

# Reduction numbers and balanced ideals

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Let  $R$  be a Noetherian local ring and let  $I$  be an ideal in  $R$ . The ideal  $I$  is called balanced if the colon ideal  $J : I$  is independent of the choice of the minimal reduction  $J$  of  $I$ . Under suitable assumptions, Ulrich showed that  $I$  is balanced if and only if the reduction number,  $r(I)$ , of  $I$  is at most the “expected” one, namely  $\ell(I) - \text{ht } I + 1$ , where  $\ell(I)$  is the analytic spread of  $I$ . In this article we propose a generalization of balanced. We prove under suitable assumptions that if either  $R$  is one-dimensional or the associated graded ring of  $I$  is Cohen–Macaulay, then  $J^{n+1} : I^n$  is independent of the choice of the minimal reduction  $J$  of  $I$  if and only if  $r(I) \leq \ell(I) - \text{ht } I + n$ .

## 1. Introduction

Let  $R$  be a Noetherian ring and let  $I$  be an ideal in  $R$ . The Rees algebra  $\mathcal{R}(I)$  and the associated graded ring  $\text{gr}_I(R)$  of  $I$  are

$$\mathcal{R}(I) = R[It] = \bigoplus_{i \geq 0} I^i t^i \quad \text{and} \quad \text{gr}_I(R) = R[It]/IR[It] = \bigoplus_{i \geq 0} I^i / I^{i+1}.$$

The projective spectrums of  $\mathcal{R}(I)$  and  $\text{gr}_I(R)$  are the blowup of  $\text{Spec}(R)$  along  $V(I)$  and the normal cone of  $I$ , respectively. When studying various algebraic properties of these blowups a natural question to consider is which properties of the ring  $R$  are transferred to these graded algebras. When  $R$  is a local Cohen–Macaulay ring and  $I$  an ideal of positive height then if  $\mathcal{R}(I)$  is Cohen–Macaulay then so is  $\text{gr}_I(R)$  [Huneke 1982]. The converse does not hold true in general. A celebrated theorem of Goto and Shimoda illustrates the intricate relationship between the Cohen–Macaulay property of these blowup algebras and the reduction number of  $I$ . It states that when  $(R, \mathfrak{m})$  is a local Cohen–Macaulay ring, with infinite residue field, dimension  $d > 0$ , and  $I$  an  $\mathfrak{m}$ -primary ideal, then  $\mathcal{R}(I)$  is Cohen–Macaulay if and only if  $\text{gr}_I(R)$  is Cohen–Macaulay and the reduction number of  $I$  is at most  $d - 1$  [Goto and Shimoda 1982]. This theorem has inspired the work of many researchers and many generalizations of

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it appeared in the literature in the late 1980s and early 1990s; see, for example, [Grothe et al. 1984; Huckaba and Huneke 1992; 1993; Goto and Huckaba 1994; Johnston and Katz 1995; Aberbach et al. 1995; Simis et al. 1995].

Recall that an ideal  $J$  is a *reduction* of  $I$  if  $J \subset I$  and  $\mathcal{R}(I)$  is integral over  $\mathcal{R}(J)$  or equivalently if  $J \subset I$  and  $I^{n+1} = JI^n$  for some nonnegative integer  $n$ , see also Section 2. The smallest nonnegative integer such that the equality  $I^{n+1} = JI^n$  holds is called the *reduction number of  $I$  with respect to  $J$*  and is denoted by  $r_J(I)$ . When the ring is local then we consider *minimal* reductions, where minimality is taken with respect to inclusion. In this case the *reduction number of  $I$* , denoted by  $r(I)$ , is the minimum among all  $r_J(I)$ , where  $J$  ranges over all minimal reductions of  $I$ .

We say that  $I$  is *balanced* if the colon ideal  $J : I$  is independent of the minimal reduction  $J$  of  $I$  [Ulrich 1996, Theorem 4.8]. More precisely the definition of balanced is given below.

**Definition 1.1** [Ulrich 1996, Definition 3.1]. Let  $R$  be a Noetherian local ring, let  $I$  be an ideal, and let  $s$  be a positive integer. For a generating sequence  $f_1, \dots, f_n$  of  $I$ , let  $X$  be an  $n \times n$  matrix of indeterminates, and write

$$[a_1, \dots, a_n] = [f_1, \dots, f_n] \cdot X \quad \text{and} \quad S = R(X).$$

We say that  $I$  is  *$s$ -balanced* if there exist  $n \geq s$  and  $f_1, \dots, f_n$  as above such that  $(a_{i_1}, \dots, a_{i_s})S : IS$  yields the same  $S$ -ideal for every subset

$$\{i_1, \dots, i_s\} \subset \{1, \dots, n\}.$$

An ideal  $I$  satisfies the condition  $G_s$  for some integer  $s$  if  $\mu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}}$  for every  $\mathfrak{p} \in V(I)$  with  $\dim R_{\mathfrak{p}} \leq s - 1$ . The condition  $G_s$  is local and rather mild. For example when  $R$  is a Noetherian local ring with maximal ideal  $\mathfrak{m}$  and dimension  $d$ , then any  $\mathfrak{m}$ -primary ideal satisfies  $G_d$ . We say that an ideal  $I$  satisfies  $G_{\infty}$  if  $I$  satisfies  $G_s$  for every  $s$ .

Let  $R$  be a local Gorenstein ring with infinite residue field and let  $I$  be an ideal with  $g = \text{ht } I > 0$ . Suppose that  $I$  satisfies  $G_{\ell}$  and that

$$\text{depth } R/I^j \geq \dim R/I - j + 1,$$

for all  $1 \leq j \leq \ell - g + 1$ , where  $\ell = \ell(I)$  is the analytic spread of  $I$ . In general, there are many classes of ideals that satisfy both the depth condition and  $G_{\ell}$ , for example ideals in the linkage class of a complete intersection satisfy these conditions; see [Corso et al. 2002] for more information.

A result of Johnson and Ulrich states that under the above conditions if  $r(I) \leq \ell - g + 1$  then  $\text{gr}_J(R)$  is Cohen–Macaulay. If in addition the height of  $I$  is at least 2 this also forces  $\mathcal{R}(I)$  to be Cohen–Macaulay [Johnson and Ulrich

1996]. Moreover, the Castelnuovo–Mumford regularity of  $\mathcal{R}(I)$  and  $\text{gr}_I(R)$  can be calculated if  $r(I) \leq \ell - g + 1$ . The number  $\ell(I) - \text{ht } I + 1$  is known as the *expected reduction number* of  $I$ . This number was introduced by Ulrich in [Ulrich 1996], where he shows that under these assumptions an ideal  $I$  has reduction number at most the expected one if and only if the ideal is balanced [Ulrich 1996, Theorem 4.8].

We usually say that  $I$  is balanced if  $I$  is  $\ell(I)$ -balanced, where  $\ell(I)$  is the analytic spread of  $I$ . It turns out that ideals that have the expected reduction number have many good properties. It is then natural to ask what can be a reasonable bound for the reduction number if the ideal is not balanced. The purpose of this article is to suggest a generalization of the notion of balanced and to establish bounds on the reduction number of an ideal in that case. We propose the condition

$$J^{n+1} : I^n \text{ is independent of the minimal reduction } J$$

as a possible generalization of balanced.

We show that when the dimension of the ring  $R$  is one then  $J^{n+1} : I^n$  is independent of  $J$  if and only if  $n \geq r(I)$ , Theorem 3.2. In the case of higher dimensions, we are able to show that the independence of the colon ideal  $J^{n+1} : I^n$  from the choice of the minimal reduction  $J$  of  $I$  is equivalent to

$$r(I) \leq \ell(I) - \text{ht } I + n,$$

where  $\ell(I)$  is the analytic spread of  $I$ , provided that  $\text{gr}_I(R)$  is Cohen–Macaulay, Theorem 3.7.

Next we discuss an application of the characterization of balanced ideals as in [Ulrich 1996, Theorem 4.8]. Corso, Polini, and Ulrich [Corso et al. 2002] make use of the notion of balanced in order to establish a formula for the core of  $I$ . We recall here that  $\text{core}(I)$  is the intersection of all the reductions of  $I$ , see Section 2 for more details. Their theorem states that under the same assumptions as before one has that  $\text{core}(I) = J(J : I) = J^2 : I$  for all minimal reductions  $J$  of  $I$  if and only if  $r(I) \leq \ell - g + 1$  [Corso et al. 2002, Theorem 2.6]. Therefore in this case the ideal  $I$  is balanced if and only if  $\text{core}(I) = J^2 : I$  for all minimal reductions  $J$  of  $I$ . Most notably, we see how the balanced condition,  $J : I$  being independent of  $J$ , is intertwined with the formula for the core.

**Theorem 1.2** [Polini and Ulrich 2005, Theorem 4.5]. *Let  $R$  be a local Gorenstein ring with infinite residue field  $k$ . Let  $I$  be an ideal with  $g = \text{ht } I > 0$  and suppose that  $I$  satisfies  $G_\ell$  and that  $\text{depth } R/I^j \geq \dim R/I - j + 1$  for all  $1 \leq j \leq \ell - g$ , where  $\ell = \ell(I)$  is the analytic spread of  $I$ . Let  $J$  be a minimal reduction of  $I$ . If either  $\text{char } k = 0$ , or  $\text{char } k > r_J(I) - \ell + g$ , then  $\text{core}(I) = J^{n+1} : I^n$  for all  $n \geq \max\{r_J(I) - \ell + g, 0\}$ .*

As one can see in Theorem 1.2 the characteristic of the residue field plays an important role when computing the core of an ideal. When appropriate we will be assuming that the characteristic of the residue field is 0. In particular, under the set up of Theorem 1.2 the ideal  $J^{n+1} : I^n$  is independent of the minimal reduction  $J$  of  $I$ , since the formula for the core is independent of the choice of minimal reduction  $J$  of  $I$ . Therefore, if  $n \geq \max\{r_J(I) - \ell + g, 0\}$ , then  $J^{n+1} : I^n$  is independent of the minimal reduction  $J$  of  $I$ . Then it is natural to ask under which assumptions the converse holds true. This question is answered in part in Theorems 3.2, 3.7. Finally, our results state that when  $n = 1$  then  $I$  is balanced if and only if  $r(I) \leq \ell(I) - g + 1$  and therefore we recover [Ulrich 1996, Theorem 4.8].

## 2. Background

Let  $R$  be a Noetherian ring and  $I$  an ideal in  $R$ . Recall that an ideal  $J$  is a reduction of  $I$  if  $J \subset I$  and  $\mathcal{R}(I)$  is integral over  $\mathcal{R}(J)$  or equivalently if  $J \subset I$  and  $I^{n+1} = JI^n$  for some nonnegative integer  $n$ . When the ring is local then we consider minimal reductions, where minimality is taken with respect to inclusion. Northcott and Rees proved that if  $R$  is a Noetherian local ring with maximal ideal  $\mathfrak{m}$  and infinite residue field then minimal reductions exist and either there are infinitely many or the ideal is basic, that is, it is the only reduction of itself [Northcott and Rees 1954]. They show that minimal reductions correspond to Noether normalizations of the *special fiber ring*,  $\mathcal{F}(I) = \mathcal{R}(I) \otimes R/\mathfrak{m}$ , of  $I$ .

The concept of a reduction of an ideal was first introduced by Northcott and Rees [1954] in order to facilitate the study of ideals and their powers. Reductions are in general smaller ideals with the same asymptotic behavior as the ideal  $I$  itself. For example, all minimal reductions of  $I$  have the same height and the same radical as  $I$ . Moreover, every minimal reduction  $J$  of  $I$  has the same minimal number of generators  $\ell(I)$ , where  $\ell(I)$  is the *analytic spread* of  $I$  and is defined to be the Krull dimension of the special fiber ring  $\mathcal{F}(I)$  of  $I$ .

Let  $J$  be a minimal reduction of an ideal  $I$  in a Noetherian local ring. The *reduction number of  $I$  with respect to  $J$* , denoted by  $r_J(I)$ , is the smallest  $n$  for which the equality  $I^{n+1} = JI^n$  holds. In some sense the reduction number  $r_J(I)$  measures how closely related  $J$  and  $I$  are. The *reduction number  $r(I)$*  of  $I$  is the minimum of the reduction numbers  $r_J(I)$ , where  $J$  ranges over all minimal reductions of  $I$ .

In general, since an ideal has infinitely many reductions it is natural to consider the *core* of the ideal, namely the intersection of all the (minimal) reductions of the ideal [Rees and Sally 1988]. Several authors have determined formulas that describe the core in various settings; see, for example, [Huneke and Swanson

1995; Corso et al. 2001; 2002; Hyry and Smith 2003; 2004; Huneke and Trung 2005; Polini and Ulrich 2005; 2007; Fouli et al. 2008; 2010; Fouli and Morey 2012; Smith 2011].

The core has many connections to geometry. For instance, Hyry and Smith have discovered a connection with a conjecture of Kawamata on the nonvanishing of global sections of line bundles [Hyry and Smith 2004]. They prove that the validity of the conjecture is equivalent to a statement about core.

In a recent paper with Polini and Ulrich we have uncovered yet another such connection with geometry. A scheme  $X = \{P_1, \dots, P_s\}$  of  $s$  reduced points in  $\mathbb{P}_k^n$  is said to have the *Cayley–Bacharach* property if each subscheme of the form  $X \setminus \{P_i\} \subset \mathbb{P}_k^n$  has the same Hilbert function. It turns out that the structure of the core completely characterizes this property, namely  $X$  has the Cayley–Bacharach property if and only if  $\text{core}(\mathfrak{m}) = \mathfrak{m}^{a+2}$ , where  $\mathfrak{m}$  is the homogeneous maximal ideal of the homogeneous coordinate ring  $R$  of  $X$  and  $a$  is the  $a$ -invariant of  $R$  [Fouli et al. 2010].

We now discuss the notion of ideals of linear type. Let  $R$  be a Noetherian ring and  $I$  an ideal generated by  $f_1, \dots, f_n$ . Then there is an epimorphism  $\phi : S = R[T_1, \dots, T_n] \rightarrow \mathcal{R}(I)$  given by  $\phi(T_i) = f_i t$ . Let  $J = \ker \phi$  and notice that  $J$  is a graded ideal. Let  $J = \bigoplus_{i=1}^{\infty} J_i$ . Then  $\mathcal{R}(I) \simeq S/J$  and the ideal  $J$  is often referred to as the defining ideal of  $\mathcal{R}(I)$ . When  $J = J_1$  then  $I$  is called an ideal of *linear type*. It turns out that when  $I$  is an ideal of linear type then  $I$  is basic. The converse is not true in general.

The following is a well known result and we include it here for ease of reference.

**Lemma 2.1.** *Let  $R$  be a local Gorenstein ring and  $I$  an ideal with  $g = \text{ht } I > 0$ ,  $\ell = \ell(I)$ , and let  $J$  be a minimal reduction of  $I$ . Assume that  $I$  satisfies  $G_\ell$  and  $\text{depth } R/I^j \geq \dim R/I - j + 1$  for  $1 \leq j \leq \ell - g$ . Then for every integer  $n \geq 0$  and every integer  $i \geq 0$ ,*

$$J^{n+i} : J^n = J^i.$$

*Proof.* First we note that  $\text{ht } J = \text{ht } I > 0$ . Then  $I$  satisfies  $\text{AN}_{\ell-1}$ , by Theorem 2.9 of [Ulrich 1994]. Here  $\text{AN}$  stands for the Artin–Nagata property as in that reference. Using  $s = \ell - 1$  in Ulrich’s Theorem 1.11 we obtain  $\text{ht } J : I \geq \ell$  and hence  $J$  satisfies  $G_\infty$ . Therefore,  $J$  satisfies  $\text{AN}_{\ell-1}$ , by Remark 1.12 of [Ulrich 1994]. Using Theorem 1.8 of the same article we also obtain that  $J$  satisfies sliding depth. Therefore  $\text{gr}_J(R)$  is Cohen–Macaulay by [Herzog et al. 1983, Theorem 9.1]. Then the cancellation is clear, because  $\text{grade}(\text{gr}_J(R)_+) > 0$ , since  $\text{gr}_J(R)$  is Cohen–Macaulay and  $\text{ht } J > 0$ .  $\square$

We conclude this section with the following remark.

**Remark 2.2.** Let  $R$  be a local Gorenstein ring and  $I$  an ideal with  $g = \text{ht } I > 0$ ,  $\ell = \ell(I)$ , and let  $J$  be a minimal reduction of  $I$ . Assume that  $I$  satisfies  $G_\ell$  and  $\text{depth } R/I^j \geq \dim R/I - j + 1$  for  $1 \leq j \leq \ell - g$ . Then  $\{J^{i+1} : I^i\}_{i \in \mathbb{N}}$  is a decreasing sequence of ideals. To see this observe that for all  $i \geq 0$

$$J^{i+1} : I^i = (J^{i+2} : J) : I^i = J^{i+2} : JI^i \supset J^{i+2} : I^{i+1},$$

where the first equality holds according to Lemma 2.1.

### 3. Main results

We begin our investigation by considering the one-dimensional case. The first Lemma is analogous to [Ulrich 1996, Lemma 4.7].

**Lemma 3.1.** *Let  $R$  be an one-dimensional local Cohen–Macaulay ring with canonical module  $\omega_R$  and let  $I$  be an ideal with  $\text{ht } I > 0$ . Assume that  $I^i I^{-n} = a^i I^{-n}$  for some  $a \in I$  and for some positive integers  $i$  and  $n$ , and that  $I^{r-1} \cong \omega_R$  for some positive integer  $r$ . Then  $I^{r+n} = aI^{r+n-1}$ .*

*Proof.* First note that  $a$  is a non-zero-divisor in  $R$ . Furthermore, we may assume  $I^{r-1} = \omega_R$ . Since  $I^i I^{-n} \supseteq a^{i-1} I I^{-n} \supseteq a^i I^{-n}$  and  $I^i I^{-n} = a^i I^{-n}$  we have that  $a^{i-1} I I^{-n} = a^i I^{-n}$ . Hence  $I I^{-n} = a I^{-n}$  since  $a$  is a non-zero-divisor. Then for all  $j > 0$  it follows that  $I^j I^{-n} = a I^{j-1} I^{-n} = \dots = a^j I^{-n}$ . For  $j = r + n$  we obtain  $a^{-r} I^{r+n} I^{-n} = a^n I^{-n}$  which yields the following inclusions of fractional ideals:

$$\begin{aligned} a^{-r} I^{r+n} &\subset a^n I^{-n} : I^{-n} \subset R : I^{-n} = (\omega_R : \omega_R) : I^{-n} \\ &= \omega_R : (\omega_R I^{-n}) \stackrel{(*)}{\subset} \omega_R : (a^{r-1} I^{-n}) \\ &= a^{-r+1} \omega_R : (R : I^n) = a^{-r+1} \omega_R : ((\omega_R : \omega_R) : I^n) \\ &= a^{-r+1} \omega_R : (\omega_R : \omega_R I^n) \stackrel{(**)}{=} a^{-r+1} \omega_R I^n = a^{-r+1} I^{r+n-1}, \end{aligned}$$

where  $(*)$  holds since  $a^{r-1} \in \omega_R$  and  $(**)$  holds since  $\dim R = 1$ . Multiplication by  $a^r$  implies that  $I^{r+n} \subset aI^{r+n-1}$  and thus  $I^{r+n} = aI^{r+n-1}$ .  $\square$

Using Lemma 3.1 we are able to extend [Corso et al. 2002, Theorem 2.6] in the case of a one-dimensional ring.

**Theorem 3.2.** *Let  $R$  be a one-dimensional local Gorenstein ring with residue field of characteristic 0. Let  $I$  be an ideal with  $\text{ht } I > 0$  and  $J$  a minimal reduction of  $I$ . Then the following are equivalent for a positive integer  $n$ :*

- (a)  $J^{n+1} : I^n$  is independent of  $J$ ;
- (b)  $\text{core}(I) = J^{n+1} : I^n$  for some  $J$ ;
- (c)  $n \geq r(I)$ .

*Proof.* Notice that  $\text{ht } I = 1 = \ell(I)$  since  $\dim R = 1$ . By [Huckaba 1987, Theorem 2.1] we have that  $r_J(I)$  is independent of the minimal reduction  $J$  of  $I$ . Hence  $r_J(I) = r(I)$ . Let  $r = r_J(I) = r(I)$ .

Suppose that  $n \geq r$ . Then by [Polini and Ulrich 2005, Theorem 4.5, Remark 4.8] we have that  $\text{core}(I) = J^{n+1} : I^n$  and  $J^{n+1} : I^n$  is independent of the minimal reduction  $J$  of  $I$ . This establishes the implications (c)  $\Rightarrow$  (a) and (c)  $\Rightarrow$  (b).

To prove (b)  $\Rightarrow$  (c) suppose that  $\text{core}(I) = J^{n+1} : I^n$ . By [Polini and Ulrich 2005, Theorem 4.5] we know that  $\text{core}(I) = J^{m+1} : I^m$  for  $m \geq r$ . Let  $m \geq \max\{r, n\}$ . Then

$$\begin{aligned} \text{core}(I) &= J^{m+1} : I^m \subset J^{m+1} : J^{m-n} I^n \\ &= (J^{m+1} : J^{m-n}) : I^n \stackrel{(1)}{=} J^{n+1} : I^n = \text{core}(I), \end{aligned}$$

where (1) holds since  $J$  is generated by a single regular element. Therefore  $J^{m+1} : I^m = J^{m+1} : J^{m-n} I^n$ . Since  $R$  is Gorenstein then by linkage we have  $I^m = J^{m-n} I^n$ . Hence  $n \geq r$ .

Finally, in order to prove that (a)  $\Rightarrow$  (c) notice that there exists  $m \gg 0$  such that for general linear combinations  $f_1, \dots, f_m$  of the generators of  $I$ , we have that  $(f_i)$  forms a reduction of  $I$  for  $1 \leq i \leq m$  and  $I^{n+1} = (f_1^{n+1}, \dots, f_m^{n+1})$  since  $\text{char } k = 0$ . For example one may take  $m = e(R)$ , the multiplicity of the ring  $R$ . Let  $J = (a)$ . Then for all  $1 \leq i \leq m$ ,

$$a^{n+1} I^{-n} = a^{n+1} : I^n = f_i^{n+1} : I^n = f_i^{n+1} I^{-n}.$$

Hence  $a^{n+1} I^{-n} = I^{n+1} I^{-n}$ . Then by Lemma 3.1 we obtain  $I^{n+1} = a I^n$  and thus  $n \geq r$ .  $\square$

Next we give a description for the canonical module of the extended Rees ring.

**Remark 3.3.** Let  $R$  be a local Gorenstein ring and  $I$  an ideal with  $g = \text{ht } I > 0$ ,  $\ell = \ell(I)$ , and  $J$  a minimal reduction of  $I$ . Write  $B = R[It, t^{-1}]$ . Assume that  $I$  satisfies  $G_\ell$  and  $\text{depth } R/I^j \geq \dim R/I - j + 1$  for  $1 \leq j \leq \ell - g$ . We fix a graded canonical module for the ring  $B$  such that  $\omega_B \subset R[t, t^{-1}]$  and  $[\omega_B]_i = R t^i$  for all  $i \ll 0$ . Notice that this uniquely determines  $\omega_B$  as a submodule of  $R[t, t^{-1}]$ . According to [Polini and Ulrich 2005, Remark 2.2] we have the following description of  $\omega_B$ . For all  $n \geq \max\{r_J(I) - \ell + g, 0\}$

$$\omega_B = \bigoplus_{i \in \mathbb{Z}} (J^i :_R I^n) t^{i-n+g-1} = \dots \oplus R t^{g-n-1} \oplus (J : I^n) t^{g-n} \oplus \dots$$

Let  $R$  be a Noetherian local ring that is an epimorphic image of a local Gorenstein ring. Let  $B$  be a  $\mathbb{Z}$ -graded Noetherian  $R$ -algebra with  $B_0 = R$  and

unique homogeneous maximal ideal  $\mathfrak{m}$ . We also assume that  $B/\mathfrak{m}$  is a field. Let  $\omega_B$  be the graded canonical module of  $B$ . Recall that the  $a$ -invariant of  $B$  is  $a(B) = -\text{indeg}(\omega_B \otimes_B B/\mathfrak{m})$ . Notice that if  $B$  is positively graded then  $a(B) = -\text{indeg}\omega_B$ .

In the setting of Theorem 3.2 the reduction numbers were independent of the choice of minimal reduction as seen in the proof of Theorem 3.2. In the next proposition we provide conditions that guarantee the independence of the reduction numbers. This result was already known in the case  $I$  is equimultiple and  $\text{depth } \text{gr}_I(R)_+ \geq \dim R - 1$  by [Huckaba 1987, Theorem 2.1]. In the case that  $I$  is an  $\mathfrak{m}$ -primary ideal this result was also obtained by Trung [1987, Theorem 1.2]. Our setup is more general.

**Proposition 3.4.** *Let  $R$  be a local Gorenstein ring with infinite residue field. Let  $I$  be an ideal with  $g = \text{ht } I > 0$  and  $\ell = \ell(I)$ . Assume that  $I$  satisfies  $G_\ell$  and  $\text{depth } R/I^j \geq \dim R/I - j + 1$  for  $1 \leq j \leq \ell - g$ . We further assume that  $\text{gr}_I(R)$  is Cohen–Macaulay. Then  $r(I) = r_J(I)$  for every minimal reduction  $J$  of  $I$ .*

*Proof.* According to [Johnson and Ulrich 1996, Corollary 5.5] either  $r(I) = 0$  or  $r(I) > \ell - g$ . If  $r(I) = 0$  then there is nothing to show. So we assume that  $r(I) > \ell - g$ . Let  $J$  be a minimal reduction of  $I$ . Then  $r_J(I) > \ell - g$ .

Let  $\mathfrak{p} \in \text{Spec}(R)$  such that  $\mathfrak{p} \supset I$  and  $\ell(I_{\mathfrak{p}}) = \text{ht } \mathfrak{p} < \ell$ . Then  $I_{\mathfrak{p}}$  is of linear type and thus  $r(I_{\mathfrak{p}}) = 0$ , according to [Ulrich 1994, Proposition 1.11]. Thus  $r(I_{\mathfrak{p}}) - \text{ht } \mathfrak{p} \leq -g < r_J(I) - \ell$  and hence  $a(\text{gr}_I(R)) = r_J(I) - \ell$  by [Aberbach et al. 1995, Corollary 4.5]. But the  $a$ -invariant of  $\text{gr}_I(R)$  is independent of the choice of the minimal reduction  $J$  and thus  $r_J(I)$  is independent of  $J$ . Hence  $r_J(I) = r(I)$ .  $\square$

The next result is essentially obtained in [Ulrich 1996, Corollary 2.4] but we are able to weaken the assumptions on the depth condition.

**Proposition 3.5.** *Let  $R$  be a local Gorenstein ring with residue field of characteristic 0. Let  $I$  be an ideal with  $g = \text{ht } I > 0$ ,  $\ell = \ell(I)$ , and let  $J$  be a minimal reduction of  $I$ . Suppose that  $I$  satisfies  $G_\ell$  and  $\text{depth } R/I^j \geq \dim R/I - j + 1$  for  $1 \leq j \leq \ell - g$ . We further assume that  $\text{gr}_I(R)$  is Cohen–Macaulay. Then*

- (a)  $J : I^n \neq R$  for all  $n \leq \max\{r(I) - \ell + g, 0\}$ ,
- (b)  $\max\{r(I) - \ell + g, 0\} = \min\{i \mid I^{i+1} \subset \text{core}(I)\}$ .

*Proof.* Write  $G = \text{gr}_I(R)$  and  $B = R[It, t^{-1}]$ . As  $G$  is Cohen–Macaulay then so is  $B$ , since  $G = \text{gr}_I(R) \cong B/(t^{-1})$ . According to Proposition 3.4 one has that  $r_J(I) = r(I)$ . Furthermore  $a(G) = \max\{r(I) - \ell, -g\}$  by [Simis et al. 1995, Theorem 3.5]. On the other hand,  $a(G) = a(B) - 1$  since  $G$  is Cohen–Macaulay and  $G \cong B/(t^{-1})$ . Therefore  $a(B) = m - g + 1$ , where  $m = \max\{r(I) - \ell(I) + g, 0\}$ .

Hence

$$[\omega_B]_{g-m-1} = R \quad \text{and} \quad J : I^m = [\omega_B]_{g-m} \neq R,$$

by Remark 3.3, since  $r_J(I) = r(I)$ . Hence  $J : I^n \subset J : I^m \neq R$  for all  $n \leq m$ . This proves (a).

For part (b) we claim that  $m = \min\{i \mid I^{i+1} \subset \text{core}(I)\}$ . To see this observe that  $J \subset J : I^m$  and  $J : I^m$  is independent of  $J$  by [Polini and Ulrich 2005, Remark 2.3], since  $r_J(I) = r(I)$ . Thus  $I \subset J : I^m$  and hence  $I^{m+1} \subset J$ . Consequently  $I^{m+1} \subset \text{core}(I)$ . But since  $J : I^m \neq R$  we have that  $I^m \not\subset J$  and therefore  $I^m \not\subset \text{core}(I)$ .  $\square$

In order to extend Theorem 3.2 we prove the first two statements are equivalent in higher dimensions without any additional assumptions.

**Proposition 3.6.** *Let  $R$  be a local Gorenstein ring with residue field of characteristic 0. Let  $I$  be an ideal with  $g = \text{ht } I > 0$ ,  $\ell = \ell(I)$ , and let  $J$  be a minimal reduction of  $I$ . Suppose that  $I$  satisfies  $G_\ell$  and  $\text{depth } R/I^j \geq \dim R/I - j + 1$  for  $1 \leq j \leq \ell - g$ . Then the following are equivalent for an integer  $n$ :*

- (a)  $J^{n+1} : I^n$  is independent of  $J$ ;
- (b)  $\text{core}(I) = J^{n+1} : I^n$  for every  $J$ .

*Proof.* By [Polini and Ulrich 2005, Theorem 4.5] we have that

$$\text{core}(I) = J^{m+1} : I^m$$

for  $m \gg 0$  and any minimal reduction  $J$  of  $I$ .

Suppose that  $J^{n+1} : I^n$  is independent of  $J$ . Notice that

$$J^{n+1} : I^n \subset J^{n+1} : J^n = J,$$

where the equality holds by Lemma 2.1. Since  $J^{n+1} : I^n$  is independent of  $J$  it follows that  $J^{n+1} : I^n \subset \text{core}(I) = J^{m+1} : I^m$  for  $m \gg 0$ . By Remark 2.2 we have that  $\{J^{i+1} : I^i\}_{i \in \mathbb{N}}$  is a decreasing sequence of ideals and hence it follows that  $\text{core}(I) = J^{n+1} : I^n$  for every minimal reduction  $J$  of  $I$ .

The other implication is clear since the formula for  $\text{core}(I)$  is independent of the choice of the minimal reduction  $J$  of  $I$ .  $\square$

We are now ready to prove the main result of this article. If we assume that the associated graded ring of  $I$  is Cohen–Macaulay then we obtain a generalization to Theorem 3.2 in higher dimensions.

**Theorem 3.7.** *Let  $R$  be a local Gorenstein ring with residue field of characteristic 0. Let  $I$  be an ideal with  $g = \text{ht } I > 0$ ,  $\ell = \ell(I)$ , and let  $J$  be a minimal reduction of  $I$ . Suppose that  $I$  satisfies  $G_\ell$  and  $\text{depth } R/I^j \geq \dim R/I - j + 1$  for  $1 \leq j \leq \ell - g$ . We further assume that  $\text{gr}_I(R)$  is Cohen–Macaulay. Then the following are equivalent for an integer  $n$ :*

- (a)  $J^{n+1} : I^n$  is independent of  $J$ ;
- (b)  $\text{core}(I) = J^{n+1} : I^n$  for every  $J$ ;
- (c)  $n \geq \max\{r(I) - \ell + g, 0\}$ .

*Proof.* The first two statements are equivalent as seen in Proposition 3.6. Write  $G = \text{gr}_I(R)$  and  $B = R[It, t^{-1}]$ . Since  $G$  is Cohen–Macaulay then so is  $B$  since  $G = \text{gr}_I(R) \simeq B/(t^{-1})$ . Notice that  $r_J(I) = r(I)$  by Proposition 3.4.

Let  $m = \max\{r(I) - \ell + g, 0\}$  and suppose that  $n \geq m$ . Then  $\text{core}(I) = J^{n+1} : I^n$  for any minimal reduction  $J$  of  $I$  according to [Polini and Ulrich 2005, Theorem 4.5], since  $r_J(I) = r(I)$ .

Finally, suppose that  $\text{core}(I) = J^{n+1} : I^n$ . Then  $J^{n+1} \subset \text{core}(I)$  for every minimal reduction  $J$  of  $I$ . Since  $\text{char } k = 0$  we obtain that  $I^{n+1} \subset \text{core}(I)$ . Therefore  $n \geq m$  by Proposition 3.5.  $\square$

The following example is due to Angela Kohlhass. It establishes that without the Cohen–Macaulay assumption on the associated graded ring the result of Theorem 3.7 does not hold in general.

**Example 3.8** (A. Kohlhass). Let  $R = k[[x, y]]$  be a power series ring over a field  $k$  of characteristic 0. Let  $I = (x^{10}, x^4y^5, y^9)$  and  $J$  a general minimal reduction of  $I$ . Then  $I$  is  $\mathfrak{m}$ -primary, where  $\mathfrak{m}$  is the maximal ideal of  $R$ ,  $r(I) = 4$ , and  $\text{depth } \text{gr}_I(R) = 0$ . It turns out that  $J^4 : I^3 = \text{core}(I) = J^5 : I^4$ .

**Remark 3.9.** We remark that in Example 3.8 the associated graded ring of the ideal  $I$  has depth 0 and the ideal  $J^4 : I^3$  is independent of the choice of the minimal reduction  $J$  of  $I$ , whereas  $r(I) = 4$ . This shows that in general Theorem 3.7 does not hold without any assumptions on  $\text{gr}_I(R)$ . It is conceivable that when  $\text{depth } \text{gr}_I(R) \geq \dim R - 1$  then a similar statement might hold.

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