

# The interplay of algebra and geometry in the setting of regular algebras

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This article aims to motivate and describe the geometric techniques introduced by M. Artin, J. Tate and M. Van den Bergh in the 1980s at a level accessible to graduate students. Additionally, some advances in the subject since the early 1990s are discussed, including a recent generalization of complete intersection to the noncommutative setting, and the notion of graded skew Clifford algebra and its application to classifying quadratic regular algebras of global dimension at most three. The article concludes by listing some open problems.

## Introduction

Many noncommutative algebraists in the 1980s were aware of the successful marriage of algebra and algebraic geometry in the commutative setting and wished to duplicate that relationship in the noncommutative setting. One such line of study was the search for a subclass of noncommutative algebras that “behave” enough like polynomial rings that a geometric theory could be developed for them. One proposal for such a class of algebras are the *regular* algebras, introduced in [Artin and Schelter 1987], that were investigated using new geometric techniques in the pivotal papers of M. Artin, J. Tate and M. Van den Bergh [Artin et al. 1990; 1991].

About the same time, advances in quantum mechanics in the 20th century had produced many new noncommutative algebras on which traditional techniques had only yielded limited success, so a need had arisen to find new techniques to study such algebras (see [Reshetikhin et al. 1989; Kapustin et al. 2001; Sklyanin 1982; 1985; Sudbery 1993]). One such algebra was the Sklyanin algebra, which had emerged from the study of quantum statistical mechanics [Sklyanin 1982; 1985]. By the early 1990s, T. Levasseur, S. P. Smith, J. T. Stafford and others

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had solved the ten-year old open problem of completely classifying all the finite-dimensional irreducible representations (simple modules) over the Sklyanin algebra, and their methods were the geometric techniques developed by Artin, Tate and Van den Bergh [Levasseur and Smith 1993; Smith and Stafford 1992; Smith and Staniszki 1993].

Concurrent with the above developments, another approach was considered via differential geometry and deformation theory to study the algebras produced by quantum physics. That approach is the study of certain noncommutative algebras via Poisson geometry (see [Drinfeld 1987]). At the heart of both approaches are homological and categorical techniques, so it is perhaps no surprise that the two approaches have much overlap; often, certain geometric objects from one approach are in one-to-one correspondence with various geometric objects from the other approach (depending on the algebra being studied — cf. [Vancliff 1995; 1999; 2000]). A survey of recent advances in Poisson geometry may be found in [Goodearl 2010].

Given the above developments, the early 1990s welcomed a new era in the field of noncommutative algebra in which geometric techniques took center stage. Since that time, the subject has spawned many new ideas and directions, as demonstrated by the MSRI programs in 2000 and 2013.

This article is based on a talk given by the author in the *Connections for Women* workshop held at MSRI in January 2013 and it has two objectives. The first is to motivate and describe the geometric techniques of Artin, Tate and Van den Bergh at a level accessible to graduate students, and the second is to discuss some developments towards the attempted classification of quadratic regular algebras of global dimension four, while listing open problems. An outline of the article is as follows.

Section 1 concerns the motivation and development of the subject, with emphasis on quadratic regular algebras of global dimension four. Section 2 discusses constructions of certain types of quadratic regular algebras of arbitrary finite global dimension, with focus on graded Clifford algebras and graded skew Clifford algebras. This section also discusses a new type of symmetry for square matrices called  $\mu$ -symmetry. We conclude this section by revisiting the classification of quadratic regular algebras of global dimension at most three, since almost all such algebras may be formed from regular graded skew Clifford algebras. In Section 3, we discuss geometric techniques that apply to graded Clifford algebras and graded skew Clifford algebras in order to determine when those algebras are regular. This section also considers the issue of complete intersection in the noncommutative setting. We conclude with Section 4 which lists some open problems and related topics.

Although the main objects of study from [Artin et al. 1990; 1991] are discussed in this article, several topics from the same are omitted; for surveys of those topics, the reader is referred to [Stafford 2002; Stafford and van den Bergh 2001] and to D. Rogalski’s lecture notes [2014] from the graduate workshop “Noncommutative Algebraic Geometry” at MSRI in June 2012.

## 1. The geometric objects

In this section, we discuss the motivation and development of the subject, with emphasis on quadratic regular algebras of global dimension at most four.

Throughout this section,  $\mathbb{k}$  denotes an algebraically closed field and, for any graded algebra  $B$ , the span of the homogeneous elements of degree  $i$  will be denoted by  $B_i$ .

**1A. Motivation.** Consider the  $\mathbb{k}$ -algebra,  $S$ , on generators  $z_1, \dots, z_n$  with defining relations

$$z_j z_i = \mu_{ij} z_i z_j, \quad \text{for all distinct } i, j,$$

where  $0 \neq \mu_{ij} \in \mathbb{k}$  for all  $i, j$ , and  $\mu_{ij} \mu_{ji} = 1$  for all distinct  $i, j$ . If  $\mu_{ij} = 1$ , for all  $i, j$ , then  $S$  is the commutative polynomial ring and has a rich subject of algebraic geometry associated with it; in particular, by the (projective) Nullstellensatz, the points of  $\mathbb{P}(S_1^*)$  are in one-to-one correspondence with certain ideals of  $S$  via  $(\alpha_1, \dots, \alpha_n) \leftrightarrow (\alpha_1 z_1 - \alpha_1 z_i, \dots, \alpha_i z_n - \alpha_n z_i)$ , where  $\alpha_i \neq 0$ . Before continuing, we first observe that for such an ideal  $I$ , the graded module  $S/I$  has the property that its Hilbert series is  $H(t) = 1/(1-t)$  and that  $S/I$  is a 1-critical (with respect to GK-dimension) graded cyclic module over  $S$ .

However, if  $\mu_{ij} \neq 1$  for any  $i, j$ , then  $S$  still “feels” close to commutative, and one would expect there to be a way to relate algebraic geometry to it. The geometric objects in [Artin et al. 1990] are modeled on the module  $S/I$  above; instead of using actual points or lines etc, certain graded modules are used as follows.

### 1B. Points, lines, etc.

**Definition 1.1.** [Artin et al. 1990] Let  $A = \bigoplus_{i=0}^{\infty} A_i$  denote an  $\mathbb{N}$ -graded, connected (meaning  $A_0 = \mathbb{k}$ )  $\mathbb{k}$ -algebra generated by  $A_1$  where  $\dim(A_1) = n < \infty$ . A graded right  $A$ -module  $M = \bigoplus_{i=0}^{\infty} M_i$  is called a *right point module* (respectively, *line module*) if

- (a)  $M$  is cyclic with  $M = M_0 A$ , and
- (b)  $\dim_{\mathbb{k}}(M_i) = 1$  for all  $i$  (respectively,  $\dim_{\mathbb{k}}(M_i) = i + 1$ ) for all  $i$ .

If  $A$  is the polynomial ring  $S$ , then the module  $S/I$  from Section 1A is a point module. In general, one may associate some geometry to point and line modules

as follows. Condition (a) implies that  $A$  maps onto  $M$  via  $a \mapsto ma$ , for all  $a \in A$ , where  $\{m\}$  is a  $\mathbb{k}$ -basis for  $M_0$ , and this map restricts via the grading to a linear map  $\theta : A_1 \rightarrow M_1$ . Let  $K \subset A_1$  denote the kernel of  $\theta$ . Condition (b) implies that  $\dim_{\mathbb{k}}(K) = n - 1$  (respectively,  $n - 2$ ), so that  $K^\perp \subset A_1^*$  has dimension one (respectively, two). Thus,  $\mathbb{P}(K^\perp)$  is a point (respectively, a line) in the geometric space  $\mathbb{P}(A_1^*)$ .

The Hilbert series of a point module is  $H(t) = 1/(1 - t)$ , whereas the Hilbert series of a line module is  $1/(1 - t)^2$ . Hence, a plane module is defined as in Definition 1.1 but condition (b) is replaced by the requirement that the module have Hilbert series  $1/(1 - t)^3$  (see [Artin et al. 1990]). Similarly, one may define  $d$ -linear modules, where the definition is modeled on Definition 1.1, but the module has Hilbert series  $1/(1 - t)^{d+1}$  (see [Shelton and Vancliff 2002]).

For many algebras,  $d$ -linear modules are  $(d + 1)$ -critical with respect to GK-dimension. This leads to the following generalization of a point module.

**Definition 1.2** [Cassidy and Vancliff 2014]. With  $A$  as in Definition 1.1, we define a *right base-point module* over  $A$  to be a graded 1-critical (with respect to GK-dimension) right  $A$ -module  $M$  such that  $M = \bigoplus_{i=0}^{\infty} M_i = M_0A$  and  $M$  has Hilbert series  $H_M(t) = c/(1 - t)$  for some  $c \in \mathbb{N}$ .

If  $c = 1$  in Definition 1.2, then the module is a point module; whereas if  $c \geq 2$ , then the module is called a fat point module [Artin 1992]. The only base-point modules over the polynomial ring are point modules. On the other hand, in general, the algebra  $S$  from Section 1A can have fat point modules, so fat point modules are viewed as generalizations of points, and this is made more precise in [Artin 1992].

Artin, Tate and Van den Bergh [1990] proved that, under certain conditions, the point modules are parametrized by a scheme; that is, there is a scheme that represents the functor of point modules. Later, in [Vancliff and Van Rompay 1997], this scheme was called the *point scheme*. A decade later, in [Shelton and Vancliff 2002], it was proved by B. Shelton and the author that (under certain conditions)  $d$ -linear modules are parametrized by a scheme; that is, there is a scheme that represents the functor of  $d$ -linear modules. If  $d = 0$ , then this scheme is isomorphic to the point scheme; if  $d = 1$ , the scheme is called the *line scheme*.

By factoring out a nonzero graded submodule from a point module, one obtains a truncated point module as follows.

**Definition 1.3** [Artin et al. 1990]. With  $A$  as in Definition 1.1, we define a *truncated right point module of length  $m$*  to be a graded right  $A$ -module  $M = \bigoplus_{i=0}^{m-1} M_i$  such that  $M$  is cyclic,  $M = M_0A$  and  $\dim_{\mathbb{k}}(M_i) = 1$  for all  $i = 0, \dots, m - 1$ .

For many quadratic algebras  $A$ , there exists a one-to-one correspondence between the truncated point modules over  $A$  of length three and the point modules over  $A$ . Moreover, if the algebra  $A$  in Definition 1.3 is quadratic, then the truncated point modules of length three are in one-to-one correspondence with the zero locus in  $\mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$  of the defining relations of  $A$ . To see this, we fix a  $\mathbb{k}$ -basis  $\{x_1, \dots, x_n\}$  for  $A_1$ , and use  $T$  to denote the free  $\mathbb{k}$ -algebra on  $x_1, \dots, x_n$ , and let  $Z \subset \mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$  denote the zero locus of the defining relations of  $A$ . Viewing each  $x_i$  as the  $i$ -th coordinate function on  $A_1^*$ , let  $p = (\alpha_i) \in \mathbb{P}(A_1^*)$  and  $r = (\beta_i) \in \mathbb{P}(A_1^*)$ , where  $\alpha_i, \beta_i \in \mathbb{k}$  for all  $i = 1, \dots, n$ . Let  $M = \mathbb{k}v_0 \oplus \mathbb{k}v_1 \oplus \mathbb{k}v_2$  denote a three-dimensional vector space that is a  $T$ -module via the action determined by

$$v_0x_i = \alpha_i v_1, \quad v_1x_i = \beta_i v_2, \quad v_2x_i = 0,$$

for all  $i$ . It follows that  $M$  is a truncated point module over  $T$  of length three. If  $g \in T_2$ , then  $v_1g = 0 = v_2g$  and  $v_0g = g(p, r)v_2$ . In particular, if  $f \in T_2$  is a defining relation of  $A$ , then  $Mf = 0$  if and only if  $f(p, r) = 0$ . Hence,  $M$  is an  $A$ -module if and only if  $(p, r) \in Z$ . This one-to-one correspondence between  $Z$  and truncated point modules of length three also exists at the level of schemes; the reason being that the scheme  $Z$  represents the functor of truncated point modules of length three. The method of proof of this is to repeat the preceding argument for a truncated point module of length three over  $R \otimes_{\mathbb{k}} T$  and  $R \otimes_{\mathbb{k}} A$ , where  $R$  is a commutative  $\mathbb{k}$ -algebra, together with localization techniques; for details the reader is referred to [Artin et al. 1990, Proposition 3.9], its proof, and the paragraph preceding that result. This correspondence will be revisited in Section 1D.

For completeness, we finish this subsection with some technical definitions that play minor roles throughout the text. The reader is referred to [Levasseur 1992; Levasseur and Smith 1993] for details and for results concerning algebras satisfying these definitions.

**Definition 1.4** [Levasseur 1992, Definition 2.1]. A noetherian ring  $B$  is called *Auslander-regular* (respectively, *Auslander–Gorenstein*) if

- (a) the global homological dimension (respectively, (left and right) injective dimension) of  $B$  is finite, and
- (b) every finitely generated  $B$ -module  $M$  satisfies the *Auslander condition*, namely, for every  $i \geq 0$  and for every  $B$ -submodule  $N$  of  $\text{Ext}_B^i(M, B)$ , we have  $j(N) \geq i$ , where  $j(N) = \inf\{\ell : \text{Ext}_B^\ell(N, B) \neq 0\}$ .

**Definition 1.5** [Levasseur 1992, Definition 5.8]. A noetherian  $\mathbb{k}$ -algebra  $B$  of integral GK-dimension  $n$  satisfies the *Cohen–Macaulay property* if  $\text{GKdim}(M) + j(M) = n$  for all nonzero finitely generated  $B$ -modules  $M$ .

**1C. Regular algebras.** The goal of [Artin et al. 1990] was to classify, in a user-friendly way, the generic regular algebras of global dimension three that were first analyzed in [Artin and Schelter 1987]. Using geometric techniques developed for the purpose, those algebras were shown in [Artin et al. 1990] to be noetherian. Regular algebras are often viewed as noncommutative analogues of polynomial rings and are defined as follows.

**Definition 1.6** [Artin and Schelter 1987]. A finitely generated,  $\mathbb{N}$ -graded, connected  $\mathbb{k}$ -algebra  $A = \bigoplus_{i=0}^{\infty} A_i$ , generated by  $A_1$ , is *regular* (or *AS-regular*) of global dimension  $r$  if

- (a) it has global homological dimension  $r < \infty$ , and
- (b) it has polynomial growth (i.e., there exist positive real numbers  $c$  and  $\delta$  such that  $\dim_{\mathbb{k}}(A_i) \leq ci^{\delta}$  for all  $i$ ), and
- (c) it satisfies the *Gorenstein condition*, namely, a minimal projective resolution of the left trivial module  ${}_A\mathbb{k}$  consists of finitely generated modules and dualizing this resolution yields a minimal projective resolution of the right trivial module  $\mathbb{k}_A[e]$ , shifted by some degree  $e$ .

Although all three conditions in Definition 1.6 are satisfied by the polynomial ring, the main reason a regular algebra is viewed as a noncommutative analogue of a polynomial ring is due to condition (c), since it imposes a symmetry condition on the algebra that replaces the symmetry condition of commutativity. The reader should note that, in the literature, (c) is sometimes replaced by an equivalent condition that makes the symmetry property less obvious; namely,  $\text{Ext}_A^i({}_A\mathbb{k}, A) \cong \delta_r^i \mathbb{k}_A[e]$ , where  $\delta_r^i$  is the Kronecker delta. An  $\mathbb{N}$ -graded connected  $\mathbb{k}$ -algebra that is generated by degree-1 elements and which is Auslander-regular with polynomial growth is AS-regular [Levasseur 1992]. For a notion of regular algebra where the algebra is not generated by degree-1 elements; see [Cassidy 1999; 2003; Stephenson 1996; 1997; 2000a; 2000b].

**Examples 1.7.** (a) The algebra  $S$  from Section 1A is regular.

- (b) If  $\mathbb{k} = \mathbb{C}$ , then many algebras from physics are regular. In particular, homogenizations of universal enveloping algebras of finite-dimensional Lie algebras, the coordinate ring of quantum affine  $n$ -space, the coordinate ring of quantum  $m \times n$  matrices, and the coordinate ring of quantum symplectic  $n$ -space are all regular [Le Bruyn and Smith 1993; Le Bruyn and Van den Bergh 1993; Levasseur and Stafford 1993].
- (c) If the global dimension of a regular algebra is one, then the algebra is the polynomial ring on one variable. However, by [Artin and Schelter 1987], if the global dimension is two, then there are two types of such algebra as follows. For both types, the algebra has two generators,  $x, y$ , of degree one

and one defining relation  $f$ , where either  $f = xy - yx - x^2$  (*Jordan plane*) or  $f = xy - qyx$  (*quantum affine plane*), where  $q \in \mathbb{k}$  can be any nonzero scalar.

However, if the global dimension is three, then the situation is much richer; some of the algebras are quadratic with three generators and three defining relations, whereas the rest have two generators and two cubic relations [Artin and Schelter 1987]. Such algebras that are generic are classified in [Artin et al. 1990] according to their point schemes, and in all cases, the point scheme is the graph of an automorphism  $\sigma$ . Moreover, the algebra is a finite module over its center if and only if  $\sigma$  has finite order.

**1D. Global dimension four.** Although many regular algebras of global dimension four have been extensively studied, there is no classification yet. Recently, the progress towards classifying nonquadratic regular algebras of global dimension four made good headway via the work in [Lu et al. 2007; Rogalski and Zhang 2012]. However, quadratic regular algebras of global dimension four constitute most of the regular algebras of global dimension four, so their attempted classification is one of the motivating problems that drives the subject forward. We end this section by summarizing some key results for this latter case; in this setting, the algebra has four generators and six relations.

In unpublished work, Van den Bergh proved in the mid-1990s that any quadratic (not necessarily regular) algebra  $A$  on four generators with six generic defining relations has twenty (counted with multiplicity) nonisomorphic truncated point modules of length three. Hence,  $A$  has at most twenty nonisomorphic point modules. He also proved that if, additionally,  $A$  is Auslander-regular of global dimension four, then  $A$  has a 1-parameter family of line modules. For lack of a suitable reference, we outline the proof of these results. Let  $M(4, \mathbb{k})$  denote the space of  $4 \times 4$  matrices with entries in  $\mathbb{k}$ . For the first result, we write points of  $\mathbb{P}(A_1^*)$  as columns and, by mapping  $(a, b) \in \mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$  to the matrix  $ab^T \in M(4, \mathbb{k})$ , we have that  $\mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$  is isomorphic to the scheme  $\Omega_1$  of rank-1 elements in  $\mathbb{P}(M(4, \mathbb{k}))$ . Correspondingly, the defining relations of  $A$  map to homogeneous degree-1 polynomial functions on  $M(4, \mathbb{k})$ , and their zero locus  $Z' \subset \mathbb{P}(M(4, \mathbb{k}))$  can be identified with a  $\mathbb{P}^9$ . With these identifications, the zero locus  $Z \subset \mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$  of the defining relations of  $A$  is isomorphic to  $\Omega_1 \cap Z' \subset \mathbb{P}(M(4, \mathbb{k}))$ . Since  $\Omega_1$  has dimension six and degree twenty,  $\dim(Z) \geq 6 + 9 - 15 = 0$ , and, by Bézout's Theorem,  $\deg(Z) = 20$ . Hence, generically,  $Z$  is finite with twenty points, so the first result follows by using the discussion after Definition 1.3. For the second result, we identify  $A_1 \otimes_{\mathbb{k}} A_1$  with  $M(4, \mathbb{k})$ , and the assumption on regularity allows the application of [Levasseur and Smith 1993, Proposition 2.8], so that the line modules are in

one-to-one correspondence with the elements in the span of the defining relations of  $A$  that have rank at most two. In particular, we compute  $\dim(\Omega_2 \cap \Delta)$  in  $\mathbb{P}(M(4, \mathbb{k}))$ , where  $\Omega_2$  denotes the elements in  $\mathbb{P}(M(4, \mathbb{k}))$  of rank at most two and  $\Delta$  denotes the projectivization of the image in  $\mathbb{P}(M(4, \mathbb{k}))$  of the span of the defining relations of  $A$ . Since  $\Delta \cong \mathbb{P}^5$  and  $\dim(\Omega_2) = 11$ , the dimension is thus at least equal to  $11 + 5 - 15 = 1$ , so, generically,  $A$  has a 1-parameter family of line modules.

In spite of Van den Bergh's work, it was still not clear that a regular algebra satisfying the hypotheses from the preceding paragraph could have both a finite point scheme (especially one of cardinality twenty) and a 1-dimensional line scheme simultaneously. However, Vancliff, Van Rompay and Willaert [Vancliff et al. 1998] proved that there exists a quadratic regular algebra of global dimension four on four generators with six defining relations that has exactly one point module (up to isomorphism) and a 1-parameter family of line mods.

Some years later, Shelton and Vancliff [2002] proved that if a quadratic algebra on four generators with six defining relations has a *finite* scheme of truncated point modules of length three, then that scheme determines the defining relations of the algebra. One should note that this result assumes no hypothesis of regularity nor of any other homological data. Moreover, by [Van Rompay 1996] this result is false in general if the scheme is infinite, even if the algebra is assumed to be regular and noetherian.

Shelton and Vancliff [2002] also proved that if a quadratic regular algebra of global dimension four (that satisfies a few other homological conditions) has four generators and six defining relations and a 1-dimensional line scheme, then that scheme determines the defining relations of the algebra.

These last two results are counterintuitive, since they seem to be saying that if the point scheme (respectively, line scheme) is *as small as possible*, then the defining relations can be recovered from it.

However, by the start of 2001, it was still unclear whether or not any quadratic regular algebra exists that has global dimension four, four generators, six defining relations, exactly twenty nonisomorphic point modules and a 1-dimensional line scheme. Fortunately, this was resolved by Shelton and Tingey [2001] in the affirmative. Sadly, their method to produce their example used much trial and error on a computer, which they and others were unable to duplicate to produce more examples. This hurdle likely had a negative impact on the development of the subject, since it is difficult to make conjectures if there is only one known example. Hence, a quest began to find an algorithm to construct such algebras, but it was another several years before this situation was remedied, and that is discussed in the next section.



## 2. Graded Clifford algebras, graded skew Clifford algebras and quantum planes

This section describes a construction of a certain type of regular algebra of arbitrary finite global dimension; such an algebra is called a graded skew Clifford algebra as it is modeled on the construction of a graded Clifford algebra. If the global dimension is four, then this construction is able to produce regular algebras that have the desired properties described at the end of the previous section. We conclude this section by revisiting the classification of quadratic regular algebras of global dimension three, and show that almost all such algebras may be obtained from regular graded skew Clifford algebras.

We continue to assume that  $\mathbb{k}$  is algebraically closed; we additionally assume  $\text{char}(\mathbb{k}) \neq 2$ . We write  $M(n, \mathbb{k})$  for the space of  $n \times n$  matrices with entries in  $\mathbb{k}$ , and  $M_{ij}$  for the entry in the  $n \times n$  matrix  $M$  that is in row  $i$  and column  $j$ .

### 2A. Graded Clifford algebras.

**Definition 2.1** [Aubry and Lemaire 1985; Le Bruyn 1995]. Let

$$M_1, \dots, M_n \in M(n, \mathbb{k})$$

denote symmetric matrices. A *graded Clifford algebra* (GCA) is the  $\mathbb{k}$ -algebra  $C$  on degree-one generators  $x_1, \dots, x_n$  and on degree-two generators  $y_1, \dots, y_n$  with defining relations given by

- (i) (degree-2 relations)  $x_i x_j + x_j x_i = \sum_{k=1}^n (M_k)_{ij} y_k$  for all  $i, j = 1, \dots, n$ , and
- (ii) degree-3 and degree-4 relations that guarantee  $y_k$  is central in  $C$  for all  $k = 1, \dots, n$ .

In general, GCAs need not be quadratic nor regular, as demonstrated by the next example.

**Example 2.2.** Let  $M_1 = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$  and  $M_2 = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$ . The corresponding GCA is the  $\mathbb{k}$ -algebra on degree-one generators  $x_1, x_2$  with defining relations

$$x_1 x_2 + x_2 x_1 = -x_1^2 - x_2^2, \quad x_1^2 x_2 = x_2 x_1^2,$$

so this algebra is not quadratic nor regular (as  $(x_1 + x_2)^2 = 0$ ). For more details on this algebra, the reader may consult [Vancliff 2015, Example 2.4].

GCAs  $C$  are noetherian by [Artin et al. 1990, Lemma 8.2], since

$$\dim_{\mathbb{k}}(C/\langle y_1, \dots, y_n \rangle) < \infty.$$

Moreover, since each matrix  $M_k$  in the definition is symmetric, we may associate a quadratic form to  $M_k$ , and thereby associate a quadric in  $\mathbb{P}^{n-1}$  to  $M_k$  for each

$k$ . This means that for each GCA, as in Definition 2.1, there is an associated quadric system  $\mathcal{Q}$  in  $\mathbb{P}^{n-1}$ . Quadric systems are said to be base-point free if they yield a complete intersection; that is, the intersection of all the quadrics in the quadric system is empty. Although Example 2.2 demonstrates that a GCA need not be quadratic nor regular, if  $\mathcal{Q}$  is base-point free, it determines these properties of the associated GCA as follows.

**Theorem 2.3** [Aubry and Lemaire 1985; Le Bruyn 1995]. *The GCA  $C$  is quadratic, Auslander-regular of global dimension  $n$  and satisfies the Cohen–Macaulay property with Hilbert series  $1/(1-t)^n$  if and only if the associated quadric system is base-point free; in this case,  $C$  is regular and a domain.*

In spite of this result, regular GCAs of global dimension four are not candidates for generic quadratic regular algebras of global dimension four, since, although their point schemes can be finite [Stephenson and Vancliff 2007; Vancliff et al. 1998], the symmetry of their relations prevents their line schemes from having dimension one [Shelton and Vancliff 2002]. The standard argument to prove this for a quadratic regular GCA  $C$  of global dimension four exploits the symmetry of the defining relations of  $C$  to move the computation of Section 1D inside  $\mathbb{P}(W)$ , where  $W$  is the 10-dimensional subspace of  $M(4, \mathbb{k})$  consisting of all symmetric matrices. Hence, using the notation from Section 1D,  $\Delta \subset \mathbb{P}(W)$  and the line modules are parametrized by  $(\Omega_2 \cap \mathbb{P}(W)) \cap \Delta \subset \mathbb{P}(W)$ ; thus the dimension is at least  $6 + 5 - 9$ , so it is at least two.

Hence, a modification of the definition of GCA is desired in such a way that enough symmetry is retained so as to allow an analogue of Theorem 2.3 to hold, while, at the same time, losing some symmetry so that the line scheme might have dimension one.

**2B. Graded skew Clifford algebras.** In order to generalize the notion of GCA and to have a result analogous to Theorem 2.3, we need to generalize the notions of symmetric matrix and quadric system and make use of normalizing sequences. For any  $\mathbb{N}$ -graded  $\mathbb{k}$ -algebra  $B$ , a sequence  $\{g_1, \dots, g_m\}$  of homogeneous elements of positive degree is called *normalizing* if  $g_1$  is a normal element in  $B$  and, for each  $k = 1, \dots, m-1$ , the image of  $g_{k+1}$  in  $B/\langle g_1, \dots, g_k \rangle$  is a normal element.

We write  $\mathbb{k}^\times$  for  $\mathbb{k} \setminus \{0\}$ .

**Definition 2.4** [Cassidy and Vancliff 2010].

- (a) Let  $\mu \in M(n, \mathbb{k}^\times)$  satisfy  $\mu_{ij}\mu_{ji} = 1$  for all distinct  $i, j$ . We say a matrix  $M \in M(n, \mathbb{k})$  is  $\mu$ -symmetric if  $M_{ij} = \mu_{ij}M_{ji}$  for all  $i, j = 1, \dots, n$ . We write  $M^\mu(n, \mathbb{k})$  for the subspace of  $M(n, \mathbb{k})$  consisting of all  $\mu$ -symmetric matrices.

(b) Fix  $\mu$  as in (a) and additionally assume  $\mu_{ii} = 1$  for all  $i$ . Let  $M_1, \dots, M_n \in M^\mu(n, \mathbb{k})$ . A *graded skew Clifford algebra* (GSCA) associated to  $\mu$  and  $M_1, \dots, M_n$  is a graded  $\mathbb{k}$ -algebra  $A = A(\mu, M_1, \dots, M_n)$  on degree-one generators  $x_1, \dots, x_n$  and on degree-two generators  $y_1, \dots, y_n$  with defining relations given by

- (i) (degree-2 relations)  $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^n (M_k)_{ij} y_k$  for all  $i, j = 1, \dots, n$ , and
- (ii) degree-3 and degree-4 relations that guarantee the existence of a normalizing sequence  $\{y'_1, \dots, y'_n\}$  that spans  $\sum_{k=1}^n \mathbb{k} y_k$ .

Clearly, symmetric matrices and skew-symmetric matrices are  $\mu$ -symmetric matrices for appropriate  $\mu$ , and GCAs are GSCAs. Moreover, by [Artin et al. 1990, Lemma 8.2], GSCAs  $A$  are noetherian since  $\dim_{\mathbb{k}}(A/\langle y_1, \dots, y_n \rangle) < \infty$ . Furthermore, in Definition 2.4(b)(i), for all  $i, j$ , the  $ji$ -relation can be deduced from the  $ij$ -relation by the  $\mu$ -symmetry of the  $M_k$ .

**Examples 2.5.** (a) With  $\mu$  as in Definition 2.4(b), skew polynomial rings on generators  $x_1, \dots, x_n$  with relations  $x_i x_j = -\mu_{ij} x_j x_i$ , for all  $i \neq j$ , are GSCAs.

(b) (quantum affine plane) Let  $n = 2$ , and  $M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$  and  $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ . The degree-2 relations of  $A(\mu, M_1, M_2)$  have the form

$$2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2, \quad x_1 x_2 + \mu_{12} x_2 x_1 = 0,$$

so that  $\mathbb{k}\langle x_1, x_2 \rangle / \langle x_1 x_2 + \mu_{12} x_2 x_1 \rangle \twoheadrightarrow A(\mu, M_1, M_2)$ . By Theorem 2.6 below, this map is an isomorphism (see Examples 3.2(a)).

(c) (“Jordan” plane) Let  $n = 2$ , and  $M_1 = \begin{bmatrix} 2 & 1 \\ \mu_{21} & 0 \end{bmatrix}$  and  $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ . The degree-2 relations of  $A(\mu, M_1, M_2)$  have the form

$$2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2, \quad x_1 x_2 + \mu_{12} x_2 x_1 = y_1 = x_1^2,$$

so that  $\mathbb{k}\langle x_1, x_2 \rangle / \langle x_1 x_2 + \mu_{12} x_2 x_1 - x_1^2 \rangle \twoheadrightarrow A(\mu, M_1, M_2)$ . By Theorem 2.6 below, this map is an isomorphism (see Examples 3.2(b)). Depending on the choice of  $\mu_{12}$ , this family of examples contains the Jordan plane and some quantum affine planes.

(d) The quadratic regular algebra of global dimension four found by Shelton and Tingey [2001], and discussed above in Section 1D, that has exactly twenty nonisomorphic point modules and a 1-dimensional line scheme is a GSCA [Cassidy and Vancliff 2010].

One can associate a noncommutative “quadric” to each  $\mu$ -symmetric matrix  $M_k$  and, in so doing, there is also a notion of “base-point free”. These ideas are discussed in Section 3B below, and yield a generalization of Theorem 2.3:

**Theorem 2.6** [Cassidy and Vancliff 2010]. *The GSCA  $A$  is quadratic, Auslander-regular of global dimension  $n$  and satisfies the Cohen–Macaulay property with Hilbert series  $1/(1-t)^n$  if and only if the associated quadric system is normalizing and base-point free; in this case,  $A$  is regular and a domain and uniquely determined, up to isomorphism, by the data  $\mu, M_1, \dots, M_n$ .*

Theorem 2.6 allowed the production in [Cassidy and Vancliff 2010] of many algebras that are candidates for generic quadratic regular algebras of global dimension four. In particular, there exist quadratic regular GSCAs of global dimension four on four generators with six defining relations that have exactly twenty nonisomorphic point modules and a 1-dimensional line scheme.

It is an open problem to describe the 1-dimensional line schemes of the regular GSCAs of global dimension four in [Cassidy and Vancliff 2010] that have exactly twenty nonisomorphic point modules.

By Examples 1.7(c) and 2.5(b)(c), the regular algebras of global dimension at most two are GSCAs, and, by Section 2C, almost all quadratic regular algebras of global dimension three are determined by GSCAs, so GSCAs promise to be very helpful in the classification of all quadratic regular algebras of global dimension four.

**2C. Quadratic quantum planes.** In the language of [Artin 1992], a regular algebra of global dimension three that is generated by degree-1 elements is sometimes called a *quantum plane* or *quantum projective plane* or a *quantum  $\mathbb{P}^2$* . The classification of the *generic* quantum planes is in [Artin and Schelter 1987; Artin et al. 1990; 1991]. In this subsection, we summarize the results of [Nafari et al. 2011], in which all *quadratic* quantum planes are classified by using GSCAs.

We continue to assume that  $\mathbb{k}$  is algebraically closed, but its characteristic is arbitrary unless specifically stated otherwise.

Let  $D$  denote a quadratic quantum plane and let  $X \subset \mathbb{P}^2$  denote its point scheme. By [Artin et al. 1990, Proposition 4.3; Nafari et al. 2011, Lemma 2.1], there are, in total, four cases to consider:

- $X$  contains a line, or
- $X$  is a nodal cubic curve in  $\mathbb{P}^2$ , or
- $X$  is a cuspidal cubic curve in  $\mathbb{P}^2$ , or
- $X$  is a (nonsingular) elliptic curve in  $\mathbb{P}^2$ .

**Theorem 2.7** [Nafari et al. 2011]. *Suppose  $\text{char}(\mathbb{k}) \neq 2$ . If  $X$  contains a line, then either  $D$  is a twist, by an automorphism, of a GSCA, or  $D$  is a twist, by a twisting system, of an Ore extension of a regular GSCA of global dimension two.*

**Theorem 2.8** [Nafari et al. 2011]. *If  $X$  is a nodal cubic curve, then  $D$  is isomorphic to a  $\mathbb{k}$ -algebra on generators  $x_1, x_2, x_3$  with defining relations*

$$\lambda x_1 x_2 = x_2 x_1, \quad \lambda x_2 x_3 = x_3 x_2 - x_1^2, \quad \lambda x_3 x_1 = x_1 x_3 - x_2^2, \quad (*)$$

where  $\lambda \in \mathbb{k}$  and  $\lambda^3 \notin \{0, 1\}$ . Conversely, for any such  $\lambda$ , any quadratic algebra with defining relations  $(*)$  is a quantum plane and its point scheme is a nodal cubic curve in  $\mathbb{P}^2$ . Moreover, if  $\text{char}(\mathbb{k}) \neq 2$ , then  $D$  is an Ore extension of a regular GSCA of global dimension two; in particular, if  $\lambda^3 = -1$ , then  $D$  is a GSCA.

**Theorem 2.9** [Nafari et al. 2011]. *If  $\text{char}(\mathbb{k}) = 3$ , then  $X$  is not a cuspidal cubic curve in  $\mathbb{P}^2$ . If  $\text{char}(\mathbb{k}) \neq 3$  and if  $X$  is a cuspidal cubic curve in  $\mathbb{P}^2$ , then  $D$  is isomorphic to a  $\mathbb{k}$ -algebra on generators  $x_1, x_2, x_3$  with defining relations*

$$\begin{aligned} x_1 x_2 &= x_2 x_1 + x_1^2, & x_3 x_1 &= x_1 x_3 + x_1^2 + 3x_2^2, \\ x_3 x_2 &= x_2 x_3 - 3x_2^2 - 2x_1 x_3 - 2x_1 x_2. \end{aligned} \quad (\dagger)$$

Moreover, any quadratic algebra with defining relations  $(\dagger)$  is a quantum plane; it has point scheme given by a cuspidal cubic curve in  $\mathbb{P}^2$  if and only if  $\text{char}(\mathbb{k}) \neq 3$ . If  $\text{char}(\mathbb{k}) \neq 2$ , then any quadratic algebra with defining relations given by  $(\dagger)$  is an Ore extension of a regular GSCA of global dimension two.

It remains to discuss the case that  $X$  is an elliptic curve. In [Artin and Schelter 1987; Artin et al. 1990], such algebras are classified into types A, B, E, H, where some members of each type might not have an elliptic curve as their point scheme, but a generic member does.

**Theorem 2.10** [Nafari et al. 2011]. *Suppose that  $\text{char}(\mathbb{k}) \neq 2$  and that  $X$  is an elliptic curve.*

- (a) *Quadratic quantum planes of type H are GSCAs.*
- (b) *Quadratic quantum planes of type B are GSCAs.*
- (c) *As in [Artin and Schelter 1987; Artin et al. 1990], a quadratic quantum plane  $D$  of type A is given by a  $\mathbb{k}$ -algebra on generators  $x, y, z$  with defining relations*

$$axy + byx + cz^2 = 0, \quad ayz + bzy + cx^2 = 0, \quad azx + bxz + cy^2 = 0,$$

where  $a, b, c \in \mathbb{k}^\times$ ,  $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$ ,  $\text{char}(\mathbb{k}) \neq 3$ , and either  $a^3 \neq b^3$ , or  $a^3 \neq c^3$ , or  $b^3 \neq c^3$ . In the case that  $a^3 = b^3 \neq c^3$ ,  $D$  is a GSCA; whereas in the case  $a^3 \neq b^3 = c^3$  (respectively,  $a^3 = c^3 \neq b^3$ ),  $D$  is a twist, by an automorphism, of a GSCA.

In (c) of the last result, the case that  $a^3 \neq b^3 \neq c^3 \neq a^3$  is still open. Moreover, the case when  $D$  is of type E is still open, but this case only consists of one algebra, up to isomorphism and antiisomorphism. However, both type A and type E have the property that the Koszul dual of  $D$  is a quotient of a regular GSCA; so, in this sense, such algebras are weakly related to GSCAs.

### 3. Complete intersections

In this section, we define the geometric terms used in Theorem 2.6. That discussion leads naturally into a consideration of a notion of noncommutative complete intersection that mimics the commutative definition.

We continue to assume that the field  $\mathbb{k}$  is algebraically closed.

**3A. Commutative complete intersection and quadric systems.** Let  $R$  denote the commutative polynomial ring on  $n$  generators of degree one. If  $f_1, \dots, f_m$  are homogeneous elements of  $R$  of positive degree, then  $\{f_1, \dots, f_m\}$  is a regular sequence in  $R$  if and only if  $\text{GKdim}(R/\langle f_1, \dots, f_k \rangle) = n - k \geq 0$ , for all  $k = 1, \dots, m$ . Geometrically, this corresponds to the zero locus in  $\mathbb{P}(R_1^*)$  of the ideal  $J_k = \langle f_1, \dots, f_k \rangle$  having dimension  $n - 1 - k \geq -1$  for all  $k$ . If  $\{f_1, \dots, f_m\}$  is a regular sequence, then the zero locus of  $J_m$  (respectively,  $R/J_m$ ) is called a *complete intersection* (see [Eisenbud 1995]).

In the setting of Definition 2.1, a quadric system  $\mathfrak{Q}$  is associated to symmetric matrices  $M_1, \dots, M_n$ . In that setting,  $\mathfrak{Q}$  corresponds to a regular sequence in  $R$  if and only if  $\mathfrak{Q}$  is a complete intersection, that is, if and only if  $\mathfrak{Q}$  has no base points (a base point is a point that lies on all the quadrics in  $\mathfrak{Q}$ ). A noncommutative analogue of this is needed for Theorem 2.6.

**3B. Noncommutative complete intersection and quadric systems.** The following result uses the notion of base-point module defined in Definition 1.2.

**Proposition 3.1** [Cassidy and Vancliff 2010; 2014]. *Let  $S$  denote the skew polynomial ring from Section 1A, and let  $f_1, \dots, f_n$  denote homogeneous elements of  $S$  of positive degree. If  $\{f_1, \dots, f_n\}$  is a normalizing sequence in  $S$ , then the following are equivalent:*

- (a)  $\{f_1, \dots, f_n\}$  is a regular sequence in  $S$ .
- (b)  $\dim_{\mathbb{k}}(S/\langle f_1, \dots, f_n \rangle) < \infty$ .
- (c) For each  $k = 1, \dots, n$ , we have  $\text{GKdim}(S/\langle f_1, \dots, f_k \rangle) = n - k$ .
- (d) The factor ring  $S/\langle f_1, \dots, f_n \rangle$  has no right base-point modules.
- (e) The factor ring  $S/\langle f_1, \dots, f_n \rangle$  has no left base-point modules.

Such a sequence  $\{f_1, \dots, f_n\}$  (respectively,  $S/\langle f_1, \dots, f_n \rangle$ ) satisfying the equivalent conditions (a)-(e) from Proposition 3.1 is called a *complete intersection* in [Cassidy and Vancliff 2014].

In the setting of Section 2B, one associates  $S$  to the GSCA by using  $\mu$ . The isomorphism  $M^\mu(n, \mathbb{k}) \rightarrow S_2$  defined by  $M \mapsto (z_1, \dots, z_n)M(z_1, \dots, z_n)^T$  associates a quadric system  $\mathfrak{Q}$  to the  $\mu$ -symmetric matrices  $M_1, \dots, M_n$ ; that is,  $\mathfrak{Q}$  is the span in  $S_2$  of the images of the  $M_k$  under this map. If  $\mathfrak{Q}$  is given by a normalizing sequence in  $S$ , then it is called a *normalizing quadric system*. By Proposition 3.1, if  $\mathfrak{Q}$  is normalizing, then it corresponds to a regular sequence in  $S$  if and only if it is a complete intersection, that is, if and only if  $S/\langle \mathfrak{Q} \rangle$  has no right (respectively, left) base-point modules; this is the meaning of *base-point free* in Theorem 2.6.

**Examples 3.2.** (a) [Cassidy and Vancliff 2010] We revisit the quantum affine plane from Example 2.5(b), where  $n = 2$ . In that case,  $M_i \mapsto q_i = 2z_i^2 \in S_2$ , for  $i = 1, 2$ . The sequence  $\{q_1, q_2\}$  is normalizing in  $S$  and  $\dim(S/\langle q_1, q_2 \rangle) < \infty$ . Thus, by Proposition 3.1, the corresponding quadric system is base-point free.

(b) [Cassidy and Vancliff 2010] For Example 2.5(c),  $n = 2$  and  $M_1 \mapsto q_1 = 2(z_1^2 + z_1 z_2)$  and  $M_2 \mapsto q_2 = 2z_2^2$ . Here, the sequence  $\{q_2, q_1\}$  is normalizing in  $S$  and  $\dim(S/\langle q_2, q_1 \rangle) < \infty$ , so by Proposition 3.1, the corresponding quadric system is base-point free.

Proposition 3.1 has recently been extended in [Vancliff 2015] to a family of algebras that contains the skew polynomial ring  $S$  from Section 1A. In particular, an analogue of Proposition 3.1 holds for regular GSCAs, many quantum groups, and homogenizations of finite-dimensional Lie algebras.

**Theorem 3.3** [Vancliff 2015]. *Let  $A = \bigoplus_{i=0}^{\infty} A_i$  denote a connected,  $\mathbb{N}$ -graded  $\mathbb{k}$ -algebra that is generated by  $A_1$ . Suppose  $A$  is Auslander–Gorenstein of finite injective dimension and satisfies the Cohen–Macaulay property, and that there exists a normalizing sequence  $\{y_1, \dots, y_\nu\} \subset A \setminus \mathbb{k}$  consisting of homogeneous elements such that  $\text{GKdim}(A/\langle y_1, \dots, y_\nu \rangle) = 1$ . If  $\text{GKdim}(A) = n \in \mathbb{N}$ , and if  $F = \{f_1, \dots, f_n\} \subset A \setminus \mathbb{k}^\times$  is a normalizing sequence of homogeneous elements, then the following are equivalent:*

- (a)  $F$  is a regular sequence in  $A$ .
- (b)  $\dim_{\mathbb{k}}(A/\langle F \rangle) < \infty$ .
- (c) For each  $k = 1, \dots, n$ , we have  $\text{GKdim}(A/\langle f_1, \dots, f_k \rangle) = n - k$ .
- (d) The factor ring  $A/\langle F \rangle$  has no right base-point modules.
- (e) The factor ring  $A/\langle F \rangle$  has no left base-point modules.

The reader should note that other notions of complete intersection abound in the literature, with most emphasizing a homological approach, such as the recent work in [Kirkman et al. 2013].

#### 4. Conclusion

In this section, we list some open problems and related topics. The open problems are not listed in any particular order in regards to difficulty, and many challenge levels are included, with some quite computational in nature, and so accessible to junior researchers.

**4A. *Some open problems.*** (1) As stated at the end of Section 2, it is still open whether or not quadratic quantum planes of type A with  $a^3 \neq b^3 \neq c^3 \neq a^3$  are directly related to GSCAs; the analogous problem is also open for type E.

(2) Is it possible to classify cubic quantum planes by using GSCAs, or by using an appropriate analogue of a GSCA?

(3) Is it possible to classify quadratic regular algebras of global dimension four by using GSCAs? Presumably, such a classification will use both the point scheme and the line scheme.

(4) Can standard results on commutative quadratic forms and quadrics be extended to noncommutative quadratic forms and quadrics? For example, P. Veerapen and Vancliff [2013] have extended the notion of rank of a (commutative) quadratic form to noncommutative quadratic forms on  $n$  generators, where  $n = 2, 3$ ; can this be done for  $n \geq 4$ ?

(5) Can results concerning GCAs be carried over to GSCAs? In particular, Veerapen and the author applied their aforementioned generalization of rank to GSCAs in a way that is analogous to that used for the traditional notion of rank with GCAs in [Vancliff et al. 1998]. They proved [2014] that various results in [Vancliff et al. 1998] concerning point modules over GCAs apply to point modules over GSCAs.

(6) Can standard results concerning symmetric matrices be extended or generalized to  $\mu$ -symmetric matrices?

(7) Can the results in [Vancliff 2015], mentioned above at the end of Section 3, on complete intersections be extended to an even larger family of algebras?

(8) By combining results in [Cassidy and Vancliff 2010] and [Stephenson and Vancliff 2007], it is known that regular GSCAs of global dimension four can have exactly  $N$  nonisomorphic point modules, where  $N \notin \{2, 19\}$ ; it is not yet known if  $N \in \{2, 19\}$  is possible. In fact, by [Stephenson and Vancliff 2006],  $N = 2$  is possible if the algebra is quadratic and regular of global dimension four



but is not a GSCA, but it is not known if  $N = 19$  is possible, even if the algebra is not a GSCA.

(9) What is the line scheme of some known quadratic regular algebras of global dimension four? Such as those in [Cassidy and Vancliff 2010, Section 5], double Ore extensions in [Zhang and Zhang 2008; 2009], generalized Laurent polynomial rings in [Cassidy et al. 2006], etc.

(10) Does the line scheme of a generic quadratic regular algebra of global dimension four have a particular form? Perhaps a union of elliptic curves? Or, perhaps it contains at least one elliptic curve?

(11) Suppose  $A$  is as in Definition 1.1 and  $F$  is as in Theorem 3.3. Let  $I_k = \langle f_1, \dots, f_k \rangle$  for all  $k \leq n$ , and let  $\widehat{\mathcal{V}}(I_k)$  denote the set of isomorphism classes of right base-point modules over  $A/I_k$ . If  $A$  is commutative, then, for each  $k$ ,  $\widehat{\mathcal{V}}(I_k)$  is a scheme, and so has a dimension. In particular, if  $A$  is the polynomial ring, then  $F$  is regular if and only if  $\dim(\widehat{\mathcal{V}}(I_k)) = n - k - 1$ , for all  $k \leq n$ . However, if  $A$  is not commutative, is there an analogous statement and under what hypotheses on  $A$  could it hold?

**4B. Related topics.** Since the publication of [Artin et al. 1990], the subject has branched out in many directions, the key topics being: classification of regular algebras; classification of projective surfaces; seeing which commutative techniques (e.g., blowing-up, blowing-down) carry over to the noncommutative setting; and connections with differential geometry (e.g., via Poisson geometry). Module categories and homological algebra provide a unifying umbrella. These directions are highlighted in the references cited in the Introduction and throughout the text, and in some of the articles in this volume.

New directions continue to emerge, with one of the most recent trends being the study of regular algebras and Hopf algebras together via the consideration of Hopf actions on regular algebras, such as the work in [Chan et al. 2014]. However, perhaps the most recent exciting triumph of the subject is when the universal enveloping algebra of the Witt algebra was viewed through the geometric lens of [Artin et al. 1990] by Sierra and Walton [2014], enabling them to solve the long-standing problem of whether or not that algebra is noetherian.

In view of all these advances, it is now clear that the marriage of noncommutative algebra and algebraic geometry, à la [Artin et al. 1990], is a dynamic and evolving field of research.

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### References

- [Artin 1992] M. Artin, “Geometry of quantum planes”, pp. 1–15 in *Azumaya algebras, actions, and modules* (Bloomington, IN, 1990), edited by D. Haile and J. Osterburg, *Contemp. Math.* **124**, Amer. Math. Soc., Providence, RI, 1992.
- [Artin and Schelter 1987] M. Artin and W. F. Schelter, “Graded algebras of global dimension 3”, *Adv. in Math.* **66**:2 (1987), 171–216.
- [Artin et al. 1990] M. Artin, J. Tate, and M. Van den Bergh, “Some algebras associated to automorphisms of elliptic curves”, pp. 33–85 in *The Grothendieck Festschrift, I*, edited by P. Cartier et al., *Progr. Math.* **86**, Birkhäuser, Boston, 1990.
- [Artin et al. 1991] M. Artin, J. Tate, and M. Van den Bergh, “Modules over regular algebras of dimension 3”, *Invent. Math.* **106**:2 (1991), 335–388.
- [Aubry and Lemaire 1985] M. Aubry and J.-M. Lemaire, “Zero divisors in enveloping algebras of graded Lie algebras”, *J. Pure Appl. Algebra* **38**:2-3 (1985), 159–166.
- [Cassidy 1999] T. Cassidy, “Global dimension 4 extensions of Artin–Schelter regular algebras”, *J. Algebra* **220**:1 (1999), 225–254.
- [Cassidy 2003] T. Cassidy, “Central extensions of Stephenson’s algebras”, *Comm. Algebra* **31**:4 (2003), 1615–1632.
- [Cassidy and Vancliff 2010] T. Cassidy and M. Vancliff, “Generalizations of graded Clifford algebras and of complete intersections”, *J. Lond. Math. Soc. (2)* **81**:1 (2010), 91–112.
- [Cassidy and Vancliff 2014] T. Cassidy and M. Vancliff, “Corrigendum to ‘Generalizations of graded Clifford algebras and of complete intersections’”, *J. Lond. Math. Soc. (2)* **90**:2 (2014), 631–636.
- [Cassidy et al. 2006] T. Cassidy, P. Goetz, and B. Shelton, “Generalized Laurent polynomial rings as quantum projective 3-spaces”, *J. Algebra* **303**:1 (2006), 358–372.
- [Chan et al. 2014] K. Chan, C. Walton, Y. H. Wang, and J. J. Zhang, “Hopf actions on filtered regular algebras”, *J. Algebra* **397** (2014), 68–90.
- [Drinfeld 1987] V. G. Drinfeld, “Quantum groups”, pp. 798–820 in *Proceedings of the International Congress of Mathematicians* (Berkeley, 1986), vol. 1, edited by A. M. Gleason, Amer. Math. Soc., Providence, RI, 1987.
- [Eisenbud 1995] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer, New York, 1995.
- [Goodearl 2010] K. R. Goodearl, “Semiclassical limits of quantized coordinate rings”, pp. 165–204 in *Advances in ring theory*, edited by D. V. Huynh and S. R. López-Permouth, Birkhäuser/Springer, Basel, 2010.
- [Kapustin et al. 2001] A. Kapustin, A. Kuznetsov, and D. Orlov, “Noncommutative instantons and twistor transform”, *Comm. Math. Phys.* **221**:2 (2001), 385–432.

- [Kirkman et al. 2013] E. Kirkman, J. Kuzmanovich, and J. J. Zhang, “Noncommutative complete intersections”, preprint, 2013. arXiv 1302.6209
- [Le Bruyn 1995] L. Le Bruyn, “Central singularities of quantum spaces”, *J. Algebra* **177**:1 (1995), 142–153.
- [Le Bruyn and Smith 1993] L. Le Bruyn and S. P. Smith, “Homogenized  $\mathfrak{sl}(2)$ ”, *Proc. Amer. Math. Soc.* **118**:3 (1993), 725–730.
- [Le Bruyn and Van den Bergh 1993] L. Le Bruyn and M. Van den Bergh, “On quantum spaces of Lie algebras”, *Proc. Amer. Math. Soc.* **119**:2 (1993), 407–414.
- [Levasseur 1992] T. Levasseur, “Some properties of noncommutative regular graded rings”, *Glasgow Math. J.* **34**:3 (1992), 277–300.
- [Levasseur and Smith 1993] T. Levasseur and S. P. Smith, “Modules over the 4-dimensional Sklyanin algebra”, *Bull. Soc. Math. France* **121**:1 (1993), 35–90.
- [Levasseur and Stafford 1993] T. Levasseur and J. T. Stafford, “The quantum coordinate ring of the special linear group”, *J. Pure Appl. Algebra* **86**:2 (1993), 181–186.
- [Lu et al. 2007] D.-M. Lu, J. H. Palmieri, Q.-S. Wu, and J. J. Zhang, “Regular algebras of dimension 4 and their  $A_\infty$ -Ext-algebras”, *Duke Math. J.* **137**:3 (2007), 537–584.
- [Nafari et al. 2011] M. Nafari, M. Vancliff, and J. Zhang, “Classifying quadratic quantum  $\mathbb{P}^2$ s by using graded skew Clifford algebras”, *J. Algebra* **346** (2011), 152–164.
- [Reshetikhin et al. 1989] N. Y. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev, “Quantization of Lie groups and Lie algebras”, *Algebra i Analiz* **1**:1 (1989), 178–206. In Russian; translated in *Leningrad Math. J.* **1**:1 (1990), 193–225.
- [Rogalski 2014] D. Rogalski, “An introduction to noncommutative projective algebraic geometry”, preprint, 2014. arXiv 1403.3065
- [Rogalski and Zhang 2012] D. Rogalski and J. J. Zhang, “Regular algebras of dimension 4 with 3 generators”, pp. 221–241 in *New trends in noncommutative algebra*, edited by P. Ara et al., Contemp. Math. **562**, Amer. Math. Soc., Providence, RI, 2012.
- [Shelton and Tingey 2001] B. Shelton and C. Tingey, “On Koszul algebras and a new construction of Artin–Schelter regular algebras”, *J. Algebra* **241**:2 (2001), 789–798.
- [Shelton and Vancliff 2002] B. Shelton and M. Vancliff, “Schemes of line modules, I”, *J. London Math. Soc.* (2) **65**:3 (2002), 575–590.
- [Sierra and Walton 2014] S. J. Sierra and C. Walton, “The universal enveloping algebra of the Witt algebra is not noetherian”, *Adv. Math.* **262** (2014), 239–260.
- [Sklyanin 1982] E. K. Sklyanin, “Some algebraic structures connected with the Yang–Baxter equation”, *Funktsional. Anal. i Prilozhen.* **16**:4 (1982), 27–34. In Russian; translated in *Functional Anal. Appl.* **16**:4 (1982), 263–270.
- [Sklyanin 1985] E. K. Sklyanin, “On an algebra generated by quadratic relations”, *Uspekhi Mat. Nauk* **40**:2 (1985), 214. Abstract of lecture given at the Moscow Mathematical Society. In Russian.
- [Smith and Stafford 1992] S. P. Smith and J. T. Stafford, “Regularity of the four-dimensional Sklyanin algebra”, *Compositio Math.* **83**:3 (1992), 259–289.
- [Smith and Staniszkis 1993] S. P. Smith and J. M. Staniszkis, “Irreducible representations of the 4-dimensional Sklyanin algebra at points of infinite order”, *J. Algebra* **160**:1 (1993), 57–86.
- [Stafford 2002] J. T. Stafford, “Noncommutative projective geometry”, pp. 93–103 in *Proceedings of the International Congress of Mathematicians, II* (Beijing, 2002), edited by T. Li, Higher Ed. Press, Beijing, 2002.

- [Stafford and van den Bergh 2001] J. T. Stafford and M. van den Bergh, “Noncommutative curves and noncommutative surfaces”, *Bull. Amer. Math. Soc. (N.S.)* **38**:2 (2001), 171–216.
- [Stephenson 1996] D. R. Stephenson, “Artin–Schelter regular algebras of global dimension three”, *J. Algebra* **183**:1 (1996), 55–73.
- [Stephenson 1997] D. R. Stephenson, “Algebras associated to elliptic curves”, *Trans. Amer. Math. Soc.* **349**:6 (1997), 2317–2340.
- [Stephenson 2000a] D. R. Stephenson, “Quantum planes of weight  $(1, 1, n)$ ”, *J. Algebra* **225**:1 (2000), 70–92.
- [Stephenson 2000b] D. R. Stephenson, “Corrigendum to ‘Quantum planes of weight  $(1, 1, n)$ ’”, *J. Algebra* **234**:1 (2000), 277–278.
- [Stephenson and Vancliff 2006] D. R. Stephenson and M. Vancliff, “Some finite quantum  $\mathbb{P}^3$ s that are infinite modules over their centers”, *J. Algebra* **297**:1 (2006), 208–215.
- [Stephenson and Vancliff 2007] D. R. Stephenson and M. Vancliff, “Constructing Clifford quantum  $\mathbb{P}^3$ s with finitely many points”, *J. Algebra* **312**:1 (2007), 86–110.
- [Sudbery 1993] A. Sudbery, “Matrix-element bialgebras determined by quadratic coordinate algebras”, *J. Algebra* **158**:2 (1993), 375–399.
- [Van Rompay 1996] K. Van Rompay, “Segre product of Artin–Schelter regular algebras of dimension 2 and embeddings in quantum  $\mathbb{P}^3$ s”, *J. Algebra* **180**:2 (1996), 483–512.
- [Vancliff 1995] M. Vancliff, “The defining relations of quantum  $n \times n$  matrices”, *J. London Math. Soc. (2)* **52**:2 (1995), 255–262.
- [Vancliff 1999] M. Vancliff, “Primitive and Poisson spectra of twists of polynomial rings”, *Algebr. Represent. Theory* **2**:3 (1999), 269–285.
- [Vancliff 2000] M. Vancliff, “Non-commutative spaces for graded quantum groups and graded Clifford algebras”, pp. 303–320 in *Clifford algebras and their applications in mathematical physics, I* (Ixtapa-Zihuatanejo, 1999), edited by R. Ablamowicz and B. Fauser, Progr. Phys. **18**, Birkhäuser, Boston, 2000.
- [Vancliff 2015] M. Vancliff, “On the notion of complete intersection outside the setting of skew polynomial rings”, *Comm. Algebra* **43**:2 (2015), 460–470.
- [Vancliff and Van Rompay 1997] M. Vancliff and K. Van Rompay, “Embedding a quantum nonsingular quadric in a quantum  $\mathbb{P}^3$ ”, *J. Algebra* **195**:1 (1997), 93–129.
- [Vancliff and Veerapen 2013] M. Vancliff and P. P. Veerapen, “Generalizing the notion of rank to noncommutative quadratic forms”, pp. 241–250 in *Noncommutative birational geometry, representations and combinatorics*, edited by A. Berenstein and V. Retakh, Contemp. Math. **592**, Amer. Math. Soc., Providence, RI, 2013.
- [Vancliff and Veerapen 2014] M. Vancliff and P. P. Veerapen, “Point modules over regular graded skew Clifford algebras”, *J. Algebra* **420** (2014), 54–64.
- [Vancliff et al. 1998] M. Vancliff, K. Van Rompay, and L. Willaert, “Some quantum  $\mathbb{P}^3$ s with finitely many points”, *Comm. Algebra* **26**:4 (1998), 1193–1208.
- [Zhang and Zhang 2008] J. J. Zhang and J. Zhang, “Double Ore extensions”, *J. Pure Appl. Algebra* **212**:12 (2008), 2668–2690.
- [Zhang and Zhang 2009] J. J. Zhang and J. Zhang, “Double extension regular algebras of type  $(14641)$ ”, *J. Algebra* **322**:2 (2009), 373–409.

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