This book introduces and explores some of the deep connections between Einstein's theory of gravitation and differential geometry. As an outgrowth of graduate summer schools, the presentation is aimed at graduate students in mathematics and mathematical physics, starting from the foundations of special and general relativity, and moving to more advanced results in geometric analysis and the Einstein constraint equations. Topics include the formulation of the Einstein field equation and the Einstein constraint equations; a treatment of the Penrose singularity theorem; an introduction to scalar curvature deformation and the conformal method; a detailed introduction to asymptotically flat spaces and the Riemannian positive mass theorem; gluing construction of initial data sets which are Schwarzschild near infinity; constant mean curvature surfaces and the center of mass for asymptotically flat initial data sets; and an introduction to the Riemannian Penrose inequality.

While the book assumes a background in differential geometry and real analysis, a number of basic results in geometry are included in the text and exercises. A brief treatment of elliptic partial differential equations is designed to help the reader navigate through the applications of geometric analysis to the Einstein constraint equations discussed in the analysis-heavy second half of the book.

There are well over 100 exercises, many woven into the fabric of the chapters as well as others collected at the end of chapters, to give readers a chance to engage and extend the text.

Mathematical Sciences Research Institute
Publications

66

Lectures on Mathematical Relativity

This series is based on work undertaken at the Simons Laufer Mathematical Sciences Institute (SLMath), formerly the Mathematical Sciences Research Institute (MSRI), in Berkeley, California. It publishes surveys and workshop proceedings of long-lasting value, as well as lecture notes and monographs by visitors to the Institute. The volumes below are published by Cambridge University Press; earlier ones may be available from Springer-Verlag.

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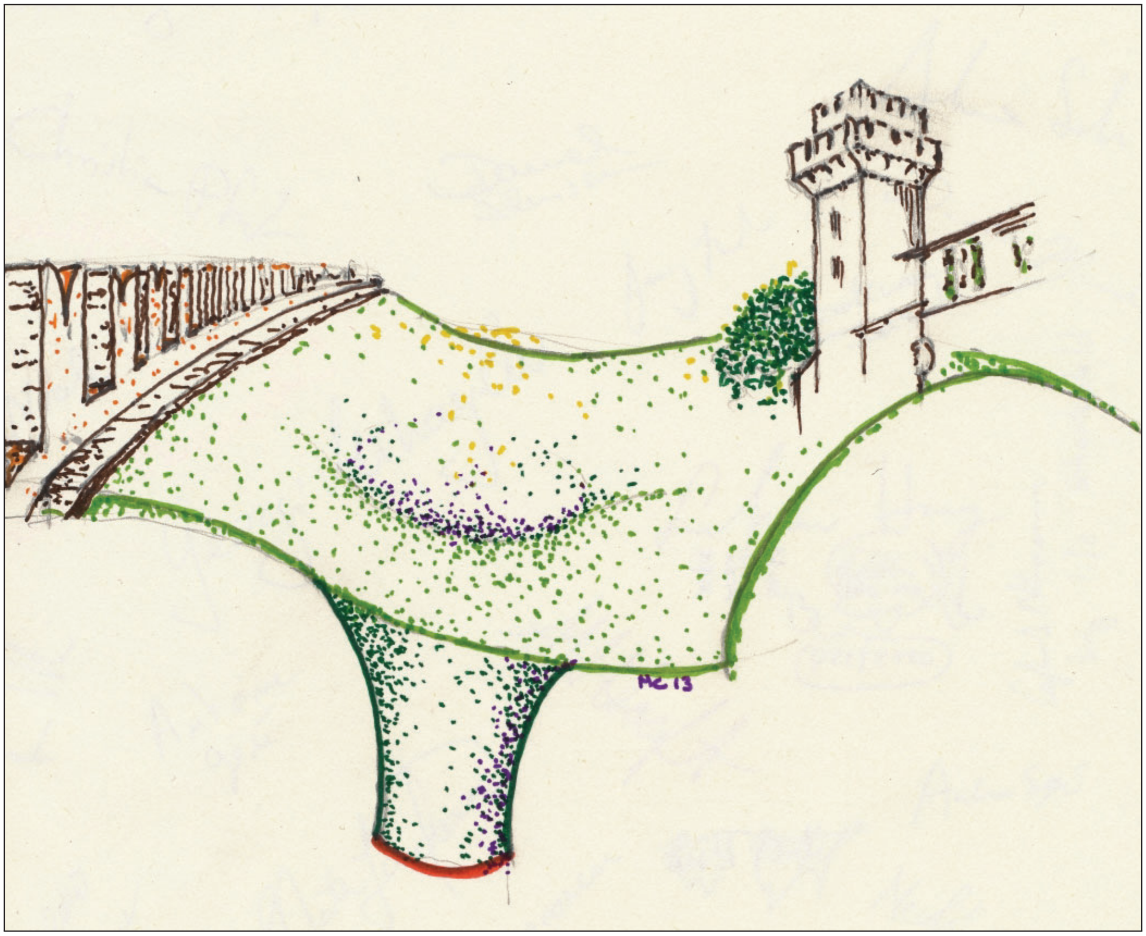

Schwarzschild meets Cortona: with thanks to Mauro Carfora

Lectures on Mathematical Relativity

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We dedicate this book to the memory of Sergio Dain.

Contents

Preface

This volume arose from the Summer Graduate Workshop in Mathematical General Relativity at the Mathematical Sciences Research Institute (MSRI, now renamed the Simons Laufer Mathematical Sciences Institute) in Berkeley, CA in 2012, and the subsequent summer school in Cortona, Italy, in 2013. The editors of the volume served as scientific organizers for the summer schools. The contributions to the volume grew out of lectures given at one or both of the schools.

We have endeavored to enhance the presentation of the material covered in the two-week summer schools to make it suitable for reading in book form, while at the same time remaining faithful to the spirit of those schools.

The advertised prerequisites for the schools, and hence this volume, included a standard first-year graduate analysis course, with elements of real and functional analysis as might be found in *Real analysis* by H. Royden [193] and *Real and complex analysis* by W. Rudin [194]. We also assumed introductory graduate courses in differential and Riemannian geometry, at the level of the following texts: *An introduction to smooth manifolds* and *Riemannian manifolds*, by J. M. Lee [141; 140]; *Riemannian geometry* by M. P. do Carmo [41]; *Riemannian geometry* by P. Peterson [182]; and of particular relevance to the summer schools, *Semi-Riemannian geometry* by B. O'Neill [174]. A graduate course in partial differential equations (PDE), at the level of *Partial differential equations* by L. C. Evans [86] and the first half of *Elliptic partial differential equations of second order* by D. Gilbarg and N. Trudinger [107], was not a requirement for the schools, and although some of the lectures needed to draw on some PDE results, students without such background could profit from the bulk of the material discussed at the schools. For this volume, however, certain PDE details in some of the presentations have been fleshed out, so those sections would be better approached with this background in hand. We have endeavored to bridge the gap by including a section introducing and motivating some of the PDE tools.

The students came to the schools with a wide range of backgrounds in mathematics, from those who had nearly completed their doctoral dissertations to those who came without the prerequisite geometry background. Through exercises and tutorial sessions, students were able to build enough intuition and computational skills to understand much of the material presented. While we decided a primer section on elliptic PDE was essential for the flow of this book, we resisted the temptation to add further sections on background geometry. That said, we do recall or develop some foundational material where needed, and some startup notations and conventions are reviewed starting on p. xix. We include exercises that were assigned before and during the schools, both to give readers a feel for the tutorials and to help focus those who are learning the topics for the first time or reviewing on the fly. We added many exercises as well, some collected at the end of chapters, some interspersed in the text. Of particular note, many exercises in Chapters 1 and 2 serve to review and extend background in geometry.

Strictly speaking, no physics background is required. We assume, as we did at the schools, a nodding acquaintance with pre-relativity physics, enough so that students can approach the development of the theory of special and general relativity with context from which to appreciate the rudiments of spacetime structure and the line of thought from Galileo to Newton to Einstein, and to motivate why the Einstein equations and the initial value constraints were to receive so much of their attention. In part for this reason, the first chapter contains some very basic material that would be included in an undergraduate course in special relativity, but we found it to be a fun way to start each school, engendering some interesting discussion amongst participants without needing much in the way of background. The first two chapters on special and general relativity may seem somewhat chatty, including some discussion of physics without always being mathematically efficient or fastidious, but we hope it helps to frame the mathematical theory. We could have cut the physics discussion short by formulating the mathematical postulates from the start with a small amount of motivation, but we decided, given the audience, to put some more time into developing these ideas from their genesis in physics. Even giving ourselves some leeway, the presentation is not too leisurely, and the lecture schedule at the schools called for covering the physics background reasonably efficiently at the beginning of the first week.

The mathematical and physical foundations of relativity have been an active topic of discussion and research for over a hundred years, and we have not tried to approach the scope of the debate (for instance, we chose not to discuss Mach's principle in depth), nor have we tried to use too fine a brush in painting the logical and philosophical distinctions, nor strained to give a serious historical account of the development of the theory. Interested readers can follow up with references such as [82; 83; 85; 161; 169; 170; 171], and with a wealth of material available online.

Even while starting off in an elementary fashion, and keeping in mind the range of student backgrounds represented, we were able to cover a reasonable amount of ground at each school. During the MSRI workshop, Pengzi Miao covered sufficient elements of causal theory to present the proof of the Penrose singularity theorem. Justin Corvino developed enough background in scalar curvature and asymptotically flat solutions of the constraint equations to be able to present a proof of the Riemannian positive mass theorem in three dimensions, while Lan-Hsuan Huang and Fernando Schwartz were able to build on this to discuss advanced aspects of the geometry of initial data sets, with Lan-Hsuan discussing constant mean curvature surfaces and the notion of center of mass, and with Fernando outlining multiple approaches to the Riemannian Penrose inequality. This volume reflects essentially the material covered during the MSRI workshop.

At Cortona, in lieu of Pengzi's lectures, Mauro Carfora (Università di Pavia) presented an engaging and marvelously illustrated development connecting the constraint equations (elliptic PDE governing initial values for the Einstein evolution) and the Ricci flow,[1] while Michael Eichmair (ETH Zürich, now at the University of Vienna) developed connections between the positive mass theorem and the geometry of initial data sets (including isoperimetry of large spheres),[2] which dovetailed beautifully with the lectures of Huang and Schwartz.

With all this background to present, the organizers decided to focus the topics lectures on the Einstein constraint equations which govern the initial data for the Einstein evolution, at the expense of not including advanced and/or current topics on the evolution problem. While this is a reasonable basis for criticism (of the schools and hence this volume), the field has developed to a point where there is room for multiple programs on each of these topics, and the relations between them; articles such as [64] and volumes such as [11; 51] indicate the considerable breadth and depth of the field.

The years just after the workshops witnessed a flurry of activity in general relativity. The centennial year of 2015 marked the hundredth anniversary of Einstein's formulation of a geometric theory of gravity governed by the Einstein equation, and was capped off with the excitement over the detection by LIGO of gravitational waves generated from black hole mergers — the discovery of which led to the 2017 Nobel Prize in physics. Roger Penrose shared the 2020

[1] A full treatment of the topic in Mauro's lectures can be found in the recent monograph [40].

[2] For this material see [32; 33; 79; 80; 81].

Nobel Prize in physics for his work on singularity formation and black holes, some of which we discuss. We hope the field will continue to develop in a robust manner, and that this work will be of some value in introducing graduate students to the field, and showing them some aspects of more advanced topics. Along these lines, we enthusiastically point the reader to the graduate text *Geometric Relativity* by Dan A. Lee (Queens College, CUNY), which has appeared recently [142], and would surely have been a recommended text for the schools.

The first two chapters of this volume present the basic background, from Minkowski spacetime and special relativity, to Einstein's equation and general relativity. Chapters 3 and 4 treat causality and the Penrose singularity theorem. Chapter 5 on the Einstein constraint equations rounds out the basic background from general relativity. Starting from Chapter 6 the text takes a sharp turn in the direction of geometric analysis. Chapter 6 includes some background motivation on elliptic PDE, with some applications to the constraint equations and scalar curvature; of note, there is an excursus on the first and second variations of area, which will appear throughout the rest of the text. Chapters 7–9 are written as topical chapters and are largely independent of each other, though one might find utility in referring to Chapter 7 for some properties of asymptotically flat spaces. That said, on a first pass, some readers might find themselves giving some of the more technical discussions in Chapter 7 a light read.

We would like to thank the graduate students for their hard work and enthusiasm at the summer schools, and in particular Alan Parry and Xin Zhou, as well as Peter McGrath and Andrea Santi, for their work as graduate assistants at the MSRI and Cortona schools, respectively. During one tutorial session, Alan introduced us to his research area, by presenting work of his thesis advisor Hubert Bray (Duke University), which modifies the Einstein–Hilbert action of general relativity with a goal to model dark matter; while we do not treat this topic in the text, we refer the interested reader to [27]; see also [30]. It has been inspiring to the scientific organizers to see so many of the students producing a staggering amount of interesting theses and papers in the years since the summer schools were held, and many have moved on to postdocs and faculty positions. In particular, Brian Allen, currently in the Department of Mathematics at Lehman College, CUNY, attended the MSRI summer school as a graduate student, and is a coauthor on Chapter 9 in this volume.

There are many people to thank for helping this project along. Giorgio Patrizio (Università di Firenze) first broached the idea of a volume after the Cortona summer school. We thank Heléne Barcelo (MSRI) for her enthusiastic support throughout the process. We also thank all the great staff at MSRI, and in particular Chris Marshall, for their support before, during and after the school, and likewise at Cortona, in particular Silvana Boscherini and Cinzia Benedetti. Funding for the schools was provided in part by National Science Foundation, the Clay Foundation, and INdAM (Istituto Nazionale di Alta Matematica), and we thank them for their generous support. Likewise we thank our respective home institutions, Lafayette College and the University of Miami. The editors shaped the book in part during their invited mini-course at the 2013 Taiwan International Conference on Geometry, at the National Taiwan University, and we would like to extend our thanks to Yng-Ing Lee for that opportunity. JC thanks Lehigh University, and especially Huai-Dong Cao, for inviting him to teach a graduate course in mathematical relativity in 2011, an experience that helped frame the approach to some of the material. JC would also like to acknowledge invitations from the Park City Math Institute, the Erwin Schrödinger Institute in Vienna, as well as from the Ravello Summer School, where he delivered mini-courses in the summers of 2013, 2014 and 2015, respectively, at which some of the presentation was honed. Of particular note is the support of Tommaso Ruggeri (Università di Bologna) for both the Cortona and Ravello summer schools. We thank Greg Galloway for reading Chapters 3 and 4 and offering some helpful feedback. JC thanks former student Kevin Manogue (Lafayette College) for feedback on Chapters 1 and 2, David Maxwell (University of Alaska, Fairbanks) for discussions on the conformal method, Farhan Abedin (Lafayette College) for reading parts of several chapters, and whose critical feedback led to a reorganization of Chapters 5–7, and finally John D. Norton (University of Pittsburgh) for several enlightening email exchanges on the foundations of general relativity. In addition to lecturing in Cortona, Mauro Carfora read several chapters in detail and offered critical advice from a physics perspective; in addition, his beautiful sketch of the palace at which the school was held adorns this volume. A huge thank you goes out to the editor Silvio Levy not only for his advice and encouragement, but for his calm patience while this project took longer than anticipated.

This book is dedicated to our friend and colleague Sergio Dain, who passed away in February 2016 at the age of 46. Sergio was an inspiration — through his work and his talks, he shared his deep insights into mathematical relativity and inspired you to be a better mathematician, while through his friendly and generous personality, interacting with him inspired you to be a better person. We lack the words to express how much he is missed.

Notation and conventions

We will often indicate conventions when they appear in the text (sometimes repeatedly), but we will mention a few here, just to get started.

While we generally use the term *smooth* to mean C^∞ (partly for definiteness), we note that often it will be obvious that a certain C^k-smoothness level is *sufficiently smooth* for the context under consideration. Subset notation $A \subset B$ also allows for $A = B$. Vectors will be denoted in various ways; standard basis vectors in coordinates x^i will be often written as partial derivative operators $\partial/\partial x^i$, so that a vector V can be written as a linear combination $V = V^i\,\partial/\partial x^i$. Here we have used the *Einstein summation convention* of summing over repeated upper and lower indices. While this convention will be in force unless otherwise noted, we will repeat it on occasion for the sake of clarity.

The term *manifold* will generally refer to a smooth manifold without boundary. A *closed manifold* will refer to a compact manifold (again, without boundary). While we assume the standard topological conditions that manifolds are Hausdorff and second countable, we are ambivalent about whether to restrict to connected manifolds: many results will not require connectedness, and for certain results that do, it is rather obvious that a statement as written would only hold on each component separately. We will try to point out where connectedness is assumed, but we trust the reader can discern if we have missed such an instance. A submanifold of codimension one is a *hypersurface*, which will generally be taken to be smoothly embedded, though we will try to point out when we allow it to be immersed, or weaken the regularity assumption (as in Chapter 3).

We will work with *semi-Riemannian* (also called *pseudo-Riemannian*) metrics on M, mostly Lorentzian or Riemannian; our signature for Lorentzian metrics is $(-, +, +, \ldots, +)$. When the spacetime is the focus, it may be given as a Lorentzian manifold (M, g), whereas at some point, the focus in the book will shift primarily to Riemannian manifolds, often construed as Riemannian hypersurfaces in a spacetime, so that the Riemannian manifold might then be given as (M, g), and the corresponding spacetime (if referenced) by $(\mathcal{S}, \bar{g})$, for example. Pay close attention to this, and also to the dimension of the spacetime.

This will be made clear in each situation, but just keep it in mind when cross-referencing formulae across chapters and sections.

When dealing with tensors, we sometimes just need the value of the tensor at a point, and sometimes we are referring to a *tensor field*; this will not always be explicitly stated, but should be clear in context. If a formula refers to derivatives of the tensor field, we will assume, unless stated otherwise, that the tensor field is smooth, or, at least smooth enough to do the indicated computations. For example, "consider a one-form θ" might really mean "consider a smooth one-form field θ". At various points we will consider fields that have less regularity (e.g., Sobolev spaces of tensor fields), and that will be made clear when needed; in particular, we will be more deliberate about emphasizing the regularity when it comes to the fore starting with the PDE discussion in Chapter 6.

Recall that a connection on the tangent bundle TM (an *affine connection*) assigns to vector fields X and Y a vector field $\nabla_X Y$, which is $C^\infty(M)$-linear (and hence tensorial) in X and $\mathbb{R}$-linear in Y, and satisfies the product rule $\nabla_X(fY) = (\nabla_X f)Y + f\nabla_X Y$ for $f \in C^\infty(M)$, where $\nabla_X f = X[f]$ is the directional derivative of f; the value of $(\nabla_X Y)|_p$ depends only on $X|_p$ and the values of Y along a curve tangent to $X|_p$. One can extend the connection to tensor fields T, defining $\nabla_X T$ by applying a product rule; e.g., if T is a one-form, $\nabla_X(T(Y)) = (\nabla_X T)(Y) + T(\nabla_X Y)$. In general, $\nabla_X T$ is a tensor of the same rank as T, and it follows easily from the definition that $\nabla_X T$ is tensorial in X. Hence we can construe ∇T as a tensor with rank higher by one: if T is an (r, s)-tensor, producing a scalar from a tuple of r one-forms and s vectors, then ∇T is an $(r, s+1)$-tensor. On a semi-Riemannian (M, g), there is a unique connection, called the *Levi-Civita connection* and denoted by ∇ (among other notations you might see in the text), which is torsion-free ($\nabla_X Y - \nabla_Y X = [X, Y]$) and satisfies $\nabla g = 0$; this will generally be the connection employed unless stated otherwise.

A metric g will often be written in bracket notation: $g(X, Y) = \langle X, Y\rangle$. In coordinates, g is given by a symmetric matrix of components g_{ij}, so that locally $g = g_{ij}\, dx^i \otimes dx^j = g_{ij}\, dx^i dx^j$, where for one-forms θ and η we define $\theta\eta = \frac{1}{2}(\theta\otimes\eta + \eta\otimes\theta)$ (whereas the *wedge product* is given by $\theta\wedge\eta = \theta\otimes\eta - \eta\otimes\theta$). Thus the Euclidean metric $g_{\mathbb{E}^n}$ on $\mathbb{R}^n$, for which the component functions x^i are *Cartesian* coordinates, is then expressed as $g_{\mathbb{E}^n} = \delta_{ij} dx^i dx^j$, for example. The nondegeneracy of g corresponds in components to the invertibility of the matrix (g_{ij}), and we write $(g^{ij}) = (g_{ij})^{-1}$, i.e., $g^{ij} g_{jk} = \delta^i_{\ k}$. There is a natural volume measure dv_g associated to g, which in local coordinates takes the form $dv_g = \sqrt{|\det(g_{ij})|}\, dx$, where dx is the Euclidean (Lebesgue) volume measure in coordinates; dv_g corresponds to a volume form ω_g in case M is orientable. We sometimes let $\det g = \det(g_{ij})$ for abbreviation, and we often let $d\sigma$ or $d\sigma_g$ be the volume measure induced on a semi-Riemannian submanifold.

Since at each point on M the metric g is nondegenerate, it can be used to change the tensor type, e.g., a vector X is associated to a dual form $X^\flat$ by $g(X, Y) = X^\flat(Y)$, and likewise a one-form α can be associated to its vector dual $\alpha^\sharp$ by $g(\alpha^\sharp, Y) = \alpha(Y)$. It is easy to check in a basis v_j for T_pM with dual basis θ^i for T_p^*M (so $\theta^i(v_j) = \delta^i{}_j$) that if $X = X^j v_j$ then $X^\flat = X_i\theta^i$ with $X_i = g_{ij}X^j$, where $g_{ij} = g(v_i, v_j)$; similarly, if $\alpha = \alpha_i\theta^i$, then $\alpha^\sharp = \alpha^j v_j$ with $\alpha^j = g^{ij}\alpha_i$. This kind of operation, known as *raising* and *lowering* of indices from the way the notation is arranged, can be performed on more general tensors T, with the positions of the indices generally indicating tensor type in lieu of the musical $\sharp$ and $\flat$ notation.

We remark on the consistency of the raising/lowering notation: if T is a $(0, 2)$-tensor with components T_{ij}, then $T^{ij} = g^{ik}g^{j\ell}T_{k\ell}$ give the components of the tensor obtained by type-changing using g, so that if $T = g$, then in fact we see $T^{ij} = g^{ij}$ (the components of the inverse matrix). Furthermore, we can extend g as a bilinear form on more general tensors, defining $\langle S, T\rangle$ to be an appropriate metric contraction of $S \otimes T$; e.g., if S and T are $(1, 2)$-tensors, then in a local basis $\langle S, T\rangle = g_{i\ell}g^{js}g^{km}S^i_{jk}T^\ell_{sm}$. We may write this in various ways, depending on context: $\langle S, T\rangle = \langle S, T\rangle_g = S \cdot_g T = S \cdot T$, and we let $|T|_g^2 = \langle T, T\rangle_g$ (this is in fact nonnegative when g is Riemannian). Note that if h is a $(0, 2)$-tensor, then $\langle g, h\rangle_g = g^{ij}h_{ij} = \mathrm{tr}_g h$, and similarly if h is a $(2, 0)$-tensor.

The *Riemann curvature tensor* will be defined via the vector field

$$R(X, Y, Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z = R(X, Y)Z,$$

with index conventions $R^\ell_{ijk}\frac{\partial}{\partial x^\ell} = R\big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\big)$, in which R is a $(1, 3)$-tensor, while the components of the corresponding $(0, 4)$-tensor are given by $R_{ijk\ell} = g_{\ell m}R^m_{ijk} = \big\langle R\big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\big), \frac{\partial}{\partial x^\ell}\big\rangle$. Different books use different conventions, so be alert! The curvature tensor enjoys a number of symmetries. Clearly, $R(X, Y, Z) = -R(Y, X, Z)$; slightly less obvious is *symmetry-by-pairs* $\langle R(V, W, Y), Z\rangle = \langle R(Y, Z, V), W\rangle$. Thus we have the component identities: $R_{k\ell ij} = R_{ijk\ell} = -R_{jik\ell} = R_{ji\ell k}$. For a nondegenerate two-plane $\Pi \subset T_pM$, the following expression is independent of basis $\{V, W\}$ for Π, and defines the *sectional curvature* $K(\Pi)$:

$$K(\Pi) = \frac{\langle R(V, W, W), V\rangle}{\langle V, V\rangle\langle W, W\rangle - \langle V, W\rangle^2}. \qquad (0.0.1)$$

For given X and Y, $R(\,\cdot\,, X, Y)$ is a linear transformation, whose trace is defined to be $\mathrm{Ric}(X, Y)$, the *Ricci curvature*. The Ricci tensor Ric (alternatively, $\mathrm{Ric}(g)$

or Ric_g) is a *symmetric* $(0,2)$-tensor (via the preceding curvature component identities), and it is generally the same tensor across texts (though a notable exception is [221], where the sign differs from ours), which means the way it is defined from the Riemann tensor may differ to account for sign. In our convention,

$$\mathrm{Ric}(X,Y) = dx^\ell\big(R\big(\tfrac{\partial}{\partial x^\ell}, X, Y\big)\big) = g^{k\ell}\big\langle R\big(\tfrac{\partial}{\partial x^\ell}, X, Y\big), \tfrac{\partial}{\partial x^k}\big\rangle,$$
$$R_{ij} = \mathrm{Ric}\big(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\big) = R^\ell_{\ \ell ij} = g^{k\ell} R_{\ell ijk}.$$

The *scalar curvature* is the metric trace of the Ricci tensor, and is given in components by $R(g) = g^{ij} R_{ij}$.

A comma is used to denote a partial derivative, whereas a semicolon is used to denote components of the covariant derivative of a tensor. For example, with $T_{ijk} = g_{km} T^m_{ij}$, we have $(\nabla T)_{ijk\ell} = T_{ijk;\ell} = (g_{km}T^m_{ij})_{;\ell} = g_{km} T^m_{ij;\ell}$, since $\nabla g = 0$. While the covariant derivative of a function f is naturally a one-form df, i.e., $\nabla f(X) = \nabla_X f = X[f] = df(X)$, sometimes ∇f is instead taken to be the vector $(df)^\sharp = \mathrm{grad}_g f$ dual to df, i.e., the *gradient* of f with respect to the metric g, so that $df(X) = g(X, \mathrm{grad}_g f)$; the meaning should be clear in context.

The *Christoffel symbols* Γ^k_{ij} for a coordinate frame are defined by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k},$$

and can be computed in terms of the metric as $\Gamma^k_{ij} = \frac{1}{2} g^{km}(g_{mj,i} + g_{im,j} - g_{ij,m})$.

If u is a smooth function on M, the *Hessian* of u is defined by $\mathrm{Hess}_g u = \nabla(du)$. It is a $(0,2)$-tensor, with $(\mathrm{Hess}_g u)_{ij} = u_{;ij}$ in components, and moreover it is symmetric (Exercise 1-9). The *Laplacian* is the trace of the Hessian:

$$\Delta_g u = \mathrm{tr}_g(\mathrm{Hess}_g u) = g^{ij} u_{;ij}.$$

In some texts, the term *Laplacian* is reserved for the case (M,g) is Riemannian, and may be defined as the *negative* of our definition. When (M,g) is Lorentzian, the trace of the Hessian is often called (again, up to a sign) the *wave operator* $\Box_g$.

Geodesic normal coordinates at a point $p \in M$ can be useful in computations. In such a coordinate system, $g_{ij}(p) = \pm\delta_{ij}$ and $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\big|_p = \Gamma^k_{ij}\big|_p \frac{\partial}{\partial x^k}\big|_p = 0$, the latter condition being equivalent to the vanishing $g_{ij,k}(p) = 0$ of all the partial derivatives of the components of g at p. Thus, for example, if T is a $(1,2)$-tensor field, then $T^k_{ij;\ell} = T^k_{ij,\ell} + \Gamma^k_{\ell m} T^m_{ij} - \Gamma^m_{\ell i} T^k_{mj} - \Gamma^m_{\ell j} T^k_{im}$, which greatly simplifies at a point p in normal coordinates. When we use an expression like "at a point in normal coordinates", we generally imply evaluating at the point p around which the normal coordinates chart is centered.

We will sometimes use "big O" notation: $f = O(h)$ means that $|f| \leq C|h|$ for some $C > 0$, where the quantities may be tensors, with corresponding norms. Generally one must pay attention to the dependence of C. If f and h are functions of x, then C might be uniformly chosen for x in a compact subset, or possibly f is a function of a tensor h, and so the C might depend on the set of tensors under consideration. Sometimes this notation also implies some bounds on derivatives of f as well, which will have to be specified in context.

Various function spaces will play a role in some of the analysis herein. We will in Chapter 6 recall basic definitions of Sobolev and Hölder spaces, and we encourage the reader to review their basic properties from references such as [2; 86; 107; 144]. We let Ω be an open subset of $\mathbb{R}^n$, sometimes called a *domain in* $\mathbb{R}^n$. For k a nonnegative integer, we let $C^k(\Omega)$ be the set of all functions u on Ω such that u and all its partials up through order k are continuous, and we let $C^\infty(\Omega) = \bigcap_{k=0}^\infty C^k(\Omega)$.