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# Growth functions

JASON P. BELL

We give a survey of the use of growth functions in algebra. In particular, we define Gelfand–Kirillov dimension and give an overview of some of the main results about this dimension, including Bergman's gap theorem, the solution of the Artin–Stafford conjecture by Smoktunowicz, and the characterization of groups of polynomially bounded growth by Gromov. In addition, we give a summary of the main ideas employed in the proof of Gromov's theorem and discuss the work of Lenagan and Smoktunowicz, which gives a counterexample to Kurosh's conjecture with polynomially bounded growth.

## 1. Introduction

The notion of growth is a fundamental object of study in the theory of groups and algebras, due to its utility in answering many basic questions in these fields. The concept of growth was introduced by Gelfand and Kirillov [1966] for algebras and by Milnor [1968] for groups, who showed that there is a strong relation between the growth of the fundamental group of a Riemannian manifold and its curvature. After the seminal works of Gelfand and Kirillov and of Milnor, the study of growth continued and many important advances were made. In particular, Borho and Kraft [1976] further developed the theory of growth in algebras, giving a systematic study of the theory of Gelfand–Kirillov dimension. In addition to this, Milnor [1968] and Wolf [1968] gave a complete characterization of solvable groups with polynomially bounded growth (see Section 2 for relevant definitions).

The reason for the importance of Gelfand–Kirillov dimension, Gelfand–Kirillov transcendence degree, and corresponding notions in the theory of groups is that it serves as a natural noncommutative analogue of Krull dimension (resp. transcendence degree) and thus provides a suitable notion of dimension for noncommutative algebras. Indeed, the first application of Gelfand–Kirillov dimension was to show that the quotient division algebras of the $m$-th and $n$-th Weyl algebras are isomorphic if and only if $m = n$, by showing that their transcendence degrees differed when $n \neq m$. Since this initial application,

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the theory of growth has expanded considerably and this notion now plays a fundamental role in both geometric group theory and noncommutative projective geometry, where it serves as a natural notion of dimension.

The objective of these notes is to give a survey of the foundational results of Gelfand–Kirillov dimension as well as related growth functions in the theory of groups and rings which have been used to answer difficult questions. In Section 2, we give an overview of the basic terminology that we will be using throughout. In Section 3, we define Gelfand–Kirillov dimension and give a survey of the most important results in the theory of growth, and in Section 4 we give some of the important results in the theory of combinatorics on words and their application to growth of algebras; in particular, we prove Bergman's gap theorem, which asserts that no algebras whose growth is subquadratic but faster than linear can exist.

In Section 5, we show that Gelfand–Kirillov dimension is a noncommutative analogue of Krull dimension for finitely generated algebras and discuss algebras of low Gelfand–Kirillov dimension. In particular, we discuss the Small–Stafford–Warfield theorem [Small et al. 1985] saying that finitely generated algebras of Gelfand–Kirillov dimension satisfy a polynomial identity and we give a brief discussion of what is known about algebras of Gelfand–Kirillov dimension two. In Section 6, we discuss the ingredients in the proof of Gromov's theorem, which beautifully characterizes the finitely generated groups whose group algebras have finite Gelfand–Kirillov dimension; namely, Gromov's theorem asserts that such groups must have a finite-index subgroup that is nilpotent. In Section 7 and Section 8, we discuss two relatively recent advances in the study of growth; in Section 7, we discuss constructions, mostly developed by Smoktunowicz, which show how to construct pathological examples of algebras of finite Gelfand–Kirillov dimension; in Section 8, we discuss Zhang's so-called lower transcendence degree and its applications to the study of division algebras. Finally, in Section 9, we give a brief overview of Artin's conjecture on the birational classification of noncommutative surfaces and its relation to growth.

## 2. Preliminaries

We begin by stating the basic definitions we will be using. We let $\mathscr{C}$ denote the class of maps $f:\mathbb{N}\to\mathbb{N}$ that are monotonically increasing and have the property that there is some positive number $C$ such that $f(n) < C^n$. We say that $f\in\mathscr{C}$ has *polynomially bounded* growth if there is some $d>0$ such that $f(n)\le n^d$ for all $n$ sufficiently large; we say that $f$ has *exponential growth* if there exists a constant $C>1$ such that $f(n)>C^n$ for all $n$ sufficiently large. If $f\in\mathscr{C}$ has neither polynomially bounded nor exponential growth then we say it has *intermediate growth*. If $f(n) = \exp(o(n))$ we say that $f(n)$ has *subexponential growth*. Note that according to these definitions, an element of $\mathscr{C}$ can have intermediate growth without having subexponential growth. As an example, let $T$ be the union of sets of the form $\{(2i)!, \ldots, (2i+1)! - 1\}$ as $i$ ranges over the natural numbers. We may define a weakly increasing map $f(n)$ by declaring that $f(0) = 1$ and $f(n+1) = 2f(n)$ if $n \in T$ and $f(n+1) = f(n)$ if $n \notin T$. Then $f((2m)!) \le 2^{(2m-1)!}$ and thus $f$ cannot have exponential growth. On the other hand, $f((2m+1)!) \ge 2^{(2m+1)!-(2m)!}$. Thus when $n = (2m+1)!$ and $m \ge 1$, we have $f(n) \ge (3/2)^n$ and so $f$ does not have subexponential growth according to our definition. (We note that some, perhaps even most, authors take subexponential growth to include growth types such as the one given in this example.)

Given $f \in \mathscr{C}$ of polynomially bounded growth. We define the *degree* of growth to be

$$\deg(f) := \limsup_{n\to\infty} \frac{\log f(n)}{\log n}.$$

We note that if $f(n)$ is asymptotic to $Cn^{\alpha}$ then this quantity is equal to $\alpha$, and so this notion coincides with our usual notion of degree in this case.

Given $f, g \in \mathscr{C}$, we say that $f$ is *asymptotically dominated* by $g$ if there natural numbers $k_1, k_2 \ge 1$ such that $f(n) \le k_1 g(k_2 n)$. If $f$ is asymptotically dominated by $g$ and $g$ is asymptotically dominated by $f$, then we say that the functions are *asymptotically equivalent*. Asymptotically equivalent functions need not be asymptotic to one another in the conventional sense, but polynomial, exponential, intermediate, and subexponential growth are preserved under this notion of asymptotic equivalence. Furthermore, asymptotically equivalent functions of polynomially bounded growth have the same degree of growth. Henceforth, we will only consider functions up to this notion of asymptotic equivalence.

Given a finitely generated group $G$ and a generating set $S$ with the properties that $1 \in S$ and if $s \in S$ then $s^{-1} \in S$, we can construct a *growth function* of $G$ with respect to the generating set $S$ as follows. We let $d_S(n)$ denote the number of distinct elements of $G$ that can be written as a product of $n$ elements of $S$. Then $d_S(n)$ is an element of $\mathscr{C}$ since $1 \in S$. For example, if $G = \mathbb{Z}^2$ with generators $x, y$, then if we take $S = \{1, x, x^{-1}, y, y^{-1}\}$ then we have $d_S(n) = \#\{x^i y^j \ : \ |i| + |j| \le n\} = (n+1)^2$. We note that if $T$ is another generating set then since $T^k \supseteq S$ and $S^k \supseteq T$ for some natural number $k$ we have $d_S(n) \le d_T(kn)$ and $d_T(n) \le d_S(kn)$ for all $n \ge 0$. Thus, although two different growth functions need not be equal, they are equal up to asymptotic equivalence. Thus we may speak unambiguously of the growth function of a finitely generated group $G$.

Similarly, if $k$ is a field and $A$ is a finitely generated $k$-algebra, we can associate a growth function as follows. Let $V$ be a finite-dimensional subspace of $A$ that generates $A$ as a $k$-algebra. Then we define

$$d_V(n) := \dim_k\left(\sum_{j=1}^{n} V^j\right).$$

Unless otherwise specified, we assume that our algebras have an identity; in this case we also assume that $1 \in V$ and so we have $d_V(n) = \dim_k(V^n)$. As in the case with groups, if $W$ is another generating subspace then we have that $d_W(n)$ and $d_V(n)$ are asymptotically equivalent and so we again speak unambiguously of the growth function of $A$. We make the remark that the growth of a group $G$ is equal to the growth of its group algebra $k[G]$ and so it is enough to consider growth of algebras.

## 3. General results for algebras of polynomially bounded growth

Given a finitely generated $k$-algebra $A$ of polynomially bounded growth. We recall that we have a degree function associated to its growth. In this setting, the degree function is called the *Gelfand–Kirillov* dimension and is denoted by $\mathrm{GKdim}(A)$. More formally, we have

$$\mathrm{GKdim}(A) := \limsup_{n\to\infty} \frac{\log \dim(V^n)}{\log n},$$

where $V$ is a finite-dimensional vector space containing 1 that generates $A$ as a $k$-algebra.

A related quantity was first used by Gelfand and Kirillov [1966] to show that $D_n \cong D_m$ if and only if $n = m$ where $D_n$ and $D_m$ are respectively the quotient division algebras of the $n$-th and $m$-th Weyl algebras. In addition they conjectured that the quotient division algebra of the enveloping algebra of a finite-dimensional algebraic Lie algebra is isomorphic to the quotient division algebra of a Weyl algebra. (This was ultimately shown to be false [Alev et al. 1996].) We note that in the nonfinitely generated case, one simply defines the GK dimension to be the supremum of the GK dimensions of all finitely generated subalgebras.

We now discuss the foundational results in the theory of growth. Before discussing these results in greater detail, we give a quick summary of these results for the reader's convenience. We let $k$ be a field and we let $A$ be a finitely generated $k$-algebra. Then we have the following:

(i) if $\mathrm{GKdim}(A) \in [0, 1)$ then $A$ is finite-dimensional;

(ii) (Bergman (see [Krause and Lenagan 2000, Theorem 2.5])) if $\mathrm{GKdim}(A) \in [1, 2)$ then $A$ has GK dimension 1;

(iii) (Small, Stafford and Warfield [Small et al. 1985]) if $A$ has GK dimension one then $A$ satisfies a polynomial identity;

(iv) (Small, Stafford, Warfield [Small et al. 1985] and van den Bergh) if $A$ is a domain of GK dimension one and $k$ is algebraically closed then $A$ is commutative;

(v) (Smoktunowicz [2005; 2006]) if $A$ is a graded domain with GK dimension in $[2, 3)$ then $A$ has GK dimension two (and in fact has quadratic growth).

(vi) (Artin and Stafford [1995]) if $A$ is a graded complex domain of GK dimension 2 that is generated in degree one then $A$ is — up to a finite-dimensional vector space — equal to the twisted homogeneous coordinate ring of a curve;

(vii) (Borho and Kraft [1976]) for each $\alpha \in [2, \infty]$ there is an algebra of GK dimension $\alpha$;

(viii) if $A$ satisfies a polynomial identity and $A$ is semiprime then its GK dimension is equal to a nonnegative integer [Krause and Lenagan 2000, Chapter 10];

(ix) if $A$ is commutative then the GK dimension is equal to the Krull dimension;

(x) (Gromov [1981]) If $A = k[G]$, where $G$ is a finitely generated group, then $A$ has finite GK dimension if and only if $G$ is nilpotent-by-finite;

(xi) (Bass and Guivarch [Krause and Lenagan 2000, Theorem 11.14]) If $A = k[G]$, where $G$ is a finitely generated nilpotent-by-finite group, then the GK dimension of $A$ is an integer given by

$$\sum_i i \cdot d_i,$$

where $d_i$ is the rank of the of the $i$-th quotient of the lower central series of $G$.

We note that (i) is immediate since if $A$ is an algebra then we either have $V^i = V^{i+1}$ for some $i$ or we have $V^n \geq n + 1$ for all $n \geq 0$. We note that for (x), Gromov [1981] proved that a finitely generated group of polynomially bounded growth is virtually nilpotent and Bass and Guivarch had given the formula for the degree of growth earlier.

## 4. Combinatorics on words

In this section, we discuss the values that can arise as the GK dimension of an algebra. Many of the foundational results for algebras and groups of low growth come from the theory of combinatorics on words. The reason for this is that given a finitely generated $k$-algebra $A$ one can associate a monomial algebra

$B$ that has the same growth as $A$ and to determine the growth of a monomial algebra depends on estimating the number of words of length $n$ that avoid a given set of forbidden subwords. We now make this more precise.

A finitely generated algebra $A$ can be written in the form $k\{x_1, \ldots, x_d\}/I$ for some ideal $I$ in the free algebra $k\{x_1, \ldots, x_d\}$. We may put a degree lexicographic ordering on the monomials of the free algebra by declaring that $x_1 \prec x_2 \prec \cdots \prec x_d$. Given an element $f \in k\{x_1, \ldots, x_d\}$, we write $f$ as a linear combination of words in $\{x_1, \ldots, x_d\}$. We then define $\mathrm{in}(f)$, the initial monomial of $f$, to be the degree lexicographic word $w$ that appears with nonzero coefficient in our expression for $f$. Then we may associate a monomial ideal $J$ to $I$ by taking the ideal generated by all initial words of elements of $I$. (Note: it is not sufficient to take the initial words of a generating set for $I$, as anyone who has worked with Gröbner bases will understand.)

The monomial algebra $B := k\{x_1, \ldots, x_d\}/J$ and $A$ then have identical growth functions, but $B$ has the advantage of having a more concrete way of studying its growth. We record this observation now.

**Remark 4.1.** Given a finitely generated associative algebra $A$, there is a finitely generated monomial algebra $B = k\{x_1, \ldots, x_d\}/I$ with identical growth; moreover, if $V$ is the image of the vector space spanned by $\{1, x_1, \ldots, x_d\}$ in $B$ then the dimension of $V^n$ is precisely the number of words over the alphabet $\{x_1, \ldots, x_d\}$ of length at most $n$ that are not in $I$.

We recall two classical results in the theory of combinatorics of words. These deal with *right infinite words*, which are as one might expect just infinite sequences over some alphabet $\Sigma$. The first result is generally called König's infinity lemma, which is very easy to prove, but is nevertheless incredibly useful.

**Theorem 4.2** (König). *Let $\Sigma$ be a finite alphabet and let $S$ be an infinite subset of $\Sigma^*$. Then there is a right infinite word $w$ over $\Sigma$ such that every subword of $w$ is a subword of some word in $S$.*

The second result is Furstenberg's theorem, which is really part of a more general theorem relating to dynamical systems. We recall that a right infinite word $w$ is *uniformly recurrent* if for any finite subword $u$ that appears in $w$, there is a natural number $N = N(u)$ with the property that in any block of $N$ consecutive letters in $w$ there must be at least one occurrence of $u$.

**Theorem 4.3** (Furstenberg). *Let $\Sigma$ be a finite alphabet and let $w$ be a right infinite word over $\Sigma$. Then there is a right infinite uniformly recurrent word $u$ over $\Sigma$ such that every subword of $u$ is also a subword of $w$.*

The first significant use of the ideas from combinatorics of words in the theory of growth is due to Bergman (see [Krause and Lenagan 2000]), who showed that there is a gap theorem for growth. It is clear that there are no algebras with GK dimension strictly between 0 and 1, but Bergman showed that there are in fact no algebras with GK dimension strictly between 1 and 2. We give a short proof of this theorem.

**Theorem 4.4** (Bergman). *There are no algebras of GK dimension strictly between* 1 *and* 2.

*Proof.* Suppose that $A$ is a finitely generated algebra of GK dimension $\alpha \in (1, 2)$. It is no loss of generality to assume that $A = k\{x_1, \dots, x_d\}/I$ where $I$ is generated by monomials. Let $\mathcal{S}$ denote the set of words over $\{x_1, \dots, x_d\}$ that are not in $I$ but have the property that all sufficiently long right extensions are in $I$. By König's infinity lemma, $\mathcal{S}$ must be a finite set. We let $J$ denote the ideal generated by $I$ and the elements of $\mathcal{S}$. Then $B = k\{x_1, \dots, x_d\}/J$ and $A$ have the same growth since there is only finite set of words over $\{x_1, \dots, x_d\}$ that are not in $I$ but are in $J$. By construction, if $W$ is a word that survives mod $J$ then $W$ has arbitrarily long right extensions that survive mod $J$.

Let $f(n)$ denote the number of words of length $n$ over $\{x_1, \dots, x_d\}$ that survive mod $J$. If $V$ is the span of the images of $1, x_1, \dots, x_d$ in $B$ then $V^n = 1 + f(1) + \cdots + f(n)$.

There are now two quick cases to consider. If $f(n+1) > f(n)$ for all $n$ then we have $f(n) \geq n+1$ for every natural number $n$ and so $\dim(V^n) \geq \binom{n+2}{2}$ which gives that $B$ has GK dimension at least two, a contradiction.

If $s = f(i+1) = f(i)$ for some $i$, then we let $W_1, \dots, W_s$ denote the set of distinct words of length $i$ that are not in $J$. By our construction of $J$, each $W_j$ is an initial subword of a word of length $i+1$ that is not in $J$. Moreover, since there are only $s$ words of length $i$, we see that each $W_j$ has a unique right extension to a word of length $i+1$ that is not in $J$. But now we use the fact that each word of length $i+1$ can be written in the form $x_k W_j$ for some $k \in \{1, \dots, d\}$ and $j \in \{1, \dots, s\}$. Since $W_j$ has a unique right extension to a word of length $i+1$ that is not in $J$ we see each word of length $i+1$ has a unique extension to a word of length $i+2$ that is not in $J$. In particular, $f(i+2) = s$. Continuing in this manner, we see that $f(n) = s$ for all $n \geq i$, and so $B$ has linear growth, which is again a contradiction. □

On the other hand, Borho and Kraft [1976] showed that no additional gaps exist in general; that is, any real number that is at least two can be realized as the GK dimension of a finite generated algebra. As an example, we show how one can get an algebra of GK dimension 2.5. We note that this example can be easily modified to get any GK dimension between two and three. Taking polynomial rings over these algebras, one can then construct examples of any GK dimension greater than or equal to two.

We let $A = k\{x, y\}/I$, where $I$ is the ideal generated by all words that have at least three copies of $x$ and all words of the form $xy^jx$ with $j$ not a perfect square. Then the set of words of length at most $n$ over $\{x, y\}$ that are not in $I$ is given by

$$\mathcal{S}_n := \{y^i x y^{j^2} x y^k : i + j^2 + k \le n-2\} \cup \{y^i x y^j : i + j \le n-1\} \cup \{y^j : j \le n\}.$$

It is straightforward to check that $\{y^i x y^j : i + j \le n-1\} \cup \{y^j : j \le n\}$ has size $\binom{n+1}{2} + n + 1$. We note that the set of nonnegative integers for which $i + j^2 + k \le n-2$ has size at most $2n^{5/2}$ since $i, k \le n-1$ and $j \le 2\sqrt{n} - 1$. Thus

$$\#\{y^i x y^{j^2} x y^k : i + j^2 + k \le n-2\} \le 2n^{5/2}.$$

Similarly, since any $i \le (n-2)/4$, $j \le \sqrt{(n-2)/4}$, $k \le (n-2)/4$ satisfies $i + j^2 + k \le n-2$ we see that

$$\#\{y^i x y^{j^2} x y^k : i + j^2 + k \le n-2\} \ge (n-2)^{5/2}/32.$$

Thus

$$\limsup_{n\to\infty} \log(\#\mathcal{S}_n)/\log n = 2.5.$$

It follows that the GK dimension of $A$ is precisely 2.5.

We note that the examples of [Borho and Kraft 1976] are very far from being Noetherian or even Goldie. Smoktunowicz [2005; 2006] showed that if one considers graded domains of GK dimension less than three, then there is a gap.

**Theorem 4.5** (Smoktunowicz). *Let $A$ be a finitely generated graded algebra whose GK dimension is in* $[2, 3)$. *Then $A$ has GK dimension* 2.

A partial result of this type had earlier been obtained by Artin and Stafford [1995], who conjectured that the Smoktunowicz gap theorem should hold. Artin and Stafford proved that finitely generated graded algebras whose GK dimension lies in $(2, 11/5)$ could not exist.

## 5. Small Gelfand–Kirillov dimension

As pointed out earlier, Gelfand–Kirillov dimension can be viewed as a noncommutative analogue of Krull dimension. Much as in the commutative setting special attention has been paid to algebras of small Krull dimension (and, correspondingly, to the study of curves and surfaces and threefolds), there has also been considerable work devoted to the study of algebras of low GK dimension. We first show that GK dimension can be viewed as a reasonable analogue of Krull dimension.

**Proposition 5.1.** *Let $A$ be a finitely generated commutative $k$-algebra. Then* $\mathrm{GKdim}(A) = \mathrm{Kdim}(A)$.

*Proof.* Let $d$ denote the Krull dimension of $A$. By Noether normalization, there exists a subalgebra $B \cong k[x_1, \ldots, x_d]$ of $A$ such that $A$ is a finite $B$-module. It is straightforward to check that $A$ and $B$ have the same GK dimension. Thus it is enough to prove that $k[x_1, \ldots, x_d]$ has GK dimension $d$.

Let $C = k[x_1, \ldots, x_d]$ and let $V = k + kx_1 + \cdots + kx_d$. Then $V^n$ has a basis given by all monomials in $x_1, \ldots, x_d$ of total degree at most $n$. Observe that the monomials $x_1^{i_1} \cdots x_d^{i_d}$ with $i_1 + \cdots + i_d \leq n$ are in one-to-one correspondence with subsets of $\{1, 2, \ldots, n+d\}$ of size $d$ via the rule

$$x_1^{i_1} \cdots x_d^{i_d} \mapsto \{i_1 + 1, i_2 + 2, \ldots, i_d + d\}.$$

Thus $V^n$ has dimension $\binom{n+d}{d}$ which is asymptotic to $n^d/d!$ as $n \to \infty$. Thus the GK dimension of the polynomial ring in $d$ variables is precisely $d$. The result follows. □

We have seen that algebras of GK dimension 0 are finite-dimensional. While the class of finite-dimensional algebras is not well-understood, the Artin–Wedderburn theorem says that in the prime case all such algebras are given by a matrix ring over a division algebra that is finite-dimensional over its center. In particular, a domain of GK dimension zero over an algebraically closed field is equal to the algebraically closed field. Small, Stafford, and Warfield [1985] proved that a finitely generated algebra of GK dimension one satisfies a polynomial identity. A particularly nice consequence of this, apparently first observed by van den Bergh, shows that a domain of GK dimension one over an algebraically closed field is necessarily commutative.

**Theorem 5.2.** *Let $k$ be an algebraically closed field and let $A$ be a finitely generated $k$-algebra that is a domain of GK dimension one. Then $A$ is commutative.*

*Proof.* Let $t \in A \setminus k$. Then $t$ is not algebraic over $k$ and hence $k[t]$ must be a polynomial ring in one variable. A theorem of Borho and Kraft [1976] shows that $A$ has a quotient division ring $D$ and that $D$ is a finite-dimensional left vector space over $k(t)$. It is straightforward to check that $k(t)$ has GK dimension 1 as a $k$-algebra and thus $D$ has GK dimension one, since it is a finite module over $k(t)$. We let $D$ act on itself, regarded as a finite-dimensional $k(t)$-vector space, by left multiplication. This gives an embedding of $D$ into $\mathrm{End}_{k(t)}(D)$, which is a matrix ring over $k(t)$. It follows that $D$ satisfies a polynomial identity. Thus $D$ is finite-dimensional over its center. But the center $Z$ of $D$ has the same GK dimension as $D$ since $[D : Z] < \infty$. Hence $Z$ is a field of transcendence degree one. By Tsen's theorem we see that $D = Z$. □

As Stafford and van den Bergh point out, intuitively, this result makes perfect sense: a one-dimensional algebra should be essentially generated by one element and since an element commutes with itself, it is quite reasonable that such algebras should be commutative.

For algebras of Gelfand–Kirillov dimension two, the picture becomes significantly more complicated. For GK dimension two, there is a natural subclass of algebras: algebras of *quadratic growth*. These are finitely generated algebras $A$ of GK dimension two with the property that there is some finite-dimensional generating subspace $V$ of $A$ that contains 1 with the property that the growth of the dimension of $V^n$ is bounded above by $Cn^2$ for some positive constant $C$ for $n$ sufficiently large.

It is known that once one abandons quadratic growth and considers all algebras of GK dimension two, pathologies arise (see, for example, [Bell 2003]). On the other hand, there are no known examples of prime Noetherian algebras of GK dimension two that do not have quadratic growth. In the case of quadratic growth, algebras appear to be very well-behaved. In [Bell 2010] we showed that a complex domain of quadratic growth is either primitive or it satisfies a polynomial identity. This says that the algebra is either very close to being commutative or, in some sense, as far from being commutative as possible.

## 6. Gromov's theorem

Gromov's theorem states that every finitely generated group of polynomially bounded growth is nilpotent-by-finite. In this section we will give a brief overview of the ideas used in the proof and discuss possible extensions. We first note that the case of solvable groups of subexponential growth had already been considered by Milnor [1968] and Wolf [1968].

**Theorem 6.1** [Milnor 1968; Wolf 1968]. *Let $G$ be a finitely generated solvable group of subexponential growth. Then $G$ is nilpotent-by-finite.*

This result has a completely elementary proof. The first main idea is that a combinatorial argument gives that if $G$ is a finitely generated group of subexponential growth then $G'$ is also finitely generated. From this, one obtains that $G$ is polycyclic. This was Milnor's contribution to the theorem. One now uses induction on the solvable length of $G$ and the fact that conjugation by elements of $G$ on a characteristic finitely generated abelian subgroup gives a linear map. By looking at the eigenvalues of this map, one sees that a dichotomy arises: if one has an eigenvalue whose modulus is strictly greater than 1 then one gets exponential growth; if all eigenvalues have modulus one then Kronecker's theorem gives that they are roots of unity and one can deduce nilpotence of a finite-index subgroup from this. This eigenvalue analysis argument was Wolf's contribution to the argument.

The second thing we point out is that the linear case of Gromov's theorem is a consequence of a well-known alternative due to Tits [1972] and the above result of Milnor and Wolf.

**Theorem 6.2** (the Tits alternative). *Let $K$ be a field and let $G$ be a finitely generated subgroup of* $\mathrm{GL}_n(K)$. *Then $G$ is either solvable-by-finite or $G$ contains a free subgroup on two generators.*

The proof of the Tits alternative is very difficult and makes use of the so-called "ping-pong" lemma to construct free subgroups. We only consider the case when $K$ is the complex numbers. We recall that a matrix $A$ has a *dominant* eigenvalue $\alpha$ if $|\alpha| > |\beta|$ whenever $\beta$ is another eigenvalue of $A$ and the kernel of $(A-\alpha I)^d$ is one-dimensional for every $d \geq 1$. One first shows that if the group $G$ has an element $A$ such that both $A$ and $A^{-1}$ have a *dominant* eigenvalue and $B$ is an element of $G$ such that neither $B$ nor its inverse send the corresponding dominant eigenvectors of $A$ and $A^{-1}$ into some proper $A$-invariant subspace, then there is some $n$ such that $A^n$ and $BA^nB^{-1}$ generate a free group.

From here, one uses different absolute values of the complex numbers and different representations of $G$ to show that if $G$ does not contain such a matrix $A$ then either $G$ has a solvable normal subgroup $N$ such that $G/N$ embeds in a subdirect product of smaller linear groups (one obtains the result by an induction on the size of the linear group in this case) or $G$ has the property that every element of $G$ has all of its eigenvalues equal to roots of unity. In this case, one can use arguments to Burnside and Schur to show that such a group is necessarily solvable-by-finite.

We note that given a finitely generated group $G$ and a finite symmetric generating set $S$ that includes 1, we can create an associated undirected Cayley graph $\Gamma = \Gamma(G, S)$ with vertices given by the elements of $G$ and in which edges $x$ and $y$ are adjacent exactly when $xs = y$ for some $s \in S$. (Our choice of $S$ creates loops in $\Gamma$.) We note that $\Gamma$ has the property that each vertex in $\Gamma$ has degree $|S|$ (and is adjacent to itself) and so the adjacency matrix has the property that each row has exactly $|S|$ ones and in particular $|S|$ as an eigenvalue. As it turns out, when $G$ is infinite there is a large eigenspace associated to the eigenvalue $|S|$.

We make this precise now. We consider the complex vector space $V$ of maps $f$ from $G$ to $\mathbb{C}$ with the properties that $\sum_{s\in S} f(xs) = |S| \cdot f(x)$ and such that for each $s \in S$ the map $f(x) - f(xs)$ is in $L^\infty(G)$. Such functions are called the *Lipschitz harmonic* functions on $G$ with respect to $S$. Kleiner [2010] shows that when $G$ is a finitely generated infinite group of polynomially bounded growth, the vector space $V$ is finite-dimensional and has dimension at least two. We record this result now.

**Theorem 6.3** [Kleiner 2010]. *Let $G$ be a finitely generated infinite group of polynomially bounded growth. Then the space of Lipshitz harmonic functions on $G$ with respect to $S$ is finite-dimensional of dimension at least two.*

This is a particularly important part of the proof and appears to be the most vulnerable in terms of trying to extend Gromov's theorem to periodic groups whose growth functions are bounded by $\exp(n^{\epsilon})$ with $\epsilon > 0$ very small. The fact that $V$ has dimension at least two is something that is true for all finitely generated infinite groups; it is the finite-dimensionality of $V$ that is really the difficult point.

We now see how this quickly gives Gromov's theorem. Before we begin, we make the remark that it is no loss of generality to replace $G$ by a finite-index subgroup since a finite-index subgroup of a finitely generated group is still finitely generated and has the same growth as the larger group.

*Proof of Gromov's theorem using Kleiner's theorem.* Let $S$ be a symmetric generating set for $G$. We let $\alpha$ denote the supremum of all real numbers $\beta$ with the property that the conclusion to Gromov's theorem holds for all finitely generated groups $G$ with growth degree $\beta$. If $\alpha = \infty$ then we are done, and so we may assume that $\alpha < \infty$. Then there exists some $d < \alpha + 1$ and some finitely generated group $G$ of growth degree $d$ that is not nilpotent-by-finite.

We let $V$ denote the space of Lipschitz harmonic functions on $G$ with respect to $S$. Note that $G$ acts on $V$ via $g \cdot f(x) = f(g^{-1}x)$ and this gives a homomorphism from $G$ into $\mathrm{GL}(V)$. We let $N$ denote the kernel of this homomorphism. Then $G/N$ is a linear group of subexponential growth and so by the Tits' alternative the image is solvable-by-finite (since a free group on two generators has exponential growth).

We note that $G/N$ must be infinite, since otherwise $N$ acts trivially on $V$ and this gives that all functions in $V$ are constant on cosets of $N$. In particular, each $f \in V$ takes at most $[G : N]$ values. But it is easy to see that this forces $f$ to be constant, since if $x \in G$ is chosen so that $|f(x)|$ is maximal then the equality $\sum_{s \in S} f(xs) = |S| f(x)$, gives that $f(xs) = f(x)$ for all $s \in S$ and since $S$ generates $G$ we see that $f$ is constant on $G$. This contradicts the fact that $V$ has dimension at least two.

Thus $G/N$ is an infinite solvable-by-finite group. By replacing $G$ by a finite-index subgroup if necessary, we may assume that $G/N$ is solvable and that $(G/N)/(G/N)'$ is infinite. In this case $G$ has a normal subgroup $H$ such that $H \supseteq N$ and $G/H \cong \mathbb{Z}$. Moreover, $H$ is finitely generated by a combinatorial argument due to Milnor and it is easy to check that the growth of $H$ has degree at most $d - 1 < \alpha$. Consequently, $H$ is nilpotent-by-finite. By assumption, $G/H$ is solvable since it is a homomorphic image of $G/N$. It is straightforward to check that $G$ is solvable-by-finite, since both $G/H$ and $H$ are finitely generated and solvable-by-finite. Thus $G$ has a finite-index subgroup that is solvable in this case. Thus the theorem reduces to the solvable case of Gromov's theorem, which is handled by the Milnor–Wolf theorem. □

If one wishes to use the methods from the proof of Gromov's theorem in other contexts there are a few things that should be pointed out.

(A) First, the proof relies on the construction of a finite-dimensional vector space $V$ of dimension at least two on which no finite-index subgroup can act trivially.

(B) Second, the proof shows that $G$ has a finite-index subgroup that surjects onto $\mathbb{Z}$ and uses an induction on the degree of growth of $G$ to finish the proof.

These two points show us that it is unreasonable to use the method to try to extend Gromov's theorem to include groups of "slow" subexponential growth, since the induction is not available in this case. The second point does lend itself to the study of periodic (torsion) groups, however. In this case the existence of a surjection onto $\mathbb{Z}$ gives an immediate proof that $G$ is not a periodic group and so the first point is the only obstruction in this case.

Another interesting question involves Noetherian group algebras. We note that an immediate consequence of Gromov's theorem is that if $G$ is a finitely generated group of polynomially bounded growth then the group algebra of $G$ is Noetherian. It is conjectured that a group algebra is Noetherian if and only if the group $G$ is polycyclic-by-finite. It seems plausible that one can use the Noetherian property to handle (A); the problem, however, is that the induction step in (B) is not available. Thus we pose the following question, which in theory makes the induction step in (B) doable.

**Question 1.** Suppose that $k[G]$ is a Noetherian group algebra of finite Krull dimension. Is $G$ polycyclic-by-finite?

## 7. The Kurosh problem and growth

Due to constructions of Smoktunowicz, there has been renewed interest in the construction of finitely generated algebraic algebras that are not finite-dimensional (see, for example, [Bell and Small 2002; Bell et al. 2012; Lenagan and Smoktunowicz 2007; Lenagan et al. 2012; Smoktunowicz 2000; 2002; 2009]). The first examples of such algebras were constructed by Golod and Sharafervich [Golod 1964; Golod and Shafarevich 1964], who used a simple combinatorial criterion that guaranteed that algebras with certain presentations are infinite-dimensional. Their construction provided a counterexample to Kurosh's conjecture, which asserts that finitely generated algebraic algebras should be finite-dimensional over their base fields. By using their construction in a clever way, Golod and Shafarevich were also able to give a counterexample to the celebrated Burnside problem, which is the group-theoretic analogue of the Kurosh conjecture and asks whether or not finitely generated torsion groups are necessarily finite.

The connection between Burnside-type problems in the theory of groups and Kurosh-type problems in ring theory has led to many interesting conjectures in both fields, which have arisen naturally from results in one field or the other.

As mentioned in the preceding section, Gromov's theorem gives a concrete description of groups of polynomially bounded growth. As finitely generated nilpotent torsion groups are finite, Gromov's theorem thus immediately gives the result that a finitely generated torsion group of polynomially bounded growth is finite.

In light of the result of Gromov and its consequence for the Burnside problem, Small asked whether a finitely generated algebraic algebra of polynomially bounded growth should be finite-dimensional [Lenagan et al. 2012]. Surprisingly, Lenagan and Smoktunowicz [2007] were able to give a counterexample, by constructing a finitely generated nil algebra of Gelfand–Kirillov dimension at most 20. We point out that their construction only works over countable base fields, and it is still an open question as to whether the Kurosh problem for algebras of polynomially bounded growth should hold over uncountable fields.

Recently, Lenagan, Smoktunowicz, and Young [Lenagan et al. 2012] showed that the bound on Gelfand–Kirillov dimension could be lowered from 20 to 3. On the other hand, it is known that the bound cannot be made lower than 2, as finitely generated algebras of Gelfand–Kirillov dimension strictly less than two satisfy a polynomial identity [Small et al. 1985; Krause and Lenagan 2000, Theorem 2.5, p. 18] and the Kurosh conjecture holds for the class of algebras satisfying a polynomial identity [Herstein 1994, Section 6.4].

The fact that these constructions do not work over an uncountable base field is not surprising, as many results have appeared over the years which show there is a real dichotomy that exists regarding Kurosh-type problems when one considers base fields. For example, algebraic algebras over uncountable fields have what is known as the *linearly bounded degree* property (see, for example, [Jacobson 1964, p. 249, Definition 1]). The linearly bounded degree property simply says that given a fixed finite-dimensional subspace of an algebraic algebra, there is a natural number $d$, depending on the subspace, such that all elements in this subspace have degree at most $d$. Smoktunowicz [2000] has given an example of a nil algebra over a countable base field with the property that the ring of polynomials over this algebra is not nil and hence this algebra cannot have linearly bounded degree.

This distinction, and the fact that the elements in a finitely generated algebra over a countable base field can be enumerated, has led to a relative dearth of interesting examples of algebraic algebras over uncountable base fields. Indeed, over uncountable fields there has not been much progress since the original construction of Golod and Shafarevich.

In a [Bell and Young 2011], counterexamples to the Kurosh conjecture of subexponential growth over general fields were found.

**Theorem 7.1** [Bell and Young 2011]. *Let $K$ be a field and let*

$$\alpha : [0, \infty) \to [0, \infty)$$

*be a weakly increasing function tending to $\infty$. Then there is a finitely generated connected graded infinite-dimensional $K$-algebra*

$$B = \bigoplus_{n \geq 0} B(n)$$

*such that the homogeneous maximal ideal $\oplus_{n\geq 1} B(n)$ is nil, and* $\dim(B(n)) \leq n^{\alpha(n)}$ *for all sufficiently large $n$.*

Equivalently, Theorem 7.1 says that if $\beta(n)$ is any monotonically increasing function that grows subexponentially but superpolynomially in $n$, in the sense that $n^d/\beta(n) \to 0$ as $n \to \infty$ for every $d \geq 0$, then one can find a connected graded $K$-algebra $B$ whose homogeneous maximal ideal is nil and has the property that the growth function of $B$ is asymptotically dominated by $\beta(n)$.

One should contrast this situation with the situation in group theory, where considerably less is known about the possible growth types of finitely generated torsion groups of superpolynomial growth. There have been many constructions of *branch groups*, which provide examples of groups that have subexponential but superpolynomial growth. The first such construction was done by Grigorchuk [1980], and estimates of Bartoldi [2005] show that the growth of Branch group constructions is at least $\exp(\sqrt{n})$, but significantly less than $\exp(cn)$ for any $c > 0$. We now discuss the method used by Lenagan and Smoktunowicz [2007].

**7.1.** ***The bottleneck method.*** The bottleneck method is a technique for producing nil algebras that has been largely honed by Smoktunowicz [Lenagan and Smoktunowicz 2007; Lenagan et al. 2012; Smoktunowicz 2000; 2002; 2009] in producing a sequence of counterexamples to Kurosh-type problems. It is the modification of this method given in her paper with Lenagan, however, that has inspired this choice of name. To understand the basic philosophy of the technique, let us suppose that one wishes to construct an algebra of polynomially bounded growth with certain additional properties. Typically, people would do this by imposing a finite number of cleverly chosen relations of small degree that had the effect of curtailing the growth. This method is often used, but it is of apparently no help if one wishes to construct nil algebras of polynomially bounded growth.

Lenagan and Smoktunowicz take a more Malthusian approach to manufacturing growth. In this setting, they take a sparse sequence of natural numbers $n_1 < n_2 < n_3 < \cdots$ with $n_1$ very large. They then take a free algebra $A = k\{x, y\}$ over a countable field $k$. Since $k$ is countable, we can enumerate the elements of $A$. We let $A = \{f_1, f_2, \ldots\}$. We note that $A$ has exponential growth and there are $2^n$ words over the alphabet $\{x, y\}$ of length $n$. They then impose a large number of homogeneous relations of degree $n_1$ with the property that $f_1$ is nilpotent modulo these relations. In fact, more relations are imposed than necessary and this has the effect of creating a "bottleneck" at degree $n_1$. While many relations are imposed, the relations are chosen so that if no further relations were imposed, the algebra would sill have exponential growth. In general, at each level $n_i$, they create another bottleneck by imposing homogeneous relations of degree $n_i$ such that $f_i$ is nilpotent modulo these relations.

There is a lot of care required: if the bottlenecks are too sparse, the algebra will have intermediate growth; if the bottlenecks are too narrow, the algebra will be finite-dimensional. The fact that one is dealing with two-sided ideals, which are much more difficult to control than their one-sided counterparts makes the construction especially difficult.

Despite these obstacles, Lenagan and Smoktunowicz were able to construct a nil algebra of polynomially bounded growth over any countable field. Later, in a paper with Young, they showed that one can construct a nil algebra whose GK dimension is at most three.

Let $K$ be a field and let $A = K\{x, y\}$ denote the free $K$-algebra on two generators $x$ and $y$. Then $A$ is an $\mathbb{N}$-graded algebra, and we let $A(n)$ denote the $K$-subspace spanned by all words over $x$ and $y$ of length $n$. We give the key proposition of [Lenagan and Smoktunowicz 2007, Theorem 3], which we have expressed in a slightly more general form (see [Bell and Young 2011]).

**Proposition 7.2.** *Let $f, g : \mathbb{N} \to \mathbb{N}$ be two maps satisfying*

(i) $f(i-1) < f(i) - g(i) - 1$ *for all natural numbers $i$*,

(ii) *for each natural number $i$, there is a subspace $W_i \subseteq A(2^{f(i)})$ whose dimension is at most $2^{2^{g(i)}} - 2$,*

*and let*

$$T = \bigcup_i \{f(i) - g(i) - 1,\ f(i) - g(i),\ \ldots,\ f(i) - 1\}.$$

*Then for each natural number $n$, there exist $K$-vector subspaces $U(2^n)$ and $V(2^n)$ of $A(2^n)$ satisfying the following properties:*

(1) $U(2^n) \oplus V(2^n) = A(2^n)$ *for every natural number* $n$;

(2) $\dim(V(2^n)) = 2$ *whenever* $n \notin T$;

(3) $\dim(V(2^{n+j})) = 2^{2^j}$ *whenever* $n = f(i) - g(i) - 1$ *and* $0 \le j \le g(i)$;

(4) *for each natural number* $n$, $V(2^n)$ *has a basis consisting of words over* $x$ *and* $y$;

(5) *for each natural number* $i$, $W_i \subseteq U(2^{f(i)})$;

(6) $A(2^n)U(2^n) + U(2^n)A(2^n) \subseteq U(2^{n+1})$ *for every natural number* $n$;

(7) $V(2^{n+1}) \subseteq V(2^n)V(2^n)$ *for every natural number* $n$;

(8) *if* $n \notin T$ *then there is some word* $w \in V(2^n)$ *such that* $wA(2^n) \subseteq U(2^{n+1})$.

One should think of the subspaces $U(n)$ and $V(n)$ as follows. Condition (6) says that the sum of the $U(n)$ is in some sense very close to being a two-sided ideal. It is not a two-sided ideal, but Lenagan and Smoktunowicz show that there is a homogeneous two-sided ideal $I$ which is a close approximation to this space. Then one should think of the image of the sum of the $V(n)$ when we mod out by this ideal as being very close to a basis for the factor ring $A/I$.

The fact that there are infinitely many $n \notin T$, and conditions (2) and (3), say that the growth of $A/I$ should be small if $g(n)$ grows sufficiently slowly compared to $f(n)$. The role of the subspaces $W_i$ is that they correspond to homogeneous relations introduced. Thus if we are not introducing too many relations and we have that the dimension of $W_i$ is bounded by $2^{2^{g(i)}} - 2$, then we can hope to find an infinite-dimensional algebra with slow growth in which the images of all relations coming from the subspaces $W_i$ are zero.

The above proposition allows one, over countable fields, to construct infinite-dimensional nil algebras of finite Gelfand–Kirillov dimension by picking sparse small sets of relations that imply nilpotence of elements in the algebra and augmenting this with sets that give slow growth.

## 8. Lower transcendence degree

Transcendence degree for fields is an important invariant and has proved incredibly useful in algebraic geometry. In the noncommutative setting, many different transcendence degrees have been proposed [Gelfand and Kirillov 1966; Borho and Kraft 1976; Resco 1980; Schofield 1984; Stafford 1983; Zhang 1998; 1996; Yekutieli and Zhang 2006], many of which possess some of the desirable properties that one would hope for a noncommutative analogue of transcendence degree to possess. None of these invariants, however, has proved as versatile as the ordinary transcendence degree has in the commutative setting, as there has always been the fundamental problem: they are either difficult to compute in practice or are not powerful enough to say anything meaningful about division subalgebras.

The first such invariant was defined by Gelfand and Kirillov [1966], who, as we noted earlier, used their Gelfand–Kirillov transcendence degree to prove that if the quotient division algebras of the $n$-th and $m$-th Weyl algebras are isomorphic, then $n = m$. Gelfand–Kirillov transcendence degree is obtained from Gelfand–Kirillov dimension in a natural way.

The *Gelfand–Kirillov transcendence degree* for a division algebra $D$ with center $k$ is defined to be

$$\mathrm{Tdeg}(A) = \sup_V \inf_b \limsup_{n\to\infty} \frac{\log \dim_k (k + bV)^n}{\log n},$$

where $V$ ranges over all finite-dimensional $k$-vector subspaces of $D$ and $b$ ranges over all nonzero elements of $D$.

Zhang [1998] introduced a combinatorial invariant, which he called the *lower transcendence degree* of a division algebra $D$, which he denoted $\mathrm{Ld}(D)$. To give a concrete description of the main principle behind lower transcendence degree, one can consider the problem of assigning a dimension to a geometric object. If one has a $d$-dimensional hypercube, one can recover $d$ by noting that upon dilating the hypercube by a factor of 2, the volume increases by a factor of $2^d$. In some sense, Gelfand–Kirillov dimension works according to this principle. There is, however, another way of extracting $d$. One can note that if $S$ is the surface area of the $d$-dimensional hypercube and $V$ is its volume, then $S$ is proportional to $V^{(d-1)/d}$. This is the basic principle upon which lower transcendence degree rests.

Zhang showed that this degree had many of the basic properties that one would expect a transcendence degree to have. In particular, he showed that if $k$ is a field, $A$ is a $k$-algebra that is an Ore domain of finite GK dimension, and $D$ is the quotient division algebra of $A$, then

$$\mathrm{GKdim}(A) \geq \mathrm{Ld}(D),$$

and thus the invariant is well-behaved under localization.

We now define lower transcendence degree. Given a field $k$ and a $k$-algebra $A$, we say that a $k$-vector subspace $V$ of $A$ is a *subframe* of $A$ if $V$ is finite-dimensional and contains 1; we say that $V$ is a *frame* if $V$ is a subframe and $V$ generates $A$ as a $k$-algebra.

If $V$ is a subframe of $A$, we define $\mathrm{VDI}(V)$ to be the supremum over all nonnegative numbers $d$ such that there exists a positive constant $C$ such that

$$\dim_k(VW) \geq \dim_k(W) + C\dim(W)^{(d-1)/d}$$

for every subframe $W$ of $D$. (If no nonnegative $d$ exists, we take $\mathrm{VDI}(V)$ to be zero.) VDI stands for "volume difference inequality" and it gives a measure of the growth of an algebra. Note that

$$\dim_k(VW) - \dim_k(W)$$

is really just $\dim_k(VW/W)$ and thus this is in some sense giving the dimension of the boundary of $W$ with respect to $V$. In terms of the hypercube analogy, this quantity corresponds to the surface area. The quantity $\dim(W)$ corresponds to the volume under this analogy and so the exponent $(d-1)/d$ is telling us what the right notion of "dimension" should be in this case.

We then define the *lower transcendence degree* of $A$ by

$$\mathrm{Ld}(A) = \sup_V \mathrm{VDI}(V),$$

where $V$ ranges over all subframes of $A$.

The definition, while technical, gives a powerful invariant that Zhang [1998] has used to answer many difficult problems about division algebras. Zhang showed that if $A$ is an Ore domain of finite GK dimension and $D$ is the quotient division algebra of $A$ then $\mathrm{Ld}(D) \le \mathrm{GKdim}(A)$. Moreover, equality holds for many classes of rings. In particular, if $A$ is a commutative domain over a field $k$, then equality holds and so Lower transcendence degree agrees with ordinary transcendence degree.

Lower transcendence degree has the nice additional property that if $D$ is a division algebra and $E$ is a division subalgebra of $D$ then $\mathrm{Ld}(E) \le \mathrm{Ld}(D)$. Using this fact, Zhang was able to answer a conjecture of Small's by proving the following theorem.

**Theorem 8.1** (Zhang). *Let $A$ be a finitely generated $k$-algebra that is a domain of Gelfand–Kirillov dimension $d$. If $K$ is a maximal subfield of $Q(A)$, then $K$ has transcendence degree at most $d$ as an extension of $k$.*

*Proof.* We have

$$\mathrm{Trdeg}(K) = \mathrm{Ld}(K) \le \mathrm{Ld}(D) \le \mathrm{GKdim}(A) = d.$$

The result follows. □

The author [Bell 2012], extended Zhang's result by showing that if $A$ does not satisfy a polynomial identity then one in fact has $\mathrm{Trdeg}(K) \le d-1$. In particular, this shows that quotient division algebras of domains of GK dimension two are either finite-dimensional over their centers or have the property that all maximal subfields have transcendence degree one.

Despite some of the successes of lower transcendence degree, there are still some basic questions about the invariant that remain unanswered. We give a few of the basic questions.

**Question 2.** Is there a Bergman-style gap theorem for lower transcendence degree? More specifically, if $D$ is a division ring and $\mathrm{Ld}(D) \in [1, 2)$, does it follow that $\mathrm{Ld}(D) = 1$?

**Question 3.** Let $A$ be a finitely generated Ore domain and let $D = Q(A)$. If $\mathrm{Ld}(D) = 1$, is it true that $D$ is finite-dimensional over its center?

We note that a positive solution to these questions would immediately give Smoktunowicz's graded gap theorem. The reason for this is that if $A$ is a finitely generated graded domain whose GK dimension lies in $[2, 3)$ then $A$ has a graded quotient division ring $Q_{\mathrm{gr}}(A) = D[t, t^{-1}; \sigma]$. (Here $D$ is obtained by taking the degree zero part of the algebra obtained by inverting the nonzero homogeneous elements of $A$.) Zhang proves that in such situations one has $\mathrm{Ld}(D) \le \mathrm{Ld}(A) - 1 \le \mathrm{GKdim}(A) - 1$. Thus $\mathrm{Ld}(D) = 1$ and so $D$ is a finite-dimensional over its center, which is a field of transcendence degree one. The arguments of [Artin and Stafford 1995] now show that $A$ has GK dimension exactly two.

## 9. Division algebras of transcendence degree two: Artin's conjecture

One of the truly difficult problems that pertains to growth is to give a so-called birational classification of finitely generated complex domains of Lower transcendence degree two. This is strongly related to Artin's conjecture, which we now briefly describe.

Artin [1997] gives a proposed birational classification of a certain class of graded domains of GK dimension 3. We note that Artin warns the reader that "everything should be taken with a grain of salt," as far as his proposed classification is concerned, and some people believe there are additional division rings yet to be found that are missing from the list.

To begin, we let $A$ be a graded Noetherian domain of GK dimension 3 that is generated in degree 1 and we let $\mathscr{C}$ denote the category of finitely generated graded right $A$-modules modulo the subcategory of torsion modules. (This can be thought of as the category of "tails" of finitely generated graded $A$-modules.) We let $\mathrm{Proj}(A)$ denote the triple $(\mathscr{C}, \mathscr{O}, s)$, where $\mathscr{O}$ is the image of the right module $A$ in $\mathscr{C}$ and $s$ is the autoequivalence of $\mathscr{C}$ defined by the shift operator on graded modules.

We note that $A$ has a *graded quotient division ring*, denoted by $Q_{\mathrm{gr}}(A)$, which is formed by inverting the nonzero homogeneous elements of $A$. Then there is a division ring $D$ and automorphism $\sigma$ of $D$ such that

$$Q_{\mathrm{gr}}(A) \cong D[t, t^{-1}; \sigma].$$

We think of $D$ as being the function field of $X = \mathrm{Proj}(A)$.

Artin gives a proposed classification of the type of division rings $D$ that can occur when $A$ is a complex Noetherian domain of GK dimension 3. (There are also a few other technical homological assumptions that he assumes the algebra possesses, but we shall ignore these and instead refer the interested reader to Artin's paper [1997].) If $A$ has these properties and $Q_{\mathrm{gr}}(A) \cong D[t, t^{-1}; \sigma]$, then Artin asserts that up to isomorphism the possible division rings $D$ must satisfy at least one of the conditions on the list:

(1) $D$ is finite-dimensional over its center, which is a finitely generated extension of $\mathbb{C}$ of transcendence degree 2;

(2) $D$ is birationally isomorphic to a quantum plane; that is, $D$ is isomorphic to the quotient division ring of the complex domain generated by $x$ and $y$ with relation $xy = qyx$ for some nonzero $q \in \mathbb{C}$;

(3) $D$ is isomorphic to the Skylanin division ring (see [Artin 1997] for relevant definitions);

(4) $D$ is isomorphic to the quotient division ring of the Weyl algebra;

(5) $D$ is birationally isomorphic to $K[t; \sigma]$ or $K[t; \delta]$, where $K$ is a finitely generated field extension of $\mathbb{C}$ of positive genus and $\sigma$ and $\delta$ are respectively an automorphism and a derivation of $K$.

In some cases one can homogenize the relations in a domain of GK dimension 2 using a central indeterminate and obtain a graded domain of GK dimension 3 satisfying the conditions that Artin assumes. We also note that many of the division rings on Artin's list are quotient division rings of Noetherian domains of GK dimension 2. In this sense, there is a strong relationship between a birational classification of Noetherian domains of GK dimension 2 and graded Noetherian domains of GK dimension 3.

We summarize some of the work that has been obtained via growth methods on Artin's conjecture.

**Theorem 9.1.** *Let $A$ be a graded Noetherian complex domain of GK dimension* 3 *that is generated in degree* 1 *and let $D$ be the degree zero part of the homogeneous quotient division ring of $A$. If $D$ is not finite-dimensional over its center*, *then*:

(1) *all maximal subfields of $D$ are finitely generated and of transcendence degree one over $\mathbb{C}$*;

(2) *$D$ contains a free algebra on two generators.*

*Proof.* The fact that the subfields are finitely generated follows from the fact that $A$ is a finitely generated complex Noetherian domain and a theorem of the author [Bell 2007] which shows that all subfields of $Q(A)$ are finitely generated in this case. We note that if one uses the *strong lower transcendence degree*, $\mathrm{Ld}^*$, one can show that $\mathrm{Ld}^*(D) \le \mathrm{Ld}^*(A) - 1 \le 2$, since $Q_{\mathrm{gr}}(A) \cong D[t, t^{-1}; \sigma]$. Since any subfield $K$ of $D$ necessarily has the property that $D$ is infinite-dimensional as a left and right $K$-vector space, we see that $\mathrm{Ld}(K) \le 1$ by [Bell 2010]. It follows that $K$ has transcendence degree one over $\mathbb{C}$.

The fact that $D$ contains a free algebra on two generators follows from work in progress of the author and Dan Rogalski [Bell and Rogalski ≥ 2015] (see also [Bell and Rogalski 2014]), since all subfields of $D$ are finitely generated and the base field is the complex numbers. □

We note that in the case that $D$ is the quotient division algebra of the Weyl algebra, (3) had been proven by Makar-Limanov [1983; 1996].

One of the questions, which could provide an important invariant is the types of subfields that can occur in the division algebras on Artin's list. By Theorem 9.1, all subfields have transcendence degree one over $\mathbb{C}$ and are finitely generated. Mironov [2011] has shown that one can find subfields of all genera inside the quotient division algebra of the Weyl algebra. We ask whether an analogous result holds for the quantum plane.

**Question 4.** Let $D_q$ denote the quotient division algebra of the quantum plane. If $q$ is not a root of unity, can $D_q$ contain the function field of a smooth curve $X$ of positive genus over $\mathbb{C}$?

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jpbell@uwaterloo.ca *Department of Pure Mathematics, University of Waterloo, 200 University Avenue, Waterloo ON N2L 3G1, Canada*