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An algorithm for computing indistinguishability quotients in misére impartial combinatorial games

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This paper advances the theory of impartial misère octal games by developing an algorithm for finding certain infinite quotient monoids. The notion of a misère quotient monoid was introduced by Thane Plambeck, who also, together with Aaron Siegel, gave an algorithm for finding finite misère quotients. This paper examines the periodicity of outcomes when changing the number of heaps of various sizes. The quotient monoid for misère 0.3122 up to heaps of size 7 is found. It is the first example of an infinite misère quotient monoid.

1. Introduction

This paper gives an algorithm for computing certain misère indistinguishability quotient monoids. The approach employed here is not the genus theory of [Berlekamp et al. 2003, Chapter 13], but rather the quotient monoid approach introduced by Thane Plambeck [2005].

The algorithm described here was initially designed to analyze octal games, but is also valid for a broader class of games which will be called "heap rulesets".

2. Heap rulesets

The notion of a heap ruleset comes from Nim, which is played with heaps of beans. The rules of Nim, and its variations, specify how a player may remove beans from a heap. The terminology below is influenced greatly by play of Nim, and readers may wish to keep games like Nim in mind when reading this paper. However, the collection of heap rulesets includes many other impartial games whose standard descriptions do not involve heaps. Chomp and Cram are examples.

We take a different perspective than is customary in describing our games. We will begin with the set $\mathcal{H} = \{H_1, H_2, \dots\}$ called the *heap alphabet*. The positions that can occur during the course of play are arbitrary finite multisets of $\mathcal{H}$. The

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set of all such positions is called the *free heap monoid* (terminology developed in private conversations with Ezra Miller), and is denoted $\mathcal{A}$. Elements of $\mathcal{A}$ will be represented as vectors $(x_1, x_2, \dots)$ where x_i denotes the number of copies of H_i. Note that $\mathcal{A}$ is isomorphic to $\mathbb{N}^\infty = \bigoplus_{n\in\mathbb{Z}^+}\mathbb{N}^n$, which is a commutative monoid, with addition being the standard vector addition. $\mathcal{H}$ is a generating set for $\mathcal{A}$. If we take the set consisting of only the first n elements of $\mathcal{H}$, we have a *partial heap alphabet* which we will denote $\mathcal{H}_n = \{H_1, \dots, H_n\}$. The *size* of a heap is its index, that is H_i is the heap of size i. A *move* will change a heap of size i into a finite (perhaps empty) collection of heaps, all of which have size $< i$. The collection of moves allowed for a particular contest will be denoted Γ. A heap alphabet together with a set of legal moves Γ, will be called a *ruleset*. In essence, the ruleset specifies "how the game is played". The term *game* (misused several times above), is synonymous with position, that is, a game is a particular element of $\mathcal{A}$. Moves will be represented as vectors in $\mathbb{Z}^\infty = \bigoplus_{n\in\mathbb{Z}^+}\mathbb{Z}^n$, the condition stating that the rightmost non-zero coordinate of a move must be -1 and the prior coordinates are all non-negative.

3. Quotient maps and outcome functions

The goal is to be able to look at a game $G = (x_1, x_2, \dots) \in \mathcal{A}$ and determine whether there is a winning strategy for the **N**ext player to move, (an N-position) or not, in which case it is a game you would like to move to as the **P**revious player, (a P-position). We define the game outcome function $o^- : \mathcal{A} \to \{N, P\}$ accordingly.

The outcome of all games under a particular ruleset $(\mathcal{H}, \Gamma)$ can be determined recursively as follows:

Let G be a game and let $G + \Gamma$ be the set of options of G. If for all options $H \in G + \Gamma$, $o^-(H) = N$, then G is a P-position. If there exists any option $H \in G + \Gamma$ with $o^-(H) = P$, then G is an N-position. That is,

$$o^-(G) = \begin{cases} N & \text{if there exists an option } H \text{ of } G \text{ with } o^-(H) = P, \\ P & \text{if for all options } H \text{ of } G,\ o^-(H) = N \end{cases}$$

Note that we have a slight problem with this algorithm. Some elements of $G+\Gamma$ may have negative coordinates and therefore are not in $\mathcal{A}$. They correspond to moves which change a heap of size i into some smaller heaps, but no heap of size i was present in G. These moves do not exist in the actual course of play. We get around this difficulty by defining $o^-(\gamma) = N$ if any coordinate of γ is negative. The addition of extra moves to N-positions does not alter the above algorithm, which searches for the existence of a P-position amongst the options.

Remembering the outcome of every game in $\mathcal{A}$ is cumbersome, and so some simplifications are necessary.

Definition 1. Two games $G, H \in \mathcal{A}$ are *indistinguishable* if $o^-(G+X) = o^-(H+X)$ for all games $X \in \mathcal{A}$.

Indistinguishablility is a congruence relation on $\mathcal{A}$ and the congruence classes form the quotient monoid $\mathcal{Q}$. (When the free heap monoid $\mathcal{A}$ corresponds to a partial heap alphabet $\mathcal{H}_n$, it is customary to call the quotient a *partial quotient* and denote it $\mathcal{Q}_n$.) The induced map from $\mathcal{A}$ to $\mathcal{Q}$ is denoted Φ and is called the *quotient map*. The quotient map induces a quotient outcome function $\mathcal{O} : \mathcal{Q} \to \{N, P\}$ making the following diagram commute.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "o^-"] \arrow[dr, "\Phi"'] & \{N, P\}. \\ & \mathcal{Q} \arrow[u, "\mathcal{O}"'] \end{tikzcd}$$

This represents a notational change from earlier work. What Plambeck and Siegel [2008] call the "P-portion of $\mathcal{Q}$" and denote by $\mathcal{P}$ is here the inverse image of P under the quotient outcome map and is denoted $\mathcal{O}^{-1}(P)$.

It is customary to use multiplicative notation in $\mathcal{Q}$, which can be achieved by first passing to the multiplicative monoid $\mathcal{M}$ with generators $\{a, b, c, \dots\}$ replacing $\{H_1, H_2, H_3, \dots\}$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "o^-"] \arrow[d] \arrow[dr, "\Phi"'] & \{N, P\}. \\ \mathcal{M} \arrow[r] & \mathcal{Q} \arrow[u, "\mathcal{O}"'] \end{tikzcd}$$

4. Computation of periodicity of outcomes and the corresponding quotients for finite heap alphabets

For a finite heap alphabet $\mathcal{H}_n$, when the quotient $\mathcal{Q}$ is finite, we are able to find relations of the form $\alpha^r = \alpha^{r+d}$ with $d > 0$ for each free heap monoid generator α according to the following algorithm.

4.1. ***Algorithm for computing periodicity.*** We wish to find for all $x_i \geq 0$, $1 \leq i \leq n-1$, the smallest values of the preperiod, r_n, and the period, d_n, so that for all $k \geq 0$, $o^-(x_1, x_2, \dots, x_{n-1}, r_n + k) = o^-(x_1, x_2, \dots, x_{n-1}, r_n + d_n + k)$. We achieve this in the following manner.

To find the periodicity for the n-th heap size:

(1) Begin with the preperiods $(r_1, \dots, r_{n-1})$ and periods $(d_1, \dots, d_{n-1})$ calculated for the previous heap sizes so that $o^-(r_1 + k_1, r_2 + k_2, \dots, r_{n-1} + k_{n-1}) =$

$o^-(r_1+d_1+k_1, r_2+d_2+k_2, \dots, r_{n-1}+d_{n-1}+k_{n-1})$ for all $k_1, \dots, k_{n-1} \geq 0$. Set $m = 1$.

(2) Calculate outcomes for games with $x_n = m$ up to the periodicity of the previous heap sizes, that is, for $x_i \leq r_i + d_i$. Verify for each heap size, that the previous periodicity still holds. If not, calculate additional outcomes in order to update the preperiods and periods.

(3) Check to see if the outcomes for $x_n = m$ agree with the outcomes for $x_n = l$, for some $l < m$. If so, then $r_n = l$ and $d_n = m - l$, if not, increase m by 1 and repeat.

4.2. ***Periodicity theorem.*** Games can be ordered using the *colexicographical order*, in which G precedes H, denoted $G < H$, if the rightmost coordinate in which G and H differ is smaller in G. Note that if $G < H$, then starting at the game G, no sequence of moves can arrive at H.

The following theorem is central to the algorithm. An example which shows the use of this theorem in the algorithm appears in the following subsection.

Theorem 2. *Fix a heap ruleset* $(\mathcal{H}_n, \Gamma)$, *under misère play, and fix values for* $i, y_{i+1}, y_{i+2}, \dots, y_n$. *Suppose that the following assumptions are satisfied for some* r_i, d_i:

(i) *The outcomes for games of the form*

$$G = (x_1, x_2, \dots, x_{i-1}, r_i, y_{i+1}, y_{i+2}, \dots, y_n)$$

agree with the outcomes of

$$G^* = (x_1, x_2, \dots, x_{i-1}, r_i + d_i, y_{i+1}, y_{i+2}, \dots, y_n)$$

for all $x_1, x_2, \dots, x_{i-1}$.

(ii) *The outcomes for games of the form*

$$K = (x_1, x_2, \dots, x_{i-1}, r_i + u, x_{i+1}, x_{i+2}, \dots, x_n)$$

agree with the outcomes of

$$K^* = (x_1, x_2, \dots, x_{i-1}, r_i + d_i + u, x_{i+1}, x_{i+2}, \dots, x_n)$$

for all $x_1, x_2, \dots, x_{i-1}$, *for* $(x_{i+1}, x_{i+2}, \dots, x_n)$ *preceding* $(y_{i+1}, y_{i+2}, \dots, y_n)$ *in the colexicographic order and all* $u \geq 0$.

Then $o^-(G + vH_i) = o^-(G^* + vH_i)$ *for all* $v, x_1, x_2, \dots, x_{i-1} \geq 0$.

Proof by induction. The statement is true for $v = 0$ by supposition. Assume the statement is true for v. All moves from the heaps $H_{i+1}, \dots H_n$ correspond to options with the same outcomes for $G + (v+1)H_i$ and $G^* + (v+1)H_i$ by

(ii) and all moves from H_i correspond to options with the same outcomes for $G+(v+1)H_i$ and $G^*+(v+1)H_i$ by the induction hypothesis. These are the only moves available when $x_1=0, x_2=0, \ldots x_{i-1}=0$. Since the options have the same outcomes, the games themselves have the same outcome. We then induct on x_1, leaving $x_2=0, x_3=0, \ldots, x_{i-1}=0$. Since the only new options are to move from H_1, the outcomes of the new options will be the same by induction, and the games themselves again have the same outcome. Now we have the statement true for arbitrary x_1, $x_2=0$ and $x_3=0, x_4=0, \ldots x_{i-1}=0$. We next induct on x_2. The new options make use of H_2 which are known to have the same value by a similar argument. Continue inducting on each heap size up to $i-1$ to complete the proof. □

4.3. ***Example.*** As a first example, we will compute several partial quotients for the quaternary ruleset 0.3122.

First partial quotient of 0.3122. This first quotient shows the fundamental nature of the periodicity of outcomes.

For the first partial heap alphabet $\mathcal{H}_1$, there is only one legal move, namely $\Gamma=\{(-1)\}$. The outcome of the game (0) is defined to be N in misère play. $o^-(1)=P$, since the only move is to the N-position (0) and $o^-(2)=N$, since there is a move to the P-position (1).

What about the sum of several copies of H_1? That is, what is the outcome of (x_1) for arbitrary x_1. Since the only move is (-1), the outcome of (x_1) will be P if $o^-(x_1-1)=N$ and vice versa. The outcomes are periodic with period two.

$$\begin{array}{rcccccccc} x_1 = & 0 & 1 & 2 & 3 & 4 & 5 & \ldots \\ o^-(x_1) = & N & P & N & P & N & P & \ldots \end{array}$$

The key fact is that the outcome of (3) is a function of $o^-(2)$, just as the outcome of (1) is a function of $o^-(0)$. In fact, $o^-(j+1)$ is a function of $o^-(j)$ for all $j\geq 0$. Moreover, the outcomes of $(j+k)$ for $j,k>0$ are all determined by a single outcome, $o^-(j)$. Since the outcome of (2) is the same as the outcome of (0), then the outcome of $(2+k)$ must be the same as the outcome of $(0+k)$ for all $k\geq 0$.

We will record this periodicity for the partial heap alphabet $\mathcal{H}_n$ as follows: R_n and D_n are n-tuples where R_n records the preperiod in each component and D_n records the period. That is, when $R_n=(r_1,r_2,\ldots,r_n)$ and $D_n=(d_1,d_2,\ldots,d_n)$, if you have at least r_i heaps of size i, adding d_i more heaps of size i won't change the outcome.

Thus in the current example, $R_1=(0)$ and $D_1=(2)$

This periodicity gives us a candidate for the quotient monoid, namely $\mathcal{Q}^*=\langle a \mid a^2=1\rangle$. The relation $a^2=a^0$ is clearly valid from the periodicity. The candidate quotient is therefore a refinement of the true quotient $\mathcal{Q}$ in the sense that every game associated to a particular element of $\mathcal{Q}^*$ belongs to a single indistinguishability congruence class in $\mathcal{Q}$. However, $\mathcal{Q}^*$ may have too many elements by having further partitioned the congruence classes in $\mathcal{Q}$. In general, we need to check that there do not exist distinct elements $p, q \in \mathcal{Q}^*$ where for all $x \in \mathcal{Q}^*$, $\mathcal{O}(px) = \mathcal{O}(qx)$. If we find two such elements, then we add the relation $p = q$ to create a new candidate quotient. When the full periodicity R_n, D_n has been found for a partial heap alphabet, the initial candidate quotient is finite, as will be the list of relations to check. Each successive quotient will have strictly smaller cardinality than the previous quotient, and therefore this verification will converge to the true quotient $\mathcal{Q}$ in a finite number of steps.

In this case, since the N-position (0) is mapped to 1 in $\mathcal{Q}^*$ and the P-position (1) is mapped to a, then $\mathcal{O}(1) = N$ and $\mathcal{O}(a) = P$. Thus 1 and a are distinguishable and the candidate $\mathcal{Q}^*$ is as small as possible, that is $\mathcal{Q}^* = \mathcal{Q}_1$.

Second partial quotient of 0.3122. This second step shows the recursive nature of the algorithm, and provides an example where the periodicity of outcomes occurs after a preperiod.

For the second partial quotient $\mathcal{Q}_2$, the legal moves are

$$\Gamma = \{(-1, 0),\ (0, -1),\ (1, -1)\}.$$

We already know the outcomes of games of the form $(x_1, 0)$ and notice that $o^-(x_1, 1) = N$ for all $x_1 \geq 0$ since $(x_1, 1)$ has options to $(x_1, 0)$ and $(x_1 + 1, 0)$, one of which is a P-position. Moving on to the next row, we compute the first three outcomes.

$$\begin{array}{rcccccccc} x_1 = & 0 & 1 & 2 & 3 & 4 & 5 & \dots \\ o^-(x_1, 0) = & N & P & N & P & N & P & \dots \\ o^-(x_1, 1) = & N & N & N & N & N & N & \dots \\ o^-(x_1, 2) = & P & N & P & & & & \end{array}$$

We use the theorem to claim that this row must also be periodic. Compare the options of $(3, 2)$ versus the options of $(1, 2)$. The moves from $(3, 2)$ which make use of H_2, that is, the moves $(1, -1)$ and $(0, -1)$ to the positions $(4, 1)$ and $(3, 1)$ respectively, are moves with the same outcomes as the options of $(1, 2)$ to $(2, 0)$ and $(1, 0)$, since the previous row was periodic. (Part (ii) of the theorem.) Also, $o^-(2, 2) = o^-(0, 2)$. Since the outcomes of the options of $(3, 2)$ agree with the outcomes of the options of $(1, 2)$, then $o^-(3, 2) = o^-(1, 2)$. We then proceed inductively using part (i) of the theorem. For all $k > 0$, $o^-(3 + k, 2) = o^-(1 + k, 2)$ since using the move $(1, -1)$ from each game has $o^-(4+k, 1) = o^-(2+k, 1)$ by the previous row's periodicity. Likewise, using the move $(0, -1)$ has $o^-(3 + k, 1) = o^-(1 + k, 1)$ by the previous row's periodicity.

Finally, using the move $(-1,0)$ has $o^-(3+k-1,1) = o^-(1+k-1,1)$ by induction.

We then repeat this process to calculate the outcome of further rows.

$$\begin{array}{rcccccccc}
x_1 = & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\
o^-(x_1,0) = & N & P & N & P & N & P & \cdots \\
o^-(x_1,1) = & N & N & N & N & N & N & \cdots \\
o^-(x_1,2) = & P & N & P & N & P & N & \cdots \\
o^-(x_1,3) = & N & N & N & N & N & N & \cdots
\end{array}$$

Notice that for *all* $x_1 \geq 0$ we have $o^-(x_1,3) = o^-(x_1,1)$. We now claim to have established periodicity in the second dimension. The outcomes of $(x_1,4)$ for all $x_1 \geq 0$ are determined by the outcomes of the previous row $(x_2 = 3)$, just as the outcomes of $(x_1,2)$ for all $x_1 \geq 0$ were determined by the previous row $(x_2 = 1)$. Since $o^-(x_1,3) = o^-(x_1,1)$, then $o^-(x_1,4) = o^-(x_1,2)$. By induction, $o^-(x_1, 3+k) = o^-(x_1, 1+k)$ for all $x_1 \geq 0$ and $k \geq 0$.

Computing the quotient monoid $\mathcal{Q}_2$, the periodicity $R_2 = (0,1)$ $D_2 = (2,2)$ gives the candidate quotient $\mathcal{Q}^* = \langle a, b \mid a^2 = 1, b^3 = b\rangle$.

Again, we check to see if we can find $p, q \in \mathcal{Q}^*$ with $\mathcal{O}(px) = \mathcal{O}(qx)$ for all $x \in \mathcal{Q}^*$. In fact, all six elements are distinguishable (for instance, a and b^2 are distinguishable, since $\mathcal{O}[a(b^2)] = N$, but $\mathcal{O}[b^2(b^2)] = \mathcal{O}(b^4) = \mathcal{O}(b^2) = P$.) Therefore $\mathcal{Q}^*$ is the true quotient $\mathcal{Q}_2$

In the second row, we verified periodicity by comparing the options of $(3,2)$ versus the options of $(1,2)$, but the periodicity occurs from the beginning of the row. The difficulty in comparing $(2,2)$ and $(0,2)$ is that from $(0,2)$, the move $(-1,0)$ takes us off the board to $(-1,2) \notin \mathcal{A}$. We can fix this problem by extending the domain of outcome function o^- to include vectors in $\mathbb{Z}^n$ by declaring that $o^-(x_1, x_2, \ldots x_n) = N$ if any $x_i < 0$. Adding in extra illegal moves to N-positions will not alter the existence or non-existence of legal options to a P-position.

$$\begin{array}{rcccccccc}
x_1 = & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\
o^-(x_1,-1) = & \boldsymbol{N} & \boldsymbol{N} & \boldsymbol{N} & \boldsymbol{N} & \boldsymbol{N} & \boldsymbol{N} & \boldsymbol{N} & \cdots \\
o^-(x_1,0) = & \boldsymbol{N} & N & P & N & P & N & P & \cdots \\
o^-(x_1,1) = & \boldsymbol{N} & N & N & N & N & N & N & \cdots \\
o^-(x_1,2) = & \boldsymbol{N} & P & N & P & N & P & N & \cdots \\
o^-(x_1,3) = & \boldsymbol{N} & N & N & N & N & N & N & \cdots \\
\vdots & & & & & & & &
\end{array}$$

Third and fourth partial quotients of 0.3122. Calculation of the third partial quotient is similar to the second. The fourth partial quotient gives an example of step 2) of the algorithm updating the prior periodicity.

We can continue to calculate outcomes for $x_1 = 0, 1, 2$, and the reader unfamiliar with *reversibility* may wish to continue to include these outcomes. However, in order to decrease the amount of data we need to keep track of, we will employ the following simplification. It is a global condition on any ruleset $(\mathcal{H}, \Gamma)$, for any game whose only option is to the identity, (H_1 in this case), that two copies of that game are reversible to, and therefore indistinguishable from, the identity. (See [Berlekamp et al. 2003, Chapter 13].) Therefore we only need to keep track of columns 0, 1 and do not need to verify the periodicity for heaps of size 1. Column 2 will always agree with column 0.

For $\mathcal{H}_4$, the legal moves are $\Gamma = \{(-1, 0, 0, 0),\ (0, -1, 0, 0),\ (1, -1, 0, 0),$ $(0, 1, -1, 0),\ (1, 0, 0, -1),\ (0, 0, 1, -1)\}$

We calculate the outcomes for $x_3 = 1$ up to the previous periodicity on x_2 which requires us to calculate rows 0, 1, 2, 3. The periodicity for heaps of size 2 checks, that is $o^-(x_1, 3, 1) = o^-(x_1, 1, 1)$, but the outcomes for $(x_1, x_2, 1)$ do not agree with the outcomes of $(x_1, x_2, 0)$, so we continue to $x_3 = 2$.

$\mathbf{x_3 = 0}$	$x_1 =$	0	1	$\mathbf{x_3 = 1}$	$x_1 =$	0	1	$\mathbf{x_3 = 2}$	$x_1 =$	0	1
$x_2 = 0$		N	P			P	N			P	N
$x_2 = 1$		N	N			N	N			N	N
$x_2 = 2$		P	N			P	N			P	N
$x_2 = 3$		N	N			N	N			N	N

Now the outcomes for $(x_1, x_2, 2)$ agree with those for $(x_1, x_2, 1)$ so $R_3 = (0, 1, 1)$; $D_3 = (2, 2, 1)$.

When $x_4 = 1$ and $x_3 = 0$, the outcomes for $x_2 = 3$ no longer agree with the outcomes of $x_2 = 1$, so we need to calculate additional outcomes. The new preperiod for x_2 must be greater than the old preperiod and the new period must be a multiple of the old period. In this case the new preperiod is 2 and the period remains 2.

$\mathbf{x_4 = 1},\ \mathbf{x_3 = 0}$	$x_1 =$	0	1
$x_2 = 0$		N	P
$x_2 = 1$		N	N
$x_2 = 2$		N	N
$x_2 = 3$		P	N
$x_2 = 4$		N	N

We continue calculating outcomes until the full periodicity $R_4 = (0, 2, 1, 0)$, $D_4 = (2, 2, 1, 2)$ is reached. (See figure on the next page.)

The candidate quotient arrived at from the periodicity is

$$\mathcal{Q}^* = \langle a, b, c, d \mid a^2 = 1, b^4 = b^2, c^2 = c, d^2 = 1 \rangle.$$

$\mathbf{x_4 = 0}$

$\mathbf{x_3 = 0}$ $x_1 =$	0	1	$\mathbf{x_3 = 1}$ $x_1 =$	0	1	$\mathbf{x_3 = 2}$ $x_1 =$	0	1
$x_2 = 0$	N	P		P	N		P	N
$x_2 = 1$	N	N		N	N		N	N
$x_2 = 2$	P	N		P	N		P	N
$x_2 = 3$	N	N		N	N		N	N
$x_2 = 4$	P	N		P	N		P	N

$\mathbf{x_4 = 1}$

$\mathbf{x_3 = 0}$ $x_1 =$	0	1	$\mathbf{x_3 = 1}$ $x_1 =$	0	1	$\mathbf{x_3 = 2}$ $x_1 =$	0	1
$x_2 = 0$	N	P		N	N		N	N
$x_2 = 1$	N	N		P	N		P	N
$x_2 = 2$	N	N		N	N		N	N
$x_2 = 3$	P	N		P	N		P	N
$x_2 = 4$	N	N		N	N		N	N

$\mathbf{x_4 = 2}$

$\mathbf{x_3 = 0}$ $x_1 =$	0	1	$\mathbf{x_3 = 1}$ $x_1 =$	0	1	$\mathbf{x_3 = 2}$ $x_1 =$	0	1
$x_2 = 0$	N	P		P	N		P	N
$x_2 = 1$	N	N		N	N		N	N
$x_2 = 2$	P	N		P	N		P	N
$x_2 = 3$	N	N		N	N		N	N
$x_2 = 4$	P	N		P	N		P	N

A check of outcomes reveals that $c = b^2$ and $b^2 d = b^3$, so that

$$\mathcal{Q}_4 = \langle a, b, d \mid a^2 = 1, b^4 = b^2, b^2 d = b^3, d^2 = 1 \rangle,$$

with outcome map $\Phi(H_1) = a$, $\Phi(H_2) = b$, $\Phi(H_3) = b^2$, $\Phi(H_4) = d$ and $\mathcal{O}^{-1}(P) = \{a, b^2, ad\}$.

5. Infinite quotients

If the quotient $\mathcal{Q}$ is infinite for a particular partial heap alphabet, the algorithm will fail to terminate. Similar techniques searching for periodicity in directions involving more than one heap size may be employed to discover these infinite monoids.

One example is the fifth partial quotient of the quaternary game 0.31011. The moves are given as follows:

$$\Gamma = \{(-1, 0, 0, 0, 0),\ (1, -1, 0, 0, 0),\ (0, -1, 0, 0, 0),\ (0, 1, -1, 0, 0),\\ (0, 0, 1, -1, 0),\ (0, 0, 0, -1, 0),\ (0, 0, 0, 1, -1),\ (0, 0, 0, 0, -1)\}$$

For each value of x_5, the first three preperiods are $r_1 = 0$, $r_2 = 2$, $r_3 = 3$ and the first three periods are $d_1 = 2$, $d_2 = 2$, $d_3 = 1$. However, r_4 continues to increase with x_5. A "diagonal" periodicity is produced, so that for all games G with $x_4 + x_5 > 11$, $o^-(G) = o^-(G + 2H_4 + 2H_5)$. One verifies this by

(1) finding preperiods and periods in the first four dimensions for $x_5 = 0, 1, 2$;

(2) verifying that

(i) $o^-(x_1, x_2, x_3, 12, 0) = o^-(x_1, x_2, x_3, 14, 0) = o^-(x_1, x_2, x_3, 14, 2) = o^-(x_1, x_2, x_3, 16, 2)$,
(ii) $o^-(x_1, x_2, x_3, 13, 0) = o^-(x_1, x_2, x_3, 15, 2)$,
(iii) $o^-(x_1, x_2, x_3, 11, 1) = o^-(x_1, x_2, x_3, 13, 1)$;

(3) verifying for $a + b = 12$, $o^-(x_1, x_2, x_3, a, b) = o^-(x_1, x_2, x_3, a + 2, b + 2)$;

(4) verifying that

(i) $o^-(x_1, x_2, x_3, 0, 12) = o^-(x_1, x_2, x_3, 0, 14) = o^-(x_1, x_2, x_3, 2, 16)$,
(ii) $o^-(x_1, x_2, x_3, 1, 12) = o^-(x_1, x_2, x_3, 1, 14) = o^-(x_1, x_2, x_3, 1, 16)$,
(iii) $o^-(x_1, x_2, x_3, 1, 13) = o^-(x_1, x_2, x_3, 1, 15)$,
(iv) $o^-(x_1, x_2, x_3, 0, 13) = o^-(x_1, x_2, x_3, 2, 15)$

This yields the candidate quotient $\mathcal{Q}_5^* = \langle a, b, c, d, e \mid a^2 = 1;\ b^4 = b^2;\ c^4 = c^3;\ e^2d^{14} = d^{12};\ e^3d^{13} = ed^{11};\ e^4d^{12} = e^2d^{10};\ e^5d^{11} = e^3d^9;\ e^6d^{10} = e^4d^8;\ e^7d^9 = e^5d^7;\ e^8d^8 = e^6d^6;\ e^9d^7 = e^7d^5;\ e^{10}d^6 = e^8d^4;\ e^{11}d^5 = e^9d^3;\ e^{12}d^4 = e^{10}d^2;\ e^{13}d^3 = e^{11}d;\ e^{14}d^2 = e^{12}\rangle$.

Additional relations can be found, giving a new candidate quotient $\mathcal{Q}_5^{**} = \langle a, b, c, d, e \mid a^2 = 1;\ b^4 = b^2;\ cb^2 = c;\ c^4 = c^3;\ dc^3 = c^3a;\ d^2b^2 = d^2;\ ec^3 = c^3b;\ e^2b^2 = e^2;\ e^2d^6 = d^4;\ e^7d^5 = e^5d^3;\ e^8d^4 = e^6d^2;\ e^9d^3 = e^7d;\ e^{10}d^2 = d^8\rangle$.

Another example for which the verification has been performed is the seventh partial quotient of the quaternary game 0.3122: $\mathcal{Q}_7 = \langle a, b, c, d, e, f \mid a^2 = 1;\ b^5 = b^3;\ cb^3 = b^4;\ c^2b^2 = b^2;\ c^3b = cb;\ c^4 = c^2;\ d^2b^4 = d^2b^2;\ d^2cb^2 = d^2b^3;\ d^3b^2 = d^2b^2;\ eb^2 = b^4a;\ e^2 = b^4;\ fb^2 = d^2b^3a;\ fd^3cb = d^2c^2ba;\ fd^3c^2 = d^2c^3a;\ f^2ed^2c^2 = fedc^3a;\ f^3dcb = f^2c^2ba;\ f^3dc^2 = f^2c^3a;\ f^4d^6 = d^2;\ f^5ed^5 = fed;\ f^6d^4 = f^2\rangle$, with outcome map Φ having values

$$\Phi(H_1) = a, \quad \Phi(H_2) = b, \quad \Phi(H_3) = b^2d^2, \quad \Phi(H_4) = c,$$
$$\Phi(H_5) = d, \quad \Phi(H_6) = e, \quad \Phi(H_7) = f.$$

6. Further work

The two examples of infinite quotients were discovered on a case-by-case basis. Automation of the process to search for "diagonal" periodicities would greatly enhance our ability to find other infinite quotients. Included in finding such an algorithm is a determination of exactly which directions are possible directions for periodicity given a ruleset.

An algorithm which exhausts the list of possible relations in the infinite case is also needed.

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