category a **V-model category** is a model category M that is tensored, cotensored, and V -enriched in such a way that $(\otimes, \{, \}, \text{Map})$ is a Quillen two-variable adjunction and the maps

$$QI \otimes m \rightarrow I \otimes m \cong m$$

are weak equivalences if m is cofibrant.

Exercise 2.4.26. In a locally small category M with products and coproducts the hom bifunctor is part of a two-variable adjunction:

$$-* -: \text{Set} \times M \rightarrow M, \quad \{-,-\}: \text{Set}^{\text{op}} \times M \rightarrow M, \quad \text{Hom}: M^{\text{op}} \times M \rightarrow \text{Set}.$$

Equipping Set with the model structure whose weak equivalences are all maps, whose cofibrations are monomorphisms, and whose fibrations are epimorphisms, prove that

¹¹ If the monoidal product is symmetric then of course these two conditions are equivalent and if it is closed then they are also equivalent to a dual one involving the internal hom [130, 4.2.7].

- (i) Set is a cartesian monoidal model category.
- (ii) Any model category M is a Set-model category.

Example 2.4.27. Quillen's model structure of Theorem 2.3.30 is a closed symmetric monoidal model category. The term **simplicial model category** refers to a model category enriched over this model structure.

Exercise 2.4.28. Show that if M is a simplicial model category then the full simplicial subcategory M_{cf} is Kan-complex enriched.

The conditions (2.4) on the cofibrant replacement of the monoidal unit are implied by the Quillen two-variable adjunction if the monoidal unit is cofibrant and are necessary for the proof of Theorem 2.4.29, which shows that the homotopy categories are again closed monoidal and enriched.

Theorem 2.4.29 ([130, 4.3.2,4]).

- (i) *The homotopy category of a closed symmetric monoidal model category is a closed monoidal category with tensor and hom given by the derived adjunction*

$$(\mathbb{L}\otimes, \mathbb{R}\text{Map}, \mathbb{R}\text{Map}): \text{HoV} \times \text{HoV} \rightarrow \text{HoV}$$

and monoidal unit QI .

- (ii) *If M is a V -model category, then HoM is the underlying category of a HoV -enriched, tensored, and cotensored category with enrichment given by the total derived two-variable adjunction*

$$(\mathbb{L}\otimes, \mathbb{R}\{, \}, \mathbb{R}\text{Map}): \text{HoV} \times \text{HoM} \rightarrow \text{HoM}.$$

In particular:

Corollary 2.4.30. *The homotopy category of spaces is cartesian closed. If M is a simplicial model category, then HoM is enriched, tensored, and cotensored over the homotopy category of spaces.*

2.4.5 Quillen equivalences between homotopy theories

Two model categories present equivalent homotopy theories if there exists a finite sequence of model categories and a zigzag of Quillen equivalences between them, in a sense we now define. A Quillen adjunction defines a Quillen equivalence just when the derived adjunction of Theorem 2.4.13 defines an **adjoint equivalence**: an adjunction with invertible unit and counit. There are several equivalent characterizations of this situation.

Definition 2.4.31 ([229, §I.4]). A Quillen adjunction between a pair of model categories

$$\begin{array}{ccc} & F & \\ M & \xrightarrow{\quad} & N \\ & G & \end{array}$$

defines a **Quillen equivalence** if any, and hence all, of the following equivalent conditions are satisfied:

- (i) The total left derived functor $\mathbf{L}F: \mathbf{HoM} \rightarrow \mathbf{HoN}$ defines an equivalence of categories.
- (ii) The total right derived functor $\mathbf{R}G: \mathbf{HoN} \rightarrow \mathbf{HoM}$ defines an equivalence of categories.
- (iii) For every cofibrant object $A \in \mathbf{M}$ and every fibrant object $X \in \mathbf{N}$, a map $f^\sharp: FA \rightarrow X$ is a weak equivalence in \mathbf{N} if and only if its transpose $f^b: A \rightarrow GX$ is a weak equivalence in \mathbf{M} .
- (iv) For every cofibrant object $A \in \mathbf{M}$, the composite $A \rightarrow GFA \rightarrow GRFA$ of the unit with fibrant replacement is a weak equivalence in \mathbf{M} , and for every fibrant object $X \in \mathbf{N}$, the composite $FQGX \rightarrow FGX \rightarrow X$ of the counit with cofibrant replacement is a weak equivalence in \mathbf{N} .

Famously, the formalism of Quillen equivalences enables a proof that the homotopy theory of spaces is equivalent to the homotopy theory of simplicial sets.

Theorem 2.4.32 (Quillen [229, §II.3]). *The homotopy theory of simplicial sets is equivalent to the homotopy theory of topological spaces via the geometric realization \dashv total singular complex adjunction*

$$\begin{array}{ccc}
 & \dashv & \\
 \text{sSet} & \xrightarrow{\quad} & \text{Top} \\
 & \perp & \\
 & \xleftarrow{\text{Sing}} &
 \end{array}$$

2.4.6 Extending homological algebra to homotopical algebra

Derived functors are endemic to homological algebra. Quillen’s homotopical algebra can be understood to subsume classical homological algebra in the following sense. The category of chain complexes of modules over a fixed ring (or valued in an arbitrary abelian category) admits a homotopical structure where the weak equivalences are quasi-isomorphisms. Relative to an appropriately defined model structure, the left and right derived functors of homological algebra can be viewed as special cases of the construction of derived functors of left or right Quillen functors in Corollary 2.4.10 or in the more general context of Theorem 2.4.7.

The following theorem describes an equivalent presentation of the homotopy theory just discussed.

Theorem 2.4.33 (Schwede–Shiplay after Dold–Kan). *The homotopy theory of simplicial modules over a commutative ring, with fibrations and weak equivalences as on underlying simplicial sets, is equivalent to the homotopy theory of non-negatively graded chain complexes of modules, as presented by the projective model structure whose weak equivalences are the quasi-isomorphisms, fibrations are the chain maps which are surjective in positive dimensions, and cofibrations are the monomorphisms with dimensionwise projective cokernel.*

Proof. For details of the model structure on simplicial objects see [229, II.4, II.6] and on chain complexes see [130, 2.3.11, 4.2.13]. The proof that the functors Γ, N in the Dold–Kan equivalence are each both left and right Quillen equivalences can be found in [265, §4.1] or is safely left as an exercise to the reader. \square

The Dold–Kan Quillen equivalence of Theorem 2.4.33 suggests that simplicial methods might replace homological ones in non-abelian contexts. Let M be any category of “algebras” such as monoids, groups, rings (or their commutative variants), or modules or algebras over a fixed ring; technically M may be any category of models for a *Lawvere theory* [154], which specifies finite operations of any arity and relations between the composites of these operations.

Theorem 2.4.34 (Quillen [229, §II.4]). *For any category M of “algebras” — a category of models for a Lawvere theory — the category $M^{\Delta^{\text{op}}}$ of simplicial algebras admits a simplicial model structure whose*

- *weak equivalences are those maps that are weak homotopy equivalences on underlying simplicial sets,*
- *fibrations are those maps that are Kan fibrations on underlying simplicial sets, and*
- *cofibrations are retracts of free maps.*

2.5 Homotopy limits and colimits

Limits and colimits provide fundamental tools for constructing new mathematical objects from existing ones, so it is important to understand these constructions in the homotopical context. There are a variety of possible meanings of a homotopical notion of limit or colimit including:

- (i) limits or colimits in the homotopy category of a model category;
- (ii) limit or colimit constructions that are “homotopy invariant,” with weakly equivalent inputs giving rise to weakly equivalent outputs;
- (iii) derived functors of the limit or colimit functors; and finally
- (iv) limits or colimits whose universal properties are (perhaps weakly) enriched over simplicial sets or topological spaces.

We will explore these possibilities in turn. We begin in §2.5.1 by observing that the homotopy category has few genuine limits and colimits but does have “weak” ones in the case where the category is enriched, tensored, and cotensored over spaces. For the reason explained in Remark 2.5.5, homotopy limits or colimits rarely satisfy condition (i).

In §2.5.2, we define homotopy limits and colimits as derived functors, which in particular give “homotopy invariant” constructions, and introduce hypotheses on the ambient model category that ensure that these homotopy limit and colimit functors always exist. In §2.5.3 we consider particular diagram shapes, the so-called Reedy categories, for which homotopy limits and colimits exist in any model category. Finally,

in §2.5.4 we permit ourselves a tour through the general theory of weighted limits and colimits as a means of elucidating these results and introducing families of Quillen bifunctors that deserve to be better known. This allows us to finally explain the sense in which homotopy limits or colimits in a simplicial model category satisfy properties (ii)-(iv) and in particular have an enriched universal property which may be understood as saying they “represent homotopy coherent cones” over or under the diagram.

2.5.1 Weak limits and colimits in the homotopy category

Consider a category M that is enriched over spaces — either topological spaces or simplicial sets will do — meaning that for each pair of objects x, y , there is a mapping space $\text{Map}(x, y)$ whose points are the usual set $M(x, y)$ of arrows from x to y . We may define a homotopy category of M using the construction of Definition 2.3.24.

Definition 2.5.1. If M is a simplicially enriched category its **homotopy category** $\text{h}M$ has

- objects the same objects as M and
- hom-sets $\text{h}M(x, y) := \pi_0 \text{Map}(x, y)$ taken to be the path components of the mapping spaces.

Thus, a morphism from x to y in $\text{h}M$ is a homotopy class of vertices in the simplicial set $\text{Map}(x, y)$, where two vertices are homotopic if and only if they can be connected by a finite zigzag of 1-simplices.

A product of a family of objects m_α in a category M is given by a representation m for the functor displayed on the right:

$$M(-, m) \xrightarrow{\cong} \prod_{\alpha} M(-, m_{\alpha}).$$

By the Yoneda lemma, a representation consists of an object $m \in M$ together with maps $m \rightarrow m_\alpha$ for each α that are universal in the sense that for any collection $x \rightarrow m_\alpha \in M$, each of these arrows factors uniquely along a common map $x \rightarrow m$. But if M is enriched over spaces, we might instead require only that the triangles

$$\begin{array}{ccc} x & & \\ \exists \downarrow & \searrow & \\ m & \longrightarrow & m_{\alpha} \end{array} \quad (2.5)$$

commute “up to homotopy” in the sense of a path in the space $\text{Map}(x, m_\alpha)$ whose underlying set of points is $M(x, m_\alpha)$. Now we can define the **homotopy product** to be an object m equipped with a natural weak homotopy equivalence

$$\text{Map}(x, m) \rightarrow \prod_{\alpha} \text{Map}(x, m_{\alpha})$$

for each $x \in M$. Surjectivity on path components implies the existence and homotopy commutativity of the triangles (2.5).

Exercise 2.5.2. Use the fact that π_0 commutes with products and is homotopical to show, unusually for homotopy limits, that the homotopy product is a product in the homotopy category \mathbf{hM} . Similarly, a homotopy coproduct is a coproduct in the homotopy category.

For non-discrete diagram shapes, the homotopy category of a category enriched in spaces¹² will no longer have genuine limits or colimits but in the presence of tensors in the colimit case and cotensors in the limit case it will have weak ones.

Theorem 2.5.3 ([295, 11.1]). *If \mathbf{M} is cocomplete and also enriched and tensored over spaces, its homotopy category \mathbf{hM} has all weak colimits: given any small diagram $F: \mathbf{D} \rightarrow \mathbf{hM}$, there is a cone under F through which every other cone factors, although not necessarily uniquely.*

In general, the colimit of a diagram F of shape \mathbf{D} may be constructed as the reflexive coequalizer of the diagram

$$\coprod_{a,b \in \mathbf{D}} \mathbf{D}(a,b) \times Fa \begin{array}{c} \xrightarrow{\text{ev}} \\ \xleftarrow{\text{id}} \\ \xrightarrow{\text{proj}} \end{array} \coprod_{a \in \mathbf{D}} Fa$$

Note that this construction does not actually require the diagram F to be a functor; it suffices for the diagram to define a reflexive directed graph in the target category. In the case of a diagram valued in \mathbf{hM} , the weak colimit will be constructed as a “homotopy reflexive coequalizer”¹³ of a lifted reflexive directed graph in \mathbf{M} .

Proof. Any diagram $F: \mathbf{D} \rightarrow \mathbf{hM}$ may be lifted to a reflexive directed graph $F: \mathbf{D} \rightarrow \mathbf{M}$, choosing representatives for each homotopy class of morphisms in such a way that identities are chosen to represent identities. Using these lifted maps and writing I for the interval, define the weak colimit of $F: \mathbf{D} \rightarrow \mathbf{hM}$ to be a quotient of the coproduct

$$\left(\coprod_{a,b \in \mathbf{D}} \mathbf{D}(a,b) \times I \times Fa \right) \sqcup \left(\coprod_{a \in \mathbf{D}} Fa \right)$$

modulo three identifications:

$$\begin{array}{ccc} \coprod_{a,b \in \mathbf{D}} (\mathbf{D}(a,b) \times \{0\} \times Fa \sqcup \mathbf{D}(a,b) \times \{1\} \times Fa) \sqcup \coprod_{a \in \mathbf{D}} I \times Fa & \xrightarrow{(\text{ev} \sqcup \text{proj}) \sqcup \text{proj}} & \coprod_{a \in \mathbf{D}} Fa \\ \text{(incl} \sqcup \text{incl)} \sqcup \text{id} \downarrow & & \downarrow \\ \coprod_{a,b \in \mathbf{D}} \mathbf{D}(a,b) \times I \times Fa & \dashrightarrow & \text{wcolim } F \end{array}$$

The right-hand vertical map defines the legs of the colimit cone, which commute in \mathbf{hM} via the witnessing homotopies given by the bottom horizontal map.

Now consider a cone in \mathbf{hM} under F with nadir X . We may regard the data of this

¹² Here we can take our enrichment over topological spaces or over simplicial sets, the latter being more general [237, 3.7.15-16].

¹³ Succinctly, it may be defined as the weighted colimit of this reflexive coequalizer diagram weighted by the truncated cosimplicial object $* \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} I$ whose leftwards maps are the endpoint inclusions into the closed interval I ; see §2.5.4.

cone as a diagram $D \times 2 \rightarrow \mathbf{hM}$ that restricts along $\{0\} \hookrightarrow 2$ to F and along $\{1\} \hookrightarrow 2$ to the constant diagram at X . This data may be lifted to a reflexive directed graph $D \times 2 \rightarrow \mathbf{M}$ whose lift over 0 agrees with the previously specified lift F and whose lift over 1 is constant at X . This defines a cone under the pushout diagram, inducing the required map $\mathrm{wcolim} F \rightarrow X$. \square

2.5.2 Homotopy limits and colimits of general shapes

In general, limit and colimit constructions in a homotopical category fail to be weak equivalence invariant. Famously the n -sphere can be formed by gluing together two disks along their boundary spheres $S^n \cong D^n \cup_{S^{n-1}} D^n$. The diagram

$$\begin{array}{ccccc}
 D^n & \longleftarrow & S^{n-1} & \longrightarrow & D^n \\
 \downarrow \wr & & \parallel & & \downarrow \wr \\
 * & \longleftarrow & S^{n-1} & \longrightarrow & *
 \end{array} \tag{2.6}$$

reveals that the pushout functor fails to preserve componentwise homotopy equivalences.

When a functor fails to be homotopical, the next best option is to replace it by a derived functor. Because colimits are left adjoints, one might hope that $\mathrm{colim}: \mathbf{M}^D \rightarrow \mathbf{M}$ has a left derived functor and dually that $\mathrm{lim}: \mathbf{M}^D \rightarrow \mathbf{M}$ has a right derived functor, leading us to the following definition:

Definition 2.5.4. Let \mathbf{M} be a homotopical category and let D be a small category. The **homotopy colimit functor**, when it exists, is a left derived functor $\mathbb{L}\mathrm{colim}: \mathbf{M}^D \rightarrow \mathbf{M}$, while the **homotopy limit functor**, when it exists, is a right derived functor $\mathbb{R}\mathrm{lim}: \mathbf{M}^D \rightarrow \mathbf{M}$.

We always take the weak equivalences in the category \mathbf{M}^D of diagrams of shape D in a homotopical category \mathbf{M} to be defined pointwise. By the universal property of localization, there is a canonical map

$$\begin{array}{ccc}
 \mathbf{M}^D & \xrightarrow{\gamma^D} & (\mathrm{HoM})^D \\
 \gamma \downarrow & \dashrightarrow & \\
 \mathrm{Ho}(\mathbf{M}^D) & &
 \end{array} \tag{2.7}$$

but it is not typically an equivalence of categories. Indeed, some of the pioneering forays into abstract homotopy theory [295, 72, 90] were motivated by attempts to understand the essential image of the functor $\mathrm{Ho}(\mathbf{M}^D) \rightarrow (\mathrm{HoM})^D$, the objects in $(\mathrm{HoM})^D$ being homotopy commutative diagrams while the isomorphism classes of objects in $\mathrm{Ho}(\mathbf{M}^D)$ being somewhat more mysterious; see §2.6.2.

Remark 2.5.5. The diagonal functor $\Delta: \mathbf{M} \rightarrow \mathbf{M}^D$ is homotopical and hence acts as its own left and right derived functors. By Theorem 2.4.13 applied to a Quillen adjunction to be constructed in the proof of Theorem 2.5.7, the total derived functor

$\mathbb{L}\text{colim}: \text{Ho}(M^D) \rightarrow \text{Ho}M$ is left adjoint to $\Delta: \text{Ho}M \rightarrow \text{Ho}(M^D)$, but unless the comparison of (2.7) is an equivalence, this is not the same as the diagonal functor $\Delta: \text{Ho}M \rightarrow \text{Ho}(M)^D$. Hence, homotopy colimits are not typically colimits in the homotopy category.¹⁴

In the presence of suitable model structures, Corollary 2.4.10 can be used to prove that the homotopy limit and colimit functors exist.

Definition 2.5.6. Let M be a model category and let D be a small category.

- (i) The **projective model structure** on M^D has weak equivalences and fibrations defined pointwise in M .
- (ii) The **injective model structure** on M^D has weak equivalences and cofibrations defined pointwise in M .

When M is a *combinatorial model category*, both model structures are guaranteed to exist. More generally, when M is an *accessible model category* these model structures exist [118, 3.4.1]. Of course, the projective and injective model structures might happen to exist on M^D , perhaps for particular diagram shapes D , in the absence of these hypotheses.

Theorem 2.5.7. *Let M be a model category and let D be a small category.*

- (i) *Whenever the projective model structure on M^D exists, the homotopy colimit functor $\mathbb{L}\text{colim}: M^D \rightarrow M$ exists and may be computed as the colimit of a projective cofibrant replacement of the original diagram.*
- (ii) *Whenever the injective model structure on M^D exists, the homotopy limit functor $\mathbb{R}\text{lim}: M^D \rightarrow M$ exists and may be computed as the limit of an injective fibrant replacement of the original diagram.*

Proof. This follows from Corollary 2.4.10 once we verify that the colimit and limit functors are respectively left and right Quillen with respect to the projective and injective model structures. These functors are, respectively, left and right adjoint to the constant diagram functor $\Delta: M \rightarrow M^D$, so by Definition 2.4.11 it suffices to verify that this functor is right Quillen with respect to the projective model structure and also left Quillen with respect to the injective model structure. But these model structures are designed so that this is the case. \square

Exercise 2.5.8.

- (i) Show that any pushout diagram $B \longleftarrow A \longrightarrow C$ comprised of a pair of cofibrations between cofibrant objects is projectively cofibrant. Conclude that the pushout of cofibrations between cofibrant objects is a homotopy pushout and use this to compute the homotopy pushout of (2.6).

¹⁴ The comparison (2.7) is an equivalence when D is discrete; this is why homotopy products and homotopy coproducts *are* products and coproducts in the homotopy category.

- (ii) Argue that for a generic pushout diagram $Y \longleftarrow X \longrightarrow Z$, its homotopy pushout may be constructed by taking a cofibrant replacement $q: X' \rightarrow X$ of X and then factoring the composites hq and kq as a cofibration followed by a trivial fibration:

$$\begin{array}{ccccc} Y' & \longleftarrow & X' & \longrightarrow & Z' \\ \wr \downarrow & & \wr \downarrow q & & \downarrow \wr \\ Y & \longleftarrow & X & \longrightarrow & Z \end{array}$$

and then taking the ordinary pushout of this projective cofibrant replacement formed by the top row.

Exercise 2.5.9.

- (i) Verify that any ω -indexed diagram

$$A_0 \xrightarrow{f_{01}} A_1 \xrightarrow{f_{12}} A_2 \xrightarrow{f_{23}} \dots$$

of cofibrations between cofibrant objects is projectively cofibrant. Conclude that the sequential colimit of a diagram of cofibrations between cofibrant objects is a homotopy colimit.

- (ii) Argue that for a generic sequential diagram

$$X_0 \xrightarrow{f_{01}} X_1 \xrightarrow{f_{12}} X_2 \xrightarrow{f_{23}} \dots$$

its projective cofibrant replacement may be formed by first replacing X_0 by a cofibrant object Q_0 , then inductively factoring the resulting composite map $Q_n \rightarrow X_{n+1}$ into a cofibration followed by a trivial fibration:

$$\begin{array}{ccccccc} G & & Q_0 & \xrightarrow{g_{01}} & Q_1 & \xrightarrow{g_{12}} & Q_2 & \xrightarrow{g_{23}} & \dots \\ q \downarrow \wr & & q_0 \downarrow \wr & & q_1 \downarrow \wr & & q_2 \downarrow \wr & & \\ F & & X_0 & \xrightarrow{f_{01}} & X_1 & \xrightarrow{f_{12}} & X_2 & \xrightarrow{f_{23}} & \dots \end{array}$$

Conclude that the homotopy sequential colimit is formed as the sequential colimit of this top row.

2.5.3 Homotopy limits and colimits of Reedy diagrams

In fact, even if the projective model structures do not exist, certain diagram shapes allow us to construct functorial “projective cofibrant replacements” in any model category nonetheless, for example by following the prescriptions of Exercise 2.5.9. Dual “injective fibrant replacements” for pullback or inverse limit diagrams exist similarly. This happens when the categories indexing these diagrams are *Reedy categories*.

If M is any model category and D is any Reedy category, then the category M^D of Reedy diagrams admits a model structure. If the indexing category D satisfies the appropriate half of a dual pair of conditions listed in Proposition 2.5.22, then the colimit or limit functors $\text{colim}, \text{lim}: M^D \rightarrow M$ are left or right Quillen. In such

contexts, homotopy colimits and homotopy limits can be computed by applying Corollary 2.4.10.

The history of the abstract notion of Reedy categories is entertaining. The category Δ^{op} is an example of what is now called a *Reedy category*. The eponymous model structure on simplicial objects taking values in any model category was introduced in an unpublished but widely disseminated manuscript by Reedy [232]. Reedy notes that a dual model structure exists for cosimplicial objects, which, in the case of cosimplicial simplicial sets, coincides with a model structure introduced by Bousfield and Kan to define homotopy limits [58, §X]. The general definition, unifying these examples and many others, is due to Kan and appeared in the early drafts of the book that eventually became [92]. Various draft versions circulated in the mid 1990s and contributed to the published accounts [124, chapter 15] and [130, chapter 5]. The final [92] in turn references these sources in order to “review the notion of a Reedy category” that originated in an early draft of that same manuscript.

Definition 2.5.10. A **Reedy structure** on a small category A consists of a **degree function** $\text{deg}: \text{ob}A \rightarrow \omega$ together with a pair of wide subcategories \vec{A} and \overleftarrow{A} of **degree-increasing** and **degree-decreasing** arrows respectively, so that:

- (i) The degree of the domain of every non-identity morphism in \vec{A} is strictly less than the degree of the codomain, and the degree of the domain of every non-identity morphism in \overleftarrow{A} is strictly greater than the degree of the codomain.
- (ii) Every $f \in \text{mor}A$ may be factored uniquely as

$$\begin{array}{ccc}
 \bullet & \xrightarrow{f} & \bullet \\
 \swarrow \overleftarrow{f} & & \nearrow \overrightarrow{f} \\
 & \bullet &
 \end{array} \quad (2.8)$$

Example 2.5.11.

- (i) Discrete categories are Reedy categories, with all objects having degree zero.
- (ii) If A is a Reedy category, then so is A^{op} : its Reedy structure has the same degree function but has the degree-increasing and degree-decreasing arrows interchanged.
- (iii) Finite posets are Reedy categories with all morphisms degree-increasing. Declare any minimal element to have degree zero and define the degree of a generic object $d \in D$ to be the length of the maximal-length path of non-identity arrows from an element of degree zero to d . This example can be extended without change to include infinite posets such as ω provided that each object has finite degree.
- (iv) The previous example gives the category $b \leftarrow a \rightarrow c$ a Reedy structure in which $\text{deg}(a) = 0$ and $\text{deg}(b) = \text{deg}(c) = 1$. There is another Reedy category structure in which $\text{deg}(b) = 0$, $\text{deg}(a) = 1$, and $\text{deg}(c) = 2$.
- (iv) The category $a \rightrightarrows b$ is a Reedy category with $\text{deg}(a) = 0$, $\text{deg}(b) = 1$, and both non-identity arrows said to strictly raise degrees.

- (vi) The category Δ of finite non-empty ordinals and the category Δ_+ of finite ordinals and order-preserving maps both support canonical Reedy category structures, for which we take the degree-increasing maps to be the subcategories of face operators (monomorphisms) and the degree-decreasing maps to be the subcategories of degeneracy operators (epimorphisms).

Exercise 2.5.12.

- (i) Show that every morphism f factors uniquely through an object of minimum degree and this factorization is the **Reedy factorization** of (2.8).
(ii) Show that the Reedy category axioms prohibit any non-identity isomorphisms.

Remark 2.5.13. The notion of a Reedy category has been usefully extended by Berger and Moerdijk to include examples such as finite sets or finite pointed sets that do have non-identity automorphisms. All of the results to be described here have analogues in this more general context, but for ease of exposition we leave these details to [38].

To focus attention on our goal, we now introduce the *Reedy model structure*, which serves as motivation for some auxiliary constructions we have yet to introduce.

Theorem 2.5.14 (Reedy, Kan [245, §7]). *Let M be a model category and let D be a Reedy category. Then the category M^D admits a model structure whose*

- *weak equivalences are the pointwise weak equivalences, and*
- *weak factorization systems $(\mathcal{C} \cap \mathcal{W}[D], \mathcal{F}[D])$ and $(\mathcal{C}[D], \mathcal{F} \cap \mathcal{W}[D])$ are the Reedy weak factorization systems.*

In the **Reedy weak factorization system** $(\mathcal{L}[D], \mathcal{R}[D])$ defined relative to a weak factorization system $(\mathcal{L}, \mathcal{R})$ on M , a natural transformation $f: X \rightarrow Y \in M^D$ is in $\mathcal{L}[D]$ or $\mathcal{R}[D]$, respectively, if and only if, for each $d \in D$, the *relative latching map* $X^d \cup_{L^d X} L^d Y \rightarrow Y^d$ is in \mathcal{L} or the *relative matching map* $X^d \rightarrow Y^d \times_{M^d Y} M^d X$ is in \mathcal{R} . The most efficient definition of these latching and matching objects $L^d X$ and $M^d X$ appearing in Example 2.5.17 makes use of the theory of weighted colimits and limits, a subject to which we now turn.

2.5.4 Quillen adjunctions for weighted limits and colimits

Ordinary limits and colimits are objects representing the functor of cones with a given summit over or under a fixed diagram. Weighted limits and colimits are defined analogously, except that the cones over or under a diagram might have exotic “shapes.” These shapes are allowed to vary with the objects indexing the diagram. More formally, the **weight** — a functor which specifies the “shape” of a cone over a diagram indexed by D or a cone under a diagram indexed by D^{op} — takes the form of a functor in Set^D in the unenriched context or V^D in the V -enriched context.

Definition 2.5.15 (weighted limits and colimits, axiomatically). For a general small category D and bicomplete category M , the weighted limit and weighted colimit define bifunctors

$$\{-, -\}^D: (\text{Set}^D)^{\text{op}} \times M^D \rightarrow M \quad \text{and} \quad - *_{\text{D}} -: \text{Set}^D \times M^{D^{\text{op}}} \rightarrow M$$

which are characterized by the following pair of axioms.

- (i) Weighted (co)limits with representable weights evaluate at the representing object:

$$\{D(d, -), X\}^D \cong X(d) \quad \text{and} \quad D(-, d) *_{\mathbb{D}} Y \cong Y(d).$$

- (ii) The weighted (co)limit bifunctors are cocontinuous in the weight: for any diagram $X \in M^{\mathbb{D}}$, the functor $- *_{\mathbb{D}} X$ preserves colimits, while the functor $\{-, X\}^{\mathbb{D}}$ carries colimits to limits.

We interpret axiom (ii) to mean that weights can be “made to order”: a weight constructed as a colimit of representables — as all Set -valued functors are — will stipulate the expected universal property.

Let M be any locally small category with products and coproducts. For any set S , the S -fold product and coproduct define cotensor and tensor bifunctors

$$\{-, -\}: \text{Set}^{\text{op}} \times M \rightarrow M \quad \text{and} \quad - *_{-}: \text{Set} \times M \rightarrow M,$$

which form a two-variable adjunction with $\text{Hom}: M^{\text{op}} \times M \rightarrow \text{Set}$; cf. Exercise 2.4.26.

Definition 2.5.16 (weighted limits and colimits, constructively). The weighted colimit is a functor tensor product and the weighted limit is a functor cotensor product:

$$\{W, X\}^{\mathbb{D}} \cong \int_{d \in \mathbb{D}} \{W(d), X(d)\}, \quad W *_{\mathbb{D}} Y \cong \int^{d \in \mathbb{D}} W(d) * Y(d).$$

The limit $\{W, X\}^{\mathbb{D}}$ of the diagram X weighted by W and the colimit $W *_{\mathbb{D}} Y$ of Y weighted by W are characterized by the universal properties:

$$M(M, \{W, X\}^{\mathbb{D}}) \cong \text{Set}^{\mathbb{D}}(W, M(M, X)), \quad M(W *_{\mathbb{D}} Y, M) \cong \text{Set}^{\mathbb{D}^{\text{op}}}(W, M(Y, M)).$$

Example 2.5.17. Let A be a Reedy category and write $A_{\leq n}$ for the full subcategory of objects of degree at most n . Restriction along the inclusion $A_{\leq n} \hookrightarrow A$ followed by left Kan extension defines a comonad $\text{sk}_n: \text{Set}^A \rightarrow \text{Set}^A$.

Let $a \in A$ be an object of degree n and define

$$\partial A(a, -) := \text{sk}_{n-1} A(a, -) \in \text{Set}^A \quad \text{and} \quad \partial A(-, a) := \text{sk}_{n-1} A(-, a) \in \text{Set}^{A^{\text{op}}},$$

where $A(a, -)$ and $A(-, a)$ denote the co- and contravariant functors represented by a , respectively. For any $X \in M^A$, the **latching** and **matching** objects are defined by

$$L^a X := \partial A(-, a) *_{\mathbb{A}} X \quad \text{and} \quad M^a X := \{\partial A(a, -), X\}.$$

Exercise 2.5.18 (enriched weighted limits and colimits). For the reader who knows some enriched category theory, generalize Definitions 2.5.15 and 2.5.16 to the V -enriched context to define weighted limit and weighted colimit bifunctors

$$\{-, -\}^A: (V^A)^{\text{op}} \times M^A \rightarrow M \quad \text{and} \quad - \otimes_{\mathbb{A}} -: V^A \times M^{A^{\text{op}}} \rightarrow M$$

in any V -enriched, tensored, and cotensored category M whose underlying unenriched category is complete and cocomplete.

Recall the notion of Quillen two-variable adjunction, the prototypical example being the tensor-cotensor-hom of a V -model category M .

Theorem 2.5.19 ([239, 7.1]). *Let A be a Reedy category and let $\otimes: K \times L \rightarrow M$ be a left Quillen bifunctor between model categories. Then the functor tensor product*

$$\otimes_A: K^{A^{\text{op}}} \times L^A \rightarrow M$$

is left Quillen with respect to the Reedy model structures.

A dual result holds for functor cotensor products formed relative to a right Quillen bifunctor. In particular, if M is a V -model category, then its tensor, cotensor, and hom define a Quillen two-variable adjunction, and so in particular:

Corollary 2.5.20. *Let M be a V -model category and let A be a Reedy category. Then for any Reedy cofibrant weight $W \in V^A$, the weighted colimit and weighted limit functors*

$$W *_A -: M^A \rightarrow M \quad \text{and} \quad \{W, -\}^A: M^{A^{\text{op}}} \rightarrow M$$

are respectively left and right Quillen with respect to the Reedy model structures on M^A and $M^{A^{\text{op}}}$.

Example 2.5.21 (geometric realization and totalization). The Yoneda embedding defines a Reedy cofibrant weight $\Delta^\bullet \in \text{sSet}^\Delta$. The weighted colimit and weighted limit functors

$$\Delta^\bullet *_\Delta^{\text{op}} -: M^{\Delta^{\text{op}}} \rightarrow M \quad \text{and} \quad \{\Delta^\bullet, -\}^\Delta: M^\Delta \rightarrow M$$

typically go by the names **geometric realization** and **totalization**. Corollary 2.5.20 proves that if M is a simplicial model category, then these functors are left and right Quillen.

By Exercise 2.4.26, Corollary 2.5.20 also has implications in the case of an unenriched model category M , in which case “Reedy cofibrant” should be read as “Reedy monomorphic.” Ordinary limits and colimits are weighted limits and colimits where the weight is the terminal functor, constant at the singleton set.

Proposition 2.5.22 (homotopy limits and colimits of Reedy shape).

- (i) *If A is a Reedy category with the property that the constant A -indexed diagram at any cofibrant object in any model category is Reedy cofibrant, then the limit functor $\lim: M^A \rightarrow M$ is right Quillen.*
- (ii) *If A is a Reedy category with the property that the constant A -indexed diagram at any fibrant object in any model category is Reedy fibrant, then the colimit functor $\text{colim}: M^A \rightarrow M$ is left Quillen.*

Proof. Taking the terminal weight 1 in Set^A , the weighted limit reduces to the ordinary limit functor. The functor $1 \in \text{Set}^A$ is Reedy monomorphic just when, for each $a \in A$, the category of elements for the weight $\partial A(-, a)$ is either empty or connected. This is the case if and only if A has “cofibrant constants,” meaning that the constant A -indexed diagram at any cofibrant object in any model category is Reedy

cofibrant. Thus, we conclude that if A has cofibrant constants, then the limit functor $\lim: M^A \rightarrow M$ is right Quillen. See [245, §9] for more discussion. \square

There is an analogous result for projective and injective model structures which the author first saw formulated in this way by Gambino in the context of a simplicial model category.

Theorem 2.5.23 ([101]). *If M is a V -model category and D is a small category, then the weighted colimit functor*

$$-\otimes_D -: V^D \times M^{D^{\text{op}}} \rightarrow M$$

is left Quillen if the domain has the (injective, projective) or (projective, injective) model structure. Similarly, the weighted limit functor

$$\{-, -\}^D: (V^D)^{\text{op}} \times M^D \rightarrow M$$

is right Quillen if the domain has the (projective, projective) or (injective, injective) model structure.

Proof. By Definition 2.4.22 we can prove both statements in adjoint form. The weighted colimit bifunctor of Exercise 2.5.18 has a right adjoint (used to express the defining universal property of the weighted colimit)

$$\text{Map}(-, -): (M^{D^{\text{op}}})^{\text{op}} \times M \rightarrow V^D$$

which sends $F \in M^{D^{\text{op}}}$ and $m \in M$ to $\text{Map}(F-, m) \in V^D$.

To prove the statement when V^D has the projective and $M^{D^{\text{op}}}$ has the injective model structure, we must show that this is a right Quillen bifunctor with respect to the pointwise (trivial) cofibrations in $M^{D^{\text{op}}}$, (trivial) fibrations in M , and pointwise (trivial) fibrations in V^D . Because the limits involved in the definition of right Quillen bifunctors are also formed pointwise, this follows immediately from the corresponding property of the simplicial hom bifunctor, which was part of the definition of a simplicial model category. The other cases are similar. \square

The upshot of Theorem 2.5.23 is that there are two approaches to constructing a homotopy colimit: fattening up the diagram, as is achieved by the derived functors of §2.5.2, or fattening up the weight. The famous Bousfield–Kan formulae for homotopy limits and colimits in the context of a simplicial model category define them to be weighted limits and colimits for a particular weight constructed as a projective cofibrant replacement of the terminal weight; see [58] or [237, §11.5]. The Quillen two-variable adjunction of Theorem 2.5.23 can be derived as in Theorem 2.4.29 to express a homotopically enriched universal property of the weighted limit or colimit, as representing “homotopy coherent” cones over or under a diagram, an intuition to be explored in the next section.

2.6 Simplicial localizations

Quillen’s model categories provide a robust axiomatic framework within which to “do homotopy theory.” But the constructions of §2.5 imply that the homotopy theories presented by model categories have all homotopy limits and homotopy colimits, which need not be the case in general. In this section we introduce a framework, originally developed by Dwyer and Kan and re-conceptualized by Bergner, which allows us to extend our notion of equivalence between homotopy theories introduced in §2.4.5 to a more flexible notion of *DK-equivalence* (after Dwyer and Kan) that identifies when any two homotopical categories are equivalent.

A mere equivalence of categories of fractions is insufficient to detect an equivalence of homotopy theories; instead a construction that takes into account the “higher-dimensional” homotopical structure is required. To that end, Dwyer and Kan build, from any homotopical category (K, \mathcal{W}) , a simplicial category $\mathcal{L}^H(K, \mathcal{W})$ called the *hammock localization* [88] and demonstrate that their construction has a number of good properties that we tour in §2.6.1:

- The homotopy category $\mathfrak{h}\mathcal{L}^H(K, \mathcal{W})$ is equivalent to the category of fractions $K[\mathcal{W}]^{-1}$ (Proposition 2.6.5).
- If (K, \mathcal{W}) underlies a simplicial model category then the Kan complex enriched category $K_{\text{cf}} \subset K$ is DK-equivalent to $\mathcal{L}^H(K, \mathcal{W})$ (Proposition 2.6.7).
- More generally, $\mathcal{L}^H(K, \mathcal{W})$ provides a not-necessarily simplicial model category (K, \mathcal{W}) with function complexes that have the correct mapping type even if the model structure is not simplicial (Proposition 2.6.6).
- If two model categories are Quillen equivalent, then their hammock localizations are DK-equivalent (Proposition 2.6.8).

The DK-equivalences are those simplicial functors that are bijective on homotopy equivalence classes of objects and define local equivalences of the mapping spaces constructed by the hammock localization. Zooming out a categorical level, the Bergner model structure on simplicially enriched categories gives a presentation of the homotopical category of homotopy theories, with the DK-equivalences as its weak equivalences. This is the subject of §2.6.2.

2.6.1 The hammock localization

There are two equivalent ways to present the data of a simplicially enriched category, either as a category equipped with a simplicial set of morphisms between each pair of objects, or a simplicial diagram of categories K_n of *n*-**arrows**, each of which is equipped with a constant set of objects.

Exercise 2.6.1. Prove that the following are equivalent:

- (i) A simplicially enriched category with objects $\text{ob}K$.
- (ii) A simplicial object $K_\bullet: \Delta^{\text{op}} \rightarrow \text{Cat}$ in which each of the categories K_n has objects $\text{ob}K$ and each functor $K_n \rightarrow K_m$ is the identity on objects.

Exercise 2.6.4. Verify that the composite of the functor $K \rightarrow \mathcal{L}^H(K, \mathcal{W})$ just described with the quotient functor $\mathcal{L}^H(K, \mathcal{W}) \rightarrow \mathfrak{h}\mathcal{L}^H(K, \mathcal{W})$ that collapses each mapping space onto its set of path components inverts the weak equivalences in K , sending each to an isomorphism in the homotopy category $\mathfrak{h}\mathcal{L}^H(K, \mathcal{W})$.

In the hammock localization

cancelation in any dimension is achieved not by “imposing relations” in the same dimension, but by “imposing homotopy relations”, i.e. adding maps, in the next dimension, [87, §3]

in contrast with the category of fractions constructed in §2.2. By considering the effect of these “homotopy relations,” it is straightforward to see that the induced functor from the category of fractions to the homotopy category of the hammock localization is an isomorphism of categories.

Proposition 2.6.5 (Dwyer–Kan [88, 3.2]). *The canonical functor $K \rightarrow \mathcal{L}^H(K, \mathcal{W})$ induces an isomorphism of categories $K[\mathcal{W}^{-1}] \cong \mathfrak{h}\mathcal{L}^H(K, \mathcal{W})$.*

Proof. The comparison functor $K[\mathcal{W}^{-1}] \rightarrow \mathfrak{h}\mathcal{L}^H(K, \mathcal{W})$ induced by Exercise 2.6.4 and by the universal property of Proposition 2.2.2 is clearly bijective on objects and full, homotopy classes in $\mathfrak{h}\mathcal{L}^H(K, \mathcal{W})$ being represented by zigzags whose backwards maps lie in \mathcal{W} . To see that this functor is faithful it suffices to consider a 1-simplex in $\text{Map}(X, Y)$:

$$\begin{array}{ccccccc}
 & & A_{0,1} & \text{---} & A_{0,2} & \text{---} & \cdots & \text{---} & A_{0,n-1} & & \\
 & \swarrow & \downarrow \wr & & \downarrow \wr & & & & \downarrow \wr & \searrow & \\
 X & & & & & & & & & & Y \\
 & \searrow & A_{1,1} & \text{---} & A_{1,2} & \text{---} & \cdots & \text{---} & A_{1,n-1} & & \\
 & & & & & & & & & &
 \end{array}$$

and argue that the top and bottom zigzags define the same morphism in $K[\mathcal{W}^{-1}]$. This is an easy exercise in diagram chasing, applying the rules of Definition 2.2.1. \square

The previous result applies to a model category (M, \mathcal{W}) , in which case we see that $\mathcal{L}^H(M, \mathcal{W})$ is a higher-dimensional incarnation of the homotopy category, equipping $M[\mathcal{W}^{-1}]$ with mapping spaces whose path components correspond to arrows in the category of fractions. A further justification that the mapping spaces of the hammock localization have the correct homotopy type, not just the correct sets of path components, proceeds as follows. A **simplicial resolution** of $Y \in M$ is a Reedy fibrant simplicial object Y_\bullet together with a weak equivalence $Y \xrightarrow{\sim} Y_0$. **Cosimplicial resolutions** $X^\bullet \rightarrow X$ are defined dually. Every object has a simplicial and cosimplicial resolution, defined as the Reedy fibrant replacement of the constant simplicial object in $M^{\Delta^{\text{op}}}$ and the Reedy cofibrant replacement of the constant cosimplicial object in M^Δ , respectively.

Proposition 2.6.6 (Dwyer–Kan [88, 4.4]). *For any cosimplicial resolution $X^\bullet \rightarrow X$ and simplicial resolution $Y \rightarrow Y_\bullet$, the diagonal of the bisimplicial set $M(X^\bullet, Y_\bullet)$ has the same homotopy type of $\text{Map}_{\mathcal{L}^H(M, \mathcal{W})}(X, Y)$, and if X or Y are respectively cofibrant or fibrant the simplicial sets $M(X, Y_\bullet)$ and $M(X^\bullet, Y)$ do as well.*

As a corollary of this result one can show:

Proposition 2.6.7 (Dwyer–Kan [88, 4.7, 4.8]). *Let (M, \mathcal{W}) be the homotopical category underlying a simplicial model category M . Then for cofibrant X and fibrant Y , $\text{Map}_M(X, Y)$ and $\text{Map}_{\mathcal{L}^H(M, \mathcal{W})}(X, Y)$ have the same homotopy type and hence the simplicial categories M_{cf} and $\mathcal{L}^H(M, \mathcal{W})$ are DK-equivalent.*

The statement of this result requires some explanation. If K is a simplicial category whose underlying category of 0-arrows K_0 has a subcategory of weak equivalences \mathcal{W} , then these weak equivalences degenerate to define homotopical categories (K_n, \mathcal{W}) for each category of n -arrows in K . For each n we may form the hammock localization $\mathcal{L}^H(K_n, \mathcal{W})$. As n varies, this gives a bisimplicial sets of mapping complexes for each fixed pair of objects of K . The mapping complexes in the hammock localization $\mathcal{L}^H(K, \mathcal{W})$ are defined to be the diagonals of these bisimplicial sets. In the case of a simplicial model category M , the hammock localization $\mathcal{L}^H(M, \mathcal{W})$ is DK-equivalent to the hammock localization $\mathcal{L}^H(M_0, \mathcal{W})$ of the underlying unenriched homotopical category.

Proposition 2.6.8 ([88, 5.4]). *A Quillen equivalence*

$$\begin{array}{ccc} & F & \\ & \curvearrowright & \\ M & & N \\ & \curvearrowleft & \\ & G & \end{array}$$

induces DK-equivalences

$$\mathcal{L}^H(M_c, \mathcal{W}) \xrightarrow{\sim} \mathcal{L}^H(N_c, \mathcal{W}) \quad \text{and} \quad \mathcal{L}^H(N_f, \mathcal{W}) \xrightarrow{\sim} \mathcal{L}^H(M_f, \mathcal{W}).$$

Moreover, for any model category the inclusions

$$\mathcal{L}^H(M_c, \mathcal{C} \cap \mathcal{W}) \xrightarrow{\sim} \mathcal{L}^H(M_c, \mathcal{W}_c) \xrightarrow{\sim} \mathcal{L}^H(M, \mathcal{W})$$

are DK-equivalences and hence $\mathcal{L}^H(M, \mathcal{W})$ and $\mathcal{L}^H(N, \mathcal{W})$ are DK-equivalent.

2.6.2 A model structure for homotopy coherent diagrams

Several of Dwyer and Kan’s proofs of the results in the previous subsection make use of a model structure on the category of simplicial categories with a fixed set of objects and with identity-on-objects functors. But this restriction to categories with the same objects is somewhat unnatural. The Bergner model structure is the extension of Dwyer and Kan’s model structure that drops that restriction, unifying the notions of DK-equivalence, free simplicial category (also known as “simplicial computad”), and Kan complex enriched simplicial category, the importance of which will be made clear in §2.7.

Theorem 2.6.9 (Bergner [41]). *There exists a model structure on the category of simplicially enriched categories given as follows:*

- *Its equivalences are the DK equivalences.*

- Its cofibrant objects are the **simplicial computads**: those simplicial categories that, when considered as a simplicial object $\mathcal{C}_\bullet: \Delta^{\text{op}} \rightarrow \text{Cat}$ have the property that
 - each category \mathcal{C}_n is freely generated by the reflexive directed graph of its **atomic arrows** (those admitting no non-trivial factorizations) and
 - the degeneracy operators $[m] \twoheadrightarrow [n]$ in Δ preserve atomic arrows.
- Fibrant objects are the **Kan complex enriched categories**: those simplicial categories whose mapping spaces are all Kan complexes.

More generally, the cofibrations in the Bergner model structure are retracts of relative simplicial computads and the fibrations are those functors that are local Kan fibrations and define isofibrations at the level of homotopy categories; see [41] for more details.

Definition 2.3.24 tells us that maps in the homotopy category of the Bergner model structure from a simplicial category A to a simplicial category K are represented by simplicial functors from a cofibrant replacement of A to a fibrant replacement of K . These are classically studied objects. Cordier and Porter after Vogt define such functors to be **homotopy coherent diagrams** of shape A in K [72].

A particular model for the cofibrant replacement of a strict 1-category A regarded as a discrete simplicial category gives some intuition for the data involved in defining a homotopy coherent diagram. This construction, introduced by Dwyer and Kan under the name “standard resolutions” [89, 2.5], can be extended to the case where A is non-discrete by applying it levelwise and taking diagonals.

Definition 2.6.10 (free resolutions). There is a comonad (F, ϵ, δ) on the category of categories that sends a small category to the free category on its underlying reflexive directed graph. Explicitly FA has the same objects as A and its non-identity arrows are strings of composable non-identity arrows of A .

Adopting the point of view of Exercise 2.6.1, we define a simplicial category $\mathcal{C}A_\bullet$ with $\text{ob}\mathcal{C}A = \text{ob}A$ and with the category of n -arrows $\mathcal{C}A_n := F^{n+1}A$. A non-identity n -arrow is a string of composable arrows in A with each arrow in the string enclosed in exactly n pairs of well-formed parentheses. In the case $n = 0$, this recovers the previous description of the non-identity 0-arrows in FA , strings of composable non-identity arrows of A .

The required identity-on-objects functors in the simplicial object $\mathcal{C}A_\bullet$ are defined by evaluating the comonad resolution for (F, ϵ, δ) on a small category A :

$$\mathcal{C}A_\bullet := \quad FA \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} F^2A \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} F^3A \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} F^4A \quad \dots$$

Explicitly, for $j \geq 1$, the face maps

$$F^k \epsilon F^j: F^{k+j+1}A \rightarrow F^{k+j}A$$

remove the parentheses that are contained in exactly k others, while $F^{k+j} \epsilon$ composes

the morphisms inside the innermost parentheses. For $j \geq 1$, the degeneracy maps

$$F^k \delta F^j: F^{k+j+1}A \rightarrow F^{k+j+2}A$$

double up the parentheses that are contained in exactly k others, while $F^{k+j}\delta$ inserts parentheses around each individual morphism.

Exercise 2.6.11. Explain the sense in which free resolutions define Bergner cofibrant replacements of strict 1-categories by:

- (i) verifying that for any A , the free resolution $\mathcal{C}A_\bullet$ is a simplicial computad, and
- (ii) defining a canonical identity-on-objects augmentation functor $\epsilon: \mathcal{C}A \rightarrow A$ and verifying that it defines a local homotopy equivalence.

The notation $\mathcal{C}A_\bullet$ for the free resolution is non-standard and will be explained in §2.7.2, where we will gain a deeper understanding of the importance of the Bergner model structure from the vantage point of $(\infty, 1)$ -categories.

2.7 Quasi-categories as $(\infty, 1)$ -categories

Any topological space Y has an associated simplicial set $\text{Sing}(Y)$ called its **total singular complex**. The vertices in $\text{Sing}(Y)$ are the points in Y and the 1-simplices are the paths; in general, an n -simplex in $\text{Sing}(Y)$ corresponds to an n -simplex in Y , that is, to a continuous map $|\Delta^n| \rightarrow Y$. In particular, a 2-simplex $|\Delta^2| \rightarrow Y$ defines a triangular shaped homotopy from the composite paths along the spine $\Lambda_1^2 \subset \Delta^2$ of the 2-simplex to the direct path from the 0th to the 2nd vertex that is contained in its 1st face.¹⁵ Since the inclusion $|\Lambda_k^n| \rightarrow |\Delta^n|$ admits a retraction, $\text{Sing}(Y)$ is a Kan complex.

The total singular complex is a higher-dimensional incarnation of some of the basic invariants of Y , which can be recovered by truncating the total singular complex at some level and replacing the top-dimensional simplices with suitably defined “homotopy classes” of such. Its set of path components is the set $\pi_0 Y$ of path components in Y . Its homotopy category, in a sense to be defined below, comprised of the vertices and homotopy classes of paths between them, is a groupoid $\pi_1 Y$ called the *fundamental groupoid* of Y . By extension, it is reasonable to think of the higher-dimensional simplices of $\text{Sing}(Y)$ as being invertible in a similar sense, with composition relations witnessed by higher cells. In this way, $\text{Sing}(Y)$ models the ∞ -groupoid associated to the topological space Y and the Quillen equivalence 2.4.32 is one incarnation of Grothendieck’s famous “homotopy hypothesis” (the moniker due to Baez), that ∞ -groupoids up to equivalence should model homotopy types [113].

In the catalog of weak higher-dimensional categories, the ∞ -groupoids define $(\infty, 0)$ -**categories**, weak categories with morphisms in each dimension, all of which are weakly invertible. In §2.7.1, we introduce **quasi-categories**, which provide a particular model for $(\infty, 1)$ -categories — infinite-dimensional categories in which every morphism above dimension 1 is invertible — in parallel with the Kan complex model

¹⁵ The simplicial n -simplex Δ^n , its boundary sphere $\partial\Delta^n$, and its horns Λ_k^n are defined in §2.3.5.

for $(\infty, 0)$ -categories. We explain the sense in which quasi-categories, which are defined to be simplicial sets with an inner horn lifting property, model $(\infty, 1)$ -categories by introducing the homotopy category of a quasi-category and constructing the hom-space between objects in a quasi-category. In §2.7.2, we explain how simplicially enriched categories like those considered in §2.6 can be converted into quasi-categories. Then in §2.7.3, we introduce a model structure whose fibrant objects are the quasi-categories due to Joyal and in this way obtain a suitable notion of (weak) equivalence between quasi-categories.

2.7.1 Quasi-categories and their homotopy categories

The **nerve** of a small category D is the simplicial set D_\bullet whose vertices D_0 are the objects of D , whose 1-simplices D_1 are the morphisms, and whose set of n -simplices D_n is the set of n composable pairs of morphisms in D . The simplicial structure defines a diagram in Set

$$\cdots \quad D_3 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} D_2 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} D_1 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} D_0$$

Truncating at level 2 we are left with precisely the data that defines a small category D as a category internal to the category of sets and in fact this higher-dimensional data is redundant in a sense: the simplicial set D_\bullet is 2-coskeletal, meaning any sphere bounding a hypothetical simplex of dimension at least 3 admits a unique filler.

The description of the nerve as an internal category relies on an isomorphism $D_2 \cong D_1 \times_{D_0} D_1$ identifying the set of 2-simplices with the pullback of the domain and codomain maps $D_1 \rightrightarrows D_0$: a composable pair of arrows is given by a pair of arrows such that the domain of the second equals the codomain of the first. Equivalently, this condition asserts that the map

$$\begin{array}{ccc} \Lambda_1^2 & \longrightarrow & D_\bullet \\ \downarrow & \nearrow \exists! & \\ \Delta^2 & & \end{array}$$

admits a unique filler. In higher dimensions, we can consider the inclusion of the spine $\Delta^1 \cup_{\Delta^0} \cdots \cup_{\Delta^0} \Delta^1 \hookrightarrow \Delta^n$ of an n -simplex, and similarly the nerve D_\bullet will admit unique extensions along these maps. From the perspective of an infinite-dimensional category, in which the higher-dimensional simplices represent data and not just conditions on the 1-simplices, it is better to consider extensions along inner horn inclusions $\Lambda_k^n \hookrightarrow \Delta^n$ for the reasons explained by the following exercise.

Exercise 2.7.1. Prove that the spine inclusions can be presented as cell complexes (see Definition 2.3.11) built from the inner horn inclusions $\{\Lambda_n^k \hookrightarrow \Delta^n\}_{n \geq 2, 0 < k < n}$ but demonstrate by example that the inner horn inclusions cannot be presented as cell complexes built from the spine inclusions.

The original definition of a simplicial set satisfying the “restricted Kan condition,”

now called a *quasi-category* (following Joyal [140]) or an ∞ -*category* (following Lurie [169]), is due to Boardman and Vogt [48]. Their motivating example appears as Corollary 2.7.9.

Definition 2.7.2. A **quasi-category** is a simplicial set X such that $X \rightarrow *$ has the right lifting property with respect to the inner horn inclusions for each $n \geq 2$, $0 < k < n$.

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array} \quad (2.9)$$

Nerves of categories are quasi-categories; in fact in this case each lift (2.9) is unique. Tautologically, Kan complexes are quasi-categories. In particular, the total singular complex of a topological space is a Kan complex and hence a quasi-category. More sophisticated examples of (frequently large) quasi-categories are produced by Theorem 2.7.8 below.

Definition 2.7.3 (the homotopy category of a quasi-category [48, 4.12]). Any quasi-category X has an associated **homotopy category** hX whose objects are the vertices of X and whose morphisms are represented by 1-simplices, which we consequently depict as arrows $f: x \rightarrow y$ from their 0th vertex to their 1st vertex. The degenerate 1-simplices serve as identities in the homotopy category, and may be depicted using an equals sign in place of the arrow.

As the name suggests, the morphisms in hX are homotopy classes of 1-simplices, where a pair of 1-simplices f and g with common boundary are **homotopic** if there exists a 2-simplex whose boundary has any of the following forms:

$$\begin{array}{cccc} \begin{array}{ccc} \bullet & & \bullet \\ \parallel & \searrow f & \\ \bullet & \xrightarrow{\sim} & \bullet \\ \parallel & \xrightarrow{g} & \bullet \end{array} & \begin{array}{ccc} \bullet & & \bullet \\ f \nearrow & \parallel & \\ \bullet & \xrightarrow{\sim} & \bullet \\ \bullet & \xrightarrow{g} & \bullet \end{array} & \begin{array}{ccc} \bullet & & \bullet \\ \parallel & \searrow g & \\ \bullet & \xrightarrow{\sim} & \bullet \\ \bullet & \xrightarrow{f} & \bullet \end{array} & \begin{array}{ccc} \bullet & & \bullet \\ g \nearrow & \parallel & \\ \bullet & \xrightarrow{\sim} & \bullet \\ \bullet & \xrightarrow{f} & \bullet \end{array} \end{array} \quad (2.10)$$

Indeed, in a quasi-category, if any of the 2-simplices (2.10) exists then there exists a 2-simplex of each type.

Generic 2-simplices in X

$$\begin{array}{ccc} \bullet & & \bullet \\ f \nearrow & & \searrow g \\ \bullet & \xrightarrow{\sim} & \bullet \\ \bullet & \xrightarrow{h} & \bullet \end{array} \quad (2.11)$$

witness that $gf = h$ in the homotopy category. Conversely, if $h = gf$ in hX and f, g, h are any 1-simplices representing these homotopy classes, there exists a 2-simplex (2.11) witnessing the composition relation.

Exercise 2.7.4.

- (i) Verify the assertions made in Definition 2.7.3 or see [169, §1.2.3].

(ii) Show that h is the left adjoint to the nerve functor:¹⁶

$$\begin{array}{ccc} & \xrightarrow{h} & \\ \text{qCat} & \perp & \text{Cat} \\ & \xleftarrow{N} & \end{array}$$

The mapping space between two objects of a quasi-category A is modeled by the Kan complex defined via the pullback

$$\begin{array}{ccc} \text{Map}_A(x, y) & \longrightarrow & A^{\Delta^1} \\ \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \xrightarrow{(x, y)} & A \times A \end{array}$$

The following proposition of Joyal is useful in proving that $\text{Map}_A(x, y)$ is a Kan complex and also characterizes the ∞ -groupoids in the quasi-categorical model of $(\infty, 1)$ -categories.

Proposition 2.7.5 (Joyal [140, 1.4]). *A quasi-category is a Kan complex if and only if its homotopy category is a groupoid.*

Definition 2.7.6 ([140, 1.6]). A 1-simplex f in a quasi-category X is an **isomorphism** if and only if it represents an isomorphism in the homotopy category, or equivalently if and only if it admits a coherent homotopy inverse:

$$\begin{array}{ccc} \mathbb{2} & \xrightarrow{f} & X \\ \downarrow & \dashrightarrow & \\ \mathbb{I} & & \end{array}$$

extending along the map $\mathbb{2} \hookrightarrow \mathbb{I}$ including the nerve of the free-living arrow into the nerve of the free-living isomorphism.

2.7.2 Quasi-categories found in nature

Borrowing notation from the simplex category Δ , we write $[n] \subset \omega$ for the ordinal category $m + 1$, the full subcategory spanned by $0, \dots, n$ in the category that indexes a countable sequence:

$$[n] := \quad 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \dots \longrightarrow n$$

These categories define the objects of a diagram $\Delta \hookrightarrow \text{Cat}$ that is a full embedding: the only functors $[m] \rightarrow [n]$ are order-preserving maps from $[m] = \{0, \dots, m\}$ to $[n] = \{0, \dots, n\}$. Applying the free resolution construction of Definition 2.6.10 to these categories we get a functor $\mathcal{C}: \Delta \rightarrow \text{sCat}$, where $\mathcal{C}[n]$ is the full simplicial subcategory of $\mathcal{C}\omega$ spanned by those objects $0, \dots, n$.

¹⁶ In fact, this pair defines a Quillen adjunction between the model structure to be introduced in Theorem 2.7.12 and the “folk” model structure on categories [237, 15.3.8].

Definition 2.7.7 (homotopy coherent realization and nerve). The homotopy coherent nerve Ω and homotopy coherent realization \mathbb{C} are the adjoint pair of functors obtained by applying Kan's construction [237, 1.5.1] to the functor $\mathbb{C}: \Delta \rightarrow \text{sCat}$ to construct an adjunction

$$\begin{array}{ccc} & \mathbb{C} & \\ & \curvearrowright & \\ \text{sSet} & \perp & \text{sCat} \\ & \curvearrowleft & \\ & \Omega & \end{array}$$

The right adjoint, called the **homotopy coherent nerve**, converts a simplicial category S into a simplicial set ΩS whose n -simplices are homotopy coherent diagrams of shape $[n]$ in S . That is,

$$\Omega S_n := \{\mathbb{C}[n] \rightarrow S\}.$$

The left adjoint is defined by pointwise left Kan extension along the Yoneda embedding:

$$\begin{array}{ccc} \Delta & \xrightarrow{\mathbb{y}} & \text{sSet} \\ & \searrow \mathbb{C} & \swarrow \mathbb{C} \\ & \text{sCat} & \end{array} \quad \cong$$

That is, $\mathbb{C}\Delta^n$ is defined to be $\mathbb{C}[n]$ — a simplicial category that we call the **homotopy coherent n -simplex** — and for a generic simplicial set X , $\mathbb{C}X$ is defined to be a colimit of the homotopy coherent simplices indexed by the category of simplices of X .¹⁷ Because of the formal similarity with the geometric realization functor, another left adjoint defined by Kan's construction, we refer to \mathbb{C} as **homotopy coherent realization**.

Many examples of quasi-categories fit into the following paradigm.

Theorem 2.7.8 ([72, 2.1]). *If S is Kan complex enriched, then ΩS is a quasi-category.*

In particular, in light of Exercise 2.4.28, the quasi-category associated to a simplicial model category M is defined to be ΩM_{cf} .

Recall from §2.6.2 that a **homotopy coherent diagram** of shape A in a Kan complex enriched category S is a functor $\mathbb{C}A \rightarrow S$. Similarly, a **homotopy coherent natural transformation** $\alpha: F \rightarrow G$ between homotopy coherent diagrams F and G of shape A is a homotopy coherent diagram of shape $A \times [1]$ that restricts on the endpoints of $[1]$ to F and G as follows:

$$\begin{array}{ccccc} \mathbb{C}A & \xrightarrow{0} & \mathbb{C}(A \times [1]) & \xleftarrow{1} & \mathbb{C}A \\ & \searrow F & \downarrow \alpha & \swarrow G & \\ & & S & & \end{array}$$

Note that the data of a pair of homotopy coherent natural transformations $\alpha: F \rightarrow G$ and $\beta: G \rightarrow H$ between homotopy coherent diagrams of shape A does not uniquely

¹⁷ The simplicial set X is obtained by gluing in a Δ^n for each n -simplex $\Delta^n \rightarrow X$ of X . The functor \mathbb{C} preserves these colimits, so $\mathbb{C}X$ is obtained by gluing in a $\mathbb{C}[n]$ for each n -simplex of X .

determine a (vertical) “composite” homotopy coherent natural transformation $F \rightarrow H$ because this data does not define a homotopy coherent diagram of shape $A \times [2]$, where $[2] = 0 \rightarrow 1 \rightarrow 2$. Here α and β define a diagram of shape $\mathcal{C}(A \times \Lambda_1^2)$ rather than a diagram of shape $\mathcal{C}(A \times [2])$, where Λ_1^2 is the shape of the generating reflexive directed graph of the category $[2]$. This observation motivated Boardman and Vogt to define, in place of a *category* of homotopy coherent diagrams and natural transformations of shape A , a *quasi-category* of homotopy coherent diagrams and natural transformations of shape A .

For any category A , let $\text{Coh}(A, S)$ denote the simplicial set whose n -simplices are homotopy coherent diagrams of shape $A \times [n]$, i.e., are simplicial functors

$$\mathcal{C}(A \times [n]) \rightarrow S.$$

Corollary 2.7.9. $\text{Coh}(A, S) \cong \Omega S^A$ is a quasi-category.

Proof. By the adjunction of Definition 2.7.7, a simplicial functor $\mathcal{C}A \rightarrow S$ is the same as a simplicial map $A \rightarrow \Omega S$. So $\text{Coh}(A, S) \cong \Omega S^A$ and since the quasi-categories define an exponential ideal in simplicial sets as a consequence of the cartesian closure of the Joyal model structure of Theorem 2.7.12, the fact that ΩS is a quasi-category implies that ΩS^A is too. \square

Remark 2.7.10 (all diagrams in homotopy coherent nerves are homotopy coherent). This corollary explains that any map of simplicial sets $X \rightarrow \Omega S$ transposes to define a simplicial functor $\mathcal{C}X \rightarrow S$, a homotopy coherent diagram of shape X in S . While not every quasi-category is isomorphic to a homotopy coherent nerve of a Kan complex enriched category, every quasi-category is equivalent to a homotopy coherent nerve; one proof appears as [244, 7.2.2]. This explains the slogan that “all diagrams in quasi-categories are homotopy coherent.”

2.7.3 The Joyal model structure

In analogy with Quillen’s model structure of Theorem 2.3.30, in which the fibrant objects are the Kan complexes and the cofibrations are the monomorphisms, we might hope that there is another model structure on sSet whose fibrant objects are the quasi-categories and with the monomorphisms as cofibrations, and indeed there is one (and by Exercise 2.3.8(2.3.8) it is unique).

The weak equivalences in this hoped-for model structure for quasi-categories can be described using a particularly nice cylinder object. Let \mathbb{I} be the nerve of the free-standing isomorphism \mathbb{I} ; the name is selected because \mathbb{I} is something like an interval.

Proposition 2.7.11. *For any simplicial set A , the evident inclusion and projection maps define a cylinder object*

$$\begin{array}{ccc} A \sqcup A & \xrightarrow{(1_A, 1_A)} & A \\ & \searrow (i_0, i_1) & \nearrow \pi \\ & & A \times \mathbb{I} \end{array}$$

Proof. The map $(i_0, i_1): A \sqcup A \rightarrow A \times \mathbb{I}$ is a monomorphism and hence a cofibration. To see that the projection is a trivial fibration, observe that it is a pullback of $\mathbb{I} \rightarrow *$ as displayed below left, and hence by Lemma 2.3.9 it suffices to prove that this latter map is a trivial fibration. To that end, we must show that there exist solutions to lifting problems displayed on the right:

$$\begin{array}{ccc} A \times \mathbb{I} & \xrightarrow{\pi} & \mathbb{I} \\ \pi \downarrow & \lrcorner & \downarrow \\ A & \longrightarrow & * \end{array} \qquad \begin{array}{ccc} \partial \Delta^n & \longrightarrow & \mathbb{I} \\ \downarrow & \nearrow \text{---} & \downarrow \\ \Delta^n & \longrightarrow & * \end{array}$$

When $n = 0$ this is true because \mathbb{I} is non-empty. For larger n , we use the fact that $\mathbb{I} \cong \text{cosk}_0 \mathbb{I}$. By adjunction, it suffices to show that \mathbb{I} lifts against $\text{sk}_0 \partial \Delta^n \rightarrow \text{sk}_0 \Delta^n$, but for $n > 0$, the 0-skeleton of Δ^n is isomorphic to that of its boundary. \square

The proof of Joyal’s model structure has been widely circulated in unpublished notes, and can also be found in [169, 2.2.5.1] or [81, 2.13].

Theorem 2.7.12 (Joyal). *There is a cartesian closed model structure on sSet whose*

- *cofibrations are monomorphisms,*
- *weak equivalences are those maps $f: A \rightarrow B$ that induce bijections on the sets*

$$\text{Hom}(B, X)_{/\sim_\ell} \rightarrow \text{Hom}(A, X)_{/\sim_\ell}$$

of maps into any quasi-category X modulo the left homotopy relation relative to the cylinder just defined,

- *fibrant objects are precisely the quasi-categories, and*
- *fibrations between fibrant objects are the **isofibrations**, those maps that lift against the inner horn inclusions and also the map $* \rightarrow \mathbb{I}$.*

By Proposition 2.3.23, a map between quasi-categories is a weak equivalence, or we say simply **equivalence** of quasi-categories, if and only if it admits an inverse equivalence $Y \rightarrow X$ together with an “invertible homotopy equivalence” using the notion of homotopy defined with the interval \mathbb{I} . A map between nerves of strict 1-categories is an equivalence of quasi-categories if and only if it is an equivalence of categories, as usually defined. In general, every categorical notion for quasi-categories restricts along the full inclusion $\text{Cat} \subset \text{qCat}$ to the classical notion. This gives another sense in which quasi-categories model the $(\infty, 1)$ -categories introduced at the start of this section. However, quasi-categories are not the only model of $(\infty, 1)$ -categories, as we shall now discover.

2.8 Models of $(\infty, 1)$ -categories

An $(\infty, 1)$ -category should have a set of objects X_0 , a space of morphisms X_1 , together with composition and identities that are at least weakly associative and unital. One

idea of how this might be presented, due to Segal [269], is to ask that $X \in \mathbf{sSet}^{\Delta^{\text{op}}}$ is a simplicial space

$$\cdots \quad \begin{array}{ccccc} & \rightrightarrows & & \rightrightarrows & \\ & \leftleftarrows & & \leftleftarrows & \\ X_3 & \rightleftarrows & X_2 & \leftleftarrows & X_1 & \leftleftarrows & X_0 \\ & \rightrightarrows & & \rightrightarrows & & \rightrightarrows & \end{array}$$

with X_0 still a set, so that for all n the map

$$X_n \rightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1 \tag{2.12}$$

induced on weighted limits from the spine inclusion $\Delta^1 \vee \cdots \vee \Delta^1 \rightarrow \Delta^n$ is a weak equivalence in a suitable sense. Segal points out that Grothendieck has observed that, in the case where the spaces X_n are discrete, these so-called Segal maps are isomorphisms if and only if X is isomorphic to the nerve of a category.

In this section we introduce various models of $(\infty, 1)$ -categories, many of which are inspired by this paradigm. Before these models make their appearance in §2.8.2, we begin in §2.8.1 with an abbreviated tour of an axiomatization due to Toën that characterizes a homotopy theory of $(\infty, 1)$ -categories. In §2.8.3, we then restrict our attention to four of the six models that are better behaved in the sense of providing easy access to the $(\infty, 1)$ -category $\text{Fun}(A, B)$ of functors between $(\infty, 1)$ -categories A and B . These models each satisfy a short list of axioms that we exploit in §2.9 to sketch a natively “model-independent” development of the category theory of $(\infty, 1)$ -categories.

2.8.1 An axiomatization of the homotopy theory of $(\infty, 1)$ -categories

The homotopy theory of ∞ -groupoids is freely generated under homotopy colimits by the point. We might try to adopt a similar “generators and relations” approach to build the homotopy theory of $(\infty, 1)$ -categories, taking the generators to be the category Δ , which freely generates simplicial spaces. The relations assert that the natural maps

$$\Delta^1 \vee \cdots \vee \Delta^1 \rightarrow \Delta^n \quad \text{and} \quad \mathbb{I} \rightarrow \Delta^0 \tag{2.13}$$

induce equivalences upon mapping into an $(\infty, 1)$ -category. This idea motivates Rezk’s complete Segal space model, which is the conceptual center of the Toën axiomatization of a model category \mathcal{M} whose fibrant objects model $(\infty, 1)$ -categories.

For simplicity we assume that \mathcal{M} is a combinatorial simplicial model category. In practice, these assumptions are relatively mild: in particular, if \mathcal{M} fails to be simplicial it is possible to define a Quillen equivalent model structure on $\mathcal{M}^{\Delta^{\text{op}}}$ that is simplicial [78]. The model category \mathcal{M} should be equipped with a functor $C: \Delta \rightarrow \mathcal{M}$ such that $C(0)$ represents a free point in \mathcal{M} while $C(1)$ represents a free arrow. This cosimplicial object is required to be a **weak cocategory**, meaning that the duals of the Segal maps are equivalences

$$C(1) \cup_{C(0)} \cdots \cup_{C(0)} C(1) \xrightarrow{\sim} C(n).$$

We state Toën’s seven axioms without defining all the terms because to do so would demand too long of an excursion, and refer the reader to [292] for more details.

Theorem 2.8.1 (Toën [292]). *Let \mathcal{M} be a combinatorial simplicial model category equipped with a functor $C: \Delta \rightarrow \mathcal{M}$ satisfying the following properties.*

- (i) *Homotopy colimits are universal over **0-local** objects, those X so that*

$$\mathrm{Map}(*, X) \xrightarrow{\sim} \mathrm{Map}(C(1), X).$$

- (ii) *Homotopy coproducts are disjoint and universal.*

(iii) *C is an **interval**: meaning $C(0)$ and the C -geometric realization of \mathbb{I} are contractible.*

(iv) *For any weak category $X \in \mathcal{M}^{\Delta^{\mathrm{op}}}$ such that X_0 and X_1 are 0-local, X is equivalent to the Čech nerve of the map $X_0 \rightarrow |X|_C$.*

(v) *For any weak category $X \in \mathcal{M}^{\Delta^{\mathrm{op}}}$ such that X_0 and X_1 are 0-local, the homotopy fiber of $X \rightarrow \mathbb{R}\mathrm{Hom}(C, |X|_C)$ is contractible.*

(vi) *The point and interval define a **generator**: $f: X \rightarrow Y$ is a weak equivalence in \mathcal{M} if and only if $\mathrm{Map}(C(0), X) \xrightarrow{\sim} \mathrm{Map}(C(0), Y)$ and $\mathrm{Map}(C(1), X) \xrightarrow{\sim} \mathrm{Map}(C(1), Y)$.*

(vii) *C is **homotopically fully faithful**: $\Delta([n], [m]) \xrightarrow{\sim} \mathrm{Map}(C(n), C(m))$*

Then the functor $X \mapsto \mathrm{Map}(C(-), X)$ defines a right Quillen equivalence from \mathcal{M} to the model structure for complete Segal spaces on the category of bisimplicial sets.

A similar axiomatization is given by Barwick and Schommer-Pries as a specialization of an axiomatization for (∞, n) -categories [26].

2.8.2 Models of $(\infty, 1)$ -categories

We now introduce six models of $(\infty, 1)$ -categories, each arising as the fibrant objects in a model category that is Quillen equivalent to all of the others. Two of these models — the quasi-categories and the Kan complex enriched categories — have been presented already in Theorems 2.7.12 and 2.6.9.

A **Segal category** is a Reedy fibrant bisimplicial set $X \in \mathrm{sSet}^{\Delta^{\mathrm{op}}}$ such that the Segal maps (2.12) are trivial fibrations and X_0 is a set.¹⁸

Theorem 2.8.2 (Hirschowitz–Simpson [125, 278], Pellissier [220], Bergner [40]). *There is a cartesian closed model structure on the category of bisimplicial sets with discrete set of objects whose*

- *cofibrations are the monomorphisms,*
- *fibrant objects are the Segal categories that are Reedy fibrant as simplicial spaces, and*
- *weak equivalences are the DK-equivalences (in a suitable sense).*

A **complete Segal space** is similarly a Reedy fibrant bisimplicial set $X \in \mathrm{sSet}^{\Delta^{\mathrm{op}}}$ such that the Segal maps (2.12) are trivial fibrations. In this model, the discreteness condition on X_0 is replaced with the so-called **completeness** condition, which is again most elegantly phrased using weighted limits: it asks either that the map $\{\mathbb{I}, X\} \rightarrow \{\Delta^0, X\} \cong X_0$ is a trivial fibration or that the map $X_0 \rightarrow \{\mathbb{I}, X\}$ is an equivalence. Intuitively this says that the spatial structure of X_0 is recovered by the ∞ -groupoid of $\{\mathbb{I}, X\}$ of isomorphisms in X .

¹⁸ In [90, §7] the Reedy fibrancy condition, which implies that the Segal maps are Kan fibrations, is dropped and the Segal maps are only required to be weak equivalences.

Theorem 2.8.3 (Rezk [233]). *There is a cartesian closed model structure on the category of bisimplicial sets whose*

- *cofibrations are the monomorphisms,*
- *fibrant objects are the complete Segal spaces, and*
- *weak equivalences are those maps $u: A \rightarrow B$ such that for every complete Segal space X , the maps $X^B \rightarrow X^A$ are weak homotopy equivalences of simplicial sets upon evaluating at 0.*

A **marked simplicial set** is a simplicial set with a collection of marked edges containing the degeneracies; maps must then preserve the markings. A quasi-category is naturally a marked simplicial set whose marked edges are precisely the isomorphisms, described in Definition 2.7.6.

Theorem 2.8.4 (Verity [293], Lurie [169]). *There is a cartesian closed model structure on the category of marked simplicial sets whose*

- *cofibrations are the monomorphisms,*
- *fibrant objects are the naturally marked quasi-categories, and*
- *weak equivalences are those maps $A \rightarrow B$ so that for all naturally marked quasi-categories X the map $X^B \rightarrow X^A$ is a homotopy equivalence of maximal sub Kan complexes.*

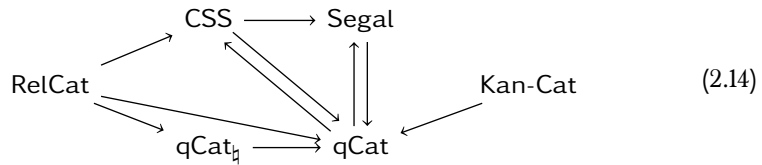
A **relative category** is a category equipped with a wide subcategory of weak equivalences. A morphism of relative categories is a homotopical functor. A weak equivalence of relative categories is a homotopical functor $F: (C, \mathcal{W}) \rightarrow (D, \mathcal{W})$ that induces a DK-equivalence on hammock localizations $\mathcal{L}^H(C, \mathcal{W}) \rightarrow \mathcal{L}^H(D, \mathcal{W})$.

Theorem 2.8.5 (Barwick-Kan [23]). *There is a model structure for relative categories whose*

- *weak equivalences are the relative DK-equivalences just defined*

and whose cofibrations and fibrant objects are somewhat complicated to describe.

Each of these model categories, represented in the diagram below by their subcategories of fibrant objects, are Quillen equivalent, connected via right Quillen equivalences as follows:¹⁹



A nice feature of the simplicial category and relative category models is that their

¹⁹ The right Quillen equivalences from relative categories are in [23]. The Quillen equivalences involving complete Segal spaces, Segal categories, and quasi-categories can all be found in [142]. Proofs that the homotopy coherent nerve defines a Quillen equivalence from simplicial categories to quasi-categories can be found in [169] and [81]. A zigzag of Quillen equivalences between simplicial categories and Segal categories is constructed in [42]. The right Quillen equivalence from naturally marked quasi-categories to the Joyal model structure can be found in [169] and [293].

objects and morphisms are strictly-defined, as honest-to-goodness enriched categories in the former case and honest-to-goodness homotopical categories in the latter. From this vantage point it is quite surprising that they are Quillen equivalent to the weaker models. But there are some costs paid to obtain this extra strictness: neither model category is cartesian closed, so both contexts lack a suitable internal hom, whereas the other four models — the quasi-categories, Segal categories, complete Segal spaces, and naturally marked quasi-categories — all form cartesian closed model categories. Consequently, in each of these models the $(\infty, 1)$ -categories define an exponential ideal: if A is fibrant and X is cofibrant, then A^X is again fibrant and moreover the maps induced on exponentials by the maps (2.13) are weak equivalences.

2.8.3 ∞ -cosmoi of $(\infty, 1)$ -categories

From the cartesian closure of the model categories for quasi-categories, Segal categories, complete Segal spaces, and naturally marked quasi-categories, it is possible to induce a secondary enrichment, in the sense of Definition 2.4.25, on these model categories:

Theorem 2.8.6 ([240, 2.2.3]). *The model structures for quasi-categories, complete Segal spaces, Segal categories, and naturally marked quasi-categories are all enriched over the model structure for quasi-categories.*

The following definition of an ∞ -cosmos collects together the properties of the fibrant objects and fibrations and weak equivalences between them in any model category that is enriched over the Joyal model structure and in which the fibrant objects are also cofibrant:

Definition 2.8.7 (∞ -cosmos). An ∞ -**cosmos** is a simplicially enriched category \mathcal{K} whose

- objects we refer to as the ∞ -**categories** in the ∞ -cosmos, whose
- hom simplicial sets $\text{Fun}(A, B)$ are all quasi-categories,

and that is equipped with a specified subcategory of **isofibrations**, denoted by “ \rightarrow ”, satisfying the following axioms:

- (i) (completeness) As a simplicially enriched category, \mathcal{K} possesses a terminal object 1 , cotensors A^U of objects A by all²⁰ simplicial sets U , and pullbacks of isofibrations along any functor.²¹
- (ii) (isofibrations) The class of isofibrations contains the isomorphisms and all of the functors $! : A \rightarrow 1$ with codomain 1 ; is stable under pullback along all

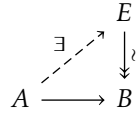
²⁰ For most purposes, it suffices to require only cotensors with finitely presented simplicial sets (those with only finitely many non-degenerate simplices).

²¹ For the theory of homotopy coherent adjunctions and monads developed in [241], limits of towers of isofibrations are also required, with the accompanying stability properties of (ii). These limits are present in all of the ∞ -cosmoi we are aware of, but will not be required for any results discussed here.

functors; and if $p: E \twoheadrightarrow B$ is an isofibration in \mathbf{K} and $i: U \hookrightarrow V$ is an inclusion of simplicial sets then the Leibniz cotensor $\widehat{\{i, p\}}: E^V \twoheadrightarrow E^U \times_{B^U} B^V$ is an isofibration. Moreover, for any object X and isofibration $p: E \twoheadrightarrow B$, $\text{Fun}(X, p): \text{Fun}(X, E) \twoheadrightarrow \text{Fun}(X, B)$ is an isofibration of quasi-categories.

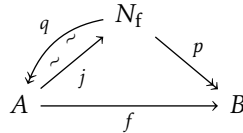
The underlying category of an ∞ -cosmos \mathbf{K} has a canonical subcategory of equivalences, denoted by “ $\xrightarrow{\sim}$ ”, satisfying the two-of-six property. A functor $f: A \rightarrow B$ is an **equivalence** just when the induced functor $\text{Fun}(X, f): \text{Fun}(X, A) \rightarrow \text{Fun}(X, B)$ is an equivalence of quasi-categories for all objects $X \in \mathbf{K}$. The **trivial fibrations**, denoted by “ $\xrightarrow{\sim\triangleright}$ ”, are those functors that are both equivalences and isofibrations. It follows from 2.8.7(i)-(ii) that:

- (iii) (cofibrancy) All objects are **cofibrant**, in the sense that they enjoy the left lifting property with respect to all trivial fibrations in \mathbf{K} :



- (iv) (trivial fibrations) The trivial fibrations define a subcategory containing the isomorphisms; are stable under pullback along all functors; and the Leibniz cotensor $\widehat{\{i, p\}}: E^V \xrightarrow{\sim\triangleright} E^U \times_{B^U} B^V$ of an isofibration $p: E \twoheadrightarrow B$ in \mathbf{K} and a monomorphism $i: U \hookrightarrow V$ between presented simplicial sets is a trivial fibration when p is a trivial fibration in \mathbf{K} or i is a trivial cofibration in the Joyal model structure on \mathbf{sSet} . Moreover, for any object X and trivial fibration $p: E \xrightarrow{\sim\triangleright} B$, $\text{Fun}(X, p): \text{Fun}(X, E) \xrightarrow{\sim\triangleright} \text{Fun}(X, B)$ is a trivial fibration of quasi-categories.

- (v) (factorization) Any functor $f: A \rightarrow B$ may be factored as $f = pj$:



where $p: N_f \twoheadrightarrow B$ is an isofibration and $j: A \xrightarrow{\sim} N_f$ is right inverse to a trivial fibration $q: N_f \xrightarrow{\sim\triangleright} A$.

It is a straightforward exercise in enriched model category theory to verify that these axioms are satisfied by the fibrant objects in any model category that is enriched over the Joyal model structure on simplicial sets, at least when all of these objects are cofibrant. Consequently:

Theorem 2.8.8 (Joyal–Tierney, Verity, Lurie, Riehl–Verity [240]). *The full subcategories \mathbf{qCat} , \mathbf{CSS} , \mathbf{Segal} , and \mathbf{qCat}_{h} all define ∞ -cosmoi.*

Moreover, each of the model categories referenced in Theorem 2.8.8 is a closed monoidal model category with respect to the cartesian product. It follows that each of these four ∞ -cosmoi is **cartesian closed** in the sense that it satisfies the extra axiom:

- (vi) (cartesian closure) The product bifunctor $- \times -: \mathbf{K} \times \mathbf{K} \rightarrow \mathbf{K}$ extends to a simplicially enriched two-variable adjunction

$$\mathrm{Fun}(A \times B, C) \cong \mathrm{Fun}(A, C^B) \cong \mathrm{Fun}(B, C^A).$$

A **cosmological functor** is a simplicial functor $F: \mathbf{K} \rightarrow \mathbf{L}$ preserving the class of isofibrations and all of the limits enumerated in Definition 2.8.7(i). A cosmological functor is a **biequivalence** when it is:

- (i) surjective on objects up to equivalence: i.e., if for every $C \in \mathbf{L}$, there is some $A \in \mathbf{K}$ so that $FA \simeq C \in \mathbf{L}$;
- (ii) a local equivalence of quasi-categories: i.e., if for every pair $A, B \in \mathbf{K}$, the map $\mathrm{Fun}(A, B) \xrightarrow{\sim} \mathrm{Fun}(FA, FB)$ is an equivalence of quasi-categories.

The inclusion $\mathrm{Cat} \hookrightarrow \mathrm{qCat}$ defines a cosmological functor but not a biequivalence, since it fails to be essentially surjective. Each right Quillen equivalence of

$$\begin{array}{ccc} \mathrm{CSS} & \longrightarrow & \mathrm{Segal} \\ & \searrow & \updownarrow \\ \mathrm{qCat}_{\mathfrak{h}} & \longrightarrow & \mathrm{qCat} \end{array}$$

defines a cosmological biequivalence.

As discussed in the next section, Theorem 2.8.8 together with an additional observation — that the ∞ -cosmoi of quasi-categories, Segal categories, complete Segal spaces, and naturally marked simplicial sets are biequivalent — forms the lynchpin of an approach to develop the basic theory of $(\infty, 1)$ -categories in a model-independent fashion. In fact, most of that development takes places in a strict 2-category that we now introduce.

Definition 2.8.9 (the homotopy 2-category of ∞ -cosmos). The **homotopy 2-category** of an ∞ -cosmos \mathbf{K} is a strict 2-category $\mathfrak{h}\mathbf{K}$ such that

- the objects of $\mathfrak{h}\mathbf{K}$ are the objects of \mathbf{K} , i.e., the ∞ -**categories**,
- the 1-cells $f: A \rightarrow B$ of $\mathfrak{h}\mathbf{K}$ are the vertices $f \in \mathrm{Fun}(A, B)$ in the mapping quasi-categories of \mathbf{K} , i.e., the ∞ -**functors**, and

- a 2-cell $A \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{array} B$ in $\mathfrak{h}\mathbf{K}$, which we call an ∞ -**natural transformation**, is represented by a 1-simplex $\alpha: f \rightarrow g \in \mathrm{Fun}(A, B)$, where a parallel pair of 1-simplices in $\mathrm{Fun}(A, B)$ represent the same 2-cell if and only if they bound a 2-simplex whose remaining outer face is degenerate.

Put concisely, the homotopy 2-category is the 2-category $\mathfrak{h}\mathbf{K}$ defined by applying the homotopy category functor $h: \mathrm{qCat} \rightarrow \mathrm{Cat}$ to the mapping quasi-categories of the ∞ -cosmos; the hom-categories in $\mathfrak{h}\mathbf{K}$ are defined by the formula

$$\mathrm{Hom}(A, B) := h\mathrm{Fun}(A, B)$$

to be the homotopy categories of the mapping quasi-categories in \mathbf{K} .

As we shall see in the next section, much of the theory of $(\infty, 1)$ -categories can be developed simply by considering them as objects in the homotopy 2-category using an appropriate weakening of standard 2-categorical techniques. A key to the feasibility of this approach is that the standard 2-categorical notion of equivalence, reviewed in Definition 2.9.2 below, coincides with the representably-defined notion of equivalence present in any ∞ -cosmos. The proof of this result should be compared with Quillen's Proposition 2.3.23.

Proposition 2.8.10. *An ∞ -functor $f: A \rightarrow B$ is an equivalence in the ∞ -cosmos \mathcal{K} if and only if it is an equivalence in the homotopy 2-category $\mathfrak{h}\mathcal{K}$.*

Proof. By definition, any equivalence $f: A \xrightarrow{\sim} B$ in the ∞ -cosmos induces an equivalence $\mathrm{Fun}(X, A) \xrightarrow{\sim} \mathrm{Fun}(X, B)$ of quasi-categories for any X , which becomes an equivalence of categories $\mathrm{Hom}(X, A) \xrightarrow{\sim} \mathrm{Hom}(X, B)$ upon applying the homotopy category functor $\mathfrak{h}: \mathfrak{q}\mathrm{Cat} \rightarrow \mathrm{Cat}$. Applying the Yoneda lemma in the homotopy 2-category $\mathfrak{h}\mathcal{K}$, it follows easily that f is an equivalence in the standard 2-categorical sense.

Conversely, as the map $\mathbb{I} \rightarrow \Delta^0$ of simplicial sets is a weak equivalence in the Joyal model structure, an argument similar to that used to prove Proposition 2.7.11 demonstrates that the cotensor $B^{\mathbb{I}}$ defines a path object for the ∞ -category B :

$$\begin{array}{ccc}
 & B^{\mathbb{I}} & \\
 \nearrow \sim & & \searrow (p_1, p_0) \\
 B & \xrightarrow{\Delta} & B \times B
 \end{array}$$

It follows from the two-of-three property that any ∞ -functor that is isomorphic in the homotopy 2-category to an equivalence in the ∞ -cosmos is again an equivalence in the ∞ -cosmos. Now it follows immediately from the two-of-six property for equivalences in the ∞ -cosmos, plus the fact that the class of equivalences includes the identities, that any 2-categorical equivalence is an equivalence in the ∞ -cosmos. \square

A consequence of Proposition 2.8.10 is that any cosmological biequivalence in particular defines an biequivalence of homotopy 2-categories, which explains the choice of terminology.

2.9 Model-independent $(\infty, 1)$ -category theory

We now develop a small portion of the theory of ∞ -categories in any ∞ -cosmos, thereby developing a theory of $(\infty, 1)$ -categories that applies equally to quasi-categories, Segal categories, complete Segal spaces, and naturally marked quasi-categories. The definitions of the basic $(\infty, 1)$ -categorical notions presented here might be viewed as “synthetic,” in the sense that they are blind to which model is being considered, in contrast with the “analytic” theory of quasi-categories first outlined in Joyal's [140] and later greatly expanded in his unpublished works and Lurie's [169, 168]. In §2.9.1, we introduce adjunctions and equivalences between ∞ -categories, which generalize the notions of Quillen adjunction and Quillen equivalence between model categories

from §2.4.3 and §2.4.5. Then in §2.9.2, we develop the theory of limits and colimits in an ∞ -category, which correspond to the homotopy limits and colimits of §2.5.

Our synthetic definitions specialize in the ∞ -cosmos of quasi-categories to notions that precisely recapture the Joyal–Lurie analytic theory; the proofs that this is the case are not discussed here, but can be found in [243, 240]. Considerably more development along these lines can be found in [242].

2.9.1 Adjunctions and equivalences

In any 2-category, in particular in the homotopy 2-category \mathfrak{hK} of an ∞ -cosmos, there are standard definitions of adjunction or equivalence, which allow us to define adjunctions and equivalences between ∞ -categories.

Definition 2.9.1. An **adjunction** between ∞ -categories consists of:

- a pair of ∞ -categories A and B ;
- a pair of ∞ -functors $f: B \rightarrow A$ and $u: A \rightarrow B$; and
- a pair of ∞ -natural transformations $\eta: \text{id}_B \Rightarrow uf$ and $\epsilon: fu \Rightarrow \text{id}_A$

so that the triangle equalities hold:

$$\begin{array}{ccc} \begin{array}{ccc} B & \xlongequal{\quad} & B \\ u \nearrow & \searrow f & \searrow \eta \\ A & \xlongequal{\quad} & A \end{array} & = & \begin{array}{c} B \\ u \left(\begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \right) u \\ A \end{array} \end{array} \qquad \begin{array}{ccc} B & \xlongequal{\quad} & B \\ f \searrow & \searrow \eta & \nearrow u \\ A & \xlongequal{\quad} & A \end{array} = \begin{array}{c} B \\ \left(\begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \right) f \\ A \end{array}$$

We write $f \dashv u$ to assert that the ∞ -functor $f: B \rightarrow A$ is **left adjoint** to the ∞ -functor $u: A \rightarrow B$, its **right adjoint**.

Definition 2.9.2. An **equivalence** between ∞ -categories consists of:

- a pair of ∞ -categories A and B ;
- a pair of ∞ -functors $f: B \rightarrow A$ and $g: A \rightarrow B$; and
- a pair of natural isomorphisms $\eta: \text{id}_B \cong gf$ and $\epsilon: fg \cong \text{id}_A$.

An ∞ -**natural isomorphism** is a 2-cell in the homotopy 2-category that admits a vertical inverse 2-cell.

We write $A \simeq B$ and say that A and B are **equivalent** if there exists an equivalence between A and B . The direction for the ∞ -natural isomorphisms comprising an equivalence is immaterial. Our notation is chosen to suggest the connection with adjunctions conveyed by the following exercise.

Exercise 2.9.3. In any 2-category, prove that:

- (i) Adjunctions compose: given adjoint ∞ -functors

$$\begin{array}{ccc} \begin{array}{ccc} C & \xrightarrow{f'} & B \\ \leftarrow \perp & & \leftarrow \perp \\ & \xrightarrow{u'} & \end{array} & \begin{array}{ccc} B & \xrightarrow{f} & A \\ \leftarrow \perp & & \leftarrow \perp \\ & \xrightarrow{u} & \end{array} & \rightsquigarrow & \begin{array}{ccc} C & \xrightarrow{ff'} & A \\ \leftarrow \perp & & \leftarrow \perp \\ & \xrightarrow{u'u} & \end{array} \end{array}$$

the composite ∞ -functors are adjoint.

- (ii) Any equivalence can always be promoted to an **adjoint equivalence** by modifying one of the ∞ -natural isomorphisms. That is, show that the ∞ -natural isomorphisms in an equivalence can be chosen so as to satisfy the triangle equalities. Conclude that if f and g are inverse equivalences then $f \dashv g$ and $g \dashv f$.

The point of Exercise 2.9.3 is that there are various diagrammatic 2-categorical proofs that can be taken off the shelf and applied to the homotopy 2-category of an ∞ -cosmos to prove theorems about adjunctions and equivalence between $(\infty, 1)$ -categories.

2.9.2 Limits and colimits

We now introduce definitions of limits and colimits for diagrams valued inside an ∞ -category. We begin by defining terminal objects, or as we shall call them “terminal elements,” to avoid an overproliferation of the generic name “objects.”

Definition 2.9.4. A **terminal element** in an ∞ -category A is a right adjoint $t: 1 \rightarrow A$ to the unique ∞ -functor $!: A \rightarrow 1$. Explicitly, the data consists of

- an element $t: 1 \rightarrow A$ and
- a ∞ -natural transformation $\eta: \text{id}_A \Rightarrow t!$ whose component ηt at the element t is an isomorphism.²²

Several basic facts about terminal elements can be deduced immediately from the general theory of adjunctions.

Exercise 2.9.5.

- (i) Terminal elements are preserved by right adjoints and by equivalences.
- (ii) If $A' \simeq A$ then A has a terminal element if and only if A' does.

Terminal elements are limits of empty diagrams. We now turn to limits of generic diagrams whose indexing shapes are given by 1-categories. For any ∞ -category A in an ∞ -cosmos \mathcal{K} , there is a 2-functor $A^{(-)}: \text{Cat}^{\text{op}} \rightarrow \mathfrak{h}\mathcal{K}$ defined by forming simplicial cotensors with nerves of categories. Using these simplicial cotensors, if J is a 1-category and A is an ∞ -category, the **∞ -category of J -indexed diagrams in A** is simply the cotensor A^J .²³

Remark 2.9.6. In the cartesian closed ∞ -cosmoi of Definition 2.8.7(vi), we also permit the indexing shape J to be another ∞ -category, in which case the internal hom A^J defines the **∞ -category of J -indexed diagrams in A** . The development of the theory of limits indexed by an ∞ -category in a cartesian closed ∞ -cosmos entirely parallels the development for limits indexed by 1-categories, a parallelism we highlight by conflating the notation of 2.8.7(i) and 2.8.7(vi).

²² If η is the unit of the adjunction $! \dashv t$, then the triangle equalities demand that $\eta t = \text{id}_t$. However, by a 2-categorical trick, to show that such an adjunction exists, it suffices to find a 2-cell η such that ηt is an isomorphism.

²³ More generally, this construction permits arbitrary simplicial sets as indexing shapes for diagrams in an ∞ -category A . In either case, the elements of A^J are to be regarded as homotopy coherent diagrams along the lines of Remark 2.7.10.

In analogy with Definition 2.9.4, we have:

Definition 2.9.7. An ∞ -category A **admits all limits of shape J** if the constant diagram ∞ -functor $\Delta: A \rightarrow A^J$, induced by the unique ∞ -functor $!: J \rightarrow 1$, has a right adjoint:

$$A \begin{array}{c} \xrightarrow{\Delta} \\ \perp \\ \xleftarrow{\lim} \end{array} A^J$$

From the vantage point of Definition 2.9.7, the following result is easy:

Exercise 2.9.8. Using the general theory of adjunctions, show that a right adjoint ∞ -functor $u: A \rightarrow B$ between ∞ -categories that admit all limits of shape J necessarily preserves them, in the sense that the ∞ -functors

$$\begin{array}{ccc} A^J & \xrightarrow{u^J} & B^J \\ \lim \downarrow & \cong & \downarrow \lim \\ A & \xrightarrow{u} & B \end{array}$$

commute up to isomorphism.

The problem with Definition 2.9.7 is that it is insufficiently general: many ∞ -categories will have certain, but not all, limits of diagrams of a particular indexing shape. With this in mind, we will now re-express Definition 2.9.7 in a form that permits its extension to cover this sort of situation. For this, we make use of the 2-categorical notion of an **absolute right lifting**, which is the “op”-dual (reversing the 1-cells but not the 2-cells) of the notion of absolute right Kan extension introduced in Definition 2.4.1.

Exercise 2.9.9. Show that in any 2-category, a 2-cell $\epsilon: fu \Rightarrow \text{id}_A$ defines the counit of an adjunction $f \dashv u$ if and only if

$$\begin{array}{ccc} & & B \\ & \nearrow u & \downarrow f \\ A & \xlongequal{\quad} & A \end{array} \quad \Downarrow \epsilon$$

defines an absolute right lifting diagram.

Applying Exercise 2.9.9, Definition 2.9.7 is equivalent to the assertion that the **limit cone**, our term for the counit of $\Delta \dashv \lim$, defines an absolute right lifting diagram:

$$\begin{array}{ccc} & & A \\ & \nearrow \lim & \downarrow \Delta \\ A^J & \xlongequal{\quad} & A^J \end{array} \quad \Downarrow \epsilon \tag{2.15}$$

Recall that the appellation “absolute” means “preserved by all functors,” in this case by restriction along any ∞ -functor $X \rightarrow A^J$. In particular, an absolute right lifting diagram (2.15) restricts to define an absolute right lifting diagram on any subobject of the ∞ -category of diagrams. This motivates the following definition.

Definition 2.9.10 (limit). A **limit** of a J -indexed diagram in A is an absolute right lifting of the diagram d through the constant diagram ∞ -functor $\Delta: A \rightarrow A^J$

$$\begin{array}{ccc}
 & A & \\
 \lim d \nearrow & \Downarrow \lambda & \downarrow \Delta \\
 1 & \xrightarrow{d} & A^J
 \end{array} \tag{2.16}$$

the 2-cell component of which defines the **limit cone** $\lambda: \Delta \lim d \Rightarrow d$.

If A has all J -indexed limits, the restriction of the absolute right lifting diagram (2.15) along the element $d: 1 \rightarrow A^J$ defines a limit for d . Interpolating between Definitions 2.9.10 and 2.9.7, we can define a **limit of a family of diagrams** to be an absolute right lifting of the family $d: K \rightarrow A^J$ through $\Delta: A \rightarrow A^J$. For instance:

Theorem 2.9.11 ([243, 5.3.1]). *For every cosimplicial object in an ∞ -category that admits a coaugmentation and a splitting, the coaugmentation defines its limit. That is, for every ∞ -category A , the ∞ -functors*

$$\begin{array}{ccc}
 & A & \\
 \text{ev}_{[-1]} \nearrow & \Downarrow \lambda & \downarrow \Delta \\
 A^{\Delta_{\perp}} & \xrightarrow{\text{res}} & A^{\Delta}
 \end{array}$$

define an absolute right lifting diagram.

Here Δ is the usual simplex category of finite non-empty ordinals and order-preserving maps. It defines a full subcategory of Δ_+ , which freely appends an initial object $[-1]$, and this in turn defines a subcategory of Δ_{\perp} , which adds an “extra degeneracy” map between each pair of consecutive ordinals. Diagrams indexed by $\Delta \subset \Delta_+ \subset \Delta_{\perp}$ are, respectively, called **cosimplicial objects**, **coaugmented cosimplicial objects**, and **split cosimplicial objects**. The limit of a cosimplicial object is often called its **totalization**.

Proof sketch. In Cat , there is a canonical 2-cell

$$\begin{array}{ccc}
 \Delta & \hookrightarrow & \Delta_{\perp} \\
 \downarrow ! & \nearrow \uparrow \lambda & \\
 \mathbb{1} & & [-1]
 \end{array}$$

because $[-1] \in \Delta_{\perp}$ is initial. This data defines an absolute right extension diagram that is moreover preserved by any 2-functor, because the universal property of the functor $[-1]: \mathbb{1} \rightarrow \Delta_{\perp}$ and the 2-cell λ is witnessed by a pair of adjunctions. The 2-functor $A^{(-)}: \text{Cat}^{\text{op}} \rightarrow \mathfrak{hK}$ converts this into the absolute right lifting diagram of the statement. \square

The most important result relating adjunctions and limits is of course this:

Theorem 2.9.12 ([243, 5.2.13]). *Right adjoints preserve limits.*

Our proof will closely follow the classical one. Given a diagram $d: 1 \rightarrow A^J$ and a right adjoint $u: A \rightarrow B$ to some ∞ -functor f , a cone with summit $b: 1 \rightarrow B$ over $u^J d$ transposes to define a cone with summit fb over d , which factors uniquely through the limit cone. This factorization transposes back across the adjunction to show that u carries the limit cone over d to a limit cone over $u^J d$.

Proof. Suppose that A admits limits of a diagram $d: 1 \rightarrow A^J$ as witnessed by an absolute right lifting diagram (2.16). Since adjunctions are preserved by all 2-functors, an adjunction $f \dashv u$ induces an adjunction $f^J \dashv u^J$. We must show that

$$\begin{array}{ccc} & A & \xrightarrow{u} B \\ \lim d \nearrow & \downarrow \Delta & \downarrow \Delta \\ 1 & \xrightarrow{d} A^J & \xrightarrow{u^J} B^J \end{array}$$

is again an absolute right lifting diagram. Given a square

$$\begin{array}{ccc} X & \xrightarrow{b} & B \\ \downarrow ! & \Downarrow \chi & \downarrow \Delta \\ 1 & \xrightarrow{d} A^J & \xrightarrow{u^J} B^J \end{array}$$

we first “transpose across the adjunction,” by composing with f and the counit.

$$\begin{array}{ccc} X & \xrightarrow{b} & B & \xrightarrow{f} & A \\ \downarrow ! & \Downarrow \chi & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta \\ 1 & \xrightarrow{d} & A^J & \xrightarrow{u^J} & B^J & \xrightarrow{f^J} & A^J \end{array} = \begin{array}{ccc} X & \xrightarrow{b} & B & \xrightarrow{f} & A \\ \downarrow ! & \Downarrow \exists! \zeta & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta \\ 1 & \xrightarrow{d} & A^J & \xrightarrow{u^J} & B^J & \xrightarrow{f^J} & A^J \end{array}$$

The universal property of the absolute right lifting diagram $\lambda: \Delta \lim \Rightarrow d$ induces a unique factorization ζ , which may then be “transposed back across the adjunction” by composing with u and the unit.

$$\begin{array}{ccc} X & \xrightarrow{b} & B & \xrightarrow{f} & A & \xrightarrow{u} & B \\ \downarrow ! & \Downarrow \exists! \zeta & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta \\ 1 & \xrightarrow{d} & A^J & \xrightarrow{u^J} & B^J & \xrightarrow{f^J} & A^J \end{array} = \begin{array}{ccc} X & \xrightarrow{b} & B & \xrightarrow{f} & A & \xrightarrow{u} & B \\ \downarrow ! & \Downarrow \chi & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta \\ 1 & \xrightarrow{d} & A^J & \xrightarrow{u^J} & B^J & \xrightarrow{f^J} & A^J \end{array}$$

$$= \begin{array}{ccc} X & \xrightarrow{b} & B & \xrightarrow{f} & A & \xrightarrow{u} & B \\ \downarrow ! & \Downarrow \chi & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta \\ 1 & \xrightarrow{d} & A^J & \xrightarrow{u^J} & B^J & \xrightarrow{f^J} & A^J \end{array} = \begin{array}{ccc} X & \xrightarrow{b} & B & \xrightarrow{f} & A & \xrightarrow{u} & B \\ \downarrow ! & \Downarrow \chi & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta \\ 1 & \xrightarrow{d} & A^J & \xrightarrow{u^J} & B^J & \xrightarrow{f^J} & A^J \end{array}$$

Here the second equality is a consequence of the 2-functoriality of the simplicial

cotensor, while the third is an application of a triangle equality for the adjunction $f^J \dashv u^J$. The pasted composite of ζ and η is the desired factorization of χ through λ .

The proof that this factorization is unique, which again parallels the classical argument, is left to the reader: the essential point is that the transposes defined via these pasting diagrams are unique. \square

Colimits are defined “co”-dually, by reversing the direction of the 2-cells but not the 1-cells. There is no need to repeat the proofs however: any ∞ -cosmos K has a co-dual ∞ -cosmos K^{co} with the same objects but in which the mapping quasi-categories are defined to be the opposites of the mapping quasi-categories in K .

2.10 Epilogue

A category K equipped with a class of “weak equivalences” \mathcal{W} — perhaps saturated in the sense of containing all of the maps inverted by the Gabriel–Zisman localization functor or perhaps merely generating the class of maps to be inverted in the category of fractions — defines a “homotopy theory,” a phrase generally used to refer to the associated homotopy category together with the homotopy types of the mapping spaces, as captured for instance by the Dwyer–Kan hammock localization. We have studied two common axiomatizations of this abstract notion: Quillen’s model categories, which present homotopy theories with all homotopy limits and homotopy colimits, and $(\infty, 1)$ -categories, which might be encoded using one of the models introduced in §2.8 or worked with model-independently in the sense outlined in §2.9.

From the point of view of comparing homotopy categories, the model-independent theory of $(\infty, 1)$ -categories has some clear advantages: equivalences between homotopy theories are directly definable (see Definition 2.9.2) instead of being presented as zigzags of DK- or Quillen equivalences. The formation of diagram categories (see Remark 2.9.6) is straightforward and homotopy limit and colimit functors become genuine adjoints (see Definition 2.9.7) and homotopy limits and colimits become genuine limits and colimits — at least in the sense appropriate to the theory of $(\infty, 1)$ -categories. So from this vantage point it is natural to ask: Do we still need model categories?²⁴ While some might find this sort of dialog depressing, in our view it does not hurt to ask.

Chris Schommer-Pries has suggested a useful analogy to contextualize the role played by model categories in the study of homotopy theories that are complete and cocomplete:

$$\left\{ \begin{array}{l} \text{model category} :: (\infty, 1)\text{-category} \\ \text{basis} :: \text{vector space} \\ \text{local coordinates} :: \text{manifold} \end{array} \right.$$

A precise statement is that combinatorial model categories present those $(\infty, 1)$ -categories that are complete and cocomplete and more generally (locally) presentable; this

²⁴ See <https://mathoverflow.net/questions/78400/do-we-still-need-model-categories>.

result is proven in [169, A.3.7.6] by applying a theorem of Dugger [77].²⁵ In general having coordinates is helpful for calculations. In particular, when working inside a particular homotopy theory as presented by a model category, you also have access to the non-bifibrant objects. For instance, the Bergner model structure of §2.6.2 is a useful context to collect results about homotopy coherent diagrams, which are defined to be maps from the cofibrant (and not typically fibrant) objects to the fibrant ones (which are not typically cofibrant).

But Quillen himself was somewhat unsatisfied with the paradigm-shifting abstract framework that he introduced, writing:

This definition of the homotopy theory associated to a model category is obviously unsatisfactory. In effect, the loop and suspension functors are a kind of primary structure on HoM and the families of fibration and cofibration sequences are a kind of secondary structure since they determine the Toda bracket Presumably there is higher order structure . . . on the homotopy category which forms part of the homotopy theory of a model category, but we have not been able to find an inclusive general definition of this structure with the property that this structure is preserved when there are adjoint functors which establish an equivalence of homotopy theories. [229, pp. 3–4]

Quillen was referring to a model category that is pointed, in the sense of having a zero object, like the role played by the singleton space in Top_* . A more modern context for the sort of stable homotopy theory that Quillen is implicitly describing is the category of spectra, the $(\infty, 1)$ -category of which has many pleasant properties collected together in the notion of a *stable ∞ -category*. We posit that these notions, which are the subject of Chapter 4 of this volume, might fulfill Quillen’s dream.

²⁵ Morally, in the sense discussed in §2.3.2, all model categories are Quillen equivalent to locally presentable ones. More precisely, the result that every cofibrantly generated (in a suitable sense of this term) model category is Quillen equivalent to a combinatorial one has been proven by Raptis and Rosicky to be equivalent to a large cardinal axiom called Vopěnka’s principle [254].