Random Matrices
MSRI Publications
Volume **65**, 2014

Asymptotic analysis of the two-matrix model with a quartic potential

MAURICE DUITS, ARNO B. J. KUIJLAARS AND MAN YUE MO

We give a summary of the recent progress made by the authors and collaborators on the asymptotic analysis of the two-matrix model with a quartic potential. The paper also contains a list of open problems.

1. Two-matrix model: introduction

The Hermitian two-matrix model is the probability measure

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(V(M_1)+W(M_2)-\tau M_1 M_2)} \, dM_1 \, dM_2 \tag{1-1}$$

defined on pairs (M_1, M_2) of $n \times n$ Hermitian matrices. Here V and W are two polynomial potentials, $\tau \neq 0$ is a coupling constant, and

$$Z_n = \int e^{-n \operatorname{Tr}(V(M_1)+W(M_2)-\tau M_1 M_2)} \, dM_1 \, dM_2$$

is a normalization constant in order to make (1-1) a probability measure.

In recent works of the authors and collaborators [Duits et al. 2011; 2012; Duits and Kuijlaars 2009; Mo 2009] the model was studied with the aim to gain understanding in the limiting behavior of the eigenvalues of M_1 as $n \to \infty$, and to find and describe new types of critical behaviors.

The results should be compared with the well known results for the Hermitian one-matrix model

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(V(M))} \, dM, \tag{1-2}$$

which we briefly summarize here. The eigenvalues of the random matrix M

Duits is supported in part by the grant KAW 2010.0063 from the Knut and Alice Wallenberg Foundation and by the Swedish Research Council (VR) Grant 2012-3128. Kuijlaars is supported by FWO grants G.0641.11 and G.0934.13, K. U. Leuven research grants OT/08/33 and OT/12/73, and by research project MTM2011-28952-C02-01 of the Spanish Ministry of Science and Innovation. Mo is supported by the EPSRC grant EP/G019843/1.

from (1-2) have the explicit joint pdf

$$\frac{1}{\widetilde{Z}_n}\prod_{j<k}(x_k-x_j)^2\prod_{j=1}^{n}e^{-nV(x_j)},$$

which yields that the eigenvalues are a determinantal point process with correlation kernel

$$K_n(x,y)=\sqrt{e^{-nV(x)}}\sqrt{e^{-nV(y)}}\sum_{k=0}^{n-1}p_{k,n}(x)p_{k,n}(y),$$

where $(p_{k,n})_k$ is the sequence of orthonormal polynomials with respect to the weight function $e^{-nV(x)}$ on the real line. As $n\to\infty$ the empirical eigenvalue distributions have an a.s. weak limit[1]

$$\frac{1}{n}\sum_{j=1}^{n}\delta_{x_j}\to\mu^*,$$

where μ^* is a nonrandom probability measure that is characterized as the minimizer of the energy functional (Coulomb gas picture)

$$E_V(\mu)=\iint\log\frac{1}{|x-y|}\,d\mu(x)\,d\mu(y)+\int V(x)\,d\mu(x) \tag{1-3}$$

when taken over all probability measures on the real line. For a polynomial V the minimizer μ^* is supported on a finite union of intervals [Deift et al. 1998]. In addition there is a polynomial Q of degree $\deg V-2$ such that

$$\xi(z)=V'(z)-\int\frac{d\mu_1^*(s)}{z-s}$$

is the solution of a quadratic equation

$$\xi^2-V'(z)\xi+Q(z)=0. \tag{1-4}$$

From this it follows that μ_1^* has a density with respect to Lebesgue measure that is real analytic in the interior of any of the intervals and that can be written as

$$\rho(x)=\frac{d\mu^*(x)}{dx}=\frac{1}{\pi}\sqrt{q^-(x)},\qquad x\in\mathbb{R},$$

where q^- denotes the negative part of the polynomial

$$q(x)=\left(\frac{V'(x)}{2}\right)^2-Q(x).$$

[1] That is, for any bounded continuous function f, we have $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}f(x_j)=\int f\,d\mu^*$ almost surely.

2. Limiting eigenvalue distribution

2.1. ***Vector equilibrium problem.*** Guionnet [2004] showed that the eigenvalues of the matrices M_1 and M_2 in the two-matrix model (1-1) have a limiting distribution as $n \to \infty$. The results of [Guionnet 2004] are in fact valid for a much greater class of random matrix models. The limiting distribution is characterized as the minimizer of a certain functional, which is however very different from the energy functional (1-3) for the one matrix model.

Our aim is to develop an analogue of the Coulomb gas picture for the eigenvalues of the matrices in the two-matrix model (1-1). We have been successful in doing this for the eigenvalues of M_1 in the case of even polynomial potentials V and W with W of degree 4. Thus our assumptions are:

- V is an even polynomial with positive leading coefficient.
- $W(y) = \frac{1}{4}y^4 + \frac{\alpha}{2}y^2$ with $\alpha \in \mathbb{R}$.
- $\tau > 0$ (without loss of generality).

We recall some notions from logarithmic potential theory [Saff and Totik 1997]: the mutual logarithmic energy

$$I(\mu, \nu) = \iint \log \frac{1}{|x-y|} \, d\mu(x) \, d\nu(y)$$

of two measures μ and ν, and the logarithmic energy

$$I(\mu) = I(\mu, \mu)$$

of a measure μ. Then the limiting mean distribution of the eigenvalues of M_1 is characterized by a vector equilibrium problem for three measures. This involves an energy functional

$$E(\mu_1, \mu_2, \mu_3) = I(\mu_1) + I(\mu_2) + I(\mu_3) - I(\mu_1, \mu_2) - I(\mu_2, \mu_3) + \int V_1(x) \, d\mu_1(x) + \int V_3(x) \, d\mu_3(x) \qquad \text{(2-1)}$$

defined on three measures μ_1, μ_2, μ_3. Note that there is an attraction between the measures μ_1 and μ_2 and between the measures μ_2 and μ_3, while there is no direct interaction between the measures μ_1 and μ_3. This type of interaction is characteristic for a Nikishin system [Nikishin and Sorokin 1991].

The energy functional (2-1) depends on the external fields V_1 and V_3 that act on the measures μ_1 and μ_3 in (2-1). The vector equilibrium problem will also have an upper constraint σ_2 for the measure μ_2. These input data take a very special form that we describe next.

External field V_1. The external field that acts on μ_1 is defined by

$$V_1(x) = V(x) + \min_{s\in\mathbb{R}}(W(s) - \tau x s), \tag{2-2}$$

where we recall that $W(s) = \frac{1}{4}s^4 + \frac{\alpha}{2}s^2$. For the case $\alpha = 0$, this is simply $V_1(x) = V(x) - \frac{3}{4}|\tau x|^{4/3}$.

External field V_3. The external field that acts on the third measure is absent if $\alpha \geq 0$, i.e.,

$$V_3(x) \equiv 0 \quad \text{if } \alpha \geq 0.$$

The function $s \in \mathbb{R} \mapsto W(s) - \tau x s$ has a global minimum at $s = s_1(x)$ and this value plays a role in the definition of V_1, see (2-2). For $\alpha < 0$, and $x \in (-x^*(\alpha), x^*(\alpha))$, where

$$x^*(\alpha) = \frac{2}{\tau}\left(\frac{-\alpha}{3}\right)^{3/2}, \quad \alpha < 0,$$

the function $s \in \mathbb{R} \mapsto W(s) - \tau x s$ has another local minimum at $s = s_2(x)$, and a local maximum at $s = s_3(x)$.

Then V_3 is defined by

$$V_3(x) = (W(s_3(x)) - \tau x s_3(x)) - (W(s_2(x)) - \tau x s_2(x)) \tag{2-3}$$

if $x \in (-x^*(\alpha), x^*(\alpha))$, and $V_3(x) \equiv 0$ otherwise.

Upper constraint σ_2. The upper constraint σ_2 that acts on the second measure is the measure on the imaginary axis with the density

$$\frac{d\sigma_2(z)}{|dz|} = \frac{\tau}{\pi} \max_{s^3+\alpha s = \tau z} \operatorname{Re} s, \quad z \in i\mathbb{R}. \tag{2-4}$$

In case $\alpha = 0$ this simplifies to

$$\frac{d\sigma_2}{|dz|} = \frac{\sqrt{3}}{2\pi}\tau^{4/3}|z|^{1/3}.$$

If $\alpha < 0$ then the density of σ_2 is positive and real analytic on the full imaginary axis. If $\alpha > 0$ then the support of σ_2 has a gap around 0:

$$\operatorname{supp}(\sigma_2) = (-i\infty, -iy^*(\alpha)] \cup [iy^*(\alpha), i\infty),$$

where

$$y^*(\alpha) = \frac{2}{\tau}\left(\frac{\alpha}{3}\right)^{3/2}, \quad \alpha > 0.$$

Theorem 1 [Duits et al. 2012, Theorem 1.1]. *There is a unique minimizer* $(\mu_1^*, \mu_2^*, \mu_3^*)$ *of the energy functional* (2-1) *subject to the following conditions, with input data* V_1, V_3, *and* σ (*as described above*):

(a) μ_1 *is a measure on* $\mathbb{R}$ *with* $\mu_1(\mathbb{R}) = 1$.

(b) μ_2 *is a measure on* $i\mathbb{R}$ *with* $\mu_2(i\mathbb{R}) = \frac{2}{3}$.

(c) μ_3 *is a measure on* $\mathbb{R}$ *with* $\mu_3(\mathbb{R}) = \frac{1}{3}$.

(d) $\mu_2 \leq \sigma_2$.

The proof of the existence of a minimizer was completed and simplified in [Hardy and Kuijlaars 2012]; see Section 4.2 below.

Now that we have existence and uniqueness, it is natural to ask about further properties of the minimizer. The three measures μ_1^*, $\sigma - \mu_2^*$ and μ_3^* are absolutely continuous with respect to the Lebesgue measure with densities that are real analytic in the interior of their supports, except possibly at the origin. Furthermore, denoting by $S(\mu)$ the support of a measure μ, we have:

- The support of μ_1^* is a finite union of bounded intervals on the real line.
- There exists $c_2 \geq 0$ such that $S(\sigma_2 - \mu_2^*) = i\mathbb{R} \setminus (-ic_2, ic_2)$, and if $c_2 > 0$ the density of $\sigma_2 - \mu_2^*$ vanishes like a square root at $\pm ic_2$.
- There exists $c_3 \geq 0$ such that $S(\mu_3^*) = \mathbb{R} \setminus (-c_3, c_3)$, and if $c_3 > 0$ the density of μ_3^* vanishes like a square root at $\pm c_3$.

In a generic situation, the density of μ_1^* is strictly positive in the interior of its support and vanishes like a square root at endpoints. In addition strict inequality holds in the variational inequality outside the support $S(\mu_1^*)$. Moreover, generically if $c_2 = 0$ the density of $\sigma - \mu_2^*$ is positive at the origin, and likewise if $c_3 = 0$ the density of μ_3^* is positive at the origin. If we are in such a generic situation, then we say that (V, W, τ) is *regular*. See [Duits et al. 2012, Section 1.5] for more details and a discussion on the singular situations that may occur.

Theorem 2 [Duits et al. 2012, Theorem 1.4]. *Let* μ_1^* *be the first component of the minimizer in Theorem* 1, *and assume that* (V, W, τ) *is regular*, *then as* $n \to \infty$ *with* $n \equiv 0 \pmod 3$, *the mean eigenvalue distribution of* M_1 *convergences to* μ_1^*.

We are convinced that the theorem is also valid in the singular cases, which correspond to phase transitions in the two-matrix model. The condition that n is a multiple of three is nonessential as well. It is imposed for convenience in the analysis.

In [Duits et al. 2012] only the convergence of mean eigenvalue distributions was considered, which is a rather weak form of convergence. However, when combined with the results of [Guionnet 2004] it will actually follow that the empirical eigenvalue distributions of M_1 tend to μ_1^* almost surely.

The analysis of [Duits et al. 2012] also proves the usual universality results for local eigenvalue statistics in Hermitian matrix ensembles, given by the sine kernel in the bulk of the spectrum and by the Airy kernel at edge points. In nonregular situations one may find Pearcey and Painlevé II kernels, while in multicritical cases new kernels may appear. This was indeed proved recently in [Duits and Geudens 2013]; see Section 4.1 below.

2.2. *Riemann surface.* A major ingredient in the asymptotic analysis in [Duits et al. 2012] is the construction of an appropriate Riemann surface (or spectral curve), which plays a role similar to the algebraic equation (1-4) in the one-matrix model. The existence of such a Riemann surface is implied by the work of Eynard [2005] on the formal two-matrix model. Our approach is different from the one of Eynard, in that we use the vector equilibrium problem to construct the Riemann surface, and in a next step we define a meromorphic function on it.

The main point is that the supports $S(\mu_1^*)$, $S(\sigma-\mu_2^*)$ and $S(\mu_3^*)$ associated to the minimizer in Theorem 1 determine the cut structure of a Riemann surface

$$\mathcal{R} = \bigcup_{j=1}^{4} \mathcal{R}_j$$

with four sheets:

$$\begin{aligned}
\mathcal{R}_1 &= \overline{\mathbb{C}} \setminus S(\mu_1^*),\\
\mathcal{R}_2 &= \mathbb{C} \setminus (S(\mu_1^*) \cup S(\sigma_2-\mu_2^*)),\\
\mathcal{R}_3 &= \mathbb{C} \setminus (S(\sigma_2-\mu_2^*) \cup S(\mu_3^*)),\\
\mathcal{R}_4 &= \mathbb{C} \setminus S(\mu_3^*).
\end{aligned}$$

The sheet $\mathcal{R}_j$ is glued to the next sheet $\mathcal{R}_{j+1}$ along the common cut in the usual crosswise manner. The meromorphic function on $\mathcal{R}$ arises in the following way:

Proposition 3 ([Duits et al. 2012, Proposition 4.8]). *The function*

$$\xi_1(z) = V'(z) - \int \frac{d\mu_1^*(x)}{z-x}, \quad z \in \mathcal{R}_1,$$

extends to a meromorphic function on the Riemann surface $\mathcal{R}$ whose only poles are at infinity. There is a pole of order deg V *at infinity on the first sheet*, *and a simple pole at the other point at infinity.*

The proof of Proposition 3 follows from the Euler–Lagrange variational conditions that are associated with the vector equilibrium problem. See Section 4.2 of [Duits et al. 2012] for explicit expressions for the meromorphic continuation of ξ_1 to the other sheets.

It follows from Proposition 3 that ξ_1 is one of the solutions of a quartic equation, which is the analogue of the quadratic equation (1-4) that is relevant in the one-matrix model.

3. About the proof

We describe the main tools that are used in the proof of Theorem 2.

3.1. ***Biorthogonal polynomials.*** We make use of the integrable structure of the two-matrix model that is described in terms of biorthogonal polynomials. In this context the biorthogonal polynomials are two sequences of monic polynomials $(p_{j,n})_j$ and $(q_{k,n})_k$ (depending on n) with $\deg p_{j,n} = j$ and $\deg q_{k,n} = k$, that satisfy

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{j,n}(x)q_{k,n}(y)e^{-n(V(x)+W(y)-\tau xy)}\,dx\,dy = h_{k,n}\delta_{j,k};$$

see [Bertola 2011; Bertola et al. 2002; 2003; Ercolani and McLaughlin 2001; Eynard and Mehta 1998]. These polynomials uniquely exist, have real and simple zeros [Ercolani and McLaughlin 2001], and in addition the zeros of $p_{j,n}$ and $p_{j+1,n}$ interlace, as well as those of $q_{k,n}$ and $q_{k+1,n}$; see [Duits et al. 2011].

There is an explicit expression for the joint pdf of the eigenvalues of M_1 and M_2:

$$\frac{1}{(n!)^2}\det\begin{pmatrix} K_n^{(1,1)}(x_i,x_j) & K_n^{(1,2)}(x_i,y_j) \\ K_n^{(2,1)}(y_1,y_j) & K_n^{(2,2)}(y_i,y_j), \end{pmatrix} \tag{3-1}$$

with four kernels that are expressed in terms of the biorthogonal polynomials and their transformed functions

$$Q_{k,n}(x) = \int_{-\infty}^{\infty} q_{k,n}(y)e^{-n(V(x)+W(y)-\tau xy)}\,dy,$$
$$P_{j,n}(y) = \int_{-\infty}^{\infty} p_{j,n}(x)e^{-n(V(x)+W(y)-\tau xy)}\,dx,$$

as follows:

$$\begin{aligned} K_n^{(1,1)}(x_1,x_2) &= \sum_{k=0}^{n-1}\frac{1}{h_{k,n}^2}p_{k,n}(x_1)Q_{k,n}(x_2), \\ K_n^{(1,2)}(x,y) &= \sum_{k=0}^{n-1}\frac{1}{h_{k,n}^2}p_{k,n}(x)q_{k,n}(y), \\ K_n^{(2,1)}(y,x) &= \sum_{k=0}^{n-1}\frac{1}{h_{k,n}^2}P_{k,n}(y)Q_{k,n}(x) - e^{-n(V(x)+W(y)-\tau xy)}, \\ K_n^{(2,2)}(y_1,y_2) &= \sum_{k=0}^{n-1}\frac{1}{h_{k,n}^2}P_{k,n}(y_1)q_{k,n}(y_2). \end{aligned} \tag{3-2}$$

The joint pdf (3-1) is determinantal, which means that eigenvalue correlation functions have determinantal expressions with the same kernels $K_n^{(i,j)}$, $i, j = 1, 2$. In particular, after averaging out the eigenvalues of M_2 we get that the eigenvalues of M_1 are a determinantal point process with kernel $K_n^{(1,1)}$.

A natural first step to compute the asymptotic behavior of the polynomials and hence the kernels, is to formulate a Riemann–Hilbert problem (≡ RH problem) for the polynomials. Several different formulations exist in the literature [Bertola et al. 2003; Ercolani and McLaughlin 2001; Kapaev 2003; Kuijlaars and McLaughlin 2005]. The analysis in [Duits and Kuijlaars 2009; Duits et al. 2012; Mo 2009] is based on the RH problem in [Kuijlaars and McLaughlin 2005] that we will discuss in the next subsection.

3.2. ***Riemann–Hilbert problem.*** It turns out that the kernel (3-2) has a special structure which relates it to multiple orthogonal polynomials and the eigenvalues of M_1 (after averaging over M_2) are an example of a multiple orthogonal polynomial ensemble [Kuijlaars 2010]. This is due to the following observation of Kuijlaars and McLaughlin [2005].

Proposition 4. *Suppose W is a polynomial of degree $r+1$, and let*

$$w_{k,n}(x) = \int_{-\infty}^{\infty} y^k e^{-n(V(x)+W(y)-\tau xy)}\,dy, \quad k = 0, \dots, r-1.$$

Then the biorthogonal polynomial $p_{j,n}$ satisfies

$$\int_{-\infty}^{\infty} p_{j,n}(x) x^l w_{k,n}(x)\,dx = 0, \quad l = 0, \dots, \left\lceil \frac{j-k}{r} \right\rceil - 1, \tag{3-3}$$

for $k = 0, \dots, r-1$.

The conditions (3-3) are known as multiple orthogonality conditions [Aptekarev 1998], and they characterize the biorthogonal polynomials.

The advantage of the formulation as multiple orthogonality is that these polynomials are characterized by a RH problem of size $(r+1)\times(r+1)$, [Van Assche et al. 2001], which we state here for the case $r = 3$ and for $j = n$ with n a multiple of three. Then the RH problem has size 4×4 and it asks for a 4×4 matrix valued function Y on $\mathbb{C}\setminus\mathbb{R}$ satisfying these conditions:

(a) $Y : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{4\times 4}$ is analytic.

(b) For $x \in \mathbb{R}$,

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_{0,n}(x) & w_{1,n}(x) & w_{2,n}(x) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $Y_+(x)$ ($Y_-(x)$) denotes the limiting value of $Y(z)$ as $z \to x$ from the upper (lower) half-plane.

(c) As $z \to \infty$,

$$Y(z) = \left(I_4 + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & z^{-n/3} & 0 & 0 \\ 0 & 0 & z^{-n/3} & 0 \\ 0 & 0 & 0 & z^{-n/3} \end{pmatrix}.$$

The RH problem has a unique solution, given by

$$Y = \begin{pmatrix} p_{n,n} & C(p_{n,n}w_{0,n}) & C(p_{n,n}w_{1,n}) & C(p_{n,n}w_{2,n}) \\ p_{n,n}^{(0)} & C(p_{n,n}^{(0)}w_{0,n}) & C(p_{n,n}^{(0)}w_{1,n}) & C(p_{n,n}^{(0)}w_{2,n}) \\ p_{n,n}^{(1)} & C(p_{n,n}^{(1)}w_{0,n}) & C(p_{n,n}^{(1)}w_{1,n}) & C(p_{n,n}^{(1)}w_{2,n}) \\ p_{n,n}^{(2)} & C(p_{n,n}^{(2)}w_{0,n}) & C(p_{n,n}^{(2)}w_{1,n}) & C(p_{n,n}^{(2)}w_{2,n}) \end{pmatrix},$$

where $p_{n,n}$ is the n-th degree biorthogonal polynomial, $p_{n,n}^{(0)}$, $p_{n,n}^{(1)}$, $p_{n,n}^{(2)}$ are three polynomials of degree $\le n-1$ that satisfy certain multiple orthogonal conditions and Cf is the Cauchy transform

$$Cf(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(x)}{x-z}\,dx.$$

By using the Christoffel–Darboux formula for multiple orthogonal polynomials [Bleher and Kuijlaars 2004; Daems and Kuijlaars 2004] the correlation kernel $K_n^{(1,1)}$ for the eigenvalues of M_1 can be expressed in terms of the solution of the RH problem:

$$K_n^{(1,1)}(x,y) = \begin{pmatrix} 0 & w_{0,n}(y) & w_{1,n}(y) & w_{2,n}(y) \end{pmatrix} \frac{Y_+^{-1}(y)Y_+(x)}{2\pi i(x-y)} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3\text{-}4)$$

Multiple orthogonal polynomials and RH problems are also used for random matrices with external source [Bleher et al. 2011; Bleher and Kuijlaars 2004] and models of nonintersecting paths [Kuijlaars et al. 2009]. In these cases, correlation kernels for the relevant statistical quantities are also expressed in terms of the corresponding RH problem through (3-4).

3.3. ***Steepest descent analysis.*** The remaining part of the proof of Theorem 2 is an asymptotic analysis of the RH problem via an extension of the Deift–Zhou steepest descent method [Deift et al. 1999; Deift and Zhou 1993]. The vector equilibrium problem and the Riemann surface play a crucial role in the transformations in this analysis. For the precise transformations and the many details that are involved we refer the reader to [Duits et al. 2012]. Following the effect of the transformations on the kernel (3-4), one finds that

$$\lim_{n\to\infty} \frac{1}{n} K_n^{(1,1)}(x,x) = \frac{d\mu_1^*(x)}{dx},$$

which is what is needed to establish the theorem.

A somewhat similar steepest descent analysis is done in [Bleher et al. 2011] for a random matrix model with external source, where vector equilibrium problems and Riemann surfaces also play an important role.

4. Further developments

4.1. *Critical behavior in the quadratic/quartic model.* For the case $V(x)=\frac{1}{2}x^2$ the spectral curve can be computed and a classification of all possible cases can be made explicitly.

Case I: $0 \in S(\mu_1^*) \cap S(\mu_3^*)$ and $0 \notin S(\sigma_2 - \mu_2^*)$.

Case II: $0 \in S(\mu_3^*)$ and $0 \notin S(\mu_1^*) \cup S(\sigma_2 - \mu_2^*)$.

Case III: $0 \in S(\sigma_2 - \mu_2^*)$ and $0 \notin S(\mu_1^*) \cup S(\mu_3^*)$.

Case IV: $0 \in S(\mu_1^*)$ and $0 \notin S(\sigma_2 - \mu_2^*) \cup S(\mu_3^*)$.

Phase transitions between the regular cases represent the critical cases.

The quadratic/quartic model depends on two parameters, namely the coupling constant τ and the number α in the quartic potential $W(y) = y^4/4 + \alpha y^2/2$. Figure 1 (taken from [Duits et al. 2011]) shows the phase diagram in the α-τ plane. Critical behavior takes place on the curves $\tau^2 = \alpha + 2$ and $\alpha\tau^2 = -1$.

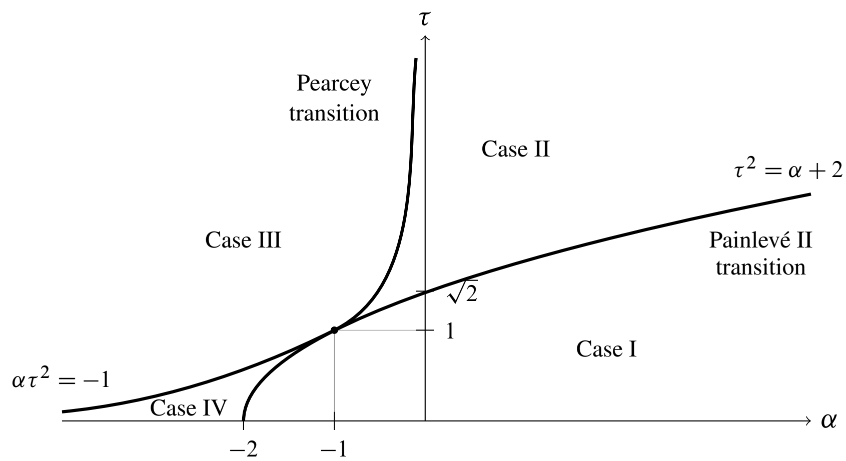

Figure 1. Phase diagram for the quadratic case $V(x) = \frac{1}{2}x^2$.

On the parabola $\tau^2 = \alpha + 2$ a gap appears around 0 in the support of either μ_1^* (if one moves from Case I to Case II) or μ_3^* (if from Case I to Case IV). This is a transition of Painlevé II type which also appears in the opening of gaps in one-matrix models [Bleher and Its 2003; Claeys and Kuijlaars 2006]. On the curve $\alpha\tau^2 = -1$ a gap appears in the support of either μ_1^* (if one moves from Case IV to Case III) or μ_3^* (if one moves from Case II to Case III), while simultaneously the gap in the support of $\sigma_2 - \mu_2^*$ closes. This is a transition of Pearcey type, which was observed before in the random matrix model with external source and in the model of nonintersecting Brownian motions [Bleher and Kuijlaars 2007; Brézin and Hikami 1998; Tracy and Widom 2006].

The phase diagram has a very special point $\alpha = -1$, $\tau = 1$ which is on both critical curves, and where all four regular cases come together. For these special values, the density of μ_1^* vanishes like a square root at the origin, which is an interior point of $S(\mu_1^*)$. The local analysis at this point was done very recently by Duits and Geudens [2013]. They found that in the asymptotic limit, the local eigenvalue correlation kernels around 0 are closely related to the limiting kernels that describe the tacnode behavior for nonintersecting Brownian motions [Adler et al. 2013; Delvaux et al. 2011; Johansson 2013]. More precisely, the kernels can be expressed in terms of an extension of the same 4×4 RH problem in [Delvaux et al. 2011]. However, they are constructed in a different way out of this 4×4 RH problem and, as a result, these kernels are not the same.

4.2. ***Vector equilibrium problems.*** The analysis in [Duits et al. 2012] of the vector equilibrium problem was not fully complete, since the lower semicontinuity of the energy functional (2-1) was implicitly assumed but not established in [Duits et al. 2012].

In [Beckermann et al. 2013; Hardy and Kuijlaars 2012] the vector equilibrium problem was studied in a more systematic way, in the more general context of an energy functional for n measures

$$E(\mu_1, \dots, \mu_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} I(\mu_i, \mu_j) + \sum_{j=1}^{n} \int V_j(x)\, d\mu_j(x), \tag{4-1}$$

where $C = (c_{ij})_{i,j=1}^n$ is a real symmetric positive definite matrix (in [Beckermann et al. 2013] also semidefinite interaction matrices are considered). The external fields $V_j : \Sigma_j \to \mathbb{R} \cup \{\infty\}$ are lower semicontinuous with domains Σ_j that are closed subsets of $\mathbb{C}$. Let $m_1, \dots, m_n$ be given positive numbers and assume that for every $i = 1, \dots, n$,

$$\liminf_{|x|\to\infty} \left(V_i(x) - \left(\sum_{j=1}^{n} c_{ij} m_j \right) \log(1 + |x|^2) \right) > -\infty.$$

Under these assumptions it is shown in [Hardy and Kuijlaars 2012] that the energy functional (4-1), restricted to the set of measures with $\mu_j(\Sigma_j) = m_j$ for $j = 1, \dots, n$,

(a) has compact sublevel sets $\{E \leq \alpha\}$ for every $\alpha \in \mathbb{R}$, (so E is in particular lower semicontinuous), and

(b) is strictly convex on the subset where it is finite.

This guarantees existence and uniqueness of a minimizer of (4-1), provided that E is not identically infinite. Existence and uniqueness of a minimizer readily extends to situations where the domain of E is further restricted by upper constraints $\mu_j \leq \sigma_j$ for $j = 1, \dots, n$, again provided that E is not identically infinite on this domain. In particular, this applies to the energy functional (2-1) for the two-matrix model with quartic potential with the constraint $\mu_2 \leq \sigma_2$ described in Section 2.

4.3. ***Open problems.*** Numerous intriguing questions and open problems arise out of our analysis.

(a) What is the motivation for the central vector equilibrium problem? In the one-matrix model there is a direct way to come from the joint eigenvalue probability density to the equilibrium problem. We do not have this direct link for the two-matrix model.

(b) How is the vector equilibrium problem related to the variational problem from [Guionnet 2004]?

(c) A possibly related question: is there a large deviation principle associated with the vector equilibrium problem? See, for example, [Anderson et al. 2010] for the large deviations interpretation of the equilibrium problem for the one-matrix model.

(d) Our analysis is restricted to even potentials V and W. This restriction provides a symmetry of the problem around zero, which is the reason why the second measure μ_2 in the vector equilibrium problem is supported on the imaginary axis. If we remove the symmetry then probably we would have to look for a contour that replaces the imaginary axis. It is likely that such a contour would be an S-curve in a certain external field, but at this moment we do not know how to handle this situation. See [Martínez-Finkelshtein and Rakhmanov 2011; Rakhmanov 2012] for important recent developments around S-curves for scalar equilibrium problems.

(e) Extension to higher degree W is wide open. If $\deg W = d$ then one would expect a vector equilibrium problem for $d-1$ measures. It may be that S-curves are needed for $d \geq 6$, even in the case of even potentials.

(f) Exploration of further critical phenomena in the two-matrix model.

References

[Adler et al. 2013] M. Adler, P. L. Ferrari, and P. van Moerbeke, “Nonintersecting random walks in the neighborhood of a symmetric tacnode”, *Ann. Probab.* **41**:4 (2013), 2599–2647.

[Anderson et al. 2010] G. W. Anderson, A. Guionnet, and O. Zeitouni, *An introduction to random matrices*, Cambridge Studies in Advanced Mathematics **118**, Cambridge University Press, Cambridge, 2010.

[Aptekarev 1998] A. I. Aptekarev, “Multiple orthogonal polynomials”, pp. 423–447 in *Proceedings of the VIIIth Symposium on Orthogonal Polynomials and Their Applications* (Seville, 1997), vol. 99, edited by A. J. Duran, 1998.

[Beckermann et al. 2013] B. Beckermann, V. Kalyagin, A. C. Matos, and F. Wielonsky, “Equilibrium problems for vector potentials with semidefinite interaction matrices and constrained masses”, *Constr. Approx.* **37**:1 (2013), 101–134.

[Bertola 2011] M. Bertola, “Two-matrix models and biorthogonal polynomials”, pp. 310–328 in *The Oxford handbook of random matrix theory*, edited by G. Akemann et al., Oxford Univ. Press, Oxford, 2011.

[Bertola et al. 2002] M. Bertola, B. Eynard, and J. Harnad, “Duality, biorthogonal polynomials and multi-matrix models”, *Comm. Math. Phys.* **229**:1 (2002), 73–120.

[Bertola et al. 2003] M. Bertola, B. Eynard, and J. Harnad, “Differential systems for biorthogonal polynomials appearing in 2-matrix models and the associated Riemann–Hilbert problem”, *Comm. Math. Phys.* **243**:2 (2003), 193–240.

[Bleher and Its 2003] P. Bleher and A. Its, “Double scaling limit in the random matrix model: the Riemann–Hilbert approach”, *Comm. Pure Appl. Math.* **56**:4 (2003), 433–516.

[Bleher and Kuijlaars 2004] P. M. Bleher and A. B. J. Kuijlaars, “Random matrices with external source and multiple orthogonal polynomials”, *Int. Math. Res. Not.* **2004**:3 (2004), 109–129.

[Bleher and Kuijlaars 2007] P. M. Bleher and A. B. J. Kuijlaars, “Large n limit of Gaussian random matrices with external source, III: double scaling limit”, *Comm. Math. Phys.* **270**:2 (2007), 481–517.

[Bleher et al. 2011] P. Bleher, S. Delvaux, and A. B. J. Kuijlaars, “Random matrix model with external source and a constrained vector equilibrium problem”, *Comm. Pure Appl. Math.* **64**:1 (2011), 116–160.

[Brézin and Hikami 1998] E. Brézin and S. Hikami, “Universal singularity at the closure of a gap in a random matrix theory”, *Phys. Rev. E* (3) **57**:4 (1998), 4140–4149.

[Claeys and Kuijlaars 2006] T. Claeys and A. B. J. Kuijlaars, “Universality of the double scaling limit in random matrix models”, *Comm. Pure Appl. Math.* **59**:11 (2006), 1573–1603.

[Daems and Kuijlaars 2004] E. Daems and A. B. J. Kuijlaars, “A Christoffel–Darboux formula for multiple orthogonal polynomials”, *J. Approx. Theory* **130**:2 (2004), 190–202.

[Deift and Zhou 1993] P. Deift and X. Zhou, “A steepest descent method for oscillatory Riemann–Hilbert problems: asymptotics for the MKdV equation”, *Ann. of Math.* (2) **137**:2 (1993), 295–368.

[Deift et al. 1998] P. Deift, T. Kriecherbauer, and K. T.-R. McLaughlin, “New results on the equilibrium measure for logarithmic potentials in the presence of an external field”, *J. Approx. Theory* **95**:3 (1998), 388–475.

[Deift et al. 1999] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, "Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory", *Comm. Pure Appl. Math.* **52**:11 (1999), 1335–1425.

[Delvaux et al. 2011] S. Delvaux, A. B. J. Kuijlaars, and L. Zhang, "Critical behavior of nonintersecting Brownian motions at a tacnode", *Comm. Pure Appl. Math.* **64**:10 (2011), 1305–1383.

[Duits and Geudens 2013] M. Duits and D. Geudens, "A critical phenomenon in the two-matrix model in the quartic/quadratic case", *Duke Math. J.* **162**:8 (2013), 1383–1462.

[Duits and Kuijlaars 2009] M. Duits and A. B. J. Kuijlaars, "Universality in the two-matrix model: a Riemann–Hilbert steepest-descent analysis", *Comm. Pure Appl. Math.* **62**:8 (2009), 1076–1153.

[Duits et al. 2011] M. Duits, D. Geudens, and A. B. J. Kuijlaars, "A vector equilibrium problem for the two-matrix model in the quartic/quadratic case", *Nonlinearity* **24**:3 (2011), 951–993.

[Duits et al. 2012] M. Duits, A. B. J. Kuijlaars, and M. Y. Mo, *The Hermitian two matrix model with an even quartic potential*, Mem. Amer. Math. Soc. **1022**, 2012.

[Ercolani and McLaughlin 2001] N. M. Ercolani and K. T.-R. McLaughlin, "Asymptotics and integrable structures for biorthogonal polynomials associated to a random two-matrix model: advances in nonlinear mathematics and science", *Phys. D* **152/153** (2001), 232–268.

[Eynard 2005] B. Eynard, *Le modèle à deux matrices, polynômes biorthogonaux, problème de Riemann–Hilbert*, Thèse d'habilitation à diriger des recherches, Université Paris 7 Denis Diderot, Paris, 2005. arXiv math-ph/0504034

[Eynard and Mehta 1998] B. Eynard and M. L. Mehta, "Matrices coupled in a chain, I: eigenvalue correlations", *J. Phys. A* **31**:19 (1998), 4449–4456.

[Guionnet 2004] A. Guionnet, "First order asymptotics of matrix integrals; a rigorous approach towards the understanding of matrix models", *Comm. Math. Phys.* **244**:3 (2004), 527–569.

[Hardy and Kuijlaars 2012] A. Hardy and A. B. J. Kuijlaars, "Weakly admissible vector equilibrium problems", *J. Approx. Theory* **164**:6 (2012), 854–868.

[Johansson 2013] K. Johansson, "Non-colliding Brownian motions and the extended tacnode process", *Comm. Math. Phys.* **319**:1 (2013), 231–267.

[Kapaev 2003] A. A. Kapaev, "Riemann–Hilbert problem for bi-orthogonal polynomials", *J. Phys. A* **36**:16 (2003), 4629–4640.

[Kuijlaars 2010] A. B. J. Kuijlaars, "Multiple orthogonal polynomial ensembles", pp. 155–176 in *Recent trends in orthogonal polynomials and approximation theory*, edited by F. M. Jorge Arvesú and A. Martínez-Finkelshtein, Contemp. Math. **507**, Amer. Math. Soc., Providence, RI, 2010.

[Kuijlaars and McLaughlin 2005] A. B. J. Kuijlaars and K. T.-R. McLaughlin, "A Riemann–Hilbert problem for biorthogonal polynomials", *J. Comput. Appl. Math.* **178**:1-2 (2005), 313–320.

[Kuijlaars et al. 2009] A. B. J. Kuijlaars, A. Martínez-Finkelshtein, and F. Wielonsky, "Non-intersecting squared Bessel paths and multiple orthogonal polynomials for modified Bessel weights", *Comm. Math. Phys.* **286**:1 (2009), 217–275.

[Martínez-Finkelshtein and Rakhmanov 2011] A. Martínez-Finkelshtein and E. A. Rakhmanov, "Critical measures, quadratic differentials, and weak limits of zeros of Stieltjes polynomials", *Comm. Math. Phys.* **302**:1 (2011), 53–111.

[Mo 2009] M. Y. Mo, "Universality in the two matrix model with a monomial quartic and a general even polynomial potential", *Comm. Math. Phys.* **291**:3 (2009), 863–894.

[Nikishin and Sorokin 1991] E. M. Nikishin and V. N. Sorokin, *Rational approximations and orthogonality*, Translations of Mathematical Monographs **92**, American Mathematical Society, Providence, RI, 1991.

[Rakhmanov 2012] E. A. Rakhmanov, "Orthogonal polynomials and S-curves", pp. 195–239 in *Recent advances in orthogonal polynomials, special functions, and their applications*, edited by J. Arvesú and G. L. Lagomasino, Contemp. Math. **578**, Amer. Math. Soc., Providence, RI, 2012.

[Saff and Totik 1997] E. B. Saff and V. Totik, *Logarithmic potentials with external fields*, Grundlehren der Math. Wissensch. **316**, Springer, Berlin, 1997.

[Tracy and Widom 2006] C. A. Tracy and H. Widom, "The Pearcey process", *Comm. Math. Phys.* **263**:2 (2006), 381–400.

[Van Assche et al. 2001] W. Van Assche, J. S. Geronimo, and A. B. J. Kuijlaars, "Riemann–Hilbert problems for multiple orthogonal polynomials", pp. 23–59 in *Special functions 2000: current perspective and future directions* (Tempe, AZ), edited by J. Bustoz et al., NATO Sci. Ser. II Math. Phys. Chem. **30**, Kluwer, Dordrecht, 2001.

duits@kth.se — *Department of Mathematics, Royal Institute of Technology (KTH), Lindstedtsvagen 25, SE-SE-10044 Stockholm, Sweden*

arno.kuijlaars@wis.kuleuven.be — *Department of Mathematics, Katholieke Universiteit Leuven, 3001 Leuven, Belgium*

Department of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, United Kingdom