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# Exact solutions of the Kardar–Parisi–Zhang equation and weak universality for directed random polymers

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We survey recent results of convergence to random matrix distributions of directed random polymer free energy fluctuations in the intermediate disorder regime. These are obtained by passing through the exact formulas for fluctuations of KPZ at finite time.

## 1. Directed random polymers

Directed random polymers were introduced in the mid eighties as models of defect lines in media with impurities (see [Kardar 2007] for a review). They became popular in physics because besides their applicability as models and inherent interest, they are a case where the replica methods developed for the more difficult spin glass models give consistent answers. We will be interested in the $1+1$ dimensional case. We are given a random environment $\xi(i, j)$ of independent identically distributed real random variables for $i, j$ in $\mathbb{Z}_+ \times \mathbb{Z}$. Given the environment, the energy of an $n$-step nearest neighbour walk $\boldsymbol{x} = (x_1, \dots, x_n)$ is

$$H_n^\xi(\boldsymbol{x}) = \sum_{i=1}^n \xi(i, x_i). \tag{1}$$

The polymer measure on such walks starting at 0 at time 0 and ending at $x$ at time $n$ is then defined by

$$P_{n,x}^{\beta,\xi}(\boldsymbol{x}) = \frac{1}{Z^{\beta,\xi}(n,x)} e^{-\beta H_n^\xi(\boldsymbol{x})}\, P(\boldsymbol{x}). \tag{2}$$

The parameter $\beta > 0$, which measures how much the path prefers to travel through areas of low energy, is called the inverse temperature. $P$ is the uniform probability measure on such walks, and $Z(n, x)$ is the partition function

$$Z^{\beta,\xi}(n,x) = \sum_{\boldsymbol{x}} e^{-\beta H_n^\xi(\boldsymbol{x})} P(\boldsymbol{x}). \tag{3}$$

This is the point-to-point free energy. If we do not specify the endpoint, we get the point-to-line free energy, which we denote by $Z^{\beta,\xi}(n)$.

What happens is that for large $n$ that path is localized about a path which is special for that $n$; it has lateral fluctuations of size $n^{2/3}$. In terms of the free energy, the key conjecture is that its fluctuations are of size $n^{1/3}$ and given by Tracy–Widom distributions. More precisely, as long as $E[\xi(i,j)_-^5] < \infty$, it is expected [Biroli et al. 2007] that there exist $c$ and $\sigma$ depending on $\beta > 0$ and the distribution of the environment such that

$$\frac{\log Z^{\beta,\xi}(n,x) - cn}{\sigma n^{1/3}} \Rightarrow F_{\mathrm{GUE}}. \tag{4}$$

Here $F_{\mathrm{GUE}}$ is the Tracy–Widom limiting distribution of the largest eigenvalue from the Gaussian orthogonal ensemble. For the point-to-line free energy $Z_{\beta,\xi}(n)$, the analogous statement is conjectured to hold except that now the asymptotic fluctuations are governed by the GOE Tracy–Widom distribution.

Remarkably, not a single case was known. The conjecture is extrapolation from the $\beta = \infty$ case with exponential or geometric distribution, where exact calculations are possible [Johansson 2000].

## 2. Continuum random polymer

The (point-to-point) continuum random polymer is the probability measure $P^{\beta,\xi}_{T,x}$ on continuous functions $x(t)$ on $[0,T]$ with $x(0) = 0$ and $x(T) = x$ and formal density

$$\exp\left(-\beta \int_0^T \xi(t,x(t))\,dt - \frac{1}{2}\int_0^T |\dot{x}(t)|^2\,dt\right), \tag{5}$$

where $\xi(t,x)$, $t \geq 0$, $x \in \mathbb{R}$ is space-time white noise.[1] One can think of it as an elastic band in a random energy background.

One can also think of the continuum random polymer as having a density

$$\exp\left(-\beta \int_0^T \xi(t,x(t))\,dt\right) \tag{6}$$

with respect to the Brownian bridge. Neither prescription makes mathematical sense; however, if one smooths out the noise, so that it does make sense, and removes the smoothing, there is a limiting measure on continuous functions $C[0,T]$ which we call $P^{\beta,\xi}_{T,x}$. Of course, the measure depends on the background randomness $\xi$ just as in the discrete case. So it is a random probability measure on $C[0,T]$. In fact, it is a Markov process, and one can define it directly as

[1] That is, the distribution valued Gaussian variable such that for smooth functions $\varphi$ of compact support in $\mathbb{R}_+ \times \mathbb{R}$, $\langle \varphi, \xi \rangle := \int_{\mathbb{R}_+\times\mathbb{R}} \varphi(t,x)\xi(t,x)\,dx\,dt$ are mean zero Gaussian with covariance $E[\langle \varphi_1, \xi\rangle\langle \varphi_1, \xi\rangle] = \langle \varphi_1, \varphi_2\rangle$.

follows. Let $z(s, x, t, y)$ denote the solution of the stochastic heat equation after time $s \geq 0$ starting with a delta function at $x$,

$$\partial_t z = \tfrac{1}{2}\partial_y^2 z - \beta\xi z, \qquad t > s, \ y \in \mathbb{R}, \tag{7}$$

$$z(s, x, s, y) = \delta_x(y). \tag{8}$$

It is important that they are all using the same noise $\xi$. Note that the stochastic heat equation is well-posed [Walsh 1986]. The solutions look locally like exponential Brownian motion in space. They are Hölder $\frac{1}{2} - \delta$ for any $\delta > 0$ in $x$ and $\frac{1}{4} - \delta$ for any $\delta > 0$ in $t$. In fact, exponential Brownian motion $e^{B(x)}$ is invariant up to multiplicative constants, that is, if one starts (7) with $e^{B(x)}$ where $B(x)$ is a two-sided Brownian motion, then there is a (random) $C(t)$ so that $C(t)z(t, x)$ is another exponential of two sided Brownian motion [Bertini and Giacomin 1997].

$P^{\beta,\xi}_{T,x}$ is then defined to be the probability measure on continuous functions $x(t)$ on $[0, T]$ with $x(0) = 0$ and $x(T) = x$ and finite dimensional distributions

$$P^{\beta,\xi}_{T,x}(x(t_1) \in dx_1, \ldots, x(t_n) \in dx_n) = \frac{z(0,0,t_1,x_1)z(t_1,x_1,t_2,x_2)\cdots z(t_{n-1},x_{n-1},t_n,x_n)z(t_n,x_n,T,x)}{z(0,0,T,x)}\,dx_1\cdots dx_n$$

for $0 < t_1 < t_2 < \cdots < t_n < T$.

One can check these are a.s. a consistent family of finite dimensional distributions. It is basically because of the Chapman–Kolmogorov equation

$$\int_{\mathbb{R}} z(s, x, \tau, u)z(\tau, u, t, y)\,du = z(s, x, t, y), \tag{9}$$

which is a consequence of the linearity of the stochastic heat equation.

We can also define the joint measure $\mathbb{P}^{\beta}_{T,x} = P^{\beta,\xi}_{T,x} \otimes Q(\xi)$ where $Q$ is the distribution of the $\xi$, that is, the probability measure of the white noise.

**Theorem 2.1** [Alberts et al. 2014]. (i) *The measures $P^{\beta,\xi}_{T,x}$ and $\mathbb{P}^{\beta}_{T,x}$ are well-defined* (*the former*, *almost surely*).

(ii) *$P^{\beta,\xi}_{T,x}$ is a Markov process supported on Hölder continuous functions of exponent $\frac{1}{2} - \delta$ for any $\delta > 0$, for $Q$ almost every $\xi$.*

(iii) *Let $t^n_k = k2^{-n}$. Then with $\mathbb{P}^{\beta}_{T,x}$ probability one*, *we have that for all $0 \leq t \leq 1$*,

$$\sum_{k=1}^{\lfloor 2^n t\rfloor} (x(t^n_k) - x(t^n_{k-1}))^2 \to t \tag{10}$$

*as $n \to \infty$*; *i.e.*, *the quadratic variation exists*, *and is the same as $\mathbb{P}^0_{T,x}$* (*Brownian bridge*).

(iv) *$P^{\beta,\xi}_{T,x}$ is singular with respect to $P^0_{T,x}$* (*Brownian bridge*) *for almost every $\xi$.*

So the continuum random polymer looks locally like, but is singular with respect to, Brownian motion. One can also define the point-to-line continuum random polymer $\mathbb{P}_T^\beta$, in the same way as in the discrete case. For large $T$, one expects $\mathrm{Var}_{\mathbb{P}_T^\beta}(x(T)) \sim T^{4/3}$ in the point-to-line case or $\mathrm{Var}_{\mathbb{P}_{T,0}^\beta}(x(T/2)) \sim T^{4/3}$ in the point-to-point case. Here the variance is over the random background as well as $P_{T,x}^{\beta,\xi}$. The conditional variance given $\xi$ should be much smaller.

## 3. Connection with KPZ

In the previous section we saw that if $z(t,x)$ is the solution of (7) with initial data (8) then $h(t,x) = -\beta^{-1}\log z(t,x)$ can be thought of as the free energy of the point-to-point continuum random polymer[2]. It is also the *Hopf–Cole solution of the Kardar–Parisi–Zhang equation*

$$\partial_t h = -\tfrac{1}{2}\beta^{-1}(\partial_x h)^2 + \tfrac{1}{2}\partial_x^2 h + \xi. \tag{11}$$

The equation was introduced by Kardar, Parisi and Zhang [1986], and has become the canonical model for random interface growth in physics. Formally, it is equivalent to the stochastic Burgers equation

$$\partial_t u = -\tfrac{1}{2}\beta^{-1}\partial_x u^2 + \tfrac{1}{2}\partial_x^2 u + \partial_x \xi, \tag{12}$$

which, if things were nice, would be satisfied by $u = \partial_x h$. Since $\log z(t,x)$ looks locally like Brownian motion, (11) is not well-posed (see [Hairer 2013] for recent progress on this question). If $\xi$ were smooth, then the Hopf–Cole transformation takes (7) to (11). For white noise $\xi$, we take $h(t,x) = -\beta^{-1}\log z(t,x)$ with $z(t,x)$ a solution of (7) to be the *definition* of the solution of (11). It is known that these are the solutions one obtains if one smooths the noise, solves the equation, and takes a limit as the smoothing is removed.[3] They are also the solutions obtained as the limit of discrete models in the weakly asymmetric limit.

To understand the weakly asymmetric limit we consider how the KPZ equation (11) rescales. Let

$$h_\epsilon(t,x) = \epsilon^a h(\epsilon^{-z} t, \epsilon^{-1} x). \tag{13}$$

Recall the white noise has the distributional scale invariance

$$\xi(t,x) \overset{\text{dist}}{=} \epsilon^{(z+1)/2}\xi(\epsilon^z t, \epsilon^1 x). \tag{14}$$

Hence, setting $\beta = 1$ for clarity,

$$\partial_t h_\epsilon = -\tfrac{1}{2}\epsilon^{2-z-a}(\partial_x h_\epsilon)^2 + \tfrac{1}{2}\epsilon^{2-z}\partial_x^2 h_\epsilon + \epsilon^{a-\frac{1}{2}z+\frac{1}{2}}\xi. \tag{15}$$

[2] Because of the conditioning it is perhaps more appropriate to call $h(t,x) - \frac{x^2}{2t} - \log\sqrt{2\pi t}$ the free energy.

[3] After subtraction of a diverging constant.

Because the paths of $h$ are locally Brownian in $x$ we are forced to take $a = \frac{1}{2}$ to see nontrivial limiting behaviour. This forces us to take

$$z = \tfrac{3}{2}. \tag{16}$$

The nontrivial limiting behaviour of models in the KPZ universality class are all obtained in this scale.

On the other hand, if we started with KPZ with a weak asymmetry

$$\partial_t h = -\tfrac{1}{2}\epsilon^{1/2}(\partial_x h)^2 + \tfrac{1}{2}\partial_x^2 h + \xi, \tag{17}$$

then a diffusive scaling,

$$h_\epsilon(t, x) = \epsilon^{1/2} h(\epsilon^{-2}t, \epsilon^{-1}x), \tag{18}$$

would bring us back to the standard KPZ equation (11). In this way, KPZ and the continuum random polymer can be obtained from discrete models having an adjustable asymmetry.

## 4. Invariance principle for directed random polymers

Consider the distribution of the rescaled polymer path

$$x_\epsilon(t) := \epsilon x_{\lfloor \epsilon^{-2} t \rfloor}, \quad 0 \le t \le T \tag{19}$$

under the measure $P^{\epsilon^{1/2}\beta,\xi}_{\epsilon^{-2}T,0}$ from (2). Note that the asymmetry here is the temperature which has been scaled into a crossover regime near zero. Again we have a joint measure on paths and noise which we call $\mathbb{P}^{\beta,\epsilon}_{T,0}$.

**Theorem 4.1** [Alberts et al. 2014]. *Assume that* $E[\xi] = 0$ *and* $E[\xi_-^8] < \infty$. *Then the* $\mathbb{P}^{\beta,\epsilon}_{T,0}$, $\epsilon > 0$ *are a tight family and the limiting measure is the continuum random polymer* $\mathbb{P}^{2^{1/2}\beta}_{T,0}$. *In particular, for the free energy* (3),

$$\log Z^{\epsilon^{1/2}\beta,\xi}(\lfloor \epsilon^{-2} t \rfloor, 0) - \epsilon^{-2}\hat{\lambda}(\epsilon^{1/2}\beta)t + \tfrac{1}{2}\log(\epsilon^{-2}/4) \to \log z_{2^{1/2}\beta}(t, x), \tag{20}$$

*where* $z_\beta$ *is the solution of the stochastic heat equation* $\partial_t z = \frac{1}{2}\partial_y^2 z - \beta\xi z$ *with initial data* $z_\beta(0, x) = \delta_0(x)$, *and*

$$\hat{\lambda}(\beta) = \tfrac{1}{2}E[\xi^2]\beta^2 + \tfrac{1}{3!}E[\xi^3]\beta^3 + \tfrac{1}{4!}E[\xi^4]\beta^4 \tag{21}$$

*are the first four terms in the expansion of the log-moment generating function of the random variables* $\xi = \xi(i, j)$.

The condition $E[\xi_-^8] < \infty$ is not optimal. One expects it to be true if $E[\xi_-^6] < \infty$ and false otherwise. The reason is that there are $\mathbb{O}(\epsilon^{-3})$ sites at play in the heat cone. Each $-\xi$ should not be larger than $\epsilon^{-1/2}$ or else it becomes an attractive point for the polymer. By Chebyshev inequality $P(\xi < -\epsilon^{-1/2}) = o(\epsilon^3)$, if $E[\xi_-^6] < \infty$, so we do not observe any such attractive points in the heat cone. The reason for the condition $E[\xi_-^8] < \infty$ is that there are no more than $\mathbb{O}(\epsilon^{-4})$ sites in all, so the argument becomes easier because we don't have to do tight estimates at the edge of the heat cone. Similar arguments, together with the conjectured localization of the polymer path lead to the conjectured $E[\xi_-^5] < \infty$ condition for the strong noise limit (4).

## 5. Asymmetric simple exclusion

The asymmetric simple exclusion process is a Markov process whose state space consists of particle configurations on $\mathbb{Z}$ with at most one particle per site. Each particle attempts to walk as a continuous time simple random walk on $\mathbb{Z}$, independently of the other particles, attempting jumps to the left as a Poisson process with rate $q$ and to the right as a Poisson process with rate $p = 1 - q$. However, the jumps only take place if the target site is unoccupied. Because of the continuous time one does not have to face the issue of possible ties. The process can be thought of as a height function $h^{\mathrm{ASEP}}(t, x)$ given as

$$h^{\mathrm{ASEP}}(t,x) = \begin{cases} 2N(t) + \sum_{0<y\le x} \hat{\eta}(t,y), & x > 0, \\ 2N(t), & x = 0, \\ 2N(t) - \sum_{0<y\le x} \hat{\eta}(t,y), & x < 0, \end{cases} \tag{22}$$

where $N(t)$ records the net number of particles to cross from site 1 to site 0 in time $t$ and where $\hat{\eta}(t, x)$ equals 1 if there is a particle at $x$ at time $t$ and $-1$ otherwise. The state space is now random walk paths in $x$, and the special definition with the $N(t)$ means that the entire dynamics for the height function is that local maxima become local minima at rate $q$ and local minima become local maxima at rate $p$, independently for different nearest neighbour pairs.

The special case in which the initial data has all sites to the right of the origin occupied and all sites to the left unoccupied is called the *corner growth model*.

**Theorem 5.1** (Tracy–Widom ASEP formula [Tracy and Widom 2009]). *Consider the corner growth model with $q > p$ such that $q + p = 1$. Let $\gamma = q - p$ and $\tau = p/q$. For $m = \lfloor \frac{1}{2}(s + x) \rfloor$, $t \ge 0$ and $x \in \mathbb{Z}$,*

$$P(h_\gamma(t,x) \ge s) = \int_{S_{\tau^+}} \frac{d\mu}{\mu} \prod_{k=0}^{\infty} (1 - \mu\tau^k) \det(I + \mu J_{t,m,x,\mu})_{L^2(\Gamma_\eta)}, \tag{23}$$

*where $S_{\tau^+}$ is a positively oriented circle centred at zero of radius strictly between $\tau$ and 1, and where the kernel of the determinant is given by*

$$J_{t,m,x,\mu}(\eta,\eta') = \int_{\Gamma_\zeta} \exp\{\Psi_{t,m,x}(\zeta) - \Psi_{t,m,x}(\eta')\} \frac{f(\mu,\zeta/\eta')}{\eta'(\zeta-\eta)}\, d\zeta. \tag{24}$$

*Here $\eta$ and $\eta'$ are on $\Gamma_\eta$, a circle centred at zero of radius strictly between $\tau$ and* 1; *the $\zeta$ integral is on $\Gamma_\zeta$, a circle centred at zero of radius strictly between* 1 *and $\tau^{-1}$; and*

$$\begin{aligned} f(\mu, z) &= \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1-\tau^k \mu} z^k, \\ \Psi_{t,m,x}(\zeta) &= \Lambda_{t,m,x}(\zeta) - \Lambda_{t,m,x}(\xi), \\ \Lambda_{t,m,x}(\zeta) &= -x \log(1-\zeta) + \frac{t\zeta}{1-\zeta} + m \log \zeta. \end{aligned} \tag{25}$$

## 6. Weakly asymmetric limit and KPZ crossover formula

It has been known since [Bertini and Giacomin 1997] that KPZ can be obtained as the weakly asymmetric limit of simple exclusion. In our case, we need this for the corner growth initial conditions which is not covered by their results.

**Theorem 6.1** [Amir et al. 2011]. *For the corner growth model*,

$$\epsilon^{1/2} h^{\mathrm{ASEP}}_{q-p=\epsilon^{1/2}}(\epsilon^{-2}t, \epsilon^{-1}x) - \tfrac{1}{2}\epsilon^{-3/2} - \tfrac{1}{8}\epsilon^{-1/2} - \log(\tfrac{1}{2}\epsilon^{-1/2}) \;\to\; -\log z(t,x), \tag{26}$$

*where $z(t,x)$ is the solution of the stochastic heat equation* (7) *with $z(0,x) = \delta_0(x)$ and $\beta = 1$.*

The expression $h(t,x) = -\log z(t,x)$ is called the *narrow wedge solution of KPZ* and governs growth models with curved initial data.

In February 2010, Amir, Corwin and the author [Amir et al. 2011] and Sasamoto and Spohn [2010] independently studied the limit (26) of (24) by steepest descent. The methods were basically the same, however Amir et al. supply a mathematical proof, while Sasamoto and Spohn used physical arguments at various points. This gives the following exact formula for KPZ. Consider the solution of the stochastic heat equation with $z(0,x) = \delta_0(x)$ and $\beta = 1$ and define $\mathscr{A}_t$ by

$$z(t,x) = \tfrac{1}{\sqrt{2\pi t}} \exp\left( -\frac{x^2}{2t} + \frac{t}{24} + t^{1/3} \mathscr{A}_t(t^{-2/3}x) \right). \tag{27}$$

It is not hard to check that for each $t$, $\mathscr{A}_t(x)$ is stationary in $x$. It is called the *crossover Airy$_2$ process*.

**Theorem 6.2** [Amir et al. 2011; Sasamoto and Spohn 2010].

$$P(t^{1/3}\mathscr{A}_t(x) \le s) = \int_{-\infty}^{\infty} e^{-e^{s-a}} \det(I - K_t) \operatorname{Tr}((I-K_t)^{-1} \operatorname{Proj}_{\mathrm{Ai}})_{L^2(t^{-1/3}a,\infty)} \, da,$$

*where*

$$K_t(x,y) = \int_{-\infty}^{\infty} \frac{1}{1 - e^{-t^{1/3}s}} \mathrm{Ai}(x+s)\mathrm{Ai}(y+s) \, ds. \tag{28}$$

*In particular*,

$$P(\mathscr{A}_t(x) \le s) \overset{t\to\infty}{\longrightarrow} F_{\text{GUE}}(s). \tag{29}$$

From (20) and (29) we obtain:

**Corollary 6.3** (weak universality for directed random polymers in $1+1$ dimensions). *Assume that $\omega$ are i.i.d. with $E[\xi_-^8] < \infty$. Then as $n \to \infty$ followed by $\beta \to \infty$,*

$$\frac{\log Z^{n^{-1/4}\beta,\xi}(n,0) - n\hat{\lambda}(\beta n^{-1/4}) + \log\sqrt{\pi n/2} + 2\beta^4/3}{2\beta^{4/3}} \overset{(d)}{\longrightarrow} F_{\text{GUE}}.$$

Here $\hat{\lambda}$ is defined in (21). As explained before the corollary is expected to be true exactly under the condition $E[\xi_-^6] < \infty$ which we hope to achieve in future work.

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