

CLT for spectra of submatrices of Wigner random matrices, II: Stochastic evolution

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We show that the global fluctuations of spectra of GOE and GUE matrices and their principal submatrices executing Dyson's Brownian motion are Gaussian in the limit of large matrix dimensions. For nested submatrices one obtains a limiting three-dimensional generalized Gaussian process; its restrictions to two-dimensional sections that are monotone in matrix sizes and time moments coincide with the two-dimensional Gaussian free field with zero boundary conditions. The proof is by moment convergence, and it extends to more general Wigner matrices and their stochastic evolution.

Introduction. The fact that the global spectral fluctuations of a GOE or a GUE random matrix evolving under Dyson's Brownian motion, are asymptotically Gaussian is well-known; see [Anderson et al. 2010, Section 4.3.3] and references therein, and also [Spohn 1998] for a general β analog. On the other hand, it was shown in [Borodin 2014] that the global fluctuations of spectra of various principal submatrices of a single GOE or GUE matrix are also Gaussian. The goal of this article is to put these two statements together.

We prove the asymptotic Gaussian behavior for submatrices of a class of stochastically evolving Wigner random matrices that includes Dyson's Brownian motion for GOE and GUE. The proof is by the method of moments, and the argument is slightly more general than the one presented in [Anderson et al. 2010] for a single Wigner matrix.

We also compute the resulting covariance kernel explicitly. In the case of nesting submatrices, it represents a three-dimensional generalized Gaussian process, where one dimension comes from the position of the spectral variable, the second dimension reflects the size of the submatrix, and the third dimension is the time variable. When restricted to the two-dimensional sections that are monotone in matrix size and time variables, it reproduces the two-dimensional Gaussian free field (GFF) with zero boundary conditions.

In the case of GUE, the appearance of GFF on monotone sections could have been predicted from the determinantal structure of the correlation functions

[Ferrari and Frings 2010; Adler et al. 2010b], and from the analysis of [Borodin and Ferrari 2014] that showed how such a structure leads to GFF covariances in the global asymptotic regime. However, the complete three-dimensional covariance structure seems to be inaccessible via that approach for example because the spectra of the full set of submatrices evolve in a non-Markovian way [Adler et al. 2010a].

Wigner matrices. Let $\{Z_{ij}(t)\}_{j>i\geq 1, t\in\mathbb{R}}$ and $\{Y_i(t)\}_{i\geq 1, t\in\mathbb{R}}$ be two families of independent identically distributed real-valued stochastic (not necessarily Markov) processes with zero mean such that for any $k \geq 1$,

$$\max_{t\in\mathbb{R}}(\mathbb{E}|Z_{12}(t)|^k, \mathbb{E}|Y_1(t)|^k) < \infty.$$

Set $c(s, t) = \frac{1}{2}\mathbb{E}Y_1(s)Y_1(t)$ and assume that

$$\begin{aligned} c(s, t) &\geq 0, & c(t, t) &\equiv 1, \\ \mathbb{E}Z_{12}(s)Z_{12}(t) &\equiv c(s, t), & \mathbb{E}Z_{12}^2(s)Z_{12}^2(t) &\equiv 2c(s, t)^2 + 1. \end{aligned}$$

Note that by Cauchy's inequality $c(s, t) \leq \sqrt{c(s, s)c(t, t)} = 1$. We say that a function $c(s, t)$ is admissible if it arises in this way.

One possibility for the above relations to be satisfied is to take all $\{2^{-\frac{1}{2}}Y_i(t)\}$ and $\{Z_{ij}(t)\}$ to be independent standard Ornstein–Uhlenbeck processes on \mathbb{R} ; then $c(s, t) = \exp(-|s - t|)$. We will refer to this possibility as to *Gaussian specialization*.

Define a *real symmetric time-dependent Wigner matrix* $X(t) = [X(i, j | t)]_{i, j \geq 1}$ by

$$X(i, j | t) = X(j, i | t) = \begin{cases} Z_{ij}(t), & i < j, \\ Y_i, & i = j. \end{cases}$$

A Hermitian variation of the same definition is as follows: Let $\{Z_{ij}\}_{j>i\geq 1}$ now be complex-valued (i.i.d., zero-mean) stochastic processes with the same uniform bound on all moments. Denote $c(s, t) = \mathbb{E}Y_1(s)Y_1(t)$ and assume that

$$\begin{aligned} c(s, t) &\geq 0, & c(t, t) &\equiv 1, \\ \mathbb{E}Z_{12}(s)Z_{12}(t) &\equiv 0, & \mathbb{E}Z_{12}(s)\overline{Z_{12}(t)} &\equiv c(s, t), \\ \mathbb{E}|Z_{12}(s)|^2|Z_{12}(t)|^2 &\equiv c(s, t)^2 + 1. \end{aligned}$$

We will also say that a function $c(s, t)$ is admissible if it arises in this way.

There is also a Gaussian specialization that corresponds to $\{Y_i(t)\}$ and

$$\{2^{\frac{1}{2}}\Re Z_{ij}(t)\}, \{2^{\frac{1}{2}}\Im Z_{ij}(t)\}$$

being independent standard Ornstein–Uhlenbeck processes on \mathbb{R} ; in that case $c(s, t) = \exp(-|s - t|)$.

Define a *Hermitian time-dependent Wigner matrix* $X(t) = [X(i, j | t)]_{i, j \geq 1}$ by

$$X(i, j | t) = \overline{X(j, i | t)} = \begin{cases} Z_{ij}(t), & i < j, \\ Y_i, & i = j. \end{cases}$$

Under the Gaussian specializations, the matrix stochastic processes defined above are called *Dyson's Brownian motions*. Traditionally one distinguishes the two cases by a parameter β that takes value 1 in the real symmetric case and value 2 in the Hermitian case. The random matrices arising at a single time moment are said to belong to the Gaussian orthogonal ensemble (GOE) in the $\beta = 1$ case, and Gaussian unitary ensemble (GUE) in the $\beta = 2$ case.

The height function. For any finite set $B \subset \{1, 2, \dots\}$ denote by X_B the $|B| \times |B|$ submatrix of a matrix X formed by the intersections of the rows and columns of X marked by elements of B .

The *height function* H associated to a time-dependent Wigner matrix X is a random integer-valued function on $\mathbb{R} \times \mathbb{R}_{\geq 1} \times \mathbb{R}$ defined by

$$H(x, y, t) = \sqrt{\frac{\beta\pi}{2}} \left\{ \text{the number of eigenvalues of } X_{\{1, 2, \dots, [y]\}}(t) \text{ that are } \geq x \right\}.$$

More generally, let $A = \{a_n\}_{n \geq 1}$ be an arbitrary sequence of pairwise distinct natural numbers. Then we define the height function H_A via

$$H_A(x, y) = \sqrt{\frac{\beta\pi}{2}} \left\{ \text{the number of eigenvalues of } X_{\{a_1, \dots, a_{[y]}\}}(t) \text{ that are } \geq x \right\}.$$

The first definition corresponds to $A = \mathbb{N}$.

The convenience of the constant prefactor $\sqrt{\beta\pi/2}$ will be evident shortly.

A three-dimensional Gaussian field. Let $c(s, t)$ be an admissible function as defined above. Set $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$ and introduce a function

$$C : (\mathbb{H} \times \mathbb{R}) \times (\mathbb{H} \times \mathbb{R}) \rightarrow \mathbb{R} \cup \{+\infty\}$$

via

$$\begin{aligned} C(z, s; w, t) &= \frac{1}{2\pi} \ln \left| \frac{c(s, t) \min(|z|^2, |w|^2) - zw}{c(s, t) \min(|z|^2, |w|^2) - z\bar{w}} \right| \\ &= \begin{cases} -\frac{1}{2\pi} \ln \left| \frac{c(s, t)z - w}{c(s, t)z - \bar{w}} \right|, & |z| \leq |w|, \\ -\frac{1}{2\pi} \ln \left| \frac{c(s, t)w - z}{c(s, t)w - \bar{z}} \right|, & |z| > |w|. \end{cases} \end{aligned}$$

It is easy to see that for any (s, t) with $c(s, t) < 1$, $C(\cdot, s; \cdot, t)$ is a continuous function on $\mathbb{H} \times \mathbb{H}$. Note also that if $c(s, t) = 1$ then

$$C(z, s; w, t) = -\frac{1}{2\pi} \ln \left| \frac{z-w}{z-\bar{w}} \right|$$

is the Green function for the Laplace operator on \mathbb{H} with Dirichlet boundary conditions. Viewed as a function in (z, w) , it represents the covariance for the two-dimensional Gaussian free field on \mathbb{H} with zero boundary conditions.

Proposition 1. *For any admissible function $c(s, t)$ as above, there exists a generalized Gaussian process on $\mathbb{H} \times \mathbb{R}$ with the covariance kernel $C(z, s; w, t)$ as above. More exactly, for any finite family of test functions $f_m(z) \in C_0(\mathbb{H} \times \mathbb{R})$ the covariance matrix*

$$\text{cov}(f_k, f_l) = \int_{\mathbb{H} \times \mathbb{R}} \int_{\mathbb{H} \times \mathbb{R}} f_k(z, s) f_l(w, t) C(z, s; w, t) dz d\bar{z} ds dw d\bar{w} dt, \\ k, l = 1, \dots, M,$$

is positive definite.

Denote the resulting generalized Gaussian process by $\mathcal{G}_{c(s,t)}$.

A proof of Proposition 1 will be given later.

Complex structure. Let A be a sequence of pairwise distinct integers. The height function $H_A(x, y, t)$ (or $H(x, y, t) = H_{\mathbb{N}}(x, y, t)$) is naturally defined on $\mathbb{R} \times \mathbb{R}_{\geq 1} \times \mathbb{R}$. Having the large parameter L , we would like to scale $(x, y) \mapsto (L^{-\frac{1}{2}}x, L^{-1}y)$, which lands us in $\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}$.

Wigner's semicircle law implies that for any $t \in \mathbb{R}$, with $L \gg 1$, $x \sim L^{\frac{1}{2}}$, $y \sim L$, after rescaling with overwhelming probability the eigenvalues (or, equivalently, the places of growth of the height function in x -direction) are concentrated in the domain

$$\{(x, y) \in \mathbb{R} \times \mathbb{R}_{>0} \mid -2\sqrt{y} \leq x \leq 2\sqrt{y}\}.$$

Let us identify the interior of this domain with \mathbb{H} via the map

$$\Omega : (x, y) \mapsto \frac{x}{2} + i\sqrt{y - \left(\frac{x}{2}\right)^2}.$$

Its inverse has the form

$$\Omega^{-1}(z) = (x(z), y(z)) = (2\Re(z), |z|^2).$$

Note that this map sends the boundary of the domain to the real line.

Thanks to Ω we can now speak of the height function H_A as being defined on $\mathbb{H} \times \mathbb{R}$; we will use the notation

$$H_A^\Omega(z; t) = H_A(L^{\frac{1}{2}}x(z), Ly(z), t), \quad z \in \mathbb{H}.$$

Note that we have incorporated rescaling in this definition.

Main result. Let X be a (real symmetric or Hermitian) time-dependent Wigner matrix. We argue that the centralized random height function

$$H^\Omega(z; t) - \mathbb{E}H^\Omega(z; t), \quad z \in \mathbb{H}, t \in \mathbb{R},$$

viewed as distribution, converges as $L \rightarrow \infty$ to the generalized Gaussian process $\mathcal{G}_{c(s,t)}$ with $c(s, t) = \frac{1}{2}\beta \mathbb{E}Y_1(s)Y_1(t)$.

One needs to verify the convergence on a suitable set of test functions. The exact statement that we prove is the following.

Theorem 2. Pick $\tau \in \mathbb{R}$, $y > 0$, and $k \in \mathbb{Z}_{\geq 0}$. Define a moment of the random height function by

$$M_{\tau,y,k} = \int_{-\infty}^{+\infty} x^k (H(L^{\frac{1}{2}}x, Ly, \tau) - \mathbb{E}H(L^{\frac{1}{2}}x, Ly, \tau)) dx.$$

Then as $L \rightarrow \infty$, these moments converge, in the sense of finite dimensional distributions, to the moments of $\mathcal{G}_{c(s,t)}$ defined as

$$\mathcal{M}_{\tau,y,k} = \int_{\substack{z \in \mathbb{H} \\ |z|^2=y}} (x(z))^k \mathcal{G}_{c(s,t)}(z; \tau) \frac{dx(z)}{dz} dz.$$

Monotone sections as two-dimensional Gaussian free fields. Consider a time-dependent Wigner matrix and assume that the function $c(s, t) = \frac{1}{2}\beta \mathbb{E}Y_1(s)Y_1(t)$ is continuous and that it has the following monotonicity property: For any $s \in \mathbb{R}$, $c(s, t)$ is strictly increasing in $t \in (-\infty, s]$ and it is strictly decreasing in $t \in [s, +\infty)$. In other words, as time distance between matrices grows, the correlation decays. Further, assume that $c(s, t) \neq 0$ for any $s, t \in \mathbb{R}$.

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous nonincreasing and a continuous nondecreasing functions, and assume that for at least one of these functions the monotonicity is strict.

Our goal is to consider the joint fluctuations of spectra of matrices

$$X_{\{1,2,\dots,[L\phi(t)]\}}(\psi(t)), \quad t \in \mathbb{R}, \quad (1)$$

where $L \gg 1$ is a large parameter. By Wigner's semicircle law, the spectrum of such a matrix scaled by $L^{\frac{1}{2}}$ is concentrated on $[-2\sqrt{\phi(t)}, 2\sqrt{\phi(t)}]$.

The two extreme cases are $\phi(t) \equiv \text{const}$ (the size of the matrices is fixed and the time is moving) and $\psi(t) \equiv \text{const}$ (the time moment is fixed and the size of the matrices is changing).

Let us choose a reference time moment $t_0 \in \mathbb{R}$ and introduce a map

$$\Xi : \{(x, t) \in \mathbb{R} \times \mathbb{R} \mid -2\sqrt{\phi(t)} < x < 2\sqrt{\phi(t)}\} \rightarrow \mathbb{H}$$

as

$$\Xi(x, t) = \begin{cases} c(\psi(t_0), \psi(t)) \left(\frac{x}{2} + i\sqrt{\phi(t) - \left(\frac{x}{2}\right)^2} \right), & t \geq t_0, \\ \frac{1}{c(\psi(t), \psi(t_0))} \left(\frac{x}{2} + i\sqrt{\phi(t) - \left(\frac{x}{2}\right)^2} \right), & t < t_0. \end{cases}$$

The continuity and monotonicity assumptions on c , ϕ , and ψ are needed for Ξ to be a bijection. Hence, its inverse is correctly defined, denote it as $\Xi^{-1}(\zeta) = (x(\zeta), t(\zeta))$.

We can now view the height function H for matrices (1) as a function on \mathbb{H} via

$$H^\Xi(\zeta) = H(L^{\frac{1}{2}}x(\zeta), L\phi(t(\zeta)), \psi(t(\zeta))).$$

Our main result implies that the centralized height function

$$H^\Xi(\zeta) - \mathbb{E}H^\Xi(\zeta), \quad \zeta \in \mathbb{H},$$

viewed as a distribution, converges as $L \rightarrow \infty$ to the Gaussian free field on \mathbb{H} in the sense of Theorem 2.

Moments as traces. Let us rescale the variable $x = L^{-\frac{1}{2}}u$ in the definition of $M_{\tau, y, k}$ and then integrate by parts. Since the derivative of the height function $H(u, [Ly], t)$ in u is

$$\frac{d}{du}H(u, [Ly], t) = -\sqrt{\frac{\beta\pi}{2}} \sum_{s=1}^{[Ly]} \delta(u - \lambda_s),$$

where $\{\lambda_s\}_{1 \leq s \leq [Ly]}$ are the eigenvalues of $X_{\{1, \dots, [Ly]\}}(t)$, we obtain

$$\begin{aligned} M_{\tau, y, k} &= L^{-\frac{k+1}{2}} \sqrt{\frac{\beta\pi}{2}} \left(\sum_{s=1}^{[Ly]} \frac{\lambda_s^{k+1}}{k+1} - \mathbb{E} \sum_{s=1}^{[Ly]} \frac{\lambda_s^{k+1}}{k+1} \right) \\ &= \frac{L^{-\frac{k+1}{2}}}{k+1} \sqrt{\frac{\beta\pi}{2}} \left(\text{Tr}(X_{\{1, \dots, [Ly]\}}^{k+1}(t)) - \mathbb{E} \text{Tr}(X_{\{1, \dots, [Ly]\}}^{k+1}(t)) \right). \end{aligned}$$

We can now reformulate the statement of Theorem 2 as follows.

Theorem 2'. Let $X(t)$ be a time-dependent (real symmetric or Hermitian) Wigner matrix with

$$c(s, t) = \frac{\beta}{2} \mathbb{E}Y_1(s)Y_1(t).$$

Let $k_1, \dots, k_m \geq 1$ be integers, t_1, \dots, t_m real numbers, and y_1, \dots, y_m positive real numbers. The m -dimensional random vector

$$\left(L^{-\frac{k_p}{2}} \left(\text{Tr}(X_{\{1, \dots, [Ly_p]\}}^{k_p}(t_p)) - \mathbb{E} \text{Tr}(X_{\{1, \dots, [Ly_p]\}}^{k_p}(t_p)) \right) \right)_{p=1}^m$$

converges (in distribution and with all moments) to the zero mean m -dimensional Gaussian random vector $(\xi_p)_{p=1}^m$ with covariance

$$\begin{aligned} \mathbb{E}\xi_p\xi_q &= \frac{2k_pk_q}{\beta\pi} \oint_{\substack{|z|^2=b_p \\ \Im z>0}} \oint_{\substack{|w|^2=b_q \\ \Im w>0}} (x(z))^{k_p-1}(x(w))^{k_q-1} \\ &\quad \times \frac{1}{2\pi} \ln \left| \frac{c(t_p, t_q) \min(y_p, y_q) - zw}{c(t_p, t_q) \min(y_p, y_q) - z\bar{w}} \right| \frac{dx(z)}{dz} \frac{dx(w)}{dw} dz dw. \end{aligned}$$

More general submatrices. In the spirit of [Borodin 2014], we will actually prove a more general claim that involves arbitrary sequences of symmetric submatrices of the Wigner matrix that are sufficiently well-behaved. The exact statement is as follows.

Theorem 2''. *Let $X(t)$ be a time-dependent (real symmetric or Hermitian) Wigner matrix with*

$$c(s, t) = \frac{\beta}{2} \mathbb{E}Y_1(s)Y_1(t).$$

Let $k_1, \dots, k_m \geq 1$ be integers, $t_1, \dots, t_m \in \mathbb{R}$, and let B_1, \dots, B_m be subsets of \mathbb{N} dependent on the large parameter L so that there exist limits

$$b_p = \lim_{L \rightarrow \infty} \frac{|B_p|}{L} > 0, \quad b_{pq} = \lim_{L \rightarrow \infty} \frac{|B_p \cap B_q|}{L}, \quad p, q = 1, \dots, m.$$

Then the m -dimensional random vector

$$\left(L^{-\frac{k_p}{2}} (\text{Tr}(X_{B_p}^{k_p}(t_p)) - \mathbb{E} \text{Tr}(X_{B_p}^{k_p}(t_p))) \right)_{p=1}^m \quad (2)$$

converges (in distribution and with all moments) to the zero mean m -dimensional Gaussian random variable $(\xi_p)_{p=1}^m$ with covariance

$$\begin{aligned} \mathbb{E}\xi_p\xi_q &= \frac{2k_pk_q}{\beta\pi} \oint_{\substack{|z|^2=b_p \\ \Im z>0}} \oint_{\substack{|w|^2=b_q \\ \Im w>0}} (x(z))^{k_p-1}(x(w))^{k_q-1} \\ &\quad \times \frac{1}{2\pi} \ln \left| \frac{c(t_p, t_q)b_{pq} - zw}{c(t_p, t_q)b_{pq} - z\bar{w}} \right| \frac{dx(z)}{dz} \frac{dx(w)}{dw} dz dw. \quad (3) \end{aligned}$$

Theorem 2'' can also be viewed as the moment convergence of the centralized height function $H_A(x, y, t)$ to a limiting generalized Gaussian process but we do not give further details here. The static variant of this convergence is discussed in [Borodin 2014].

Proof of Theorem 2''. The argument closely follows that given in Section 2.1.7 of [Anderson et al. 2010] in the case of one set $B_j \equiv B$, and the proof of Theorem 2' in [Borodin 2014] in the static case. One proves the convergence of moments,

which is sufficient to also claim the convergence in distribution for Gaussian limits.

Any joint moment of the coordinates of (2) is written as a finite combination of contributions corresponding to suitably defined graphs that are in their turn associated to words. This reduction is explained in Section 2.1.7 of [Anderson et al. 2010]. The key fact in the real symmetric case is that averages of products of powers of matrix elements that involve at least one matrix element with exponent 1 vanish. The time-dependent analog of this fact is that averages of products of powers of matrix elements taken at different time moments that involve one matrix element with exponent 1 *at only one time moment* vanish. This clearly holds by independence of matrix elements and our zero mean assumption. In the static Hermitian case, one needs in addition that $\mathbb{E}Z_{12}^2 = 0$. The time-dependent analog reads $\mathbb{E}Z_{12}(s)Z_{12}(t) = 0$ for any $s, t \in \mathbb{R}$, which is one of our assumptions. This allows the exact same reduction to go through in the time-dependent setting.

The only difference of the multiset case from the one-set case is that one needs to keep track of the *alphabets* the words are built from: A word corresponding to coordinate number p of (2) would have to be built from the alphabet that coincides with the set B_p . Equivalently, the corresponding graphs will have their vertices labeled by elements of B_p .

Since all sizes $|B_p|$ have order L , and $|B_1 \cup \dots \cup B_m| = O(L)$, and also the moments of matrix elements at all times are uniformly bounded, the estimate showing that all contributions not coming from matchings are negligible in [Anderson et al. 2010, Lemma 2.1.34] carries over without difficulty. It only remains to compute the covariance.

We start with the case of the real symmetric Wigner matrices.

In the one-set case, the limits of the variances of the coordinates of (2) are given by (2.1.44) in [Anderson et al. 2010]. It reads (with $k = k_p$ for a p between 1 and m)

$$2k^2 C_{\frac{k-1}{2}}^2 + k^2 C_{\frac{k}{2}}^2 + \sum_{r=3}^{\infty} \frac{2k^2}{r} \left(\sum_{\substack{k_i \geq 0 \\ \sum_{i=1}^r k_i = k-r}} \prod_{i=1}^r C_{k_i} \right)^2, \quad (4)$$

where $\{C_k\}_{k \geq 0}$ are the Catalan numbers, and we set $C_a = 0$ if $a \notin \{0, 1, 2, \dots\}$. The Catalan number C_k counts the number of rooted planar trees with k edges, and different terms of (4) have the following interpretation (see [Anderson et al. 2010] for detailed explanations):

- The first term comes from two trees with $(k-1)/2$ edges each that hang from a common vertex; the factor k^2 originates from choices of certain starting points on each tree united with the common vertex, and the extra 2 is actually $\mathbb{E}Y_1^2$.

- The second term comes from two trees with $k/2$ edges each that are glued along one edge. There are $k/2$ choices of this edge for each of the trees, there is an additional $2 = \mathbb{E}Z_{12}^4 - 1$, and another additional 2 responsible of the choice of the orientation of the gluing.
- The third term comes from two graphs each of which is a cycle of length r with pendant trees hanging off each of the vertices of the cycle; the total number of edges in the extra trees being $(k-r)/2$ (this must be an integer). As for the first term, there is an extra $k^2 = k \cdot k$ coming from the choice of the starting points and also an extra 2 for the choice of the gluing orientation along the cycle.

For each of the three terms the total number of vertices in the resulting graph is equal to k , and if one labels each vertex with a letter from an alphabet of cardinality $|B|$ this would yield a factor of

$$|B|(|B| - 1) \cdots (|B| - k + 1) = |B|^k + O(|B|^{k-1}).$$

Normalization by $|B|^k$ yields (4).

In the general case, in order to evaluate the covariance

$$L^{-\frac{k_p+k_q}{2}} \mathbb{E}[(\text{Tr}(X_{B_p}^{k_p}(t_p)) - \mathbb{E} \text{Tr}(X_{B_p}^{k_p}(t_p)))(\text{Tr}(X_{B_q}^{k_q}(t_q)) - \mathbb{E} \text{Tr}(X_{B_q}^{k_q}(t_q)))] \quad (5)$$

in the limit, we need to employ the same graph counting, except for the two graphs being glued now correspond to different values k_p and k_q of k , and their vertices are marked by letters of different alphabets B_p and B_q .

- The first term gives $2k_p k_q C_{\frac{k_p-1}{2}} C_{\frac{k_q-1}{2}}$ for the graph counting, and an extra

$$|B_p \cap B_q| \cdot (|B_p| - 1)(|B_p| - 2) \cdots (|B_p| - \frac{1}{2}(k_p + 1)) \\ \cdot (|B_q| - 1)(|B_q| - 2) \cdots (|B_q| - \frac{1}{2}(k_q + 1))$$

for the vertex labeling (the factor $|B_p \cap B_q|$ comes from the only common vertex). Moreover, $\mathbb{E}Y_1^2$ is replaced by $\mathbb{E}Y_1(t_p)Y_1(t_q) = c(t_p, t_q)$. Normalized by $L^{-\frac{k_p+k_q}{2}}$ this yields

$$2k_p k_q C_{\frac{k_p-1}{2}} C_{\frac{k_q-1}{2}} c(t_p, t_q) b_{pq} b_p^{\frac{k_p-1}{2}} b_q^{\frac{k_q-1}{2}}.$$

- The second term has

$$k_p k_q C_{\frac{k_p}{2}} C_{\frac{k_q}{2}}$$

from the graph counting and

$$c_{pq}^2 b_p^{\frac{k_p}{2}-1} b_q^{\frac{k_q}{2}-1}$$

from the label counting. In addition, $\mathbb{E}Z_{12}^4 - 1$ is replaced by

$$\mathbb{E}Z_{12}^2(t_p)Z_{12}^2(t_q) - 1 = 2c^2(t_p, t_q).$$

The total contribution is

$$k_p k_q C_{\frac{k_p}{2}} C_{\frac{k_q}{2}} (c(t_p, t_q) b_{pq})^2 b_p^{\frac{k_p}{2}-1} b_q^{\frac{k_q}{2}-1}.$$

• For the third term in the same way we obtain

$$\sum_{r=3}^{\infty} \frac{2k_p k_q}{r} \left(\sum_{\substack{s_i \geq 0 \\ 2 \sum_{i=1}^r s_i = k_p - r}} \prod_{i=1}^r C_{s_i} \right) \left(\sum_{\substack{t_i \geq 0 \\ 2 \sum_{i=1}^r t_i = k_q - r}} \prod_{i=1}^r C_{t_i} \right) (c(t_p, t_q) b_{pq})^r b_p^{\frac{k_p-r}{2}} b_q^{\frac{k_q-r}{2}}$$

where $c^r(t_p, t_q)$ appeared as $(\mathbb{E}Z_{12}(t_p)Z_{12}(t_q))^r$, which in its turn came from the edges of the r -cycle.

Thus, the asymptotic value of the covariance (5) is

$$\begin{aligned} & 2k_p k_q C_{\frac{k_p-1}{2}} C_{\frac{k_q-1}{2}} (c(t_p, t_q) b_{pq}) b_p^{\frac{k_p-1}{2}} b_q^{\frac{k_q-1}{2}} \\ & + k_p k_q C_{\frac{k_p}{2}} C_{\frac{k_q}{2}} (c(t_p, t_q) b_{pq})^2 b_p^{\frac{k_p}{2}-1} b_q^{\frac{k_q}{2}-1} \\ & + \sum_{r=3}^{\infty} \frac{2k_p k_q}{r} \left(\sum_{\substack{s_i \geq 0 \\ 2 \sum_{i=1}^r s_i = k_p - r}} \prod_{i=1}^r C_{s_i} \right) \left(\sum_{\substack{t_i \geq 0 \\ 2 \sum_{i=1}^r t_i = k_q - r}} \prod_{i=1}^r C_{t_i} \right) (c(t_p, t_q) b_{pq})^r b_p^{\frac{k_p-r}{2}} b_q^{\frac{k_q-r}{2}}. \end{aligned}$$

We now use the fact that for any $S = 0, 1, 2, \dots$,

$$\sum_{\substack{s_i \geq 0 \\ \sum_{i=1}^r s_i = S}} \prod_{i=1}^r C_{s_i} = \binom{2S+r}{S} \frac{r}{2S+r};$$

see [Graham et al. 1989, (5.70)]. This allows us to rewrite the asymptotic covariance in terms of binomial coefficients: $\text{interdisplaylinepenalty0}$

$$\begin{aligned} & 2 \binom{k_p}{(k_p-1)/2} \binom{k_q}{(k_q-1)/2} (c(t_p, t_q) b_{pq}) b_p^{\frac{k_p-1}{2}} b_q^{\frac{k_q-1}{2}} \\ & + 4 \binom{k_p}{k_p/2-1} \binom{k_q}{k_q/2-1} (c(t_p, t_q) b_{pq})^2 b_p^{\frac{k_p-2}{2}} b_q^{\frac{k_q-2}{2}} \\ & + \sum_{r=3}^{\infty} 2r \binom{k_p}{(k_p-r)/2} \binom{k_q}{(k_q-r)/2} (c(t_p, t_q) b_{pq})^r b_p^{\frac{k_p-r}{2}} b_q^{\frac{k_q-r}{2}} \\ & = \sum_{r=1}^{\infty} 2r \binom{k_p}{(k_p-r)/2} \binom{k_q}{(k_q-r)/2} (c(t_p, t_q) b_{pq})^r b_p^{\frac{k_p-r}{2}} b_q^{\frac{k_q-r}{2}}. \end{aligned}$$

Using the binomial theorem, we can write this expression as a double contour integral

$$\frac{2}{(2\pi i)^2} \iint_{K_1=|z| < |w|=K_2} \left(z + \frac{b_p}{z}\right)^{k_p} \left(w + \frac{b_q}{w}\right)^{k_q} \frac{c(t_p, t_q) b_{pq}}{b_p} \frac{dz dw}{\left(\frac{c(t_p, t_q) b_{pq}}{b_p} z - w\right)^2}, \quad (6)$$

with K_1, K_2 fixed. Note that $c(t_p, t_q) b_{pq}/b_p < 1$, so $(c(t_p, t_q) b_{pq} z/b_p - w)^{-2}$ on our domain of integration can be expanded using a derivative of the geometric series.

Consider the right-hand side of (3) and assume that $|z|^2 = b_p < b_q = |w|^2$. Observe that

$$\begin{aligned} & 2 \ln \left| \frac{c(t_p, t_q) b_{pq} - zw}{c(t_p, t_q) b_{pq} - z\bar{w}} \right| \\ &= -2 \ln \left| \frac{\frac{c(t_p, t_q) b_{pq}}{b_p} \bar{z} - w}{\frac{c(t_p, t_q) b_{pq}}{b_p} \bar{z} - \bar{w}} \right| \\ &= -\ln \left(\frac{c(t_p, t_q) b_{pq}}{b_p} z - w \right) + \ln \left(\frac{c(t_p, t_q) b_{pq}}{b_p} z - \bar{w} \right) \\ &\quad + \ln \left(\frac{c(t_p, t_q) b_{pq}}{b_p} \bar{z} - w \right) - \ln \left(\frac{c(t_p, t_q) b_{pq}}{b_p} \bar{z} - \bar{w} \right). \end{aligned}$$

This allows us to rewrite the right-hand side of (3) as a double contour integral over complete circles in the form

$$-\frac{k_p k_q}{2\beta\pi^2} \oint_{|z|^2=b_p} \oint_{|w|^2=b_q} (x(z))^{k_p-1} (x(w))^{k_q-1} \ln \left(\frac{c(t_p, t_q) b_{pq}}{b_p} z - w \right) \cdot \frac{dx(z)}{dz} \frac{dx(w)}{dw} dz dw.$$

Recalling that $\beta = 1$ and noting that

$$k_p (x(z))^{k_p-1} \frac{dx(z)}{dz} = \frac{d(x(z))^{k_p}}{dz}, \quad k_q (x(w))^{k_q-1} \frac{dx(w)}{dw} = \frac{d(x(w))^{k_q}}{dw},$$

we integrate by parts in z and w and recover (6). The proof for $b_p = b_q$ is obtained by continuity of both sides, and to see that the needed identity holds for $b_p > b_q$ it suffices to observe that both sides are symmetric in p and q .

The argument in the case of Hermitian Wigner matrices is exactly the same, except in the combinatorial part for the first term the factor 2 is missing due to the change in $\mathbb{E}Y_1(s)Y_1(t)$, in the second term 2 is missing due to the change in $\mathbb{E}|Z_{12}(s)|^2|Z_{12}(t)|^2$, and in the third term 2 is missing because there is no choice in the orientation of two r -cycles that are being glued together. \square

Proof of Proposition 1. We need to show that for any complex numbers $\{u_k\}_{k=1}^M$,

$$\sum_{k,l=1}^M u_k \bar{u}_l \int_{\mathbb{H} \times \mathbb{R}} \int_{\mathbb{H} \times \mathbb{R}} f_k(z, s) f_l(w, t) C(z, s; w, t) dz d\bar{z} ds dw d\bar{w} dt \geq 0.$$

We can approximate the integration over the three-dimensional domains by finite sums of one-dimensional integrals over semicircles of the form $|z| = \text{const}$, $s = \text{const}$. On each semicircle we further uniformly approximate the (continuous) integrand by a polynomial in $\Re(z)$. Finally, for the polynomials the nonnegativity follows from Theorem 2'. \square

Chebyshev polynomials. One way to describe the limiting covariance structure in the one-matrix static case is to show that traces of the Chebyshev polynomials of the matrix are asymptotically independent; see [Johansson 1998]. A similar effect takes place for time-dependent submatrices as well.

For $n = 0, 1, 2, \dots$, let $T_n(x)$ be the n -th degree Chebyshev polynomial of the first kind:

$$T_n(x) = \cos(n \arccos x),$$

or equivalently,

$$T_n(\cos x) = \cos(nx).$$

For any $a > 0$, let $T_n^a(x) = T_n(x/a)$ be the rescaled version of T_n .

Proposition 5. *In the assumptions of Theorem 2'', for any $p, q = 1, \dots, m$*

$$\begin{aligned} \lim_{L \rightarrow \infty} \mathbb{E} \left[\left(\text{Tr}(T_{k_p}^{2\sqrt{b_p L^{k_p}}}(X_{B_p}(t_p))) - \mathbb{E} \text{Tr}(T_{k_p}^{2\sqrt{b_p L^{k_p}}}(X_{B_p}(t_p))) \right) \right. \\ \left. \times \left(\text{Tr}(T_{k_q}^{2\sqrt{b_q L^{k_q}}}(X_{B_q}(t_q))) - \mathbb{E} \text{Tr}(T_{k_q}^{2\sqrt{b_q L^{k_q}}}(X_{B_q}(t_q))) \right) \right] \\ = \delta_{k_p k_q} \frac{k_p}{2\beta} \left(\frac{c(t_p, t_q) b_{pq}}{\sqrt{b_p b_q}} \right)^{k_p}. \end{aligned}$$

Proof. Using (6) and assuming $b_p < b_q$ we obtain that the needed limit equals

$$\begin{aligned} \frac{2}{\beta(2\pi i)^2} \iint_{b_p=|z| < |w|=b_q} T_{k_p}(\cos \arg z) T_{k_q}(\cos \arg w) \frac{c(t_p, t_q) b_{pq}}{b_p} \frac{dz dw}{\left(\frac{c(t_p, t_q) b_{pq}}{b_p} z - w \right)^2} \\ = \frac{1}{2\beta(2\pi i)^2} \iint_{b_p=|z| < |w|=b_q} \left(\left(\frac{z}{\sqrt{b_p}} \right)^{k_p} + \left(\frac{\sqrt{b_p}}{z} \right)^{k_p} \right) \left(\left(\frac{w}{\sqrt{b_q}} \right)^{k_q} + \left(\frac{\sqrt{b_q}}{w} \right)^{k_q} \right) \\ \times \frac{c(t_p, t_q) b_{pq}}{b_p} \frac{dz dw}{\left(\frac{c(t_p, t_q) b_{pq}}{b_p} z - w \right)^2}. \end{aligned}$$

Writing $(c(t_p, t_q)b_{pq}z/b_p - w)^{-2}$ as a series in z/w we arrive at the result. Continuity and symmetry of both sides of the limiting relation removes the assumption $b_p < b_q$. \square

Note that in the Gaussian specialization (when $c(s, t) = \exp(-|s - t|)$) and for a single size L time-dependent Wigner matrix (i.e., $b_p = b_q = b_{pq} = 1$), the centralized traces of Chebyshev polynomials of this matrix evolve as independent Ornstein–Uhlenbeck processes with speeds equal to the degrees of the polynomials.

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