Commutative Algebra and Noncommutative Algebraic Geometry, I
MSRI Publications
Volume **67**, 2015

From Briançon–Skoda to Scherk–Varchenko

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To the memory of Egbert Brieskorn

In this survey paper we try to explain how the monodromy theorem for isolated hypersurface singularities led to unexpected conjectures by J. Scherk relating the smallest power r for which f^r belongs to the jacobian ideal J_f to the size of the Jordan blocks in the vanishing cohomology. These were proven by A. Varchenko using his asymptotic mixed Hodge structure on the vanishing cohomology.

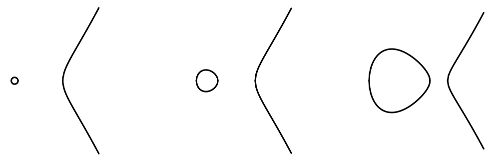
1. The monodromy transformation

The study of the ramification of integrals depending on parameters has a history that can be traced back at least to the work of Euler, Legendre and Gauss, but it seems that the systematic study of the topology of algebraic varieties and their period integrals has its roots in the nineteenth century in the work of Poincaré and Picard. To see what is involved, let us start with a well-known and basic example.

The equation

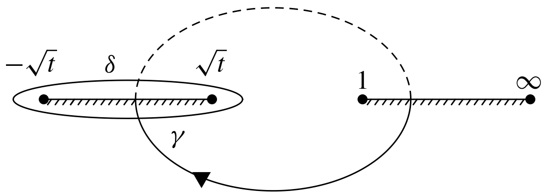
$$y^2 = (t - x^2)(1 - x)$$

describes an affine part of an elliptic curve E_t depending on a parameter t.

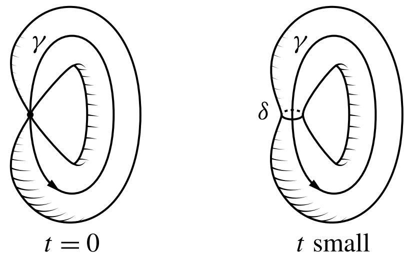


The small loop in the picture that runs between $x = -\sqrt{t}$ and $x = \sqrt{t}$ shrinks to a point for $t = 0$: it is a *vanishing cycle*. The projection to the x-line represents E_t as a double cover of the Riemann sphere, ramified over the four points

$-\sqrt{t}, \sqrt{t}, 1, \infty$:



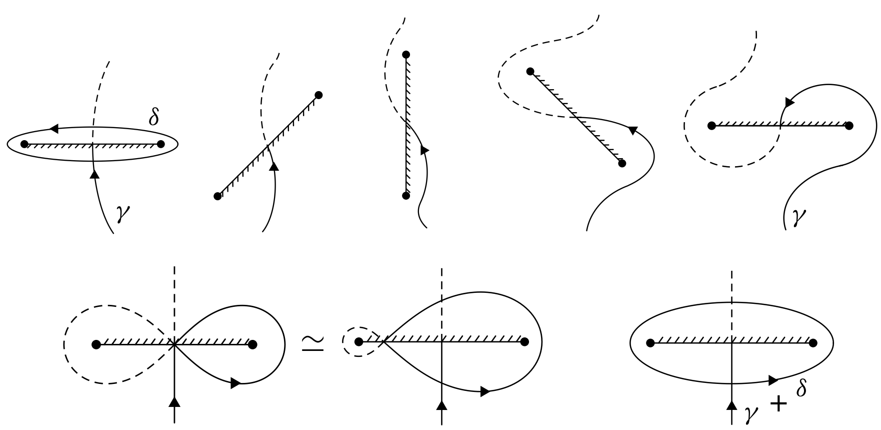
From this one can see that for general t the topology of E_t is that of a 2-torus, but for $t \to 0$, this torus degenerates to a pinched torus:



One can pick a basis for $H_1(E_t)$ consisting of the vanishing cycle $\delta = \delta(t) \in H^1(E_t)$ that runs around the points $\pm\sqrt{t}$ and a cycle $\gamma = \gamma(t)$ that survives the contraction of the vanishing cycle, but gets pinched. When we make a small detour $t = \epsilon \exp(i\theta)$, $\theta \in [0, 2\pi]$ in the complex plane around the point $t = 0$, the two branch-points $\pm\sqrt{t}$ get interchanged. When we follow the cycles by parallel transport, we obtain a *monodromy-transformation*

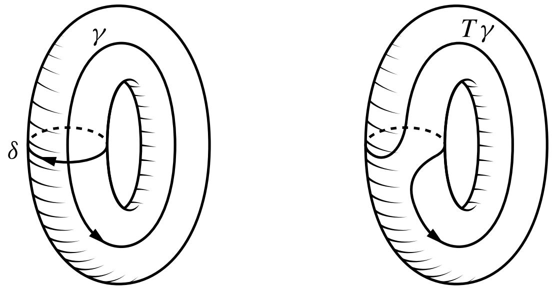
$$T : H_1(E_t) \to H_1(E_t).$$

For the cycles δ and γ we find $T\delta = \delta$, and $T\gamma = \gamma + \delta$, as indicated by the following pictures.



Hence the monodromy is represented by the matrix

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (T-1)^2 = 0.$$



The behaviour of the cycles is reflected in the behaviour of the *period integrals*

$$\Phi_\Gamma(t) = \int_\Gamma \eta_t, \quad \eta_t = \frac{dx}{\sqrt{(t-x^2)(1-x)}}.$$

These satisfy the linear differential equation

$$(16\Theta^2 - t(4\Theta+1)(4\Theta+3))\Phi_\Gamma(t) = 0,$$

where $\Theta := t\partial/\partial t$. For the above cycles δ, γ one finds the following series expansions:

$$\Phi_\delta(t) = 2\pi\left(1 + \tfrac{3}{16}t + \tfrac{105}{1024}t^2 + \cdots\right),$$
$$2\pi i\,\Phi_\gamma(t) = \log(t)\Phi_\delta(t) + 2\pi\left(\tfrac{5}{8}t + \tfrac{389}{1024}t^2 + \cdots\right).$$

The analytic continuation of these period integrals exactly reflect the monodromy behaviour of the cycles δ and γ: continuation around $t = 0$ gives

$$\Phi_\delta \to \Phi_\delta, \quad \Phi_\gamma \to \Phi_\gamma + \Phi_\delta.$$

This example turns out to be part of a much more general story: for families of curves of higher genus acquiring nodes as singularities the situation is very similar and was first described in [Picard and Simart 1897, Tome I, Chapter IV, Section 19]. For an excellent account see [Brieskorn and Knörrer 1981, Section 9.3], where also an example similar to the above one is worked out in detail. The generalisation to the case of n-dimensional varieties Y_t acquiring an ordinary double point was first described by Lefschetz [1924].

The effect of the monodromy can be described by the *Picard–Lefschetz formula*

$$T : H^n(Y_t) \to H^n(Y_t), \quad \gamma \mapsto \gamma \pm \langle \gamma, \delta \rangle \delta,$$

where $\langle -, - \rangle$ denotes the intersection of cycles on Y_t, and the sign is found to be $(-1)^{(n+1)(n+2)/2}$ [Lamotke 1981; Vassiliev 2002].

In general, a holomorphic one-parameter family of compact complex n-dimensional manifolds degenerating over 0 is described by a smooth $n+1$-dimensional complex manifold $\mathcal{Y}$ with a proper holomorphic map $f := \mathcal{Y} \to D$ to the disc D, submersive on $\mathcal{Y}^* = \mathcal{Y} \setminus f^{-1}(0)$. By the Ehresmann fibration theorem, the family $f^* : \mathcal{Y}^* \to D^*$ is a differentiable fibre bundle over the punctured disc D^*. As D^* contracts to a circle, this fibre bundle is described by a geometric monodromy transformation $Y_t \to Y_t$, which induces a cohomological monodromy transformation T.

The monodromy theorem. *The cohomological monodromy transformation*

$$T : H^q(Y_t) \to H^q(Y_t)$$

is quasiunipotent. More precisely, there exists an integer e such that

$$(T^e - 1)^{q+1} = 0.$$

So the eigenvalues of T are roots of unity and the size of the Jordan blocks is bounded by $q+1$. One can write $T = S \cdot U = U \cdot S$ where S is semisimple and U is unipotent. The nilpotent operator $1 - U$ has the same Jordan type as $1 - T^e$ or as the *monodromy logarithm*

$$N := \log(U) = (U-1) - \tfrac{1}{2}(U-1)^2 + \tfrac{1}{3}(U-1)^3 + \cdots .$$

The first proof of this fundamental theorem appeared in the (unpublished) Berkeley thesis of Landman [1966] (see also [Landman 1973]). A further topological proof was given by Clemens [1969]. Many alternative proofs, avoiding resolutions of singularities and using arithmetical or Hodge theoretical arguments were given by Deligne, Grothendieck, Katz, and Borel; see [Deligne and Katz 1973; Katz 1970; 1971].

2. Isolated hypersurface singularities

Locally around any point of $\mathcal{Y}$, the map can be described by a germ

$$f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$$

determined by a convergent power series

$$f \in S := \mathbb{C}\{x_0, x_1, \ldots, x_n\}.$$

One speaks of an *isolated singularity* if the equations

$$\frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$$

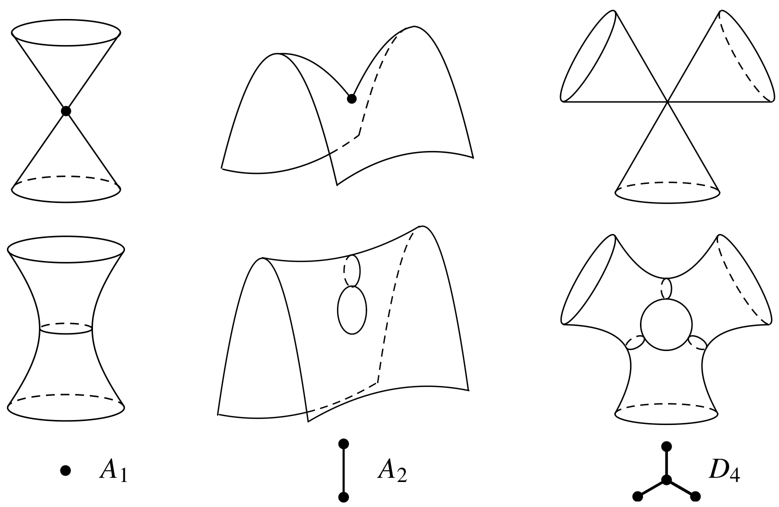
have only 0 as a common solution in a neighbourhood of 0. This is equivalent to the condition that the *Jacobi ring*

$$Q_f := S/J_f, \quad J_f = (\partial_0 f, \partial_1 f, \dots, \partial_n f)$$

is of finite $\mathbb{C}$-dimension. One says that two singularities f and g are *right-equivalent*, notation $f \sim g$, if one can find a coordinate transformation

$$(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$$

that maps f to g. The classification up to right equivalence then starts with the famous ADE list, [Arnold 1975]. Here some pictures of some well-known singularities, together with a deformation that explains their name.



2.1. ***Milnor fibration.*** An isolated singularity always possesses a so-called *good representative*; see [Looijenga 1984, p. 21]. By this we mean the following. First one picks $\epsilon > 0$ so small that for all $0 < \epsilon' \leq \epsilon$ the boundary $\partial B_{\epsilon'}$ is transverse to the special fibre $f^{-1}(0)$. One obtains a smooth orientable differentiable manifold

$$L = \partial B_\epsilon \cap f^{-1}(0)$$

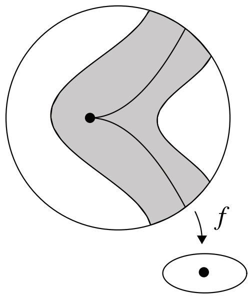
of dimension $2n - 1$, called the *link* of the singularity. Then one picks $\eta > 0$ such that for all t with $0 < |t| \leq \eta$ the fibre $f^{-1}(t)$ is transverse to ∂B_ϵ. We put $B = B_\epsilon = \{|x| \leq \epsilon\}$ and $D = D_\eta = \{|t| \leq \eta\}$ and let $\mathcal{X} := B \cap f^{-1}(D)$, so that f determines a map $\mathcal{X} \to D$, called good representative of the germ f. Furthermore, in such a situation we set $D^* := D \setminus \{0\}$, $\mathcal{X}^* := \mathcal{X} \setminus f^{-1}(0)$, and we obtain a map

$$f^* : \mathcal{X}^* \to D^*.$$

Again, the Ehresmann fibration theorem shows that

$$f^* : \mathcal{X}^* \to D^*$$

is a C^∞-fibre bundle. This fibration is now commonly called the *Milnor fibration*, its fibre $X_t := f^{-1}(t)$ the *Milnor fibre*.



Theorem. *The Milnor fibre has the homotopy type of a bouquet of n-spheres.*

$$X_t \approx \bigvee_{i=1}^{\mu} S^n.$$

The number μ *of spheres*, *called the Milnor number*, *can be computed as*

$$\mu = \dim(S/J_f).$$

The spheres appearing in the first part of the statement are contracted upon approaching the fibre over 0, and are called, extending the terminology used by Lefschetz, the *vanishing cycles* of the singularity. A consequence of the bouquet-theorem is that the Milnor fibre only has one interesting cohomology group $H^n(X_t, \mathbb{Z})$, which is free of rank μ.

Although all Milnor fibres X_t are diffeomorphic, one can not speak about "the" Milnor fibre, as the manifold X_t depends on t. For some constructions it is convenient to use the *canonical Milnor fibre* X_∞, defined as the pull-back of $\mathcal{X}^*$ over the universal covering $\widetilde{D} \to D^*$ of the punctured disc

$$X_\infty = \mathcal{X}^* \times_{D^*} \widetilde{D}.$$

Then X_∞ contracts to each of the Milnor fibres X_t and we have a single group $H^n(X_\infty, \mathbb{Z})$ isomorphic to each of the $H^n(X_t, \mathbb{Z})$.

2.2. ***Exotic spheres.*** One of the strong motivations to study the differential topological properties of isolated hypersurface singularities came from the discoveries of Hirzebruch [1964] and Brieskorn [1966a; 1966b] that the link L

of such singularity can be a sphere with an exotic differentiable structure. The so-called *Brieskorn–Pham* polynomials of the form

$$f = x_0^{a_0} + x_1^{a_1} + \cdots + x_n^{a_n}$$

played an important role in that story. The Milnor number of f is easily seen to be

$$\mu = (a_0 - 1)(a_1 - 1) \cdots (a_n - 1).$$

Furthermore, Pham [1965] determined the cohomological monodromy T of this singularity. It is of finite order

$$e := \operatorname{lcm}(a_0, a_1, \ldots, a_n),$$

and the eigenvalues of T on $H^n(X_t)$ are the numbers

$$\omega_0 \omega_1 \ldots \omega_n,$$

where ω_i runs over all a_i-th roots of unity. A closer analysis of the topology of the Milnor fibration (see [Milnor 1968, p. 65]) shows that the link L of an isolated singularity has the integral homology of a sphere if and only if $\det(I - T) = \pm 1$, from which one can conclude for $n \neq 2$ that L in fact is *homeomorphic* to a sphere. Brieskorn [1966b] used this to show, for example, that the link of

$$x_0^2 + x_1^2 + x_2^2 + x_3^3 + x_4^{6k-1}$$

for $k = 1, 2, \ldots, 28$ represents the 28 distinct differentiable structures on the 7-sphere S^7. In fact, all exotic spheres that bound a parallelizable manifold appear as links of such *Brieskorn–Pham* singularities.

2.3. ***The Brieskorn lattice.*** Brieskorn [1970] described a method to determine the cohomological monodromy of an isolated hypersurface singularity and used it to give a proof of the monodromy theorem for isolated hypersurface singularities, thus answering a question of Milnor.

Monodromy Theorem for Isolated Hypersurface Singularities. *The cohomological monodromy transformation*

$$T : H^n(X_t) \to H^n(X_t)$$

is quasiunipotent: *there exists e such that*

$$(T^e - 1)^{n+1} = 0.$$

The idea is to look at the *cohomology bundle* over D^* with fibres $H^n(X_t, \mathbb{C})$, the cohomology of the Milnor fibre. This bundle comes with a natural flat connection defined by parallel-transport of (co)cycles: the *Gauss–Manin connection*. Brieskorn then develops a de Rham description to represent sections of this cohomology bundle and gives an explicit description of Gauss–Manin connection in local terms. The resulting system of linear differential equations describe the variation of the period integrals over the vanishing cycles and the monodromy of this differential system is identified with the cohomological monodromy T. In more detail it works as follows.

A (germ of a) differential form

$$\omega \in \Omega^{n+1} := S\, dx_0\, dx_1 \dots dx_n = \mathbb{C}\{x_0, x_1, \dots, x_n\}\, dx_0\, dx_1 \dots dx_n$$

determines a section of the cohomology bundle: we obtain a family of closed differential forms on the Milnor fibres X_t by

$$\eta_t = \mathrm{Res}_{X_t}\left(\frac{\omega}{f-t}\right).$$

The forms ω that belong to the subspace $df \wedge d\Omega^{n-1}$ give rise to forms that are exact on the fibres and hence the *Brieskorn lattice* defined by

$$\mathcal{H} := \Omega^{n+1}/df \wedge d\Omega^{n-1}$$

can be thought to give families of cohomology classes on the Milnor fibration. It is called H'' in [Brieskorn 1970].

On $\mathcal{H}$ there are various important structures. First it has a natural structure as a $\mathbb{C}\{t\}$-module: the action of t on $\mathcal{H}$ is realised by multiplication of differential forms by f. In fact one has:

Theorem. *$\mathcal{H}$ is a free $\mathbb{C}\{t\}$-module of rank μ.*

The statement about the rank is due to Brieskorn, the freeness is due to Sebastiani [1970]. His and other proofs use integration and no completely algebraic proof is known to me.

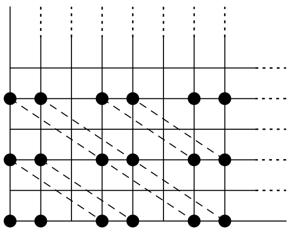
Example. Consider $f = y^2 + x^3$. In the diagram at the top of the next page the dots indicate nonzero monomials

$$x^a y^b\, dx\, dy$$

in the Brieskorn lattice $\mathcal{H}$. The dotted lines indicate relations between these monomials in $\mathcal{H}$, coming from

$$df \wedge d(x^p y^q) = (3q x^{2+p} y^{q-1} - 2p y^{q+1} x^{p-1})\, dx\, dy.$$

We can use the monomials $dx\, dy$ and $x\, dx\, dy$ as a $\mathbb{C}\{t\}$-basis of $\mathcal{H}$.



(See example on previous page).

The Brieskorn lattice $\mathcal{H}$ carries another operation called ∂^{-1}, which Brieskorn identifies as the *inverse* of the Gauss–Manin connection. It is defined as follows: if the $(n+1)$-form $\omega \in \Omega^{n+1}$ on $(\mathbb{C}^{n+1}, 0)$ represents an element of $\mathcal{H}$, we can write it as $d\eta$ for some $\eta \in \Omega^n$. One now sets

$$\partial^{-1}\omega := df \wedge \eta.$$

It is easy to check that this gives a well-defined operation on $\mathcal{H}$, which satisfies

$$t\partial^{-1} - \partial^{-1}t = \partial^{-2}.$$

The map $\partial^{-1} : \mathcal{H} \to \mathcal{H}$ is *injective* and the cokernel can be identified with

$$\mathcal{H}/\partial^{-1}\mathcal{H} = \Omega^{n+1}/df \wedge \Omega^n =: Q^f,$$

which after a choice of a volume form is isomorphic to $Q_f = S/J_f$, the Jacobi ring of $\mathbb{C}$-dimension μ. When we choose a basis $\omega_1, \omega_2, \dots, \omega_\mu$ of $\mathcal{H}$ as $\mathbb{C}\{t\}$-module, we can write out the action of ∂^{-1} in this basis

$$\partial^{-1}\begin{pmatrix}\omega_1\\ \omega_2\\ \dots\\ \omega_\mu\end{pmatrix} = B(t)\begin{pmatrix}\omega_1\\ \omega_2\\ \dots\\ \omega_\mu\end{pmatrix},$$

from which one obtains a meromorphic connection matrix

$$A(t) = B(t)^{-1}(1 - B'(t))$$

for $\mathcal{H}$:

$$\partial\begin{pmatrix}\omega_1\\ \omega_2\\ \dots\\ \omega_\mu\end{pmatrix} = A(t)\begin{pmatrix}\omega_1\\ \omega_2\\ \dots\\ \omega_\mu\end{pmatrix}.$$

If $\delta(t)$ denotes a (multivalued) horizontal family of cycles in $H_n(X_t)$ the (in general multivalued) *period integral* is

$$\Phi(t) = \int_{\delta(t)} \eta_t.$$

For such period integrals on can prove an estimate of the form

$$|\Phi(t)| \leq O(t^{-N}),$$

which implies the *regularity theorem*: the resulting differential system is *regular singular*, hence can be transformed into a system with first order pole:

$$A(t) = \frac{A_{-1}}{t} + A_0 + A_1 t + \cdots .$$

And the monodromy $\exp(2\pi A_{-1})$ is identified with the (complexification) of the cohomological monodromy T. In this way we have a theoretical method to determine the cohomological monodromy transformation T (up to conjugacy). As T is an automorphism of the lattice $H^n(X_t, \mathbb{Z})$, the characteristic polynomial has integer coefficients, and it follows that the eigenvalues of T are algebraic numbers. From the fact that the construction is "algebraically defined", the eigenvalues α of A_{-1} are algebraic too. As by the theorem of Gelfond–Schneider for an irrational algebraic number α, the number

$$\exp(2\pi i \alpha)$$

is transcendental, Brieskorn concluded that the eigenvalues of the monodromy are roots of unity!

The period integrals expand in series of the following sort

$$\Phi(t) = \sum_{\alpha,k} A_{\alpha,k} t^{\alpha} (\log t)^k.$$

It was shown by Malgrange [1974] that in fact $A_{\alpha,k} = 0$ for $\alpha \leq -1$, which provides an alternative proof of the fact that $\mathcal{H}$ is $\mathbb{C}\{t\}$-free. (The reason is that elements $\omega \in \mathcal{H}$ in the kernel of multiplication by t belong to the space C^{-1}, defined in Section 5.)

Gauss–Manin system. It has become customary to embed the Brieskorn lattice $\mathcal{H}$ into the *Gauss–Manin system* $\mathcal{G}$ of f. This is explained by Pham [1979, pp. 153–167]: one considers the de Rham complex $\Omega^\bullet$ of (germs) of differential forms on $(\mathbb{C}^{n+1}, 0)$ and let D be a variable. By $\Omega^\bullet[D]$ we denote the set of polynomials with coefficients in $\Omega^\bullet$. On it we have a the twisted differential $\underline{d} := d + D df \wedge$:

$$\underline{d}(\omega D^k) := d\omega D^k + df \wedge \omega D^{k+1}.$$

The *Gauss–Manin system* $\mathcal{G}$ is defined as the $(n+1)$-cohomology group of the *twisted de Rham complex*:

$$\mathcal{G} := H^{n+1}(\Omega^\bullet[D], d + Ddf\wedge).$$

The element ωD^k can be thought of as standing for the family of differential forms

$$\mathrm{Res}_{X_t}\left(\frac{k!\omega}{(f-t)^{k+1}}\right)$$

on the Milnor fibres X_t. On $\mathcal{G}$ one has actions of t and ∂

$$t(\omega D^k) = f\omega D^k - k\omega D^{k-1}, \quad \partial(\omega D^k) = \omega D^{k+1},$$

which are easily checked to satisfy

$$\partial t - t\partial = 1,$$

so $\mathcal{G}$ becomes a module over $\mathcal{D} := \mathbb{C}\{t\}[\partial]$. The map $\omega \in \Omega^{n+1} \mapsto \omega D^0$ induces a well-defined embedding

$$\mathcal{H} \hookrightarrow \mathcal{G}.$$

In fact, ∂ is invertible on $\mathcal{G}$, and the restriction of the inverse ∂^{-1} coincides with the operation on $\mathcal{H}$ defined earlier.

M. Schulze has implemented Brieskorn's algorithm in [Schulze 2003]. From the computational point of view it is useful to change, as advocated by Pham [1979], to the *microlocal* point of view, that is using $s = \partial^{-1}$ as expansion parameter. This boils down to looking at the incomplete Laplace transform of the period integrals, that is to the associated *oscillatory integral*. The relevant formula is

$$\int_{\Gamma(t)} e^{-f/s}\omega = \int_0^t e^{-u/s} \int_{\delta(t)} \mathrm{Res}\left(\frac{\omega}{f-u}\right) du,$$

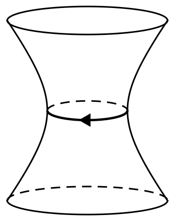
where $\Gamma(t)$ is trace of the vanishing cycle, also known as *Lefschetz thimble*.

3. Questions and answers

Griffiths [1970, pp. 249–250] reports on a question raised by Brieskorn and related to him by Deligne.

Problem. Is the P.-L. transformation $T : H^n(X_t) \to H^n(X_t)$ of finite order?

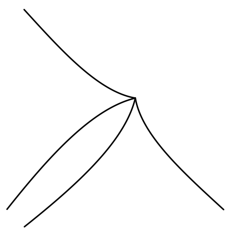
Here "P.-L." of course stands for "Picard–Lefschetz". Although the monodromy transformation in the global case usually has Jordan blocks, the transformation on the vanishing cohomology of the simplest singularities like the ordinary node or the Brieskorn–Pham singularities have finite order. Lê proved in 1971 that the monodromy is of finite order for *irreducible* curve singularities [Lê 1974]. There were serious attempts to prove the result in general.



So it came somewhat as a surprise when A'Campo [1973] published the first examples of plane curve singularities where the monodromy transformations on the cohomology of the Milnor fibre had a Jordan block.

Example (A'Campo). Consider the curve singularity that consists of two cusps, with distinct tangent cones.

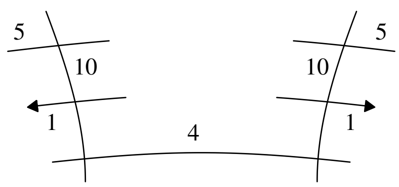
$$f = (x^2 + y^3)(y^2 + x^3) = x^2y^2 + x^5 + y^5 + x^3y^3 \sim x^2y^2 + x^5 + y^5.$$



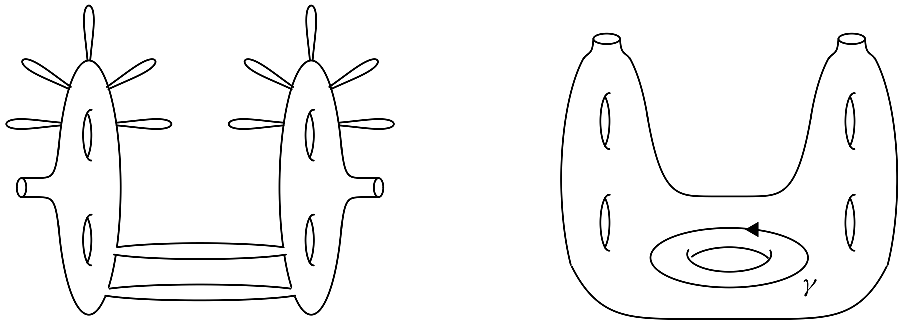
It has $\mu = 11$ and the monodromy satisfies

$$T^{10} - 1 \neq 0, \quad (T^{10} - 1)^2 = 0.$$

A good embedded resolution of $f^{-1}(0)$ is obtained by blowing up the origin and then twice in the strict transform of the two cusps. We obtain a chain of 5 exceptional divisors, with multiplicities 5, 10, 4, 10, 5; the strict transforms of the cusps pass through the components with multiplicity 10.



The Milnor fibre $f = t$ as a subset of the embedded resolution is a curve very close to the union of the exceptional curves and the strict transform of the two cusps. The multiplicity of each component indicates how often the Milnor fibre runs along the divisor. From this information one can build a topological model of the Milnor fibre. Usually one first performs a *semistable reduction*, which in this case amounts to replacing t by t^{10} and which comes down to taking a 10-fold cyclic cover of the embedded resolution. As Milnor fibre one obtains a Riemann surface consisting of two Riemann surfaces of genus 2, with a boundary, and glued together via two cylinders. The cycle γ indicated on the right has $(T^{10}-1)\gamma \neq 0$.



For more details we refer to [A'Campo 1973] and [Brieskorn and Knörrer 1981, p. 751].

A'Campo raised the problem of finding examples of singularities in $n+1$ variables whose cohomological monodromy had a Jordan block of maximal size $n+1$. Such examples were described by Malgrange [1973] in a letter to the editors, published front-to-back to the paper of A'Campo. Malgrange credits Hörmander for the idea.

Example [Malgrange 1973]. The singularity

$$f = (x_0 x_1 \dots x_n)^2 + x_0^{2n+4} + x_1^{2n+4} + \dots + x_n^{2n+4}$$

has a Jordan block of maximal size $n+1$. Let

$$E(t) := \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \mid f \leq t\}.$$

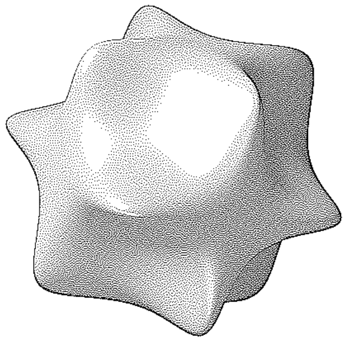
For t small enough, this is a topological ball; its boundary

$$\delta(t) := \partial E(t)$$

is a vanishing cycle that for $n=2$ looks like the picture at the top of the next page.

Clearly:

$$\int_{\delta(t)} x_0 \, dx_1 \dots dx_n = \int_{E(t)} dx_0 \, dx_1 \dots dx_n = \mathrm{Vol}(E(t)).$$



$$100(xyz)^2 + x^8 + y^8 + z^8 = 1.$$

Now Malgrange computes

$$\mathrm{Vol}(E(t)) \sim Ct^{1/2} \log^n(t),$$

where $C \neq 0$. This shows that the vanishing cycle $\delta(t) \in H_n(X_t, \mathbb{Z})$ sits in a Jordan block of size $n+1$.

4. The Briançon–Skoda theorem and Scherk's conjecture

According to C. T. C Wall [1971] it was Mather who asked about the smallest r for which

$$f^r \in J_f.$$

Around the same time as A'Campo and Malgrange found the examples of singularities with maximal Jordan blocks in their vanishing cohomology, a strange algebraic theorem was discovered, whose proof required deep results from complex analysis.

Recall that the *integral closure* $\bar{I}$ of an ideal $I \subset S = \mathbb{C}\{x_0, x_1, \ldots, x_n\}$ consists of all functions h that satisfies an *integrality equation over* I: $h \in \bar{I}$ if and only if for some n there exist $a_k \in I^k$, $k = 1, 2, \ldots, n$ such that

$$h^n + a_1 h^{n-1} + \cdots + a_n = 0.$$

This ideal can be characterised in various other ways. For example, one has $f \in \bar{I}$ if and only if

$$\gamma^*(f) \in \gamma^* I$$

for each curve germ $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^{n+1}, 0)$ [Lipman and Teissier 1981].

Theorem [Skoda and Briançon 1974]. *If I is generated by k elements then*

$$\bar{I}^{\min(k,n+1)} \subset I.$$

This is a completely algebraic statement, but its proof was not. Lipman and Teissier [1981] wrote: "The absence of an algebraic proof has been for algebraists something like a scandal — perhaps even an insult — and certainly a challenge."

In any case, as $f \in \overline{J_f}$, it follows from this theorem that for any $f \in S$ one has

$$f^{n+1} \in J_f,$$

or equivalently, the operator

$$[f] : Q_f \to Q_f$$

induced by multiplication with f on the Jacobi ring has index of nilpotency bounded by $n+1$:

$$[f]^{n+1} = 0.$$

In [Skoda and Briançon 1974] it is also remarked that this estimate on the exponent is optimal. As an example, they give

$$f = (x_0 x_1 \dots x_n)^3 + z_0^{3n+2} + z_1^{3n+2} + \dots + z_n^{3n+2},$$

for which $f^n \notin J_f$.

So we see that to an isolated hypersurface singularity $f \in S$, one can associate two natural vector-spaces of dimension μ, each with a nilpotent endomorphism. On one hand, we have the topological space $H_f := H^n(X_\infty, \mathbb{C})$ with the endomorphism N, the monodromy logarithm. On the other hand, we have the purely algebraic Q_f with the endomorphism $[f]$. The monodromy theorem tells us that $N^{n+1} = 0$, while the theorem of Briançon–Skoda tells that $[f]^{n+1} = 0$. According to Scherk, it was Brieskorn who asked about a possible relation between the two appearances of $n+1$ in these theorems.

Conjecture 1 [Scherk 1978]. *For any isolated hypersurface singularity the following holds*: *If $f^{r+1} \in J_f$, then the Jordan normal form of the monodromy has blocks of size at most* $(r+1)$.

In case $r = 0$ the conjecture follows from the following two theorems.

Theorem [Saito 1971]. *If $f \in J_f$, one can find a coordinate system in which f is represented as a* quasihomogeneous polynomial.

Recall that a polynomial f is called *quasihomogeneous* if one can find positive rational weights $w_0, w_1, \dots, w_n$ such that

$$f(\lambda^{w_0} x_0, \lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = f(x_0, x_1, \dots, x_n).$$

This is the case if and only if all monomials $x^a = x_0^{a_0} x_1^{a_1} \dots x_n^{a_n}$ appearing in f with nonzero coefficient lie in the hyperplane

$$w_0 a_0 + w_1 a_1 + \dots + w_n a_n = 1.$$

Theorem. *For a quasihomogeneous singularity with weights $w_0, w_1, \dots, w_n$, the cohomological monodromy is finite of order d, which is the least common multiple of the denominators of the w_i.*

This generalises the result of Pham on the Brieskorn–Pham singularities and can be found in [Milnor 1968, p. 71].

In the example of the $T_{p,q,r}$-singularities ($\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$), given by

$$f(x, y, z) = x^p + y^q + z^r + xyz,$$

one has $f \notin J_f$, $f^2 \in J_f$ and indeed the monodromy has a single 2×2-block for the eigenvalue 1. In this way the conjecture may also be seen as a *refinement* of the usual monodromy theorem for isolated hypersurfaces. On the other hand, the converse of the statement is certainly not true. Scherk gives the example

$$f_a = y^6 + x^4 y + a x^5.$$

For $a = 0$ the singularity is quasihomogeneous, so $f \in J_f$ and the monodromy is of finite order. For $a \neq 0$, the singularity is no longer quasihomogeneous and so we have $f \notin J_f$, but the monodromy is still of finite order, as the topology of the singularity does not depend on a. Similarly, one could take any quasihomogeneous singularity f and add a nontrivial term of quasihomogeneous degree > 1.

Scherk gave a proof of his conjecture [1980], using a globalisation of the Milnor fibre to a smooth projective hypersurface and using the resulting variation of Hodge structures. In that paper he also formulated a strengthening of his conjecture:

Conjecture 2 [Scherk 1980]. *For an isolated hypersurface singularity f and any integer k the following inequality takes place*:

$$\dim \operatorname{Ker}([f]^k : S/J_f \to S/J_f) \leq \dim \ker(N^k : H^n(X_t) \to H^n(X_t)).$$

5. Period integrals and mixed Hodge structures

The second conjecture of Scherk was proven by Varchenko in [1981] as a consequence of a stronger theorem.

Theorem [Varchenko 1981]. *Consider an isolated hypersurface singularity $f \in S$. There exists a filtration $V^\bullet$ on S/J_f with the property that*

$$[f] : V^\alpha \mapsto V^{\alpha+1},$$

and such that

$$\{f\} := Gr_V^\bullet[f] : Gr_V^\bullet S/J_f \to Gr_V^{\bullet+1} S/J_f$$

and

$$N : H^n(X_\infty, \mathbb{C}) \to H^n(X_\infty, \mathbb{C})$$

have the same Jordan normal form.

As by going to an associated graded of a filtration kernels only can get bigger, one obtains:

$$\dim \ker[f]^k \leq \dim \ker N^k.$$

The construction of the filtration $V^\bullet$ is a bit involved. It lives naturally on the Gauss–Manin system $\mathcal{G}$ and the Brieskorn lattice

$$\mathcal{H} = \Omega^{n+1}/df \wedge d\Omega^{n-1},$$

and induces a filtration on the quotient

$$Q^f = \Omega^{n+1}/df \wedge \Omega^n = \mathcal{H}/\partial^{-1}\mathcal{H}.$$

For a differential form $\omega \in \Omega^{n+1}$ the $V^\bullet$-filtration reflects the asymptotic behaviour of the period integrals

$$\Phi(t) = \int_{\delta(t)} \mathrm{Res}\left(\frac{\omega}{f-t}\right) = \sum_{\alpha,k} A_{\alpha,k} t^\alpha log(t)^k.$$

The element ω belongs to $V^\beta\mathcal{H}$, if for all $\delta(t)$ the coefficients in the above expansion vanish for $\alpha < \beta$.

Varchenko [1980] derives the theorem from his construction of an *asymptotic mixed Hodge structure* on the vanishing cohomology $H^n(X_\infty, \mathbb{Z})$.

Recall that a mixed Hodge structure on a finite rank abelian group H is a linear algebra object that consist of two filtrations, to know an increasing *weight filtration* $W_\bullet$, defined on $H_\mathbb{Q} := H \otimes \mathbb{Q}$, and a decreasing *Hodge filtration* $F^\bullet$ defined on $H_\mathbb{C} := H \otimes \mathbb{C}$, such that $F^\bullet$ induces on the graded pieces $Gr_k^W H = W_k/W_{k-1}$ a pure Hodge structure of weight k. We refer to [Peters and Steenbrink 2008] for a more systematic account of mixed Hodge theory.

Steenbrink [1975/76; 1977] had first constructed such a mixed Hodge structure, using an embedded resolution of f. The weight-filtration is constructed using the nilpotent operator N: it is the unique increasing filtration $W_\bullet$

$$0 \subset W_0 \subset W_1 \subset W_2 \ldots \subset W_{2n-1} \subset W_{2n} = H^n(X_\infty, \mathbb{Q}),$$

such that

$$N : W_k \to W_{k-2},$$

with the property that the operator N^k induces an isomorphism from $\mathrm{Gr}^W_{n+k}H$ to $\mathrm{Gr}^W_{n-k}H$:

$$N^k : \mathrm{Gr}^W_{n+k}H \xrightarrow{\approx} \mathrm{Gr}^W_{n-k}H.$$

As this filtration is uniquely defined by the cohomological monodromy operator, it is called the *monodromy weight filtration*.

In the asymptotic mixed Hodge structure of Varchenko, the Hodge filtration $F^{\bullet}$ is related to the $V^{\bullet}$-filtration and encodes the asymptotic behaviour of the period integrals when t approaches the origin radially. So we have the nice picture that the two filtrations, in a way, arise from the decomposition in *angular* and *radial* components as the parameter $t \to 0$.

We now describe, following [Scherk and Steenbrink 1985], the construction of the asymptotic mixed Hodge structure in more detail.

The generalised eigenspaces C^{α}. The Gauss–Manin system $\mathcal{G}$ has the structure of a (finitely generated) regular singular $\mathbb{C}\{t\}[\partial]$-module. One defines the generalised α-eigenspace by

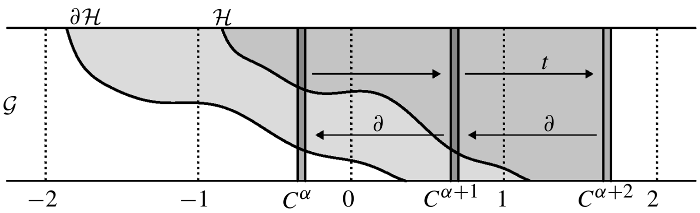
$$C^{\alpha} := \bigcup_{k>0} \ker(t\partial - \alpha)^k \subset \mathcal{G}.$$

These are finite-dimensional $\mathbb{C}$-vector spaces. Note that

$$(t\partial_t - \alpha)^{k-1}(t^{\alpha} \log^k t) = 0,$$

so that C^{α} picks out those elements of $\mathcal{G}$ that "behave like" the function $t^{\alpha} \log^k t$ for some k.

The structure of $\mathcal{G}$ can be schematically visualised as follows:



Horizontally runs the eigenvalue parameter α. The vertical bars represent the generalised eigenspaces C^{α}. Multiplication by t maps C^{α} to $C^{\alpha+1}$, whereas ∂ maps $C^{\alpha+1}$ back to C^{α}. The operators $t\partial - \alpha$ act "vertically" and are nilpotent on C^{α}. One has an isomorphism

$$H^n(X_{\infty}, \mathbb{C}) = \bigoplus_{-1<\alpha\leq 0} C^{\alpha}.$$

It follows from the regularity of the Gauss–Manin connection that the generalised $\exp(2\pi i\alpha)$-eigenspace $H^n(X_{\infty}, \mathbb{C})_{\alpha}$ of the monodromy T is isomorphic to the space C^{α} and the monodromy logarithm N identifies, up to a factor $2\pi i$, with the operator $t\partial - \alpha$.

$$\begin{array}{ccc} H^n(X_\infty, \mathbb{C})_\alpha & \xrightarrow{\approx} & C^\alpha \\ {\scriptstyle N}\downarrow & & \downarrow {\scriptstyle 2\pi i\,(t\partial-\alpha)} \\ H^n(X_\infty, \mathbb{C})_\alpha & \xrightarrow{\approx} & C^\alpha \end{array}$$

The position of the Brieskorn lattice $\mathcal{H}$ inside $\mathcal{G}$ contains important information and is indicated in the picture as the region to right of the wiggly curve. Note that $\mathcal{H} \subset V^{>-1}$, by the result of Malgrange. The action of ∂ moves $\mathcal{H}$ to the left.

The $V^\bullet$-filtration. The $V^\bullet$-filtration of $\mathcal{G}$ is defined as the $\mathbb{C}\{t\}$-span of the C^β with $\beta \geq \alpha$

$$V^\alpha \mathcal{G} := \langle C^\beta \mid \beta \geq \alpha \rangle,$$

and we have

$$C^\alpha \approx V^\alpha / V^{>\alpha}.$$

As $\mathcal{H} \subset \mathcal{G}$ we obtain by intersection a $V^\bullet$-filtration on the Brieskorn lattice $\mathcal{H}$.

On the quotient

$$Q^f = \Omega^{n+1}/df \wedge \Omega^n = \mathcal{H}/\partial^{-1}\mathcal{H},$$

one has a natural induced filtration by setting

$$V^\alpha Q^f := (V^\alpha \mathcal{H} + \partial^{-1}\mathcal{H})/\partial^{-1}\mathcal{H}.$$

For an important class of singularities the $V^\bullet$-filtration can be computed easily:

Theorem [Saito 1988]. *For a Newton nondegenerate f the $V^\bullet$-filtration on Q^f coincides with the Newton filtration $\mathcal{N}^\bullet$, shifted by one*:

$$V^\alpha Q^f = \mathcal{N}^{\alpha-1} Q^f.$$

Hodge filtration on $H^n(X_\infty)$. By applying the operator ∂ to $\mathcal{H} \subset \mathcal{G}$, we obtain a "Hodge filtration" on $\mathcal{G}$:

$$\mathcal{H} \subset \partial\mathcal{H} \subset \partial^2\mathcal{H} \subset \cdots \subset \mathcal{G}.$$

Using this, we define a filtration $F^\bullet$ on C^α by setting

$$F^p C^\alpha := (\partial^{n-p}\mathcal{H} \cap V^\alpha + V^{>\alpha})/V^{>\alpha} \subset C^\alpha$$

and

$$F^p H^n(X_\infty, \mathbb{C}) := \bigoplus_{-1<\alpha\leq 0} F^p C^\alpha.$$

Unwinding the definitions, one finds that the spaces $\mathrm{Gr}_F^p C^\alpha$ can be identified with certain $V^\bullet$-graded piece of Q^f:

$$\partial^{n-p} : \mathrm{Gr}_V^{\alpha+n-p} Q^f \xrightarrow{\approx} \mathrm{Gr}_F^p C^\alpha.$$

The main theorem on asymptotic mixed Hodge theory is the following.

Theorem. *The space $H^n(X_\infty)$, together with the monodromy weight-filtration $W_\bullet$ and the above defined Hodge filtration $F^\bullet$ define a* mixed Hodge structure, *isomorphic to the limiting mixed Hodge structure defined in* [Steenbrink 1977].

This theorem, in a slightly different form, was first proven by Varchenko [1980; 1982]. We basically followed here the presentation of [Scherk and Steenbrink 1985].

Although all the ingredients of the mixed Hodge structure can be defined locally, the proofs of the required Hodge properties use globalisation to a projective hypersurface in an essential way; apparently no purely local proof is known.

5.1. ***Varchenko's theorem.*** A feature of mixed Hodge theory is that all morphisms of mixed Hodge structures are *strictly compatible* with weight and Hodge filtration: going from a morphism $H \to H'$ of mixed Hodge structures to maps between the associated graded pieces, such as

$$\mathrm{Gr}_F^p \, \mathrm{Gr}_k^W \, H \to \mathrm{Gr}_F^p \, \mathrm{Gr}_k^W \, H',$$

preserves exactness properties. In our situation there is one particular interesting morphism of mixed Hodge structures, namely the morphism

$$N : H^n(X_\infty, \mathbb{Q}) \to H^n(X_\infty, \mathbb{Q}).$$

As by construction $N : W_k \to W_{k-2}$ and $N : F^p \to F^{p-1}$, N is a morphism of type $(-1, -1)$.

One now can argue as follows:

(1) From the strictness, the Jordan structure of N on $H := H^n(X_\infty, \mathbb{C})$ is the same as that of

$$\mathrm{Gr}_F N : \mathrm{Gr}_F^\bullet H \to \mathrm{Gr}_F^{\bullet-1} H.$$

(2) On the component C^α the map

$$\mathrm{Gr}_F N : \mathrm{Gr}_F^p C^\alpha \to \mathrm{Gr}_F^{p-1} C^\alpha.$$

is represented by

$$2\pi i (t\partial - \alpha) = 2\pi i t \partial \mod F^p.$$

(3) Identifying the Hodge spaces $\mathrm{Gr}_F^p C^\alpha$ with pieces of the $V^\bullet$-filtration on Q^f

we obtain a diagram

$$\begin{array}{ccc} \mathrm{Gr}_V^{\alpha+n-p}\, Q^f & \xrightarrow{\{f\}} & \mathrm{Gr}_V^{\alpha+n-p+1}\, Q^f \\ {\scriptstyle \partial^{n-p}}\downarrow & & \downarrow{\scriptstyle \partial^{n-p+1}} \\ \mathrm{Gr}_F^{p}\, C^{\alpha} & \xrightarrow{\mathrm{Gr}_F N} & \mathrm{Gr}_F^{p-1}\, C^{\alpha} \end{array}$$

that is commutative up to a factor $2\pi i$.

Corollary 1. *The operator* $\{f\}$ *on* $\mathrm{Gr}_V^{\bullet} Q^f$ *and* N *on* $H^n(X_\infty, \mathbb{C})$ *have the same Jordan type.*

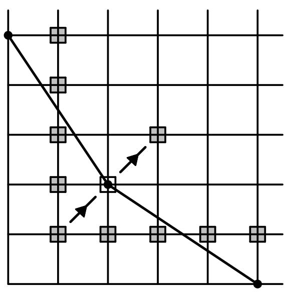
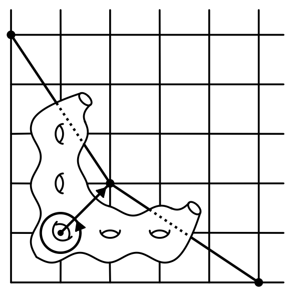
Example. We analyse the example of A'Campo in terms of the $V^{\bullet}$-filtration. As the function is Newton nondegenerate, we can use the theorem of M. Saito to identify the V-filtration with the Newton filtration (shifted by one). A basis for Q^f is given by the 11 differential forms

$$dx\,dy,\ xy\,dx\,dy,\ x^2y^2dx\,dy,$$
$$x\,dx\,dy,\ x^2dx\,dy,\ x^3dx\,dy,\ x^4dx\,dy,\quad y\,dx\,dy,\ y^2dx\,dy,\ y^3dx\,dy,\ y^4dx\,dy$$

The Newton weights of these monomials can be read off from the Newton diagram as

$$\tfrac{1}{2},\ 1,\ \tfrac{3}{2},$$
$$\tfrac{7}{10},\ \tfrac{9}{10},\ \tfrac{11}{10},\ \tfrac{13}{10},\quad \tfrac{7}{10},\ \tfrac{9}{10},\ \tfrac{11}{10},\ \tfrac{13}{10}.$$

The fractions appearing here (or diminished by 1) are called the *spectral numbers* of the singularity.

Multiplication of the monomial $dx\,dy$ of weight $\frac{1}{2}$ by f maps to the monomial $x^2y^2dx\,dy$ of weight $\frac{3}{2}$, which thus represents a nontrivial Jordan block N of the monodromy.

The picture on the right shows the Milnor fibre of f, which was described earlier and seen to be a genus 5 Riemann surface with two holes. We drew the surface around the monomials of the Newton diagram, with holes piercing through the edges of the Newton diagram. In a way that is a bit hard to explain in a precise way, one can see that the nontrivial Jordan block “hits” the cycle γ on the Riemann surface that appeared in A’Campo’s example!

This concludes our account of a unique key period in the theory of isolated hypersurface singularities. Many important developments arose out of them, e.g., M. Saito’s theory of *mixed Hodge modules* and applications to log-canonical thresholds, multiplier ideals, jumping coefficients, etc. For these more recent developments we refer to [Peters and Steenbrink 2008; Blickle and Lazarsfeld 2004; Ein et al. 2004; Mustaţă 2012].

Acknowledgement

I thank the organisers of the special program on commutative algebra at MSRI for giving me the opportunity (October 2012) to give a talk on the subject of this paper. Further thanks to Norbert A’Campo, John Scherk, Joseph Steenbrink and Bernard Teissier for useful exchanges on the topics of this paper.

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