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Experimental realization of Tracy–Widom distributions and beyond: KPZ interfaces in turbulent liquid crystal

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Analytical studies have shown the Tracy–Widom distributions and the Airy processes in the asymptotics of a few growth models in the Kardar–Parisi–Zhang (KPZ) universality class. Here the author shows evidence that these mathematical objects arise even in a real experiment: more specifically, in growing interfaces of turbulent liquid crystal. The present article is devoted to overviewing the current status of this experimental approach to the KPZ class, which directly concerns random matrix theory and related fields of mathematical physics. In particular, the author summarizes those statistical properties which were derived rigorously for simple solvable models and realized here experimentally, and those which were evidenced in the experiment and remain to be explained by further mathematical or theoretical studies.

1. Introduction

The first decade of the 21st century and a couple of preceding years have been marked by a series of remarkable analytical developments, which have revealed profound and rigorous connections among random matrix theory, combinatorial problems, and the physical problem of the fluctuating interface growth ([Baik and Rains 2001; Kriecherbauer and Krug 2010; Sasamoto and Spohn 2010a; Corwin 2012] and references therein). Their primary conclusions in terms of the interface growth problem, first pointed out by Johansson [2000] for TASEP and by Prähofer and Spohn [2000] for the PNG model, are the following [Kriecherbauer and Krug 2010; Sasamoto and Spohn 2010a; Corwin 2012]: (i) The distribution function and the spatial correlation function were obtained rigorously for the asymptotic interface fluctuations. (ii) The results depend on the global shape of the interfaces, or on the initial condition. For the two prototypical cases of the curved and flat growing interfaces, the distribution function is given by the Tracy–Widom distribution [Tracy and Widom 1994; 1996] for GUE and GOE, respectively, and the spatial two-point correlation function by the covariance of the Airy_2 and Airy_1 process [Prähofer and Spohn 2002; Sasamoto 2005], respectively. (iii) All or some of these conclusions were reached for a number of models, namely the TASEP [Johansson 2000; Borodin et al. 2007; 2008; Sasamoto 2005] and the PASEP [Tracy and Widom 2009], the PNG model [Prähofer and Spohn 2000; 2002; Borodin et al. 2008], and the KPZ equation [Sasamoto and Spohn 2010c; 2010b; Amir et al. 2011; Calabrese et al. 2010; Dotsenko 2010; Calabrese and Le Doussal 2011; Prolhac and Spohn 2011]. They are believed to be universal characteristics of the KPZ class, which is the basic universality class for describing scale-invariant growth of interfaces due to local interactions [Kardar et al. 1986].

This geometry-dependent universality of the KPZ class and the nontrivial connection to random matrix theory were recently made visible by a real experiment [Takeuchi and Sano 2010; 2012; Takeuchi et al. 2011]. Using the electrically driven convection of nematic liquid crystal [de Gennes and Prost 1995], the author and his coworker generated expanding domains of turbulence amidst another turbulent state which is only metastable (Figure 1) and measured the fluctuations of their growing interfaces. Although the growth mechanism in this experiment is far more complicated than the solvable mathematical models — it is realized by proliferation and random transport of topological defects due to local turbulent flow in the electroconvection, such microscopic difference is scaled out in the macroscopic dynamics according to the universality hypothesis. The author then indeed found the aforementioned statistical properties of the KPZ class emerge in the scaling limit [Takeuchi and Sano 2010; 2012; Takeuchi et al. 2011].

The present contribution is devoted to overviewing the current status of this experimental investigation. It summarizes, one by one, those statistical properties which were derived rigorously for the solvable models and confirmed here experimentally (Section 2) and those which were evidenced in the experiment and remain to be explained by mathematical or theoretical studies (Section 3). Note however that, because of the space constraint, the present survey does *not* cover all the experimental results obtained so far; for the complete description of the experimental system and the results, the readers are referred to the recent article by the author and the coworker [Takeuchi and Sano 2012].

2. Experimental realization of analytically solved properties

Scaling exponents. The generated growing interfaces become rougher and rougher as time elapses. In other words, the local height $h(x,t)$ measured along the average growth direction (see Figure 1) is fluctuating in both space and time and the fluctuations grow with time. The roughness can be quantified by, for example, the height-difference correlation function $C_{\mathrm{h}}(l,t) \equiv \langle [h(x+l,t) - h(x,t)]^2 \rangle$

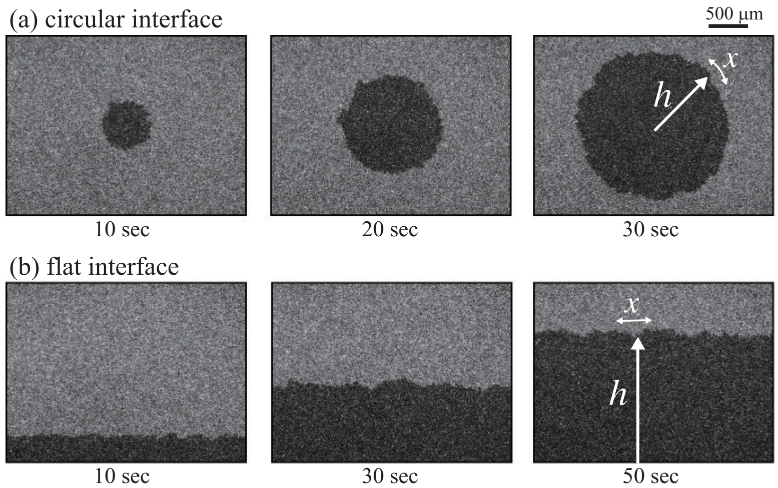

Figure 1. Growing turbulent domain (black) in the liquid-crystal convection, bordered by a circular (a) and flat (b) interface. Indicated below each image is the elapsed time for this growth process. Movies are also available as supplementary information of [Takeuchi et al. 2011]. Such interfaces were generated about a thousand times to evaluate all the statistical properties presented in this article.

with the ensemble average $\langle \cdots \rangle$. It then turned out to obey the following power law called the Family–Vicsek scaling [Family and Vicsek 1985]:

$$C_{\mathrm{h}}(l,t)^{1/2} \sim t^{\beta} F_h(lt^{-1/z}) \sim \begin{cases} l^{\alpha} & \text{for } l \ll l_*, \\ t^{\beta} & \text{for } l \gg l_*, \end{cases} \tag{1}$$

with a scaling function F_h, a crossover length scale $l_* \sim t^{1/z}$, and the KPZ characteristic exponents $\alpha = 1/2$, $\beta = 1/3$, and $z \equiv \alpha/\beta = 3/2$ for $1+1$ dimensions [Kardar et al. 1986]. The same set of the exponents was found for both circular and flat interfaces. This implies that the one-point fluctuations of the local height h can be described, for large t, as

$$h \simeq v_\infty t + (\Gamma t)^{1/3} \chi, \tag{2}$$

with two constant parameters v_∞ and Γ and with a random variable χ that captures the fluctuations of the growing interfaces. The two parameters are related to those of the KPZ equation, $\partial_t h = \nu \partial_x^2 h + (\lambda/2)(\partial_x h)^2 + \sqrt{D}\xi$ with white noise ξ, by $v_\infty = \lambda$ and $\Gamma = A^2\lambda/2$ with $A \equiv D/2\nu$ [Takeuchi and Sano 2012].

Distribution function. The random variable χ in (2) turned out to be identical to the variable $\chi_2 \equiv \chi_{\mathrm{GUE}}$ obeying the GUE Tracy–Widom distribution for the

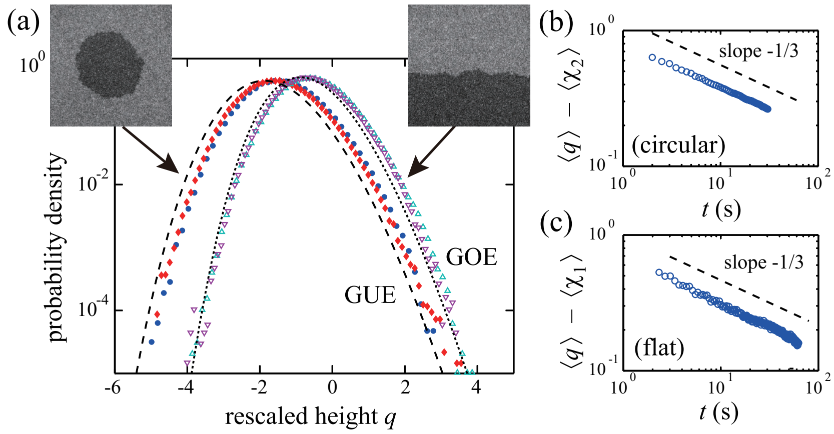

Figure 2. One-point distribution of the rescaled local height $q \equiv (h - v_\infty t)/(\Gamma t)^{1/3}$ for the circular and flat interfaces. (a) Probability density of q measured at different times, $t = 10$ s and 30 s for the circular interfaces (solid symbols) and $t = 20$ s and 60 s for the flat ones (open symbols), from right to left. The dashed and dotted curves show the GUE and GOE Tracy–Widom distributions, respectively (with the factor $2^{-2/3}$ for the latter). (b,c) Finite-time correction in the mean. The figures are reprinted from [Takeuchi and Sano 2012] with adaptations, with kind permission from Springer Science+Business Media.

circular interfaces, and to $\chi_1 \equiv 2^{-2/3}\chi_{\text{GOE}}$ with χ_{GOE} being the GOE Tracy–Widom random variable for the flat interfaces, in the limit $t \to \infty$. This is in agreement with the analytical results for all the above-mentioned solvable models [Kriecherbauer and Krug 2010; Sasamoto and Spohn 2010a; Corwin 2012]. It was shown by plotting histograms of the rescaled height $q \equiv (h - v_\infty t)/(\Gamma t)^{1/3} \simeq \chi$ [Figure 2(a)] using the experimentally measured values of the parameters v_∞ and Γ. The slight horizontal shifts visible in Figure 2(a) are due to finite-time correction in the mean $\langle q \rangle$, which decays by a power law $\langle q \rangle - \langle \chi_i \rangle \sim t^{-1/3}$ [Figure 2(b,c)] with $i = 1$ (flat) or 2 (circular). These finite-time corrections will be revisited in Section 3.

Spatial correlation function. In the solvable case, it is analytically proved that the two-point spatial correlation function

$$C_{\text{s}}(l;t) \equiv \langle h(x+l,t)h(x,t)\rangle - \langle h(x+l,t)\rangle\langle h(x,t)\rangle \tag{3}$$

is given by the covariance of the Airy$_1$ process $\mathcal{A}_1(u)$ for the flat interfaces and the Airy$_2$ process $\mathcal{A}_2(u)$ for the curved ones [Kriecherbauer and Krug 2010;

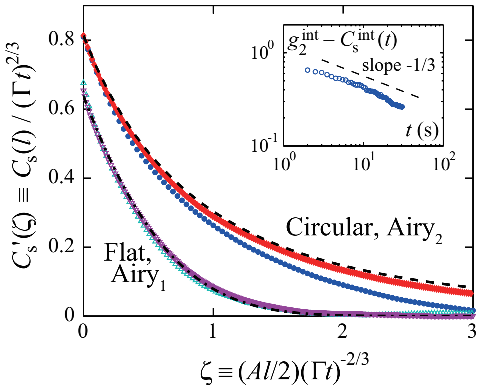

Figure 3. Two-point spatial correlation function $C_s(l; t)$ in the rescaled units. The symbols are the experimental data at $t = 10$ s and 30 s for the circular case and $t = 20$ s and 60 s for the flat one (from bottom to top for each pair). The dashed and dashed-dotted curves indicate the covariance of the Airy$_2$ and Airy$_1$ processes, respectively. The inset shows the finite-time correction for the circular case, expressed in terms of the integral $C_s^{int}(t) \equiv \int_0^\infty C'_s(\zeta; t)\, d\zeta$ and $g_2^{int} \equiv \int_0^\infty g_2(\zeta)\, d\zeta$. The figure is reprinted from [Takeuchi and Sano 2012] with adaptations, with kind permission from Springer Science+Business Media.

Sasamoto and Spohn 2010a; Corwin 2012], through the single expression

$$C_s(l; t) \simeq (\Gamma t)^{2/3} g_i\left(\tfrac{1}{2} Al(\Gamma t)^{-2/3}\right). \tag{4}$$

Here, $g_i(\zeta) \equiv \langle \mathcal{A}_i(u+\zeta)\mathcal{A}_i(u)\rangle - \langle \mathcal{A}_i(u)\rangle^2$ and the Airy processes are normalized to have the same variance as χ_i, $\langle \mathcal{A}_i^2(u)\rangle_c = \langle \chi_i^2\rangle_c$. The experimental data at large times also indicate (4) for both flat and circular interfaces (Figure 3), providing information on finite-time corrections as well.

Extreme-value statistics. Analytical studies of the curved PNG interface, or the related mathematical problems of the vicious walker and the directed polymer, have also successfully solved the asymptotic distribution for the maximal height H_{max} and, very recently, its position X_{max} in this model [Johansson 2003; Moreno Flores et al. 2013; Schehr 2012]. In the rescaled units, the two extremal quantities are described as

$$q_{max}^{(h)} \equiv (H_{max} - v_\infty t)/(\Gamma t)^{1/3} \to \max(\mathcal{A}_2(u) - u^2),$$
$$X'_{max} \equiv (AX_{max}/2)/(\Gamma t)^{2/3} \to \arg\max(\mathcal{A}_2(u) - u^2)$$

in the limit $t \to \infty$. It was proved [Johansson 2003; Moreno Flores et al. 2013; Schehr 2012] in particular that the asymptotic distribution for the rescaled maximal height is given by the GOE Tracy–Widom distribution (with the factor $2^{-2/3}$); in other words, $H_{\max} \simeq v_\infty t + (\Gamma t)^{1/3}\chi$ with χ identical to χ_1. Experimentally, this maximal height must be measured with respect to a fictitious flat substrate that includes the origin of the growing cluster, and hence $H_{\max} = \max(h \sin\phi)$ with the azimuth ϕ. In this way the GOE Tracy–Widom distribution was indeed identified in the experimental data for $H_{\max}$, with finite-time correction $\langle q_{\max}^{(\mathrm{h})} \rangle - \langle \chi_1 \rangle \sim t^{-1/3}$ [Takeuchi and Sano 2012]. The position $X_{\max}$ was also measured correspondingly and in the rescaled unit it was shown to approach the asymptotic analytical solution in [Moreno Flores et al. 2013; Schehr 2012] with increasing time [Takeuchi and Sano 2012].

3. Experimental fact for analytically unsolved properties

Finite-time corrections. The experimental data were obviously obtained at finite times and therefore allow studying the finite-time corrections from the analytical expressions derived in the asymptotic limit. For the one-point distribution, the corrections in the nth-order cumulants were found to be $\langle q^n \rangle_{\mathrm{c}} - \langle \chi_i^n \rangle_{\mathrm{c}} \sim \mathbb{O}(t^{-n/3})$ up to $n = 4$ for both flat and circular cases, except that the second- and fourth-order cumulants for the circular interfaces were too small to identify any systematic variation in time [Takeuchi and Sano 2012]. Although one can show the same exponents $\mathbb{O}(t^{-n/3})$ up to $n = 4$ for the curved exact solution of the KPZ equation [Sasamoto and Spohn 2010c; 2010b; Takeuchi and Sano 2012], for the TASEP, PASEP, and PNG model, only the corrections in the n-th-order *moments* were evaluated: $\mathbb{O}(t^{-1/3})$ for $n = 1$ and $\mathbb{O}(t^{-2/3})$ for $n \geq 2$ [Ferrari and Frings 2011; Baik and Jenkins 2013]. It would be useful to show if the corrections in the *cumulants* are in the order of $\mathbb{O}(t^{-n/3})$ or $\mathbb{O}(t^{-2/3})$ for these solvable models. From the numerical side, circular interfaces of an off-lattice Eden model showed corrections in the order of $\mathbb{O}(t^{-2/3})$ for both first- and second-order cumulants within the time window of the simulation [Takeuchi 2012]. Although one cannot exclude the possibility of crossover to $\mathbb{O}(t^{-1/3})$ for the first-order cumulant, one could also speculate that this leading term is somehow absent in the off-lattice Eden model because of some sort of symmetry. To add, in contrast to the exponents, the coefficients for these finite-time corrections are understood to be model-dependent. Indeed, that for the first-order cumulant, or the mean, $\langle q \rangle - \langle \chi_i \rangle$, is negative for the curved solution of the KPZ equation [Sasamoto and Spohn 2010c; 2010b] but positive for the TASEP [Ferrari and Frings 2011; Baik and Jenkins 2013], the simulation of the off-lattice Eden model [Takeuchi 2012], and the liquid-crystal experiment [Takeuchi and Sano 2012]. See Figure 2(b,c).

Similar data analysis was performed for the maximal height $H_{\max}$ and found, for the mean, the same exponent as for the one-point distribution:

$$\langle q_{\max}^{(h)} \rangle - \langle \chi_1 \rangle \sim \mathbb{O}(t^{-1/3})$$

for the experiment [Takeuchi and Sano 2012] and

$$\langle q_{\max}^{(h)} \rangle - \langle \chi_1 \rangle \sim \mathbb{O}(t^{-2/3})$$

for the off-lattice Eden simulation [Takeuchi 2012]. Concerning the distribution of the position $X_{\max}$, opposite signs of the corrections were found for the second- and fourth-order cumulants between the experiment and the numerically solved PNG droplet [Takeuchi and Sano 2012]. These finite-time distributions of the extremal quantities remain inaccessible by analytic means.

The finite-time corrections were also measured experimentally for the spatial correlation function $C_s(l;t)$ [Takeuchi and Sano 2012]. The corrections were quantified in terms of the integral of the rescaled correlation function, $C_s^{\text{int}}(t) \equiv \int_0^\infty C_s'(\zeta;t)\mathrm{d}\zeta$ with $C_s'(\zeta;t) \equiv C_s(l;t)/(\Gamma t)^{2/3}$ and $\zeta \equiv (Al/2)(\Gamma t)^{-2/3}$. For the circular interfaces, it was shown to approach the value of the Airy$_2$ covariance $g_2^{\text{int}} \equiv \int_0^\infty g_2(\zeta)\mathrm{d}\zeta$ as $g_2^{\text{int}} - C_s^{\text{int}} \sim \mathbb{O}(t^{-1/3})$ [Takeuchi and Sano 2012] (Figure 3 inset). The same exponent was also found numerically in the circular interfaces of the off-lattice Eden model [Takeuchi 2012], though the way the function $C_s'(\zeta;t)$ approaches $g_2(\zeta)$ appears to be different. It is therefore important to have analytical solutions for the spatial correlation function at finite times, which are not yet obtained in a controlled manner in any solvable models.

Spatial persistence probability. Although it is considered that the spatial profile of the growing interfaces itself is given by the corresponding Airy process, to the knowledge of the author, statistical quantities other than the two-point correlation function have not been explicitly calculated in the analytical studies. In other words, measuring such quantities on the spatial correlation of the interfaces can also shed light on the temporal correlation of the Airy processes, as well as that for the largest eigenvalue in Dyson's Brownian motion for GUE random matrices, which is equivalent to the Airy$_2$ process [Johansson 2003].

This strategy was also pursued in the liquid-crystal experiment [Takeuchi and Sano 2012], in which the persistence property of the height fluctuation $\delta h(x,t) \equiv h(x,t) - \langle h \rangle$ was measured. The spatial persistence probability $P_\pm^{(s)}(l;t)$ is defined as the probability that a positive (+) or negative (−) fluctuation continues over length l in a spatial profile of the interfaces at time t. The experimental data then indicated, within the experimental accuracy, exponential decay

$$P_\pm^{(s)}(l;t) \sim \mathrm{e}^{-\kappa_\pm^{(s)}\zeta}, \tag{5}$$

with $\zeta \equiv (Al/2)(\Gamma t)^{-2/3}$ for both flat and circular interfaces [Takeuchi and Sano 2012]. The decay coefficients $\kappa_\pm^{(s)}$ were however different between the two cases:

$$\begin{cases} \kappa_+^{(s)} = 1.9(3) \\ \kappa_-^{(s)} = 2.0(3) \end{cases} \quad \text{(flat)} \qquad \text{and} \qquad \begin{cases} \kappa_+^{(s)} = 1.07(8) \\ \kappa_-^{(s)} = 0.87(6) \end{cases} \quad \text{(circular)}, \tag{6}$$

where the numbers in the parentheses indicate the range of error expected in the last digit of the estimates. The exponential decay (5) in the spatial persistence probability was also identified numerically for the circular interfaces of the off-lattice Eden model, which gave $\kappa_+^{(s)} = 0.90(2)$ and $\kappa_-^{(s)} = 0.89(4)$ [Takeuchi and Sano 2012]. Since a similar set of the coefficients was numerically found in the temporal persistence probability of the GUE Dyson Brownian motion [Takeuchi and Sano 2012], namely $\kappa_+^{(s)} = 0.90(8)$ and $\kappa_-^{(s)} = 0.90(6)$, the author considers that the experimental value of $\kappa_+^{(s)}$ for the circular interfaces is somewhat affected by finite-time effect and/or experimental error. To resolve this issue, it is important to derive a theoretical expression for this persistence probability, whether rigorously or approximatively, and to provide a direct numerical evaluation with the aid of, e.g., Bornemann's method [2010] to estimate the Fredholm determinant numerically.

Temporal correlation. In contrast to the spatial correlation of the interfaces which can be dealt with in terms of the Airy processes, their temporal correlation remains inaccessible in analytical studies. Given that it is also expected to be universal in the scaling limit, explicit information provided by experimental and numerical studies may hint at the form of the solution that should be reached, if reachable, on the temporal correlation of solvable growth models.

The two-point temporal correlation function

$$C_\mathrm{t}(t, t_0) \equiv \langle h(x, t)h(x, t_0)\rangle - \langle h(x, t)\rangle\langle h(x, t_0)\rangle \tag{7}$$

was experimentally measured along the characteristic lines, or in the vertical and radial direction for the flat and circular interfaces, respectively, and turned out to be very different between the two cases [Takeuchi and Sano 2012]. For the flat case, it is governed by the scaling form $C_\mathrm{t}(t, t_0) \simeq (\Gamma^2 t_0 t)^{1/3} F_\mathrm{t}(t/t_0)$ with a scaling function $F_\mathrm{t}(t/t_0) \sim (t/t_0)^{-\bar{\lambda}}$ and $\bar{\lambda} = 1$. In contrast, for the circular case, the raw correlation function $C_\mathrm{t}(t, t_0)$ does *not* decay to zero, presumably even in the limit $t \to \infty$. The author found that the experimental curve for $C_\mathrm{t}(t, t_0)$ at each t_0 is proportional to the functional form obtained by Singha after rough theoretical approximations [Singha 2005]:

$$\frac{C_t(t,t_0)}{C_t(t_0,t_0)} \approx c(t_0) F_{\text{Singha}}(t/t_0; b(t_0)), \quad (t \neq t_0), \tag{8}$$

$$F_{\text{Singha}}(\tau; b) \equiv \frac{e^{b(1-1/\sqrt{\tau})}\Gamma(2/3, b(1-1/\sqrt{\tau}))}{\Gamma(2/3)}, \tag{9}$$

with the upper incomplete Gamma function $\Gamma(s,x)$, the Gamma function $\Gamma(s)$, and unknown parameters $b(t_0)$ and $c(t_0)$ which turned out to depend on t_0 [Takeuchi and Sano 2012]. This functional form also indicates

$$\lim_{t\to\infty} C_t(t,t_0) > 0,$$

as suggested by the experimental data. The ever-lasting temporal correlation in the circular case formally implies $\bar{\lambda} = 1/3$, in contrast with $\bar{\lambda} = 1$ for the flat case. This supports Kallabis and Krug's conjecture [1999] that the autocorrelation exponent $\bar{\lambda}$ derived for the linear growth equations:

$$\bar{\lambda} = \begin{cases} \beta + d/z & \text{(flat)}, \\ \beta & \text{(circular)}, \end{cases} \tag{10}$$

where d is the spatial dimension, also applies to the KPZ universality class. This conjecture, as well as the interesting functional form for the temporal correlation of the circular interfaces, need to be explained on the basis of more refined, hopefully rigorous, theoretical arguments.

The temporal correlation was also characterized in terms of the persistence probability. Along the characteristic lines, the temporal persistence probability $P_\pm(t,t_0)$ is defined as the joint probability that the interface fluctuation $\delta h(x,t)$ is positive (+) or negative (−) at time t_0 and maintains the same sign until time t. Experimentally, it was found to decay algebraically

$$P_\pm(t,t_0) \sim (t/t_0)^{-\theta_\pm} \tag{11}$$

with different sets of the exponents $\theta_\pm$ for the flat and circular interfaces [Takeuchi and Sano 2012]:

$$\begin{cases} \theta_+ = 1.35(5) \\ \theta_- = 1.85(10) \end{cases} \text{(flat)} \quad \text{and} \quad \begin{cases} \theta_+ = 0.81(2) \\ \theta_- = 0.80(2) \end{cases} \text{(circular)}. \tag{12}$$

It is interesting to note that θ_+ and θ_- are asymmetric in the flat case, which had also been reported in numerical work [Kallabis and Krug 1999] and associated with the nonlinearity in the KPZ equation, whereas this asymmetry is somehow canceled for the circular interfaces. This latter statement was also confirmed by the simulation of the off-lattice Eden model, which gave $\theta_+ = 0.81(3)$ and $\theta_- = 0.77(4)$ [Takeuchi 2012]. Theoretical accounts should hopefully be made on how the asymmetry $\theta_+ \neq \theta_-$, which is present in the flat case, is canceled for the circular interfaces.

4. Concluding remarks

We have briefly overviewed the main experimental results obtained for the growing interfaces of the liquid-crystal turbulence [Takeuchi and Sano 2010; 2012; Takeuchi et al. 2011]. On the one hand, this experiment provides an interesting situation where the deep and beautiful mathematical concepts developed in random matrix theory and other domains of mathematical physics arise in a real phenomenon (Section 2). It would be remarkable that we can directly look at the Tracy–Widom distributions and the Airy processes by our eyes, or more precisely by a microscope, all the more because there is no random matrix which explicitly arises in this problem. On the other hand, and more importantly for future developments, such an experimental study allows us to access statistical properties that remain unsolved in the rigorous analytical treatments (Section 3). The author believes that providing proof or theoretical accounts for those unsolved statistical properties in any solvable model will further advance our understanding on the KPZ universality class, as well as in the wide variety of related mathematical fields in this context. Finally, the author would like to refer the interested readers to the article [Takeuchi and Sano 2012], in which one can find much more complete descriptions on the experiment and the results.

Note added in proof

After submission of this article, Ferrari and Frings [2013] derived analytic expressions for the persistence probability of negative fluctuations for the Airy_1 and Airy_2 processes (corresponding to the spatial persistence probability $P_-^{(\mathrm{s})}$ for the interfaces). They numerically evaluated the decay coefficient $\kappa_-^{(\mathrm{s})}$ for the Airy_1 process and found it in agreement with the experimental value. However, in my viewpoint, two problems remain open:

(1) The persistence probability of positive fluctuations remains to be solved.

(2) The present article reported $\kappa_-^{(\mathrm{s})} \approx \kappa_+^{(\mathrm{s})}$ with the sign of the fluctuations defined with respect to the mean value $\langle h \rangle$, while Ferrari and Frings showed that $\kappa_-^{(\mathrm{s})}$ depends continuously on the reference value c used to define the sign: specifically, $\kappa_-^{(\mathrm{s})}$ decreases with increasing c. For $\kappa_+^{(\mathrm{s})}$, one naturally expects that it increases with c. Thus, it is not known whether and why $\kappa_-^{(\mathrm{s})}$ and $\kappa_+^{(\mathrm{s})}$ take the same value at $c = \langle h \rangle$, or they just happen to be close.

Likewise, after submission of the article, Alves et al. [2013] reported results of extensive simulations of the off-lattice Eden model and showed that the peculiar finite-time correction of $\mathbb{O}(t^{-2/3})$ for the mean height of this model is replaced by the usual scaling $\mathbb{O}(t^{-1/3})$ at larger times. Other open problems on finite-time corrections mentioned in Section 3 remain unsolved to the knowledge of the author.

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