1 Introduction

by Andrew J. Blumberg, Teena Gerhardt, Michael A. Hill

1.1 Goals of this book

The modern era in homotopy theory began in the 1960s with the profound realization, first codified by Boardman in his construction of *the stable category*, that the category of spaces up to stable homotopy equivalence is equipped with a rich algebraic structure, formally similar to the derived category of a commutative ring R. For example, for pointed spaces the natural map from the categorical coproduct to the categorical product becomes more and more connected as the pieces themselves become more and more connected. In the limit, this map becomes a stable equivalence, just as finitely indexed direct sums and direct products coincide for R-modules.

From this perspective, the objects of the stable category are modules over an initial commutative ring object that replaces the integers: the sphere spectrum. However, technical difficulties immediately arose. Whereas the tensor product of R-modules is an easy and familiar construction, the analogous construction of a symmetric monoidal smash product on spectra seemed to involve a huge number of ad hoc choices [1]. As a consequence, the smash product was associative and commutative only up to homotopy. The lack of a good point-set symmetric monoidal product on spectra precluded making full use of the constructions from commutative algebra in this setting — even just defining good categories of modules over a commutative ring spectrum was difficult. In many ways, finding ways to rectify this and to make the guiding metaphor provided by "modules over the sphere spectrum" precise has shaped the last 60 years of homotopy theory.

This book arose from a desire by the editors to have a reference to give to their students who have taken a standard algebraic topology sequence and who want to learn about spectra and structured ring spectra. While there are many excellent texts which introduce students to the basic ideas of homotopy theory and to spectra, there has not been a place for students to engage directly with the ideas needed to connect with commutative ring spectra and work with these objects. This book strives to provide an introduction to this whole circle of ideas, describing the tools that homotopy theorists have developed to build, explore, and use symmetric monoidal categories of spectra that refine the stable homotopy category:

1. model category structures on symmetric monoidal categories of spectra,
2. stable ∞-categories, and
3. operads and operadic algebras.

These three concepts are closely intertwined, and they all engage deeply with a fundamental principle: if the choices for some construction or map are parameterized by a space, then recording that space as part of the data makes the construction more natural.

To make this maxim precise in practice, we must keep track of the spaces of maps between objects in our categories, not just sets of maps, describing the homotopies by which two equivalent maps are seen to be equivalent. A first example of this is given by the cup product on ordinary cohomology. Students in a first algebraic topology class learn that while the cochains on a space with coefficients in a commutative ring are not a commutative ring, the cohomology of a space is canonically a graded commutative ring. Steenrod observed that over $\mathbb{F}_p$, we can keep track of cochains that enforce the symmetry between $a \smile b$ and $b \smile a$, and out of these, we can build a hierarchy of cochains and, when $a = b$, cocycles in increasingly high degree: the Steenrod reduced powers [283]. May recast this via operads: mathematical objects which exactly record spaces parametrizing particular kinds of multiplications [198, 194].

Again returning to our maxim, we want to be sure that our constructions, including of the mapping spaces, are homotopically meaningful in the sense that the resulting homotopy type of any output depends only on the homotopy types of the inputs. Model categories provide one way to ensure this, giving us not only checkable conditions to facilitate computation but also a language and explanation for fundamental constructions in homological algebra like resolutions and derived functors. More recently, ∞-categories have given another way to ensure homotopically meaningful information by recording this data from the very beginning.

Homotopy theory is at an inflection point, with much of the older literature written in the language of model categories and with newer results and machinery expressed using ∞-categories. Both approaches have distinct benefits, and we provide an introduction to both: our aim is to give people learning about stable categories and structured ring spectra a way to connect with both “neoclassical” tools and newer ones.

The book closes with applications of the tools so developed, showing how the machinery of ∞-categories allows us to fully realize Boardman’s observation and “do algebraic geometry” with commutative ring spectra. Transformative work of Goerss–Hopkins–Miller in the last 1990s ushered in the era of spectral algebraic geometry, showing first that the Lubin–Tate deformation theory of formal groups naturally lifts to a diagram of commutative ring spectra and then that the structure sheaf of the moduli stack of formal groups has an essentially unique lift to a sheaf of commutative ring spectra [106, 126]. This produced a host of new cohomology theories which are naturally tied to universal constructions in algebraic geometry and moduli problems. Additionally, it refined classical invariants of rings like modular forms to invariants of ring spectra: topological modular forms. Lurie has created a vast generalization of this, showing how one can lift algebraic geometry whole-cloth to commutative ring spectra, creating spectral algebraic geometry. The final chapter of this book provides an introduction to this new area.

1.2 Summaries of the chapters

Chapter 2 (Riehl)

The chapter begins by framing a foundational question: What do we mean by the homotopy category of a category and by derived functors? It proceeds through a historical arc: describing first categories of fractions, then moving on to Quillen's theory of model categories and their simplicial enrichments, and finally describing the newer, $(\infty, 1)$-categories. The goal here is to introduce the reader to the basic tools that will be used, fitting them into a broader narrative, demonstrating how they can be used, and connecting everything clearly to the literature for further study.

Chapter 3 (Dugger)

This chapter gives a comprehensive overview of the modern symmetric monoidal categories of spectra that were invented in the 90s: symmetric spectra, orthogonal spectra, and EKMM spectra. The technical foundations are carried out in the setting of model categories, and there is an emphasis on concrete formulas for the smash product and related constructions. The goal is for the reader to become comfortable with working in these categories of spectra.

Chapter 4 (Barwick)

This chapter returns to the construction of the category of spectra and explains the approach to spectra and stable categories more generally in the framework of $(\infty, 1)$-categories. We hope that comparing and contrasting the treatment in this chapter and the preceding one will give a flavor of the similarities and differences between the two technical approaches for abstract homotopy theory. Of necessity, many details about the underlying foundations are left to the references, but enough detail is provided to indicate how the theory works.

Chapter 5 (Mandell)

This chapter is a thorough treatment of the theory of operadic algebras in modern homotopy theory. It gives a streamlined view of the foundations, collecting in one place results that are scattered throughout the literature, with a unifying viewpoint on techniques for understanding the homotopy theory of operadic algebras and modules.

Chapter 6 (Richter)

This chapter gives a broad sampling of applications of commutative ring spectra in modern stable homotopy theory. Beginning with a treatment of the foundations, it then surveys applications in topological Hochschild homology, obstruction theory and topological André–Quillen homology, and the Picard and Brauer groups.

Chapter 7 (Lawson)

This chapter gives a detailed introduction to the theory of Bousfield localization, starting from the basic constructions and studying the multiplicative properties of the localization in the context of structured ring spectra. Bousfield localization is one of the most important techniques in the modern arsenal, and the goal of this chapter is to prepare the reader to understand how to use it.

Chapter 8 (Rezk)

This chapter draws on all of the earlier sections, showing how the machinery developed allows us to "do algebraic geometry" in a very general context. The chapter begins discussing ∞-topoi and sheaves on them, providing along the way useful tools and ways to reinterpret results to show how these constructions can be used. It then moves into more algebraic geometry notions, exploring how classical notions like étale morphism, affine and projective spaces, and stacks lift to commutative ring spectra. This culminates in a treatment of Lurie's refinement of the Goerss–Hopkins–Miller theorem that the structure sheaf of the moduli stack of elliptic curves lifts to commutative ring spectra.

1.3 Acknowledgements

We would like to thank first and foremost the authors of the chapters for their incredible work. Hélène Barcelo at MSRI and our editors at Cambridge University Press have been both enthusiastic about this project and patient. We would particularly like to thank Series Editor Silvio Levy for his extensive work on this volume. We were supported throughout the course of this work by the NSF: DMS-1812064, DMS-1810575, and DMS-1811189. This book is an outgrowth of the transformative MSRI semester on algebraic topology that took place in 2014, and we are grateful to everyone who participated in that program and honored to be able to continue the conversation.

We would also like to thank our advisors and mentors, Gunnar Carlsson, Ralph Cohen, Lars Hesselholt, Mike Hopkins, Mike Mandell, Peter May, and Haynes Miller, who taught us much of what we understand about this material and so many other things. We would also like to thank Doug Ravenel for encouraging comments and Christian Carrick, Robert Housden, Jason Schuchardt, and Andrew Smith for their comments on some of the chapters.