

The derived category of a graded Gorenstein ring

JESSE BURKE AND GREG STEVENSON

We give an exposition and generalization of Orlov's theorem on graded Gorenstein rings. We show the theorem holds for nonnegatively graded rings that are Gorenstein in an appropriate sense and whose degree zero component is an arbitrary noncommutative right noetherian ring of finite global dimension. A short treatment of some foundations for local cohomology and Grothendieck duality at this level of generality is given in order to prove the theorem. As an application we give an equivalence of the derived category of a commutative complete intersection with the homotopy category of graded matrix factorizations over a related ring.

1. Introduction

Let A be a graded Gorenstein ring. Orlov [2009] related the bounded derived category of coherent sheaves on $\text{Proj } A$ and the singularity category of graded A -modules via fully faithful functors; the exact relation depends on the a -invariant of A . This is a striking theorem that has found applications in physics, algebraic geometry and representation theory. To give an idea of the scope of the theorem: in the limiting case that A has finite global dimension (so the singularity category is trivial), it recovers (and generalizes to noncommutative rings) Beilinson's result [1978] that the derived category of $\text{Proj } A$ is generated by a finite sequence of twists of the structure sheaf.

There has been much work related to this theorem. The idea of Orlov's construction perhaps first appears in van den Bergh's paper [2004] on noncommutative crepant resolutions where he described functors similar to those considered by Orlov, in the case of torus invariants. After Orlov's paper appeared, the idea was further explored by the physicists Herbst, Hori and Page [Herbst et al. 2008]. In turn these ideas were the inspiration for two papers on the derived category of GIT quotients [Ballard et al. 2014; Halpern-Leistner 2014]. Segal and then Shipman gave geometric proofs of Orlov's theorem for commutative hypersurfaces in [Segal 2011] and [Shipman 2012]. Related results are [Baranovsky and Pecharich 2010; Isik 2013]. The theorem has been used in

similar ways in [Ballard et al. 2012; Keller et al. 2011]. Finally, it has been used in representation theory, especially in the study of weighted projective lines; see, for example, [Lenzing 2011].

Orlov assumed that A_0 a field. In this paper, that we consider largely expository, we generalize his result to show that the same relation holds when A_0 is a noncommutative noetherian ring of finite global dimension. This has an immediate application to commutative complete intersection rings and we expect there to be further applications, for instance to (higher) preprojective algebras. The structure of our proof is very close to Orlov's original arguments. We give many details and we hope that these details may help the reader (even one only interested in algebras defined over a field) to better understand Orlov's work.

The main tool in the proof is a semiorthogonal decomposition. This separates a triangulated category into an admissible subcategory and its orthogonal. Derived global sections gives an embedding of $D^b(\text{Proj } A)$ into $D^b(\text{gr}_{\geq i} A)$ as an admissible subcategory. When A_0 is a field, Orlov showed that there is an embedding of the singularity category $D_{\text{sg}}^b(\text{gr } A)$ into $D^b(\text{gr}_{\geq i} A)$. The existence of such an embedding is rather remarkable and constitutes perhaps the key insight required to prove the theorem. Orlov then used Grothendieck duality in a very clever way to compare the orthogonals of $D^b(\text{Proj } A)$ and $D_{\text{sg}}^b(\text{gr } A)$ inside of $D^b(\text{gr}_{\geq i} A)$. For example when A is Calabi–Yau, the orthogonals coincide and so there is an equivalence between $D^b(\text{Proj } A)$ and $D_{\text{sg}}^b(\text{gr } A)$.

The arguments we present here follow those of Orlov. The main addition is the observation that, when A_0 has finite global dimension, one can construct particularly nice resolutions of complexes of graded modules with bounded finitely generated cohomology. This allows us to prove Orlov's embedding $D_{\text{sg}}^b(\text{gr } A) \rightarrow D^b(\text{gr}_{\geq i} A)$ is valid in this more general setting. We also need to develop some foundations concerning local cohomology and Grothendieck duality over noncommutative rings to prove analogues of the other steps of Orlov's proof. These foundations do not seem to be contained in the literature in the form and generality that we need, although the arguments we give here are relatively straightforward generalizations of arguments by Artin and Zhang [1994].

Let us give a quick summary of the paper. The second section contains some categorical background, especially on semiorthogonal decompositions. The third section is devoted to the derived category of graded modules over a graded ring, and some standard semiorthogonal decompositions that appear there. This section contains the key observation, Lemma 3.10, needed to prove the embedding of the singularity category works for the rings we work with. The fourth section deals with local cohomology and the semiorthogonal decomposition it gives, while the fifth deals with the embedding of the singularity category, Grothendieck duality, and the semiorthogonal decomposition these give. The sixth section contains

the proof of the main result as well as a sufficient condition for a Gorenstein ring to satisfy Artin and Zhang's condition χ , which is necessary for the proof. Finally, in the last section, we apply the main theorem to give a description of the bounded derived category of a complete intersection ring in terms of graded matrix factorizations.

2. Background

We recall here some standard results on semiorthogonal decompositions of triangulated categories that we will need. Throughout this section \mathcal{T} denotes a triangulated category.

Definition 2.1. For \mathcal{D} a triangulated subcategory of \mathcal{T} , define \mathcal{D}^\perp to be the full subcategory with objects those $X \in \mathcal{T}$ such that $\text{Hom}_{\mathcal{T}}(D, X) = 0$ for all objects D of \mathcal{D} . Similarly, ${}^\perp\mathcal{D}$ has objects those X with $\text{Hom}_{\mathcal{T}}(X, D) = 0$ for all D in \mathcal{D} . Both \mathcal{D}^\perp and ${}^\perp\mathcal{D}$ are triangulated subcategories of \mathcal{T} that are closed under direct summands, that is, thick subcategories.

Definition 2.2. A triangulated subcategory \mathcal{D} of \mathcal{T} is *left admissible* in \mathcal{T} if the inclusion functor $i : \mathcal{D} \rightarrow \mathcal{T}$ has a left adjoint; \mathcal{D} is *right admissible* if i has a right adjoint.

The following criterion for admissibility can be found as [Bondal 1989, Lemma 3.1].

Lemma 2.3. *Let \mathcal{D} be a triangulated subcategory of \mathcal{T} .*

- (1) *The category \mathcal{D} is left admissible if and only if for every X in \mathcal{T} there is a triangle:*

$$E_X \rightarrow X \rightarrow D_X \rightarrow \Sigma E_X,$$

with D_X in \mathcal{D} and E_X in ${}^\perp\mathcal{D}$.

- (2) *The category \mathcal{D} is right admissible if and only if for every X in \mathcal{T} there is a triangle:*

$$D_X \rightarrow X \rightarrow E_X \rightarrow \Sigma D_X,$$

with D_X in \mathcal{D} and E_X in \mathcal{D}^\perp .

Corollary 2.4. *A subcategory \mathcal{D} of \mathcal{T} is left admissible if and only if ${}^\perp\mathcal{D}$ is right admissible. In this case $({}^\perp\mathcal{D})^\perp = \mathcal{D}$.*

Definition 2.5. A *semiorthogonal decomposition* of \mathcal{T} is a pair of subcategories \mathcal{A} and \mathcal{B} such that \mathcal{A} is left admissible and $\mathcal{B} = {}^\perp\mathcal{A}$ (equivalently, \mathcal{B} is right admissible and $\mathcal{A} = \mathcal{B}^\perp$). We write this as

$$\mathcal{T} = (\mathcal{A}, \mathcal{B}).$$

The following lemma follows from Lemma 2.3 in a straightforward way.

Lemma 2.6. *There is a semiorthogonal decomposition $\mathcal{T} = (\mathcal{A}, \mathcal{B})$ if and only if $\mathcal{B} \subseteq {}^\perp \mathcal{A}$ and for every X in \mathcal{T} there is a triangle*

$$B_X \rightarrow X \rightarrow A_X \rightarrow$$

with $B_X \in \mathcal{B}$ and $A_X \in \mathcal{A}$. We will call such a triangle the localization triangle for X .

Orlov [2009] generalized the definition of semiorthogonal decomposition to:

Definition 2.7. A sequence of full triangulated subcategories $(\mathcal{D}_1, \dots, \mathcal{D}_n)$ of \mathcal{T} is a *semiorthogonal decomposition* if for each $i = 1, \dots, n-1$, the thick subcategory generated by $\mathcal{D}_1, \dots, \mathcal{D}_i$, that we denote $\langle \mathcal{D}_1, \dots, \mathcal{D}_i \rangle$, is left admissible and

$${}^\perp \langle \mathcal{D}_1, \dots, \mathcal{D}_i \rangle = \langle \mathcal{D}_{i+1}, \dots, \mathcal{D}_n \rangle.$$

We can (and will) construct semiorthogonal decompositions inductively:

Lemma 2.8. *Let $\mathcal{T} = (\mathcal{A}, \mathcal{B})$, $\mathcal{A} = (\mathcal{D}_1, \dots, \mathcal{D}_i)$, and $\mathcal{B} = (\mathcal{D}_{i+1}, \dots, \mathcal{D}_n)$ be semiorthogonal decompositions. Then*

$$\mathcal{T} = (\mathcal{D}_1, \dots, \mathcal{D}_n)$$

is a semiorthogonal decomposition.

3. The bounded derived category of graded modules

We now provide some preliminary results on derived categories of graded modules. We begin by exhibiting some semiorthogonal decompositions of the bounded derived category that we will need in the sequel (and that are relatively straightforward generalizations of those in Orlov's work). We also prove the main technical results concerning graded projectives and graded projective resolutions that we will need.

In this section $A = \bigoplus_{i \geq 0} A_i$ is a positively graded right noetherian ring with A_0 a ring of finite global dimension.

All modules will be right modules unless otherwise stated. We denote by $\text{gr } A$ the abelian category of finitely generated graded A -modules and degree zero homogeneous maps. If $M = \bigoplus M_i$ is a graded A -module, then $M(1)$ is the graded A -module with $M(1)_i = M_{i+1}$.

We denote by $\text{gr}_{\geq i} A$ the full subcategory of $\text{gr } A$ consisting of objects M such that $M_j = 0$ for all $j < i$. This is an abelian subcategory of $\text{gr } A$ and there is an adjoint pair of functors

$$\text{gr}_{\geq i} A \begin{array}{c} \xrightarrow{\text{inc}} \\ \xleftarrow{(-)_{\geq i}} \end{array} \text{gr } A,$$

where $M_{\geq i} = \bigoplus_{j \geq i} M_j$ is right adjoint to the inclusion.

We denote by $D^b(-)$ the bounded derived category of an abelian category. Both functors of the above adjoint pair are exact and so induce functors

$$D^b(\text{gr}_{\geq i} A) \begin{array}{c} \xrightarrow{\text{inc}} \\ \xleftarrow{(-)_{\geq i}} \end{array} D^b(\text{gr} A)$$

that also form an adjoint pair. The functor induced by inclusion is fully faithful and the essential image is the full subcategory of $D^b(\text{gr} A)$ consisting of objects M such that $H^j(M) \in \text{gr}_{\geq i} A$ for all $j \in \mathbb{Z}$. We denote this subcategory also by $D^b(\text{gr}_{\geq i} A)$. It is a right admissible subcategory.

Definition 3.1. Define $\mathcal{S}_{< i}$ to be the thick subcategory generated by the objects $A_0(e)$, for all $e > -i$ and $\mathcal{S}_{\geq i}$ to be the thick subcategory generated by $A_0(e)$, for all $e \leq -i$.

Lemma 3.2. *An object M of $D^b(\text{gr} A)$ is in $\mathcal{S}_{< i}$ if and only if $M_{\geq i} \simeq 0$.*

Proof. The full subcategory with objects those M satisfying $M_{\geq i} \simeq 0$ is thick by virtue of being the kernel of an exact functor. Since $A_0(e)_{\geq i} = 0$ for all $e > -i$, we see that $\mathcal{S}_{< i}$ is contained in this thick subcategory. Thus if M is in $\mathcal{S}_{< i}$ we must have $M_{\geq i} \simeq 0$.

For the converse, we first assume M is a module, that is, concentrated in homological degree 0, and that there is an integer $e < i$ with $M_j = 0$ for all $j \neq e$. Since M_e is a finitely generated A_0 -module and A_0 has finite global dimension, M_e has a finite projective resolution over A_0 . Over A this says that M_e is in the thick subcategory generated by $A_0(-e)$, which is contained in $\mathcal{S}_{< i}$.

Now suppose M is a nonzero finitely generated graded A -module with $M_{\geq i} \simeq 0$. As M is finitely generated there is an integer j with $M_{\geq j} = M$ and we may as well choose a maximal such j , which is necessarily less than i . Consider the triangle

$$M_{\geq j+1} \rightarrow M \rightarrow M_j \rightarrow .$$

By the previous argument M_j is in $\mathcal{S}_{< i}$ and arguing inductively on the number of degrees in which M is nonzero we see that $M_{\geq j+1}$ is in $\mathcal{S}_{< i}$, and hence M is in $\mathcal{S}_{< i}$.

For an arbitrary nonzero object $M \in D^b(\text{gr} A)$, with $M_{\geq i} \simeq 0$ the result follows from induction on the number of nonvanishing cohomology modules, using the triangle

$$M^{< n} \rightarrow M \rightarrow H^n(M)[-n] \rightarrow ,$$

where $n = \max\{j \mid H^j(M) \neq 0\}$ and $M^{< n}$ is the truncation with respect to the standard t-structure. \square

Remark 3.3. It follows from the definition that $\mathcal{S}_{\geq i}$ is contained in $D^b(\text{gr}_{\geq i} A)$. We will show in Lemma 4.17 that $\mathcal{S}_{\geq i}$ is the full subcategory of $D^b(\text{gr}_{\geq i} A)$ whose objects have torsion cohomology.

Lemma 3.4. *There is a semiorthogonal decomposition*

$$D^b(\text{gr } A) = (\mathcal{S}_{< i}, D^b(\text{gr}_{\geq i} A)).$$

The localization triangle for $M \in D^b(\text{gr } A)$ is given by the canonical maps

$$M_{\geq i} \rightarrow M \rightarrow M/M_{\geq i} \rightarrow .$$

Proof. Let M be in $\mathcal{S}_{< i}$ and N be in $D^b(\text{gr}_{\geq i} A)$. Then

$$\text{Hom}_{D^b(\text{gr } A)}(N, M) \cong \text{Hom}_{D^b(\text{gr}_{\geq i} A)}(N, M_{\geq i}) = 0$$

by right adjointness of $(-)_{\geq i}$ and since $M_{\geq i} \simeq 0$. Thus $D^b(\text{gr}_{\geq i} A) \subseteq {}^\perp \mathcal{S}_{< i}$. If M is any object in $D^b(\text{gr } A)$ we have the triangle

$$M_{\geq i} \rightarrow M \rightarrow M/M_{\geq i} \rightarrow ,$$

with $M_{\geq i}$ in $D^b(\text{gr}_{\geq i} A)$ and $M/M_{\geq i}$ in $\mathcal{S}_{< i}$. Thus we may apply Lemma 2.6. \square

Definition 3.5. Define $\mathcal{P}_{< i}$ to be the thick subcategory generated by the objects $A(e)$ for all $e > -i$ and $\mathcal{P}_{\geq i}$ to the thick category generated by $A(e)$ for all $e \leq -i$.

Remark 3.6. It follows from the definition that $\mathcal{P}_{\geq i}$ is contained in $D^b(\text{gr}_{\geq i} A)$. In fact, $\mathcal{P}_{\geq i}$ is the full subcategory of $D^b(\text{gr}_{\geq i} A)$ whose objects are perfect complexes of A -modules.

Lemma 3.7. *There is a semiorthogonal decomposition*

$$D^b(\text{gr } A) = (D^b(\text{gr}_{\geq i} A), \mathcal{P}_{< i}).$$

Before the proof, we need two results on graded projective A -modules and graded projective resolutions over A .

Lemma 3.8. *Let P be a finitely generated graded projective A -module. Then there is an isomorphism, for some integers n, m_1, \dots, m_n ,*

$$P \cong \bigoplus_{i=1}^n P_i \otimes_{A_0} A(m_i),$$

where the P_i are projective right A_0 -modules.

Proof. Let P be a nonzero finitely generated graded projective A -module. Consider the graded projective A_0 -module $\bar{P} = P \otimes_A A_0$ which is nonzero by the

graded Nakayama lemma. We obtain a graded projective A -module $\bar{P} \otimes_{A_0} A$ fitting into a commutative diagram

$$\begin{array}{ccc} & \bar{P} \otimes_{A_0} A & \\ & \swarrow \text{dashed} & \downarrow \\ P & \longrightarrow & \bar{P} \longrightarrow 0 \end{array}$$

where the vertical morphism is the canonical one and the dashed arrow exists by projectivity of $\bar{P} \otimes_{A_0} A$. By construction the morphism $\bar{P} \otimes_{A_0} A \rightarrow P$ is surjective so it splits. But

$$\bar{P} \otimes_{A_0} A \otimes_A A_0 \cong \bar{P} = P \otimes_A A_0,$$

and so by another application of the graded Nakayama lemma we see $P \cong \bar{P} \otimes_{A_0} A$ is induced up from a graded projective A_0 -module proving the lemma. \square

Definition 3.9. For every graded projective A -module Q , we define summands $Q_{<i}$ and $Q_{\geq i}$ with $Q_{<i}$ in $\mathcal{P}_{<i}$ and $Q_{\geq i}$ in $\mathcal{P}_{\geq i}$ via the unique up to isomorphism split exact sequence of graded projective modules

$$0 \rightarrow Q_{<i} \rightarrow Q \rightarrow Q_{\geq i} \rightarrow 0$$

which exists by the previous lemma.

The next lemma is a key technical observation concerning the structure of resolutions over A .

Lemma 3.10. *Every object M in $D^b(\text{gr } A)$ is quasi-isomorphic to a complex of finitely generated graded projective A -modules*

$$P = \dots \rightarrow P^j \rightarrow P^{j+1} \rightarrow \dots$$

such that $P^j = 0$ for all $j \gg 0$ and for any $i \in \mathbb{Z}$ there exists a k_i with P^{-k} in $\mathcal{P}_{\geq i}$ for all $k \geq k_i$.

Proof. It is sufficient to prove the result for graded A -modules as the condition is closed under suspensions and taking cones, and every object of $D^b(\text{gr } A)$ can be written as an iterated extension of suspensions of modules using the standard t-structure. Let us introduce notation local to this proof. Given a finitely generated A -module M define the integer

$$\min(M) = \min\{i \in \mathbb{Z} \mid M_i \neq 0\}.$$

Let M be a finitely generated graded A -module and set

$$\bar{M} = M \otimes_A A_0 = M/A_{\geq 1}M,$$

which we consider as a graded A_0 -module. We may assume M has infinite projective dimension as the result is trivial in the finite projective dimension case. We will construct a projective resolution of the desired form. If \bar{M} is zero then, by Nakayama, so is M and thus we may suppose $\bar{M} \neq 0$. We choose an epimorphism from a graded projective A_0 -module $\bar{P}^0 \rightarrow \bar{M}$ by writing

$$\bar{M} \cong \bigoplus_{i=1}^n M_i(a_i),$$

taking epimorphisms $\bar{P}_i^0(a_i) \rightarrow M_i(a_i)$ where the \bar{P}_i^0 are projective A_0 -modules and setting

$$\bar{P}^0 = \bigoplus_{i=1}^n \bar{P}_i^0(a_i)$$

with the obvious morphism to \bar{M} . This gives rise to an exact sequence of graded A -modules

$$0 \rightarrow Z^0 \rightarrow P^0 \rightarrow M \rightarrow 0,$$

where $P^0 = \bar{P}^0 \otimes_{A_0} A$, with the property that

$$\min(Z^0) \geq \min(P^0) = \min(M).$$

We have assumed A_0 has finite global dimension, say d . Proceeding as above we may find projectives P^i for $i = 1, \dots, d-1$ and exact sequences

$$0 \rightarrow Z^i \rightarrow P^i \rightarrow Z^{i-1} \rightarrow 0,$$

with $\min(Z^i) \geq \min(P^i) = \min(Z^{i-1})$. Thus, restriction to the graded components in degree $j = \min(M)$ gives an exact sequence

$$0 \rightarrow Z_j^{d-1} \rightarrow P_j^{d-1} \rightarrow \dots \rightarrow P_j^0 \rightarrow M_j \rightarrow 0$$

of A_0 -modules with the P_j^i projective. As A_0 has global dimension d we see Z_j^{d-1} must be projective. Hence \bar{Z}^{d-1} can be written as $Z_j^{d-1} \oplus X$ with X living in degrees strictly greater than j . As before we can pick an epimorphism $Q \rightarrow X$ from a graded projective A_0 -module Q which lives in the same degrees as X . Setting $P^d = (Z_j^{d-1} \oplus Q) \otimes_{A_0} A$ we get a short exact sequence

$$0 \rightarrow Z^d \rightarrow P^d \rightarrow Z^{d-1} \rightarrow 0,$$

with $\min(Z^d) > \min(M)$; thus our recipe guarantees projectives with generators in degrees less than or equal to $\min(M)$ cannot occur beyond the d -th step of the resolution. We can now repeat this procedure starting at Z^d to obtain a resolution satisfying the desired properties. \square

Remark 3.11. It is easy to construct examples showing this lemma is no longer true if A_0 does not have finite global dimension. Indeed, let $A = k[x, y]/(x^2, y^2)$, with $|x| = 0$ and $|y| = 1$. The resolution of $A/(x)$ is

$$\cdots \rightarrow A \xrightarrow{x} A \xrightarrow{x} A \rightarrow 0,$$

which does not satisfy the conclusion of the previous lemma.

Proof of Lemma 3.7. Given an object M in $D^b(\text{gr } A)$, let $P \xrightarrow{\sim} M$ be a quasi-isomorphism where P is a complex of projectives as in the previous lemma. Apply the decomposition in Definition 3.9 degree-wise to P to get a triangle

$$P_{<i} \rightarrow P \rightarrow P_{\geq i} \rightarrow ,$$

where $P_{<i}$ is the subcomplex of P consisting of all projective summands generated in degrees less than i and $P_{\geq i}$ is the quotient complex consisting of all projective summands generated in degree at least i . Since P^{-k} is in $\mathcal{P}_{\geq i}$ for all $k \gg 0$, we see that $P_{<i}$ is bounded, and hence in $\mathcal{P}_{<i}$. Note that $P_{\geq i}$ has bounded finitely generated cohomology by the triangle, and so must be in $D^b(\text{gr}_{\geq i} A)$.

There are no nonzero maps from objects in $\mathcal{P}_{<i}$ to any module M in $\text{gr}_{\geq i} A$. Thus $\text{gr}_{\geq i} A$ is contained in $\mathcal{P}_{<i}^\perp$ and hence so is $D^b(\text{gr}_{\geq i} A)$ since it is generated by $\text{gr}_{\geq i} A$ and $\mathcal{P}_{<i}^\perp$ is thick. Thus $\mathcal{P}_{<i} \subseteq {}^\perp D^b(\text{gr}_{\geq i} A)$. We can now apply Lemma 2.6. \square

Remark 3.12. Let M be an object of $D^b(\text{gr } A)$ and P a projective resolution of M satisfying the conditions of Lemma 3.10. The proof shows that the localization triangle for M is given by

$$P_{<i} \rightarrow P \simeq M \rightarrow P_{\geq i} \rightarrow .$$

Remark 3.13. Although we have chosen to work throughout with the grading group \mathbb{Z} , the results are valid more generally. One can replace \mathbb{Z} by any totally ordered abelian group and work with graded rings concentrated in degrees greater than or equal to the identity.

This will also be the case for the majority of the results that follow. However, there are instances in which one does need additional hypotheses. For example in Lemma 6.5 (and the main Theorem 6.4) one must assume the order admits successors.

4. Noncommutative Proj and local cohomology

Let A be a graded noncommutative ring. Artin and Zhang in defined the category of quasicohherent sheaves on the noncommutative projective scheme $\text{Proj } A$ as the category of graded modules modulo the full subcategory of torsion modules [Artin and Zhang 1994]. (Here and throughout torsion means torsion with

respect to the two-sided ideal $A_{\geq 1}$.) In this section we adapt some of their definitions and results, in particular concerning local cohomology functors, to give a semiorthogonal decomposition of $D^b(\text{gr}_{\geq i} A)$ when A is Gorenstein.

We assume $A = \bigoplus_{i \geq 0} A_i$ is a positively graded right noetherian ring. We consider $\text{Gr } A$, the abelian category of graded right A -modules. This contains $\text{gr } A$, the category of finitely generated graded A -modules, as a full abelian subcategory.

Definition 4.1. Let M be a graded A -module. An element $m \in M$ is *torsion* if

$$m \cdot (A_{\geq n}) = 0,$$

for some $n \geq 1$. Denote by $\tau(M)$ the submodule of M consisting of all torsion elements. The module M is *torsion* if $\tau(M) = M$ and *torsion-free* if $\tau(M) = 0$. Denote by $\text{Tors } A$ the full subcategory of $\text{Gr } A$ consisting of torsion modules and set $\text{tors } A = \text{Tors } A \cap \text{gr } A$.

The subcategory $\text{Tors } A$ (respectively, $\text{tors } A$) satisfies the property that for a short exact sequence

$$0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$$

in $\text{Gr } A$ ($\text{gr } A$), we have X in $\text{Tors } A$ ($\text{tors } A$) if and only if X' and X'' are in $\text{Tors } A$ ($\text{tors } A$)—that is, they are Serre subcategories. Moreover, $\text{Tors } A$ is closed under colimits. Thus we can form the quotient categories

$$\text{Qcoh } X = \text{Gr } A / \text{Tors } A \quad \text{and} \quad \text{coh } X = \text{gr } A / \text{tors } A;$$

see, for example, [Popescu 1973, Section 4.3] for the construction. The relevant features here are that:

- (1) The categories $\text{Qcoh } X$ and $\text{coh } X$ have the same objects as $\text{Gr } A$ and $\text{gr } A$, respectively.
- (2) The categories $\text{Qcoh } X$ and $\text{coh } X$ are abelian and there are canonical exact functors $\text{Gr } A \rightarrow \text{Qcoh } X$ and $\text{gr } A \rightarrow \text{coh } X$.
- (3) A map f in $\text{Gr } A$ is an isomorphism in $\text{Qcoh } X$ if and only if $\ker f$ and $\text{coker } f$ are in $\text{Tors } A$. In particular the image of every object in $\text{Tors } A$ is isomorphic to zero in $\text{Qcoh } X$. The analogous statement holds for $\text{gr } A$.

For an object M in $\text{Gr } A$, we denote by \tilde{M} the image of M in $\text{Qcoh } X$. For future reference, we note that as $\text{Tors } A$ is closed under the grading shifts, the shifts induce automorphisms of $\text{Qcoh } X$ and $\text{coh } X$ which we also denote by $(-)(i)$.

Remark 4.2. The notation $\text{Qcoh } X$ and $\text{coh } X$ reflects that these categories should be thought of as sheaves of modules on the noncommutative projective scheme $X = \text{Proj } A$. If A is commutative and generated in degree 1, then by a

famous result of Serre, the category $\text{Qcoh } X$ (respectively $\text{coh } X$) is equivalent to the category of quasicohherent (respectively coherent) sheaves on the scheme $X = \text{Proj } A$. If A is generated in higher degrees, then $\text{coh } X$ is equivalent to the category of coherent sheaves on the Deligne–Mumford stack $\text{Proj } A$.

Definition 4.3. For M, N in $\text{Gr } A$, denote by $\underline{\text{Hom}}_{\text{Gr } A}(M, N)$ the graded abelian group

$$\underline{\text{Hom}}_{\text{Gr } A}(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\text{Gr } A}(M(-i), N) \cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\text{Gr } A}(M, N(i)).$$

If M is an A - A -bimodule, for example, $M = A$, then $\underline{\text{Hom}}_{\text{Gr } A}(M, N)$ is a graded right A -module and so is in $\text{Gr } A$.

For any integer $p \geq 0$, we have a short exact sequence of A -bimodules:

$$0 \rightarrow A_{\geq p} \rightarrow A \rightarrow A/A_{\geq p} \rightarrow 0.$$

Applying $\underline{\text{Hom}}_{\text{Gr } A}(-, -)$, we have an exact sequence of endofunctors on $\text{Gr } A$:

$$0 \rightarrow \underline{\text{Hom}}_{\text{Gr } A}(A/A_{\geq p}, -) \rightarrow \underline{\text{Hom}}_{\text{Gr } A}(A, -) \rightarrow \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, -).$$

We may take the colimit of these sequences as $p \rightarrow \infty$ to get another exact sequence of functors; the sequence remains exact as both the abelian structure and colimits for endofunctors are inherited value-wise from $\text{Gr } A$ and $\text{Gr } A$ has exact filtered colimits. Note that for any M in $\text{Gr } A$ we have isomorphisms in $\text{Gr } A$:

$$\text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A/A_{\geq p}, M) \cong \tau(M) \text{ and } \underline{\text{Hom}}_{\text{Gr } A}(A, M) \cong M.$$

This gives a functorial exact sequence

$$0 \rightarrow \tau(M) \rightarrow M \rightarrow \text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, M). \quad (4.4)$$

Proposition 4.5. *The inclusion of $\text{Tors } A$ into $\text{Gr } A$ and the corresponding quotient functor have right adjoints $\tau(-)$ and Γ_* , respectively:*

$$\text{Tors } A \begin{array}{c} \xrightarrow{\text{inc}} \\ \xleftarrow{\tau(-)} \end{array} \text{Gr } A \begin{array}{c} \xrightarrow{(\tilde{-})} \\ \xleftarrow{\Gamma_*} \end{array} \text{Qcoh } X,$$

where for M in $\text{Gr } A$, $\tau(M)$ is the torsion submodule of M and

$$\Gamma_*(\tilde{M}) = \text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, M).$$

The functor Γ_* and the inclusion of $\text{Tors } A$ are fully faithful so the corresponding counit and unit respectively are isomorphisms. The remaining counit and unit are given by (4.4).

Proof. It is easy to see that $\tau(-)$ is a right adjoint to the inclusion of $\text{Tors } A \rightarrow \text{Gr } A$ and it follows from abstract nonsense; see [Popescu 1973, Section 4.4], that there exists a right adjoint $\Gamma_* : \text{Qcoh } X \rightarrow \text{Gr } A$. We give a direct proof that the functor $\text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, -)$ induces a right adjoint, and that the unit is given by (4.4).

Claim 1: *If M is in $\text{Tors } A$, then $\text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, M) = 0$.*

Proof of claim. To see this, let ϕ be an element of $\underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, M)$, for some $p \geq 0$. As A is right noetherian, $A_{\geq p}$ is finitely generated as a right ideal by some x_1, \dots, x_k . We can find $m \geq 0$ so that $\phi(x_i) \cdot A_{\geq m} = 0$ for all i , using that M is torsion. Since $\phi(x_i \cdot A_{\geq m}) = \phi(x_i) \cdot A_{\geq m} = 0$ and, picking m larger if necessary, $A_{\geq m+p} = (x_1, \dots, x_k)A_{\geq m}$, we have $\phi|_{A_{\geq m+p}} = 0$ and so $\phi = 0$ in $\text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, M)$. \square

Claim 2: *There are no nonzero morphisms from torsion modules to modules in the image of $\text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, -)$.*

Proof of claim. Let T be in $\text{Tors } A$ and let $g : T \rightarrow \text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, N)$ be a map. For $x \in T$, let $\phi \in \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, N)$ be a representative of $g(x)$, for some p . Pick m such that $x \cdot A_{\geq m} = 0$. We have $g(x) \cdot A_{\geq m} = g(x \cdot A_{\geq m}) = 0$, and $\phi \cdot A_{\geq m}$ represents $g(x) \cdot A_{\geq m}$. However, picking a larger m if necessary, we see $\phi \cdot A_{\geq m}$ is the image of ϕ under the map

$$\underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, N) \rightarrow \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq m+p}, N),$$

and so $\phi = 0$ in $\text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A/A_{\geq p}, N)$. Thus $g(x) = 0$. \square

To see that $\text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, -)$ induces a functor $\Gamma_* : \text{Qcoh } X \rightarrow \text{Gr } A$, it is enough to show that it takes morphisms f with $\ker f$ and $\text{coker } f$ in $\text{Tors } A$ to invertible morphisms. This follows from Claims 1 and 2, and two applications of the snake lemma. To show that Γ_* is right adjoint to the quotient and (4.4) is the unit, it is enough to show that any map $f : M \rightarrow \Gamma_*(\tilde{N})$ factors through $M \rightarrow \Gamma_*(\tilde{M})$. Note that by construction, we may extend (4.4) to an exact sequence:

$$0 \rightarrow \text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A/A_{\geq p}, M) \rightarrow M \rightarrow \text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, M) \rightarrow \text{colim}_{p \rightarrow \infty} \underline{\text{Ext}}_{\text{Gr } A}^1(A/A_{\geq p}, M) \rightarrow 0. \quad (4.6)$$

Since $\underline{\text{Hom}}_{\text{Gr } A}(A/A_{\geq p}, -) \cdot A_{\geq p} = 0$, subobjects and quotients of torsion modules are torsion, and the colimit of torsion modules is torsion, we see that the last term

$$\text{colim}_{p \rightarrow \infty} \underline{\text{Ext}}_{\text{Gr } A}^1(A/A_{\geq p}, M)$$

is in $\text{Tors } A$. To see that the map

$$f : M \rightarrow \Gamma_*(\tilde{N}) = \text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, N)$$

factors through $M \rightarrow \text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, M)$, by [Popescu 1973, 4.1], it is enough to show that there are no nonzero morphisms from torsion modules to modules in the image of Γ_* , which was shown in Claim 2.

We now show Γ_* is fully faithful. Let $\eta_M : M \rightarrow \Gamma_*\tilde{M}$ be the unit of the adjunction, which is the center arrow of (4.6). Since the outer two terms of that sequence are torsion, it follows that $\tilde{\eta}_M$ is an isomorphism. Let

$$\epsilon_{\tilde{M}} : (\Gamma_*\tilde{M}) \rightarrow \tilde{M}$$

be the counit of the adjunction. By definition, the composition

$$\tilde{M} \xrightarrow{\tilde{\eta}_M} (\Gamma_*\tilde{M}) \xrightarrow{\epsilon_{\tilde{M}}} \tilde{M}$$

is the identity. Thus $\epsilon_{\tilde{M}}$ is an isomorphism and so Γ_* is fully faithful. \square

Remark 4.7. As the notation suggests, if A is a commutative ring generated in degree 1, then $\Gamma_*(\tilde{-})$ is isomorphic to $\bigoplus_{i \in \mathbb{Z}} \Gamma(\text{Proj } A, \tilde{-})(i)$ as they are both right adjoint to sheafification $\text{Gr } A \rightarrow \text{Qcoh } X$.

It is clear from the definition that the functor $\tau(-)$ takes $\text{gr } A$ to $\text{tors } A$. However, Γ_* does not necessarily take objects of $\text{coh } X$ to $\text{gr } A$:

Example 4.8. Let $A = k[x]$ with k a field, graded by $|x| = 1$. The A -module structure on

$$\text{colim}_{p \rightarrow \infty} \underline{\text{Ext}}_{\text{Gr } A}^1(A/A_{\geq p}, A)$$

is easily computed: it has a k -basis e_1, \dots, e_n, \dots with $|e_n| = -n$ and $x e_n = e_{n-1}$. In particular it is not finitely generated over A and so from (4.6) we see that $\Gamma_*(\tilde{A})$ is not either.

In the example above, $(\Gamma_*(\tilde{A}))_{\geq i}$ is finitely generated (in fact of finite length) for any $i \in \mathbb{Z}$. Artin and Zhang gave a criterion for A -modules that is equivalent to this fact being true. It is often easy to check. For instance, it holds for all modules over commutative rings.

Definition 4.9 (Artin, Zhang). An object M in $\text{gr } A$ satisfies $\chi_j(M)$ if there exists an integer n_0 such that $\underline{\text{Ext}}_{\text{Gr } A}^k(A/A_{\geq n}, M)_{\geq i}$ is a finitely generated A -module for all $i \in \mathbb{Z}$, $k \leq j$ and all $n \geq n_0$. The ring satisfies condition χ_j if $\chi_j(M)$ holds for all $M \in \text{gr } A$.

If M satisfies $\chi_1(M)$, then [Artin and Zhang 1994, 3.8.3] shows that

$$\text{colim}_{p \rightarrow \infty} \underline{\text{Ext}}_{\text{Gr } A}^1(A/A_{\geq p}, M)_{\geq i}$$

is a finitely generated A -module for all $i \in \mathbb{Z}$. Thus by (4.6), we see that $\Gamma_{\geq i}(\tilde{M}) := (\Gamma_*(\tilde{M}))_{\geq i}$ is finitely generated.

Remark 4.10. As [Artin and Zhang 1994, 3.1.4] shows, if A is commutative, then every module M satisfies $\chi_j(M)$. Indeed, we can compute the A -module $\text{Ext}_{\text{gr } A}^j(A/A_{\geq p}, M)$ using a graded free resolution of $A/A_{\geq p}$, which we can assume to be finite in each degree. If A is not commutative then we must use the bimodule structure on $A/A_{\geq p}$ to compute the A -module structure on $\text{Ext}_{\text{gr } A}^j(A/A_{\geq p}, M)$, that is, in this case we must look at the derived functor of $\text{Hom}_{\text{gr } A}(A/A_{\geq p}, -)$ (rather than deriving in the first variable) and so we cannot necessarily use a free resolution of $A/A_{\geq p}$ to compute the A -module structure of $\text{Ext}_{\text{gr } A}^j(A/A_{\geq p}, M)$. In [Stafford and Zhang 1994], an example is given of a noncommutative graded noetherian domain A such that $\chi_j(A)$ does not hold for any $j > 0$.

Recall that $\text{gr}_{\geq i} A$ is the full subcategory of $\text{gr } A$ with objects those M with $M = M_{\geq i}$. We denote by $\text{Gr}_{\geq i} A$ the analogous subcategory of $\text{Gr } A$. Let $\text{Tors}_{\geq i} A = \text{Gr}_{\geq i} A \cap \text{Tors } A$ and $\text{tors}_{\geq i} A = \text{gr}_{\geq i} A \cap \text{tors } A$. The functor $\tau(-)$ restricted to $\text{Gr}_{\geq i} A$ (respectively, $\text{gr}_{\geq i} A$) is a right adjoint of the inclusion $\text{Tors}_{\geq i} A \rightarrow \text{Gr}_{\geq i} A$ (respectively, $\text{tors}_{\geq i} A \rightarrow \text{gr}_{\geq i} A$). Moreover, it is easy to check that the composition of the functors $\text{Gr}_{\geq i} A \rightarrow \text{Gr } A \rightarrow \text{Qcoh } X$ induces an equivalence $\text{Gr}_{\geq i} A / \text{Tors}_{\geq i} A \xrightarrow{\cong} \text{Qcoh } X$ and $\Gamma_{\geq i} = (\Gamma_*(-))_{\geq i}$ is a right adjoint to the quotient map. There is also an equivalence

$$\text{gr}_{\geq i} A / \text{tors}_{\geq i} A \xrightarrow{\cong} \text{coh } X.$$

Assume that A satisfies the condition χ_1 . Then, using the above, we have the following diagram where the vertical arrows are inclusions and the horizontal arrows form adjoint pairs with the left adjoint on top:

$$\begin{array}{ccccc} \text{Tors}_{\geq i} A & \xrightleftharpoons[\tau(-)]{\text{inc}} & \text{Gr}_{\geq i} A & \xrightleftharpoons[\Gamma_{\geq i}]{\tilde{(-)}} & \text{Qcoh } X \\ \uparrow & & \uparrow & & \uparrow \\ \text{tors}_{\geq i} A & \xrightleftharpoons[\tau(-)]{\text{inc}} & \text{gr}_{\geq i} A & \xrightleftharpoons[\Gamma_{\geq i}]{\tilde{(-)}} & \text{coh } X \end{array} \quad (4.11)$$

For any M in $\text{Gr}_{\geq i} A$, the counit and the unit are given by

$$0 \rightarrow \tau(M) \rightarrow M \cong \underline{\text{Hom}}_{\text{Gr } A}(A, M) \rightarrow (\text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A_{\geq p}, M))_{\geq i}, \quad (4.12)$$

as in the case of $\text{Gr } A$. Also note that $\Gamma_{\geq i}$ is fully faithful, as it is the right adjoint of a quotient functor.

We wish to extend this diagram to functors between the bounded derived categories of the above abelian categories. The existence of a corresponding localization sequence involving the derived categories is standard but we provide some details. We start with a simple lemma.

Lemma 4.13. *Let M be an object of $\text{Gr}_{\geq i} A$ and let $M \rightarrow I$ be an injective resolution in $\text{Gr} A$. Then $M \rightarrow I_{\geq i}$ is an injective resolution in $\text{Gr}_{\geq i} A$. In particular $\text{Gr}_{\geq i} A$ has enough injectives.*

Proof. The functor $(-)_{\geq i}$ is exact and $M_{\geq i} = M$, thus $M \rightarrow I_{\geq i}$ is a quasi-isomorphism. So to complete the proof it is sufficient to show $I_{\geq i}$ is a complex of injectives.

Let J be an injective object in $\text{Gr} A$. By adjunction there is an isomorphism of functors $\text{Hom}_{\text{Gr} A}(\text{inc}(-), J) \cong \text{Hom}_{\text{Gr}_{\geq i} A}(-, J_{\geq i})$. The former functor is exact as J is injective and the inclusion is exact, and thus so is the latter showing $J_{\geq i}$ is injective in $\text{Gr}_{\geq i} A$. \square

The functors to the right in (4.11) are exact and those to the left are left exact (since they are right adjoints). Since $\text{Gr}_{\geq i} A$ has enough injectives by the above lemma and $\text{Qcoh} X$ has enough injectives by [Artin and Zhang 1994, 7.1] (in fact, by standard abstract nonsense both of these categories are Grothendieck categories and so have enough injectives), we may form $\mathbf{R}\tau(-)$ and $\mathbf{R}\Gamma_{\geq i}$, the right derived functors of $\tau(-)$ and $\Gamma_{\geq i}$, respectively. This gives two pairs of adjoint functors

$$\mathbf{D}_{\text{Tors}_{\geq i} A}(\text{Gr}_{\geq i} A) \begin{array}{c} \xleftarrow{\text{inc}} \\ \xrightarrow{\mathbf{R}\tau(-)} \end{array} \mathbf{D}(\text{Gr}_{\geq i} A) \begin{array}{c} \xleftarrow{\widetilde{(-)}} \\ \xrightarrow{\mathbf{R}\Gamma_{\geq i}} \end{array} \mathbf{D}(\text{Qcoh} X), \quad (4.14)$$

where $\mathbf{D}_{\text{Tors}_{\geq i} A}(\text{Gr}_{\geq i} A)$ is the full subcategory of $\mathbf{D}(\text{Gr}_{\geq i} A)$ consisting of complexes with torsion cohomology.

Since $\Gamma_{\geq i}$ sends injectives to injectives and is fully faithful, one checks easily that $\mathbf{R}\Gamma_{\geq i}$ is also fully faithful. In particular, $\widetilde{(-)}$ is a quotient functor. As $\widetilde{(-)}$ at the level of the abelian categories is exact, the kernel of this functor at the level of derived categories consists of precisely those complexes whose cohomology is annihilated by $\widetilde{(-)}$, that is, it is exactly $\mathbf{D}_{\text{Tors}_{\geq i} A}(\text{Gr}_{\geq i} A)$. This proves the above functors give a localization sequence of triangulated categories.

It follows that for every $M \in \mathbf{D}(\text{Gr}_{\geq i} A)$ there is a localization triangle

$$\mathbf{R}\tau(M) \rightarrow M \rightarrow \mathbf{R}\Gamma_{\geq i}(\widetilde{M}) \rightarrow, \quad (4.15)$$

where the first map is the counit of the first adjunction of (4.14) and the second map is the unit of the second adjunction of (4.14).

Remark. When A is commutative, (4.15) can be constructed using the Cech complex.

For the above adjoint pairs to restrict to the bounded derived categories of complexes of finitely generated modules, we need to place two further restrictions on A . Let $\mathbf{R}\Gamma_* : D(\text{Qcoh } X) \rightarrow D(\text{Gr } A)$ be the right derived functor of the left exact functor Γ_* .

Definition 4.16 (Artin, Zhang). The *cohomological dimension* of A is

$$\text{cd}(A) := \sup\{d \mid H^d \mathbf{R}\Gamma_*(\tilde{A}) \neq 0\}.$$

By [Artin and Zhang 1994, 7.10], if $\text{cd}(A)$ is finite, then $\mathbf{R}\Gamma_*(\tilde{M})$ is a bounded complex for every $\tilde{M} \in \text{Qcoh } X$ and so restricts to a functor

$$\mathbf{R}\Gamma_* : D^b(\text{Qcoh } X) \rightarrow D^b(\text{Gr } A).$$

Since $\Gamma_{\geq i}$ is the composition of Γ_* and the exact functor $(-)_{\geq i}$, we see that $\mathbf{R}\Gamma_{\geq i} = (\mathbf{R}\Gamma_*)_{\geq i}$ and so $\mathbf{R}\Gamma_{\geq i}$ restricts to a functor

$$\mathbf{R}\Gamma_{\geq i} : D^b(\text{Qcoh } X) \rightarrow D^b(\text{Gr}_{\geq i} A).$$

By the long exact sequence in homology induced by (4.15), we see that $\mathbf{R}\tau$ also restricts to a functor between bounded derived categories.

Now we consider finiteness. We want to compute the cohomology of $\mathbf{R}\tau(M)$. We view $\tau(-) = \text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A/A_{\geq p}, -)$ as a functor $\text{gr}_{\geq i} A \rightarrow \text{tors}_{\geq i} A$. For M in $\text{gr}_{\geq i} A$, let $M \rightarrow I_M$ be an injective resolution in $\text{Gr } A$. Then $(I_M)_{\geq i}$ is an injective resolution in $\text{Gr}_{\geq i} A$. Thus we have

$$\begin{aligned} \mathbf{R}\tau(M) &= \text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A/A_{\geq p}, (I_M)_{\geq i}) \\ &= \text{colim}_{p \rightarrow \infty} \underline{\text{Hom}}_{\text{Gr } A}(A/A_{\geq p}, I_M)_{\geq i}, \end{aligned}$$

where the second equality follows from the commutativity of the square of inclusions

$$\begin{array}{ccc} \text{Tors}_{\geq i} A & \longrightarrow & \text{Gr}_{\geq i} A \\ \downarrow & & \downarrow \\ \text{Tors } A & \longrightarrow & \text{Gr } A \end{array}$$

by taking right adjoints. This shows that

$$H^k \mathbf{R}\tau(M) \cong \text{colim}_{p \rightarrow \infty} \underline{\text{Ext}}_{\text{Gr } A}^k(A/A_{\geq p}, M)_{\geq i},$$

for all $k \geq 0$. By [Artin and Zhang 1994, 3.8.3], if M satisfies $\chi_j(M)$, then $\text{colim}_{p \rightarrow \infty} \underline{\text{Ext}}_{\text{Gr } A}^k(A/A_{\geq p}, M)_{\geq i}$ is a finitely generated A -module for all $k \leq j$ and all $p \in \mathbb{Z}$.

Assume now that A has finite cohomological dimension and satisfies χ_j for all $j \geq 0$. The above shows that $\mathbf{R}\tau(-)$ restricts to a functor

$$\mathbf{R}\tau(-) : D^b(\text{gr}_{\geq i} A) \rightarrow D_{\text{tors}_{\geq i} A}^b(\text{gr}_{\geq i} A).$$

By the long exact sequence in cohomology coming from (4.15), we see that we also have a functor

$$\mathbf{R}\Gamma_{\geq i} : D^b(\text{coh } X) \rightarrow D^b(\text{gr}_{\geq i} A).$$

This gives the following diagram of adjoint functors, where the top functors are the left adjoints:

$$D_{\text{tors}_{\geq i} A}^b(\text{gr}_{\geq i} A) \begin{array}{c} \xleftarrow{\text{inc}} \\ \xrightarrow{\mathbf{R}\tau(-)} \end{array} D^b(\text{gr}_{\geq i} A) \begin{array}{c} \xleftarrow{(\tilde{-})} \\ \xrightarrow{\mathbf{R}\Gamma_{\geq i}(-)} \end{array} D^b(\text{coh } X).$$

The functor $\mathbf{R}\Gamma_{\geq i}$ is fully faithful and its image is left admissible. Also, any object in this image is contained in $(D_{\text{tors}_{\geq i} A}^b(\text{gr}_{\geq i} A))^\perp$. Indeed, for M an object with torsion cohomology and any $N \in D^b(\text{gr}_{\geq i} A)$, we have

$$\text{Hom}_{D^b(\text{gr}_{\geq i} A)}(M, \mathbf{R}\Gamma_{\geq i} \tilde{N}) \cong \text{Hom}_{D^b(\text{coh } X)}(\tilde{M}, \tilde{N}) = 0,$$

since $\tilde{M} \simeq 0$. From this containment and the triangle (4.15), we may apply Lemma 2.6 to see that there is a semiorthogonal decomposition

$$D^b(\text{gr}_{\geq i} A) = (\mathbf{R}\Gamma_{\geq i} D^b(\text{coh } X), D_{\text{tors}_{\geq i} A}^b(\text{gr}_{\geq i} A)).$$

Recall that $\mathcal{S}_{\geq i}$ is the thick subcategory generated by $A_0(e)$ for all $e \leq -i$.

Lemma 4.17. *There is an equality $\mathcal{S}_{\geq i} = D_{\text{tors}_{\geq i} A}^b(\text{gr}_{\geq i} A)$.*

Proof. It is clear that $A_0(e)$ is in $\text{tors}_{\geq i} A$ for all $e \leq -i$, so $\mathcal{S}_{\geq i}$ is contained in $D_{\text{tors}_{\geq i} A}^b(\text{gr}_{\geq i} A)$. Given M in $D_{\text{tors}_{\geq i} A}^b(\text{gr}_{\geq i} A)$, we have that $H^*(M)$ is finitely generated and torsion, thus M must have cohomology in only finitely many degrees. Analogously to the proof of Lemma 3.2, this shows that M is in $\mathcal{S}_{\geq i}$. \square

The above shows the following:

Proposition 4.18. *Let A be a positively graded right noetherian ring that satisfies condition χ and has finite cohomological dimension. Then there is a semiorthogonal decomposition*

$$D^b(\text{gr}_{\geq i} A) = (\mathbf{R}\Gamma_{\geq i} D^b(\text{coh } X), \mathcal{S}_{\geq i}).$$

The corresponding localization triangle is given by (4.15).

5. Singularity category of a Gorenstein ring

In this section we assume that $A = \bigoplus_{i \geq 0} A_i$ is a positively graded (two-sided) noetherian ring with A_0 of finite global dimension, but not necessarily commutative. We denote by $\text{id}_A M$ the graded injective dimension of a graded module M .

Definition 5.1. A ring A is (*Artin–Schelter*) *Gorenstein* if $\mathrm{id}_A A$ and $\mathrm{id}_{A^{\mathrm{op}}} A$ are finite and

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathrm{gr} A}(A_0, A) \cong A_0[n](a) \quad \text{for some } n, a \in \mathbb{Z}$$

in both $D^b(\mathrm{gr} A)$ and $D^b(\mathrm{gr} A^{\mathrm{op}})$. The unique integer a is the a -invariant of A .

Remark 5.2. In [Minamoto and Mori 2011] a different definition of Artin–Schelter Gorenstein ring is given under the restriction that A_0 is a finite dimensional algebra over a fixed base field k . Their definition differs from ours in two ways: Minamoto and Mori require the shift occurring to match the injective dimension of A , that is, $n = -\mathrm{id}_A A$, and that rather than $\mathbf{R}\underline{\mathrm{Hom}}_{\mathrm{gr} A}(A_0, A) \cong A_0[n](a)$ one asks for an isomorphism

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathrm{gr} A}(A_0, A) \cong \mathrm{Hom}_k(A_0, k)[n](a).$$

We note both definitions restrict to the classical one in the case $A_0 = k$.

As an example, if R is a commutative regular ring of positive Krull dimension and we set $A = R[x]/(x^n)$ with x in degree 1 then A is AS-Gorenstein in our sense but in that of Minamoto and Mori. On the other hand their definition covers certain (higher) preprojective algebras which are in general excluded by ours.

From this point forward we will use the term *Gorenstein ring* to refer to a ring that is Gorenstein either in the sense of Definition 5.1 or [Minamoto and Mori 2011]. Our results hold for both definitions. We will work with the definition we give and, when necessary, point out what changes in the arguments are necessary if one uses the definition of Minamoto and Mori. In fact, the only place in which the arguments do not go through verbatim are Lemmas 6.2 and 6.6 which require minor tweaking.

The most important feature of Gorenstein rings for us is the duality given below. We will make a standard abuse of notation and not differentiate between the two duality functors notationally.

Lemma 5.3. *Assume that A is a Gorenstein ring. Then the functors*

$$\begin{aligned} D &= \mathbf{R}\underline{\mathrm{Hom}}_{\mathrm{gr} A}(-, A) : D^b(\mathrm{gr} A) \rightarrow D^b(\mathrm{gr} A^{\mathrm{op}})^{\mathrm{op}}, \\ D &= \mathbf{R}\underline{\mathrm{Hom}}_{\mathrm{gr} A^{\mathrm{op}}}(-, A) : D^b(\mathrm{gr} A^{\mathrm{op}}) \rightarrow D^b(\mathrm{gr} A)^{\mathrm{op}}, \end{aligned}$$

are quasiinverse equivalences.

Proof. We first observe that D does indeed take $D^b(\mathrm{gr} A)$ to $D^b(\mathrm{gr} A^{\mathrm{op}})^{\mathrm{op}}$. Since A has finite injective dimension as both a left and a right module over itself it is clear that D preserves boundedness of cohomology. It is also clear that D sends complexes with finitely generated cohomology groups to the same as we can

resolve any object of $D^b(\text{gr } A)$ by a complex of finitely generated projectives and A is noetherian.

These functors are adjoint so we can consider the unit of this adjunction

$$\eta: \text{Id} \rightarrow D^2,$$

and we need to show it is an equivalence. But this is again clear: for a bounded above complex of finitely generated projectives the map η is just componentwise the natural map to the double dual and finitely generated projectives are reflexive. \square

Recall that $\mathcal{P}_{\geq i}$ is the thick subcategory of $D^b(\text{gr } A)$ generated by those $A(e)$ with $e \leq -i$ and $\mathcal{P}_{< i}$ is the thick subcategory generated by the $A(e)$ with $e > -i$.

Lemma 5.4. *If A is Gorenstein, there is a semiorthogonal decomposition*

$$D^b(\text{gr}_{\geq i} A) = (\mathcal{P}_{\geq i}, (\mathcal{P}_{\geq i})^\perp \cap D^b(\text{gr}_{\geq i} A)).$$

For $M \in D^b(\text{gr}_{\geq i} A)$, the localization triangle is given by

$$D(G)_{< i} \rightarrow M \cong D(G) \rightarrow D(G)_{\geq i} \rightarrow ,$$

where the notation is as in Definition 3.9, and $G \rightarrow D(M)$ is a projective resolution of the dual of M as in Lemma 3.10.

Proof. Let M be an object of $D^b(\text{gr}_{\geq i} A)$ and let $G \rightarrow D(M)$ be a projective resolution as in Lemma 3.10, where $D(M) = \mathbf{R}\underline{\text{Hom}}_{\text{gr } A}(M, A)$ is the image of M under the duality functor. As in the proof of Lemma 3.7, there is a triangle

$$G_{< -i+1} \rightarrow G \rightarrow G_{\geq -i+1} \rightarrow ,$$

where $G_{< -i+1}$ is an object of $\mathcal{P}_{< -i+1}$ and every component of $G_{\geq -i+1}$ is generated in degree at least $-i+1$. If $P = P_0 \otimes_{A_0} A(e)$ is any indecomposable graded projective A -module, then

$$D(P) \cong \underline{\text{Hom}}_{\text{gr } A}(P, A) \cong (P_0)^* \otimes_{A_0} A(-e),$$

where $(P_0)^*$ is the A_0 -dual of P_0 . Thus

$$D(G_{< -i+1}) = D(G)_{\geq i} \quad \text{and} \quad D(G_{\geq -i+1}) = D(G)_{< i}.$$

Applying D to the triangle above gives a triangle

$$D(G)_{< i} \rightarrow D(G) \rightarrow D(G)_{\geq i} \rightarrow .$$

Note that $D(G)_{\geq i}$ is in $\mathcal{P}_{\geq i}$ and that there are isomorphisms

$$M \xrightarrow{\cong} D(D(M)) \xrightarrow{\cong} D(G).$$

We can now apply Lemma 2.6, once we show that $D(G)_{<i}$ is in

$$({}^\perp \mathcal{P}_{\geq i}) \cap D^b(\text{gr}_{\geq i} A).$$

It follows from the long exact sequence in homology of the above triangle that each of the homology groups of $D(G)_{<i}$ is generated in degrees at least i and thus $D(G)_{<i}$ is in $D^b(\text{gr}_{\geq i} A)$. That $D(G)_{<i}$ is in ${}^\perp(\mathcal{P}_{\geq i})$ follows from the fact that $\text{Hom}_{\text{gr } A}(A(e), A(f)) = 0$ for $e > f$. \square

Let us denote by \mathcal{B}_i the subcategory $({}^\perp \mathcal{P}_{\geq i}) \cap D^b(\text{gr}_{\geq i} A)$, which by the above lemma is a right admissible subcategory of $D^b(\text{gr}_{\geq i} A)$. There is a description of \mathcal{B}_i using the well-known singularity category of A .

Definition 5.5. Let A be a graded ring.

- (1) An object M in $D^b(\text{gr } A)$ is *perfect* if M is in the thick subcategory generated by $A(e)$ for all $e \in \mathbb{Z}$, that is, it is quasi-isomorphic to a bounded complex of projectives. We denote the subcategory of perfect complexes by $\text{perf}(A)$. We see from the definitions that $\text{perf}(A) = \langle \mathcal{P}_{\geq i}, \mathcal{P}_{<i} \rangle$.

- (2) The *singularity category* of A is

$$D_{\text{sg}}^b(\text{gr } A) := D^b(\text{gr } A) / \text{perf}(A).$$

Orlov showed that when A is a connected graded Gorenstein algebra over a field, there is an embedding of $D_{\text{sg}}^b(\text{gr } A)$ in $D^b(\text{gr } A)$ for every $i \in \mathbb{Z}$, and the image is equal to \mathcal{B}_i . We now show this holds in the generality in which we are working. First we recall a lemma whose proof is left to the reader.

Lemma 5.6. *Let \mathcal{A} be a left admissible subcategory in a triangulated category \mathcal{T} with $i_L : \mathcal{T} \rightarrow \mathcal{A}$ the left adjoint to the inclusion $i : \mathcal{A} \rightarrow \mathcal{T}$. Then i_L induces an equivalence*

$$\mathcal{T} / {}^\perp \mathcal{A} \rightarrow \mathcal{A},$$

with inverse equivalence the composition $\mathcal{A} \rightarrow \mathcal{T} \rightarrow \mathcal{T} / {}^\perp \mathcal{A}$. The analogous statement holds for right admissible subcategories.

Applying the above lemma to Lemma 3.7 shows that there is an equivalence

$$\psi_i : D^b(\text{gr } A) / \mathcal{P}_{<i} \xrightarrow{\cong} D^b(\text{gr}_{\geq i} A).$$

Remark 3.12 shows that $\psi_i(M) = P_{\geq i}$, where $P \rightarrow M$ is a projective resolution as in Lemma 3.10. If we apply Lemma 5.6 again to the semiorthogonal decomposition $D^b(\text{gr}_{\geq i} A) = (\mathcal{P}_{\geq i}, \mathcal{B}_i)$, we have an equivalence

$$\phi_i : D^b(\text{gr}_{\geq i} A) / \mathcal{P}_{\geq i} \xrightarrow{\cong} \mathcal{B}_i,$$

with $\phi_i(N) = D(Q)_{<i}$, where $Q \rightarrow D(N)$ is a projective resolution as in Lemma 3.10. Let us set $\mathbf{b}_i = \phi_i \circ \pi \circ \psi_i$ where $\pi : D^b(\text{gr}_{\geq i} A) \rightarrow D^b(\text{gr}_{\geq i} A)/\mathcal{P}_{\geq i}$ is the quotient functor. This gives an equivalence

$$\mathbf{b}_i : D_{\text{sg}}^b(\text{gr} A) = D^b(\text{gr} A)/\langle \mathcal{P}_{<i}, \mathcal{P}_{\geq i} \rangle \xrightarrow{\cong} \mathcal{B}_i, \quad (5.7)$$

with $\mathbf{b}_i(M) = D(Q)_{<i}$, where $Q \rightarrow D(P_{\geq i})$ and $P \rightarrow M$ are projective resolutions as in Lemma 3.10. The inverse of the equivalence is given by the composition of the inclusion and quotient $\mathcal{B}_i \rightarrow D^b(\text{gr} A) \rightarrow D^b(\text{gr} A)/\text{perf}(A)$. Moreover, \mathbf{b}_i followed by the inclusion $\mathcal{B}_i \rightarrow D^b(\text{gr}_{\geq i} A)$ is left adjoint to the quotient functor

$$D^b(\text{gr}_{\geq i} A) = D^b(\text{gr} A)/\mathcal{P}_{<i} \rightarrow D^b(\text{gr} A)/\langle \mathcal{P}_{<i}, \mathcal{P}_{\geq i} \rangle = D_{\text{sg}}^b(\text{gr} A).$$

To sum up, we have shown the following:

Proposition 5.8. *If A is a graded Gorenstein ring, the quotient*

$$D^b(\text{gr}_{\geq i} A) \rightarrow D_{\text{sg}}^b(\text{gr} A)$$

has a fully faithful left adjoint

$$\mathbf{b}_i : D_{\text{sg}}^b(\text{gr} A) \rightarrow D^b(\text{gr}_{\geq i} A).$$

The image of \mathbf{b}_i is the subcategory $\mathcal{B}_i = (\perp \mathcal{P}_{\geq i}) \cap D^b(\text{gr}_{\geq i} A)$ and there is a semiorthogonal decomposition:

$$D^b(\text{gr}_{\geq i} A) = (\mathcal{P}_{\geq i}, \mathcal{B}_i).$$

The localization triangle is described in Lemma 5.4.

6. Relating the bounded derived category of coherent sheaves and the singularity category

In this section we prove the main theorem by comparing the semiorthogonal decompositions constructed in the previous sections. We assume that $A = \bigoplus_{i \geq 0} A_i$ is a positively graded noetherian Gorenstein ring with A_0 a ring of finite global dimension, but not necessarily commutative.

Gorenstein rings often satisfy the two properties we need to apply Proposition 4.18.

Lemma 6.1. *If A is a Gorenstein ring, then A has finite cohomological dimension.*

Proof. We need to show

$$\text{cd}(A) = \sup\{d \mid H^d \mathbf{R}\Gamma_*(\tilde{A}) \neq 0\} < \infty.$$

Since A is Gorenstein we can choose a bounded injective resolution I for A as a right A -module. Hence $\mathbf{R}\tau(A) = \tau(I)$ has bounded cohomology and the localization triangle

$$\mathbf{R}\tau(A) \rightarrow A \rightarrow \mathbf{R}\Gamma_*(\tilde{A}) \rightarrow$$

then implies $\mathbf{R}\Gamma_*(\tilde{A})$ also has bounded cohomology. \square

We have remarked earlier that any commutative ring satisfies condition χ . The next lemma gives some noncommutative and not necessarily graded connected examples.

Lemma 6.2. *Let k be a commutative ring and A a flat Gorenstein k -algebra. Then A satisfies condition χ .*

Proof. As A is flat over k it follows that the enveloping algebra $A \otimes_k A^{\text{op}}$ is flat over both A and A^{op} . Thus the restriction of scalars functors induced by the maps $A \rightarrow A \otimes_k A^{\text{op}}$ and $A^{\text{op}} \rightarrow A \otimes_k A^{\text{op}}$ preserve injectives. Taking an injective resolution I of A over $A \otimes_k A^{\text{op}}$ thus gives a bimodule resolution of A which is an injective resolution as both a complex of left and of right A -modules. We may use such a resolution to compute $D = \mathbf{R}\underline{\text{Hom}}_{\text{gr } A}(-, A)$ as $\underline{\text{Hom}}_{\text{gr } A}(-, I)$, and obtain the correct A^{op} -module structure and similarly for the inverse duality functor; this is just a consequence of the fact that I and A are quasi-isomorphic as complexes of bimodules. Given a complex of injectives $M \in D^b(\text{gr } A)$ we now compute, using the duality of Lemma 5.3, that there are quasi-isomorphisms of right A_0 -modules

$$\begin{aligned} \underline{\text{Hom}}_{\text{gr } A}(A_0, M) &\cong \mathbf{R}\underline{\text{Hom}}_{\text{gr } A^{\text{op}}}(\underline{\text{Hom}}_{\text{gr } A}(M, I), \underline{\text{Hom}}_{\text{gr } A}(A_0, I)) \\ &\cong \underline{\text{Hom}}_{\text{gr } A^{\text{op}}}(P, \underline{\text{Hom}}_{\text{gr } A}(A_0, I)) \\ &\cong \underline{\text{Hom}}_{\text{gr } A^{\text{op}}}(P, {}_\nu A_0[n](a)), \end{aligned}$$

where P is a projective resolution of $\underline{\text{Hom}}_{\text{gr } A}(M, I)$ over A^{op} and ν is a twist by some, possibly nontrivial, automorphism which needs to be accounted for as we view $\underline{\text{Hom}}_{\text{gr } A}(A_0, I)$ as a bimodule rather than just a right module (see, for example, [Minamoto and Mori 2011, Lemma 2.9]). Now $\underline{\text{Hom}}_{\text{gr } A^{\text{op}}}(P, \Sigma^n {}_\nu A_0(a))$ is a complex of finitely generated A_0 -modules and so in particular has finitely generated cohomology over A_0 and hence over A . In particular, if M is an injective resolution of a right A module N this shows $\underline{\text{Ext}}_{\text{gr } A}^i(A_0, N)$ is finitely generated over A for all $i \in \mathbb{Z}$.

It only remains to observe that $A/A_{\geq n}$ has a filtration, as bimodules, by copies of $A_0(j)$ for $j \in \mathbb{Z}$ and considering the corresponding long exact sequences shows $\mathbf{R}\underline{\text{Hom}}_{\text{gr } A}(A/A_{\geq n}, M)$ has finitely generated cohomology for all $M \in D^b(\text{gr } A)$. Hence A satisfies condition χ . \square

Remark 6.3. In Lemma 6.2, if A is AS-Gorenstein in the sense of [Minamoto and Mori 2011] then one has to replace ${}_v A_0(a)$ by $\text{Hom}_k({}_v A_0(a), k)$ but this does not alter the argument as $\underline{\text{Hom}}_{\text{gr } A^{\text{op}}}(P, \Sigma^n \text{Hom}_k({}_v A_0(a), k))$ is still a complex of finitely generated A_0 -modules.

Theorem 6.4. *Let $A = \bigoplus_{i \geq 0} A_i$ be a positively graded noetherian Gorenstein ring with A_0 of finite global dimension, but not necessarily commutative. We assume in addition that A satisfies condition χ . Let a be the a -invariant of A defined in Definition 5.1.*

(1) *If $a > 0$, then for every $i \in \mathbb{Z}$ there is a semiorthogonal decomposition*

$$\text{D}^b(\text{coh } X) = (\mathcal{O}(-i - a + 1), \dots, \mathcal{O}(-i), \tilde{\mathcal{B}}_i),$$

where $\mathcal{O}(j)$ is the image of $A(j)$ in $\text{coh } X$ and \mathcal{B}_i is the image of $\text{D}_{\text{sg}}^b(\text{gr } A)$ under the fully faithful functor $\mathbf{b}_i : \text{D}_{\text{sg}}^b(\text{gr } A) \rightarrow \text{D}^b(\text{gr}_{\geq i} A)$ described in (5.7).

(2) *If $a < 0$, then for every $i \in \mathbb{Z}$ there is a semiorthogonal decomposition*

$$\text{D}_{\text{sg}}^b(\text{gr } A) = (pA_0(-i), \dots, pA_0(-i + a + 1), p\mathbf{R}\Gamma_{\geq i-a} \text{D}^b(\text{coh } X)),$$

where $p : \text{D}^b(\text{gr}_{\geq i} A) \rightarrow \text{D}_{\text{sg}}^b(\text{gr } A)$ is the canonical quotient.

(3) *If $a = 0$, then for every $i \in \mathbb{Z}$ the functors $(\tilde{})\mathbf{b}_i : \text{D}_{\text{sg}}^b(\text{gr } A) \rightarrow \text{D}^b(\text{coh } X)$ and $p\mathbf{R}\Gamma_{\geq i} : \text{D}^b(\text{coh } X) \rightarrow \text{D}_{\text{sg}}^b(\text{gr } A)$ are inverse equivalences.*

Before beginning the proof, we need two lemmas. For the rest of the section we rely heavily on notation introduced earlier: recall that $\mathcal{S}_{< i}$ (respectively $\mathcal{S}_{\geq i}$) is the thick subcategory generated by the objects $A_0(e)$, for all $e > -i$ (respectively $e \leq -i$) and $\mathcal{P}_{< i}$ (respectively $\mathcal{P}_{\geq i}$) is the thick subcategory generated by the objects $A(e)$ for all $e > -i$ (respectively $e \leq -i$).

Lemma 6.5. *Let A be a graded ring.*

(1) *For every $i \in \mathbb{Z}$ there is a semiorthogonal decomposition*

$$\mathcal{P}_{\geq i} = (\mathcal{P}_{\geq i+1}, A(-i)).$$

(2) *For every $i \in \mathbb{Z}$ there is a semiorthogonal decomposition*

$$\mathcal{S}_{< i+1} = (\mathcal{S}_{< i}, A_0(-i)).$$

Proof. It is clear that $A(-i) \subseteq {}^\perp \mathcal{P}_{\geq i+1}$. We know that any object in $\mathcal{P}_{\geq i}$ is isomorphic in $\text{D}^b(\text{gr}_{\geq i} A)$ to a bounded complex X of finitely generated graded projective modules and so we may restrict ourselves to working with such complexes. As in the proof of Lemma 3.7, using the structure of graded

projectives given in Lemma 3.8 and the notation of Definition 3.9, we see that there is a short exact sequence of complexes, split in each degree

$$0 \rightarrow X_{<i+1} \rightarrow X \rightarrow X_{\geq i+1} \rightarrow 0,$$

where $X_{<i+1}$ is the subcomplex of X which is termwise the projective summands generated in degree i and $X_{\geq i+1}$ is the quotient complex which is termwise all those projective summands generated in degree at least $i+1$. This gives a triangle

$$X_{<i+1} \rightarrow X \rightarrow X_{\geq i+1} \rightarrow ,$$

with $X_{<i+1}$ in the thick subcategory generated by $A(-i)$ and $X_{\geq i+1}$ in $\mathcal{P}_{\geq i+1}$. By Lemma 2.6 we have proved part 1.

We have $A_0(-i) \in {}^\perp \mathcal{S}_{<i}$, since $\mathbf{R}\mathrm{Hom}_{\mathrm{gr} A}(A_0(e), A_0(f)) \simeq 0$ for all $e < f$. Indeed, we may find a graded free resolution of $A_0(e)$ that exists entirely in degrees at least e . For any $X \in \mathcal{S}_{<i+1}$, we have the triangle

$$X_{\geq i} \rightarrow X \rightarrow X/X_{\geq i} \rightarrow ,$$

as in Lemma 3.4. Since $X_{\geq i+1} = 0$ we see $X_{\geq i}$ has cohomology concentrated in grading degree i and so is in the thick subcategory generated by $A_0(-i)$. On the other hand $X/X_{\geq i}$ is killed by $(-)_{\geq i}$ so is in $\mathcal{S}_{<i}$ by Lemma 3.2. Applying Lemma 2.6 now proves part 2. \square

For the sake of clarity we introduce the following notation for the next lemma. We denote by $\mathcal{S}_{<i}(A)$ and $\mathcal{S}_{<i}(A^{\mathrm{op}})$ the thick subcategories generated by the $A_0(e)$, for all $e > -i$, in $D^b(\mathrm{gr} A)$ and $D^b(\mathrm{gr} A^{\mathrm{op}})$ respectively. We use similar notation for $\mathcal{S}_{\geq i}$, $\mathcal{P}_{\geq i}$, and $\mathcal{P}_{<i}$ in order to indicate in which category we are working.

Lemma 6.6. *Under the hypothesis of Theorem 6.4, we have $\mathcal{S}_{\geq i}^\perp = {}^\perp \mathcal{P}_{\geq i+a}$ as subcategories of $D^b(\mathrm{gr}_{\geq i} A)$.*

Proof. As A is Gorenstein we have Grothendieck duality by Lemma 5.3. We note that restricting the duality functor $D = \mathbf{R}\underline{\mathrm{Hom}}_{\mathrm{gr} A}(-, A)$ to $\mathcal{S}_{\geq i}$ gives an equivalence

$$D: \mathcal{S}_{\geq i}(A) \xrightarrow{\cong} (\mathcal{S}_{<-i-a+1}(A^{\mathrm{op}}))^{\mathrm{op}}.$$

Indeed, one can see this simply by computing D applied to the generators and observing equivalences send thick subcategories to thick subcategories. Similarly we can also restrict D to get an equivalence

$$D: \mathcal{P}_{<-i-a+1}(A) \xrightarrow{\cong} (\mathcal{P}_{\geq i+a}(A^{\mathrm{op}}))^{\mathrm{op}}.$$

By Lemmas 3.4, 3.7, and the definition of a semiorthogonal decomposition, we have in $D^b(\text{gr } A)$

$${}^\perp \mathcal{S}_{<-i-a+1}(A) = D^b(\text{gr } A_{\geq -i-a+1}) = \mathcal{P}_{<-i-a+1}(A)^\perp.$$

We thus have

$$\begin{aligned} \mathcal{S}_{\geq i}(A)^\perp &\xrightarrow{\cong} ((\mathcal{S}_{<-i-a+1}(A^{\text{op}}))^{\text{op}})^\perp \\ &= ({}^\perp \mathcal{S}_{<-i-a+1}(A^{\text{op}}))^{\text{op}} \\ &= (\mathcal{P}_{<-i-a+1}(A^{\text{op}})^\perp)^{\text{op}} \\ &= {}^\perp ((\mathcal{P}_{<-i-a+1}(A^{\text{op}}))^{\text{op}}) \xrightarrow{\cong} {}^\perp \mathcal{P}_{\geq i+a}(A), \end{aligned}$$

that is, the functor D^2 , which is isomorphic to the identity functor, takes $\mathcal{S}_{\geq i}(A)^\perp$ to ${}^\perp \mathcal{P}_{\geq i+a}(A)$ and hence these categories are equal. \square

Remark 6.7. If A is AS-Gorenstein in the sense of [Minamoto and Mori 2011] then one needs a minor additional argument to prove the above lemma. We need to check $D(\mathcal{S}_{\geq i}(A))$, the thick subcategory of $D^b(\text{gr } A^{\text{op}})^{\text{op}}$ generated by the $D(A_0(e))$ for $e \leq -i$, is $(\mathcal{S}_{<-i-a+1}(A^{\text{op}}))^{\text{op}}$. By definition

$$\mathbf{R} \text{Hom}_{\text{gr } A}(A_0(e), A) \cong \text{Hom}_k(A_0, k)[-n](-e+a),$$

and it is sufficient to check this object generates the same thick subcategory as $A_0(-e+a)$ (of course we can ignore the degree shift). This follows essentially immediately from the equivalence

$$\text{Hom}_k(-, k): D^b(\text{mod } A_0) \xrightarrow{\cong} D^b(\text{mod } A_0^{\text{op}})^{\text{op}},$$

which sends the generator A_0 to $\text{Hom}_k(A_0, k)$.

Proof of Theorem 6.4. Combining the decompositions of Lemma 3.4 and Proposition 5.8 via Lemma 2.8, there is a semiorthogonal decomposition

$$D^b(\text{gr } A) = (\mathcal{S}_{<i}, \mathcal{P}_{\geq i}, \mathcal{B}_i). \quad (6.8)$$

Similarly, by Lemma 3.4, Proposition 4.18, and Lemma 2.8, there is a semiorthogonal decomposition

$$D^b(\text{gr } A) = (\mathcal{S}_{<i}, \mathbf{R}\Gamma_{\geq i} D^b(\text{coh } X), \mathcal{S}_{\geq i}).$$

Using Lemma 6.6, we see that ${}^\perp \mathcal{P}_{\geq i+a} = \mathcal{S}_{\geq i}^\perp = (\mathcal{S}_{<i}, \mathbf{R}\Gamma_{\geq i} D^b(\text{coh } X))$, and thus there is a semiorthogonal decomposition

$$D^b(\text{gr } A) = (\mathcal{P}_{\geq i+a}, \mathcal{S}_{<i}, \mathbf{R}\Gamma_{\geq i} D^b(\text{coh } X)). \quad (6.9)$$

The rest of the proof boils down to comparing the decompositions (6.8) and (6.9), depending on the sign of a .

Assume first that $a \geq 0$. Then $\mathcal{P}_{\geq i+a} \subseteq D^b(\text{gr}_{\geq i} A)$ by definition, and $D^b(\text{gr}_{\geq i} A) = {}^\perp \mathcal{S}_{< i}$ by Lemma 3.4. Hence the first two factors of (6.9) are mutually orthogonal and we may swap them to get

$$D^b(\text{gr} A) = (\mathcal{S}_{< i}, \mathcal{P}_{\geq i+a}, \mathbf{R}\Gamma_{\geq i} D^b(\text{coh} X)). \quad (6.10)$$

Comparing with (6.8) we see that

$$D^b(\text{gr}_{\geq i} A) = (\mathcal{P}_{\geq i}, \mathcal{B}_i) = (\mathcal{P}_{\geq i+a}, \mathbf{R}\Gamma_{\geq i} D^b(\text{coh} X)).$$

By Lemma 6.5 there is a decomposition

$$\mathcal{P}_{\geq i} = (\mathcal{P}_{\geq i+a}, A(-i-a+1), \dots, A(-i+1), A(-i)).$$

It follows there is an equality in $D^b(\text{gr}_{\geq i} A)$:

$$(A(-i-a+1), \dots, A(-i+1), A(-i), \mathcal{B}_i) = \mathbf{R}\Gamma_{\geq i} D^b(\text{coh} X).$$

Applying (\sim) to both sides gives the semiorthogonal decomposition

$$D^b(\text{coh} X) = (\mathcal{O}(-i-a+1), \dots, \mathcal{O}(-i), \tilde{\mathcal{B}}_i).$$

Assume now that $a \leq 0$. In this case, $\mathcal{P}_{\geq i} \subseteq \mathcal{S}_{< i}^\perp$, that is,

$$\text{Hom}_{D^b(\text{gr} A)}(\mathcal{S}_{< i}, \mathcal{P}_{\geq i}) = 0.$$

To see this, it is enough to check that $\text{Hom}_{D^b(\text{gr} A)}(A_0(e)[m], A(f)) = 0$ for all $e > -i$, $f \leq -i$ and all $m \in \mathbb{Z}$. But

$$\begin{aligned} \text{Hom}_{D^b(\text{gr} A)}(A_0(e)[m], A(f)) &\cong \text{Hom}_{D^b(\text{gr} A)}(A_0, A)(f-e)[-m] \\ &= (H^{-m} \mathbf{R}\underline{\text{Hom}}_{\text{gr} A}(A_0, A))_{f-e}. \end{aligned}$$

By the definition of Gorenstein, this is nonzero if and only if $-m = n$ and $f - e = -a$, however $f - e < 0$ and $-a \geq 0$. Thus we may switch the order of the first two factors of (6.8) and we have a semiorthogonal decomposition

$$D^b(\text{gr} A) = (\mathcal{P}_{\geq i}, \mathcal{S}_{< i}, \mathcal{B}_i). \quad (6.11)$$

We also have, substituting $i - a$ for i in (6.9),

$$D^b(\text{gr} A) = (\mathcal{P}_{\geq i}, \mathcal{S}_{< i-a}, \mathbf{R}\Gamma_{\geq i-a} D^b(\text{coh} X)).$$

This shows that

$$(\mathcal{S}_{< i}, \mathbf{b}D_{\text{sg}}^b(\text{gr} A)) = (\mathcal{S}_{< i-a}, \mathbf{R}\Gamma_{\geq i-a} D^b(\text{coh} X)).$$

By Lemma 6.5 we have

$$\mathcal{S}_{< i-a} = (\mathcal{S}_{< i}, A_0(-i), A_0(-i-1), \dots, A_0(-i+a+1)).$$

Thus we have

$$\mathcal{B}_i = (A_0(-i), A_0(-i-1), \dots, A_0(-i+a+1), \mathbf{R}\Gamma_{\geq i-a} \mathbf{D}^b(\mathrm{coh} X)),$$

and applying the functor $p : \mathbf{D}^b(\mathrm{gr}_{\geq i} A) \rightarrow \mathbf{D}_{\mathrm{sg}}^b(\mathrm{gr} A)$ gives the desired decomposition. \square

7. Complete intersection rings and matrix factorizations

In this section we apply the main theorem to relate the derived category of a commutative complete intersection ring to the homotopy category of graded matrix factorizations over a “generic hypersurface”.

7.1. Graded matrix factorizations. Let $S = \bigoplus_{i \geq 0} S_i$ be a commutative noetherian graded ring and let $W \in S_d$, for some $d \geq 1$. A *graded matrix factorization* of W is a pair of graded projective S -modules E_1, E_0 and morphisms in $\mathrm{gr} S$,

$$e_1 : E_1 \rightarrow E_0, \quad e_0 : E_0 \rightarrow E_1(d),$$

such that $e_0 e_1 = W \cdot 1_{E_1}$ and $e_1(d) e_0 = W \cdot 1_{E_0}$. A morphism h between

$$\mathbb{E} = (E_1 \xrightarrow{e_1} E_0 \xrightarrow{e_0} E_1(d)) \quad \text{and} \quad \mathbb{F} = (F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} F_1(d))$$

is a pair of maps $h_1 : E_1 \rightarrow F_1$ and $h_0 : E_0 \rightarrow F_0$ making the obvious diagrams commute. One defines a homotopy between two such maps analogously to the case of a map of complexes. The category with objects graded matrix factorizations of W and morphisms homotopy equivalence classes of morphisms of matrix factorizations is the *homotopy category of graded matrix factorizations of W* and denoted $[\mathrm{gr}\text{-mf}(S, W)]$.

Now assume that S_0 is a regular commutative ring and S is a polynomial ring over S_0 . Set $A = S/(W)$ and consider the singularity category $\mathbf{D}_{\mathrm{sg}}^b(\mathrm{gr} A)$ as defined in Definition 5.5. The assignment that sends

$$\mathbb{E} = (E_1 \xrightarrow{e_1} E_0 \xrightarrow{e_0} E_1(d))$$

to the image of $\mathrm{coker} e_1$ in $\mathbf{D}_{\mathrm{sg}}^b(\mathrm{gr} A)$ induces a functor

$$\mathrm{coker} : [\mathrm{gr}\text{-mf}(S, W)] \rightarrow \mathbf{D}_{\mathrm{sg}}^b(\mathrm{gr} A). \quad (7.1)$$

It follows from work of Eisenbud [1980] and Buchweitz [1987], and appears explicitly in [Orlov 2009, Section 3], that this functor is an equivalence of categories.

7.2. Generic hypersurface. Let $R = Q/(f)$, where Q is a commutative regular ring of finite Krull dimension, and $f = f_1, \dots, f_c$ is a Q -regular sequence. Define $S = Q[T_1, \dots, T_c]$ to be the graded polynomial ring over Q with $|T_i| = 1$. Let $W = f_1 T_1 + \dots + f_c T_c \in S_1$ and set $A = S/(W)$.

Let $Y = \text{Proj } A$ and note that there is a diagram

$$\begin{array}{ccc} \mathbb{P}_R^{c-1} = \text{Proj}(S \otimes_Q R) & \xrightarrow{\beta} & Y \longrightarrow \text{Proj } S = \mathbb{P}_Q^{c-1} \\ \pi \downarrow & & \downarrow \\ \text{Spec } R & \longrightarrow & \text{Spec } Q \end{array} \quad (7.2)$$

where the vertical arrows are the canonical proper maps and each horizontal arrow is a regular closed immersion and thus has finite Tor dimension. In particular the map $\beta : \mathbb{P}_R^{c-1} \rightarrow Y$ is a regular closed immersion of codimension $c - 1$. Orlov [2006] used this setup to show that there is an equivalence between the singularity categories of R and Y . This equivalence was used in [Burke and Walker 2015; Stevenson 2014].

Lemma 7.3. *The functor $\beta_* \pi^* : D^b(R) \rightarrow D^b(\text{coh } Y)$ is fully faithful and has a right adjoint. Thus the image \mathcal{R} is a right admissible subcategory of $D^b(\text{coh } Y)$ equivalent to $D^b(R)$. Moreover, the right orthogonal of \mathcal{R} is*

$$(\beta_* \pi^* D^b(R))^\perp = \langle \mathcal{O}_Y(-c+2), \dots, \mathcal{O}_Y(-1), \mathcal{O}_Y \rangle.$$

Proof. Orlov [2006, 2.2] shows that the functor $\beta_* \pi^* : D^b(R) \rightarrow D^b(\text{coh } Y)$ is fully faithful and has a right adjoint (the existence of a right adjoint to β_* is one formulation of Grothendieck duality in this context). He also shows in [loc.cit., 2.10] that the left orthogonal of the image is $\langle \mathcal{O}_Y(1), \dots, \mathcal{O}_Y(c-1) \rangle$; a slight reworking of this argument shows the right orthogonal is as claimed. \square

7.3. The equivalence. We continue to assume that R is a complete intersection of the form $Q/(f)$, where Q is a commutative regular ring of finite Krull dimension, and $f = f_1, \dots, f_c$ is a Q -regular sequence. Recall that $A = Q[T_1, \dots, T_c]/(f_1 T_1 + \dots + f_c T_c) = S/(W)$. We wish to apply Theorem 6.4 to this ring. We first must show A is Gorenstein. This holds by “graded local duality” as in [Bruns and Herzog 1993, Section 3.4].

Lemma 7.4. *There is an isomorphism in $D^b(\text{gr } A)$,*

$$\mathbf{R} \underline{\text{Hom}}_{\text{gr } A}(A_0, A) \cong A_0[-n](c-1),$$

where $n = \dim A$. Thus A is a graded Gorenstein ring with a -invariant $c - 1$.

Theorem 7.5. *There is an equivalence*

$$\Psi : D^b(R) \xrightarrow{\cong} [\text{gr-mf}(S, W)]$$

given by $\Psi = q(\mathbf{R}\Gamma_{\geq 0})\beta_*\pi^*$, where q is the composition

$$D^b(\text{gr } A_{\geq 0}) \xrightarrow{p} D_{\text{sg}}^b(\text{gr } A) \xrightarrow{\cong} [\text{gr-mf}(S, W)].$$

Proof. By Theorem 6.4 applied to A with $i = 0$, we know that $D^b(\text{coh } Y)$ has a semiorthogonal decomposition $(\mathcal{O}_Y(-c+2), \dots, \mathcal{O}_Y, \tilde{\mathcal{B}}_i)$ where \mathcal{B}_i is the image of $D_{\text{sg}}^b(\text{gr } A)$ under the fully faithful functor $\mathbf{b}_i : D_{\text{sg}}^b(\text{gr } A) \rightarrow D^b(\text{gr}_{\geq i} A)$ described in (5.7). Thus

$$\tilde{\mathcal{B}}_i^\perp = (\mathcal{O}_Y(-c+2), \dots, \mathcal{O}_Y),$$

which by Lemma 7.3 is also equal to \mathcal{R}^\perp , where \mathcal{R} is the image of $D^b(R)$ under $\beta_*\pi^*$. Thus $\mathcal{R} = \tilde{\mathcal{B}}_i$ and applying $q\mathbf{R}\Gamma_{\geq 0}$ to both sides we have an equivalence

$$D^b(R) \xrightarrow{q\mathbf{R}\Gamma_{\geq 0}\beta_*\pi^*} D_{\text{sg}}^b(\text{gr } A).$$

Finally, the equivalence (7.1) finishes the proof. \square

In [Burke and Walker 2015], it was shown that there is an equivalence

$$D_{\text{sg}}^b(R) \cong [MF(\mathbb{P}_Q^{c-1}, \mathcal{O}(1), W)],$$

where $[MF(\mathbb{P}_Q^{c-1}, \mathcal{O}(1), W)]$ is the homotopy category of matrix factorizations of locally free sheaves on \mathbb{P}_Q^{c-1} . This category has objects pairs of locally free sheaves $(\mathcal{E}_1, \mathcal{E}_0)$ on \mathbb{P}_Q^{c-1} and maps $e_1 : \mathcal{E}_1 \rightarrow \mathcal{E}_0$ and $e_0 : \mathcal{E}_0 \rightarrow \mathcal{E}_1(1)$ such that composition is multiplication by W . Morphisms are defined analogously as in the affine case above, however there is a further localization at objects that are locally contractible. There is an obvious functor

$$(\tilde{}) : [\text{gr-mf}(S, W)] \rightarrow [MF(\mathbb{P}_Q^{c-1}, \mathcal{O}(1), W)].$$

This equivalence fits into the following commutative diagram, where the left hand arrow is the natural projection onto the singularity category.

$$\begin{array}{ccc} D^b(R) & \xrightarrow{\cong} & [\text{gr-mf}(S, W)] \\ \downarrow & & \downarrow (\tilde{}) \\ D_{\text{sg}}^b(R) & \xrightarrow{\cong} & [MF(\mathbb{P}_Q^{c-1}, \mathcal{O}(1), W)]. \end{array}$$

Acknowledgement

Ragnar-Olaf Buchweitz gave two beautiful lectures on Orlov’s theorem at Bielefeld University in July 2011 that inspired this and from which we learned a lot. The existence of the crucial left adjoint to the projection onto the singularity category over general bases of finite global dimension is implicitly contained in Ragnar’s exposition of Orlov’s work. We thank Mark Walker for several helpful conversations on the contents of the paper.

References

- [Artin and Zhang 1994] M. Artin and J. J. Zhang, “Noncommutative projective schemes”, *Adv. Math.* **109**:2 (1994), 228–287.
- [Ballard et al. 2012] M. Ballard, D. Favero, and L. Katzarkov, “Orlov spectra: Bounds and gaps”, *Invent. Math.* **189**:2 (2012), 359–430.
- [Ballard et al. 2014] M. Ballard, D. Favero, and L. Katzarkov, “Variation of geometric invariant theory quotients and derived categories”, preprint, 2014. arXiv 1203.6643
- [Baranovsky and Pecharich 2010] V. Baranovsky and J. Pecharich, “On equivalences of derived and singular categories”, *Cent. Eur. J. Math.* **8**:1 (2010), 1–14.
- [Beilinson 1978] A. A. Beilinson, “Coherent sheaves on \mathbf{P}^n and problems in linear algebra”, *Funktsional. Anal. i Prilozhen.* **12**:3 (1978), 68–69. In Russian; translated in *Funct. Anal. Appl.* **12** (1979), 214–216.
- [van den Bergh 2004] M. van den Bergh, “Non-commutative crepant resolutions”, pp. 749–770 in *The legacy of Niels Henrik Abel*, edited by O. A. Laudal and R. Piene, Springer, Berlin, 2004.
- [Bondal 1989] A. I. Bondal, “Representations of associative algebras and coherent sheaves”, *Izv. Akad. Nauk SSSR Ser. Mat.* **53**:1 (1989), 25–44. In Russian; translated in *Math. USSR, Izv.* **34**:1 (1990), 23–42.
- [Bruns and Herzog 1993] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1993.
- [Buchweitz 1987] R.-O. Buchweitz, “Maximal Cohen–Macaulay modules and Tate-cohomology over Gorenstein rings”, 1987, <http://hdl.handle.net/1807/16682>. Unpublished manuscript.
- [Burke and Walker 2015] J. Burke and M. E. Walker, “Matrix factorizations in higher codimension”, *Trans. Amer. Math. Soc.* **367**:5 (2015), 3323–3370.
- [Eisenbud 1980] D. Eisenbud, “Homological algebra on a complete intersection, with an application to group representations”, *Trans. Amer. Math. Soc.* **260**:1 (1980), 35–64.
- [Halpern-Leistner 2014] D. Halpern-Leistner, “The derived category of a GIT quotient”, *J. Amer. Math. Soc.* (online publication October 2014).
- [Herbst et al. 2008] M. Herbst, K. Hori, and D. Page, “Phases of $n=2$ theories in $1+1$ dimensions with boundary”, preprint, 2008. arXiv 0803.2045
- [Isik 2013] M. U. Isik, “Equivalence of the derived category of a variety with a singularity category”, *Int. Math. Res. Not.* **2013**:12 (2013), 2787–2808.
- [Keller et al. 2011] B. Keller, D. Murfet, and M. van den Bergh, “On two examples by Iyama and Yoshino”, *Compos. Math.* **147**:2 (2011), 591–612.

- [Lenzing 2011] H. Lenzing, “Weighted projective lines and applications”, pp. 153–187 in *Representations of algebras and related topics*, edited by A. Skowroński and K. Yamagata, Eur. Math. Soc., Zürich, 2011.
- [Minamoto and Mori 2011] H. Minamoto and I. Mori, “The structure of AS–Gorenstein algebras”, *Adv. Math.* **226**:5 (2011), 4061–4095.
- [Orlov 2006] D. O. Orlov, “Triangulated categories of singularities, and equivalences between Landau–Ginzburg models”, *Mat. Sb.* **197**:12 (2006), 117–132. In Russian; translated in *Sb. Math.* **197**:12 (2006), 1827–1840.
- [Orlov 2009] D. Orlov, “Derived categories of coherent sheaves and triangulated categories of singularities”, pp. 503–531 in *Algebra, arithmetic, and geometry: In honor of Yu. I. Manin, II*, edited by Y. Tschinkel and Y. G. Zarhin, Progr. Math. **270**, Birkhäuser, Boston, MA, 2009.
- [Popescu 1973] N. Popescu, *Abelian categories with applications to rings and modules*, London Mathematical Society Monographs **3**, Academic Press, London, 1973.
- [Segal 2011] E. Segal, “Equivalence between GIT quotients of Landau–Ginzburg B-models”, *Comm. Math. Phys.* **304**:2 (2011), 411–432.
- [Shipman 2012] I. Shipman, “A geometric approach to Orlov’s theorem”, *Compos. Math.* **148**:5 (2012), 1365–1389.
- [Stafford and Zhang 1994] J. T. Stafford and J. J. Zhang, “Examples in non-commutative projective geometry”, *Math. Proc. Cambridge Philos. Soc.* **116**:3 (1994), 415–433.
- [Stevenson 2014] G. Stevenson, “Subcategories of singularity categories via tensor actions”, *Compos. Math.* **150**:2 (2014), 229–272.

jburke@math.ucla.edu

*Department of Mathematics,
University of California, Los Angeles, Box 951555,
Los Angeles, CA 90095-1555, United States*

gstevens@math.uni-bielefeld.de

*Fakultät für Mathematik, Universität Bielefeld, BIREP
Gruppe, Postfach 10 01 31, D-33501 Bielefeld, Germany*

