

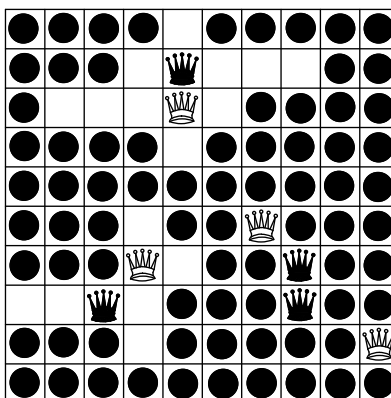
Nimbers in partizan games

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“The chess board is too small for two queens.”
 –Victor Korchnoi, challenger for World Chess Championship

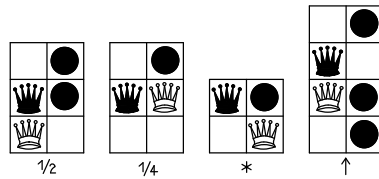
We propose some classes of games particularly useful for constructing nimbers in partizan games. We exhibit a $*4$ in Amazons with the help of a particular algebraic table.

Before reading this article one can consider the following Amazons position:

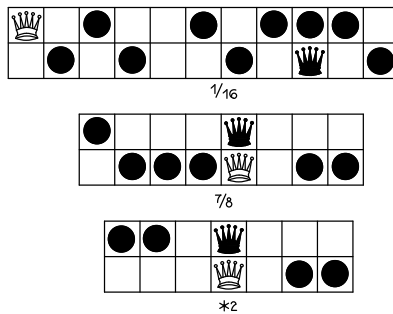


In Amazons there are a lot of hot positions: the players want to play to gain some territory. However, the position shown is not of this kind. If we use [Siegel 2011] to analyze it, this position proves to have value $*3 + *2 = *$. We will attempt to analyze the options of such a position and construct nimbers in partizan games. For instance, $*3 = \{0, *, *2 \mid 0, *, *2\}$, however, when we study the players options with [Siegel 2011] we see that the options are not $\{0, *, *2\}$. In a partizan game we have a bigger number of possible options to construct a number than in an impartial game. For instance, we know that a game like $\{\uparrow \mid 0\}$ has value $*$ too. When we think about higher stars the number of possibilities is just gigantic. So it's important to make some mathematical considerations to classify the games that “can act as numbers”. In this article, we prove some useful results about the construction of nimbers and show some interesting examples in Amazons.

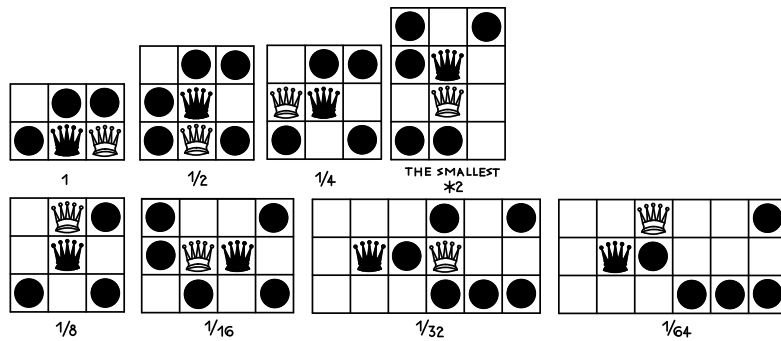
Some values in Amazons. In [Berlekamp 2000] we can see the first interesting game values in Amazons:



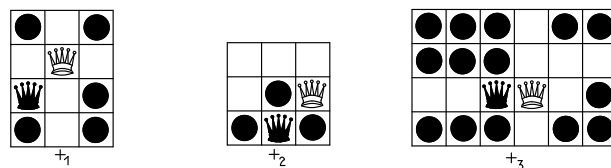
In [Snatzke 2002b], we can see more values and the first $*2$:



In [Snatzke 2002a] and [Tegos 2002], we can see a vast list of values:



to which we can add some infinitesimals...



So, we can say that Amazons is a rich combinatorial game with a vast number of interesting examples. However, there is a little drawback: Amazons is very counterintuitive. For instance, in the last picture, the game of value $+1$ is winning for Left (black). It's curious that a position with such centralized Queen is lost

for Right (white). So it's very difficult to analyze an Amazons game in chess-like fashion, using general strategic principles.

R-classes. In partizan games there is the reversibility phenomena. A reversible move for Left is one which Right can promise to respond to in such a way that prospects are at least as good as they were before. That is, a left option A of G is reversible if A has a right option, A^R , with $A^R \leq G$. In any context Right promises "if you ever choose option A of G then I will immediately move to A^R ". So, Left just chooses the option A if he intends to follow up Right's move to A^R with an immediate response to one of A^R 's left options. If he plans some other move elsewhere, he might just as well start with that. Reversibility is a simplification principle of combinatorial games.

For instance, with the reversibility simplification principle we can see why $\{\uparrow | 0\}$ has value $*$. We know that $\uparrow = \{0 | *\}$. It's easy to observe that $\{\uparrow | 0\} + * \geq 0$ (the game $\{\uparrow | 0\} + *$ is a previous player win so its value is 0). When Left chooses the option \uparrow , then Right will immediately move to $*$, which guarantees prospects at least as good as before. But Left has the option 0 immediately available so $\{\uparrow | 0\} = \{0 | 0\} = *$. Left "can go to 0 in *two tempos*".

This theoretical background is very important, but we need a more explicit way to recognize the options that can "act" like numbers by reversibility. We will propose some classes of games. First we need some notation:

Definition. Consider a game $G = \{G^L | G^R\}$. A set of games $\Delta \subseteq G^L$ has the type $\text{Mate}(G_1, \dots, G_n)$ if, for all $i \in \{1, \dots, n\}$, there exists $L \in \Delta$ with $L + G_i \geq 0$. A set of games $\Delta \subseteq G^R$ has the type $\text{Mate}(J_1, \dots, J_n)$ if, for all $i \in \{1, \dots, n\}$, exists $R \in \Delta$ with $R + J_i \leq 0$. If the set G^L has the type $\text{Mate}(G_1, \dots, G_n)$, we write $G = \{\text{Mate}(G_1, \dots, G_n) | G^R\}$. Respectively, if the set G^R has the type $\text{Mate}(J_1, \dots, J_n)$, we write $G = \{G^L | \text{Mate}(J_1, \dots, J_n)\}$.

We can now define some classes:

Definition. For $n, K \in \mathbb{N}_0$ and $n < K$ (we are preparing the construction of a $*K$), we define by recurrence the set of games as follows: $+_K R_n^{(N)}$

$$+_K R_n^{(0)} = *n,$$

and for $N > 0$, the class $+_K R_n^{(N)}$ has the form

$$\left\{ \text{Mate}(0, *, \dots, *(n-1)) \parallel \underbrace{\{G' \in +_K R_n^{(N-1)}, \Delta_1 | \text{Mate}(0, \dots, *(K-1))\}}_{\leq *K}, \underbrace{\Delta_2}_{\leq *n} \right\}.$$

The games in the set Δ_1 are smaller or confused with $*K$. The games in the set Δ_2 are smaller or confused with $*n$.

Remarks and examples. (1) When we say $G \in +_K R_n$ we mean that $\exists N_0 : G \in +_K R_n^{(N_0)}$. When we say $G \in -_K R_n^{(N)}$ we mean that $-G \in +_K R_n^{(N)}$. When we say $G \in -_K R_n$ it means that $\exists N_0 : G \in -_K R_n^{(N_0)}$.

(2) A game can be an element of a lot of classes. For instance,

$$+_1 = \{0 \mid \{0 \mid -1\}\} \in +_K R_0^{(1)}.$$

If we consider the game $G = \{2 \mid \{0 \mid -3\}, \{+_1 \mid -20\}\}$ then

$$G \in +_K R_0^{(1)} \cap +_K R_0^{(2)}.$$

(3) If $G \in +_K R_n$, we call

$$\text{depth}_{(n,K)}(G) = \text{Max}\{N \in \mathbb{N}_0 : G \in +_K R_n^{(N)}\}.$$

Lemma 1. *If $G \in +_K R_n$ then $G + *n \geq 0$.*

Proof (complete induction in $\text{depth}_{(n,K)}(G)$). If $\text{depth}_{(n,K)}(G) = 0$ then $G = *n$. In this case the result is trivial.

Suppose $G + *n \geq 0$ for all $G \in +_K R_n$ and $\text{depth}_{(n,K)}(G) \leq N - 1$. If $J \in +_K R_n$ and $\text{depth}_{(n,K)}(J) = N$ then J has the form

$$\left\{ \text{Mate}(0, *, \dots, *(n-1)) \parallel \underbrace{\{G' \in +_K R_n^{(N-1)}, \dots \mid \text{Mate}(0, \dots, *(K-1))\}}_{\trianglelefteq *K}, \underbrace{\dots}_{\trianglelefteq *n} \right\}.$$

Let's analyze the right options of the game $J + *n$. Right can move to

$$\left\{ \underbrace{G' \in +_K R_n^{(N-1)}, \dots \mid \text{Mate}(0, \dots, *(K-1))}_{\trianglelefteq *K} \right\} + *n$$

or Right can move to $J + *i$ ($i < n$). In the first case, Left chooses

$$G' \in +_K R_n^{(N-1)} + *n,$$

and wins (induction hypothesis). In the second case, Left wins because J^L is $\text{Mate}(0, \dots, *(n-1))$. \square

Lemma 2. *If $G \in +_K R_n$ then $G + *K$ is fuzzy.*

Proof (complete induction $\text{depth}_{(n,K)}(G)$). If $\text{depth}_{(n,K)}(G) = 0$ then $G = *n$. In this case the game $*n + *K$ is an easy win for next player.

Suppose $G + *K$ is fuzzy for all $G \in +_K R_n$ with $\text{depth}_{(n,K)}(G) \leq N - 1$. If $J \in +_K R_n$ and $\text{depth}_{(n,K)}(J) = N$ then we must analyze

$$\left\{ \text{Mate}(0, *, \dots, *(n-1)) \parallel \underbrace{\{G' \in +_K R_n^{(N-1)}, \dots \mid \text{Mate}(0, \dots, *(K-1))\}}_{\trianglelefteq *K}, \underbrace{\dots}_{\trianglelefteq *n} \right\} + *K.$$

If Left moves first, then he moves $*K$ to $*n$ and wins (Lemma 1). If Right moves first then he plays to

$$\left\{ G' \in +_K R_n^{(N-1)}, \underbrace{\dots}_{\leq *K} \mid \text{Mate}(0, \dots, *K - 1) \right\} + *K.$$

Now, Left must choose $G' \in +_K R_n^{(N-1)} + *K$ and Right wins (I. H.). \square

Construction theorem. If $G_i \in +_K R_i$ and $J_i \in -_K R_i, i \in \{0, \dots, K - 1\}$ then $\{G_0, \dots, G_{K-1} \mid J_0, \dots, J_{K-1}\} = *K$.

Proof. Let's play $\{G_0, \dots, G_{K-1} \mid J_0, \dots, J_{K-1}\} + *K$.

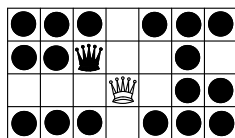
If Left moves to $\{G_0, \dots, G_{K-1} \mid J_0, \dots, J_{K-1}\} + *i, i < K$ then Right moves to $J_i + *i$ and wins (Lemma 1). If Left plays to $G_i + *K$ then the game is fuzzy by Lemma 2, so Right wins. If Right plays first, the argument is the same and Left wins, so $\{G_0, \dots, G_{K-1} \mid J_0, \dots, J_{K-1}\} + *K = 0$. \square

Examples of stars in partizan games:

We have $\uparrow \in_1 R_0$ therefore, $\{\uparrow \mid 0\} = *$.

Note that $\uparrow \notin_2 R_0$ and $\{\uparrow, * \mid 0, *\} \neq *2$.

Example in Amazons. We can use the theorem just proved to analyze canonical forms of nimbers in partizan games. Consider one important move on the top of the first position of this text (G):



If one canonicalizes this in CGSuite one gets the horrible canonical form

$$\left\{ \frac{1}{2} \mid 0, \left\{ *, \{1 \mid *, \pm\{1, \{2 \mid 0\}\}\} \mid \{0 \mid -1\}, \{*, \pm 1 \mid -1\}, \{0, * \parallel -\frac{1}{2} * \mid -3\} \right\}, \right. \\ \left. \left\{ \uparrow, \uparrow * \mid \{0 \mid -1\}, \{ * \mid -\frac{1}{4} \} \right\}, \right. \\ \left. \left\{ 0, *, *2 \mid *, \{ *, \{1 * \mid -\frac{1}{2} \} \mid -\frac{1}{2} \}, \right. \right. \\ \left. \left. \{ *, \{1 *, \{2 \mid 0\} \mid -\frac{1}{2}, \{ \pm 1, \frac{1}{2}, \{ \frac{1}{2} \mid 0 \} \mid -1 \} \} \mid -\frac{1}{2} \} \right\}, \quad (= : J) \right. \\ \left. \left\{ 0, \uparrow *, \left\{ \frac{1}{2}, \{ \frac{1}{2} \mid 0 \} \mid -1 \} \mid -1 \right\} \right\}.$$

Now we can make some considerations:

(A) $\frac{1}{2}$ is a left option so the set of left options is $\text{Mate}(0, *)$.

- (B) Consider the right option occupying two lines; call it J . We can observe that $*$ and $\{*, \{1* \mid -\frac{1}{2}\} \mid -\frac{1}{2}\}$ are right options, so J^R is $\text{Mate}(0, *, *2)$.
- (C) All the other right options of G are smaller or confused with $*2$.
- (D) In the right option of item (B), all the left options are smaller or confused with $*3$. $*2$ is a left option.

From (A), (B), (C) and (D) we can conclude that G has the form

$$\underbrace{\left\{ \frac{1}{2}, \dots \right\}}_{\text{Mate}(0, *)} \parallel \left\{ \underbrace{*2}_{\in +_3 R_2^{(0)} \leq *3}, \underbrace{0, *}_{\leq *3} \mid \underbrace{*, \{*, \{1* \mid -\frac{1}{2}\} \mid -\frac{1}{2}\}, \dots}_{\text{Mate}(0, *, *2)}, \dots \right\}, \underbrace{\dots}_{\leq *2}$$

thus, $G \in +_3 R_1^{(1)}$. In a similar way we can conclude that $G \in +_3 R_2^{(1)}$. It's very interesting because the same option can act like a $*$ and like a $*2$.

***4 in Amazons.** Theodore Tegos [2002] organized and studied a big database of Amazons game positions. About numbers, he wrote “The current challenge is to find an Amazons position whose combinatorial value is $*4$, if such position exists.”

To construct a $*4$ we will use an idea involving an algebraic table:

+	*2	*3
*2	0	*
0	*2	*3

If we observe carefully the resulting values of the table we see that those values are the “needed stuff” to construct a $*4$.

In a partizan game we can construct numbers with other options than numbers. So, there is a gigantic number of “good algebraic tables”. For instance, we know that

$$+_1 + *2 = \left\{ \underbrace{0, *, *2}_{\text{Mate}(0, *)} \parallel \underbrace{*2}_{\in +_4 R_2^{(0)}} \mid \underbrace{-1*2}_{\text{Mate}(0, *)} \right\} \in +_4 R_2^{(1)}$$

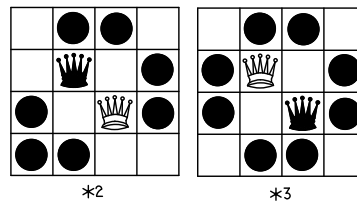
and

$$+_2 + *3 = \left\{ \underbrace{0, *, *2, *3}_{\text{Mate}(0, *, *2)} \parallel \underbrace{*3}_{\in +_4 R_3^{(0)}} \mid \underbrace{-2*3}_{\text{Mate}(0, *, *2)} \right\} \in +_4 R_3^{(1)}$$

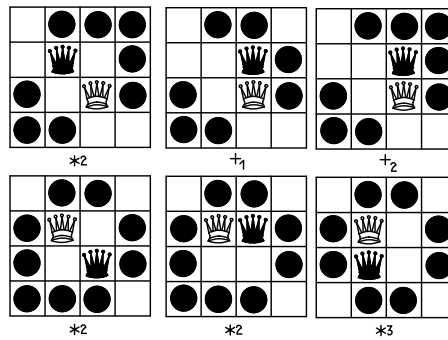
When we want to construct a $*4$, the games $+_1 + *2$ and $+_2 + *3$ acts like $*2$ and $*3$. So, the following table is also very useful to construct a $*4$:

+	*2	*3
*2	0	*
+1 or +2	$\{0, *, *2 \parallel *2 \mid -1 *2\} \in +_4R_2^{(1)}$	$\{0, *, *2, *3 \parallel *3 \mid -2 *3\} \in +_4R_3^{(1)}$

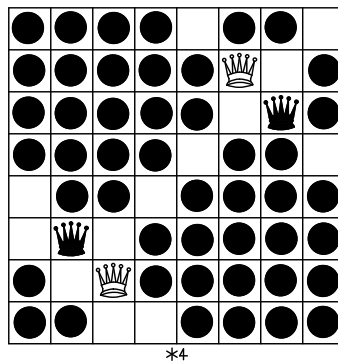
We will produce this table on the Amazons board. First, we introduce two fundamental positions:



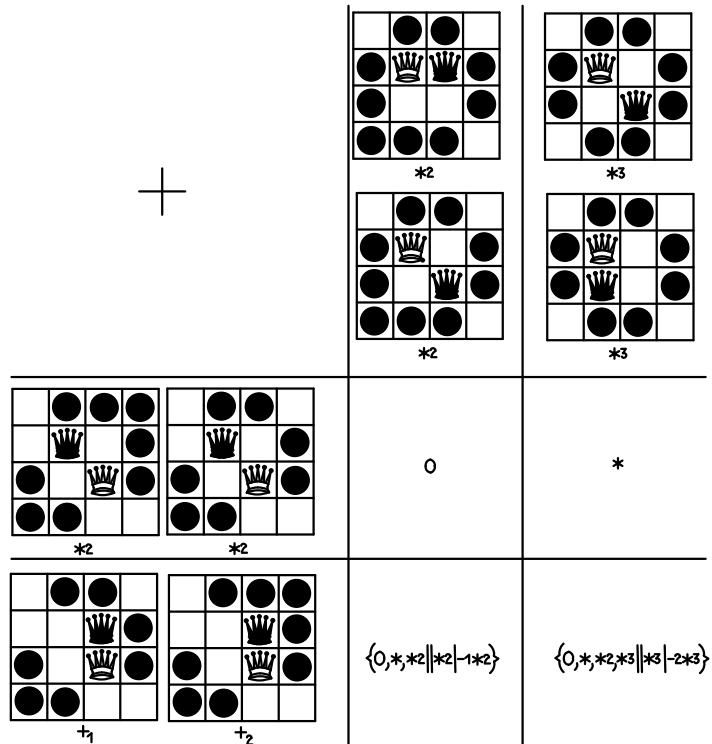
Second, we introduce some similar positions:



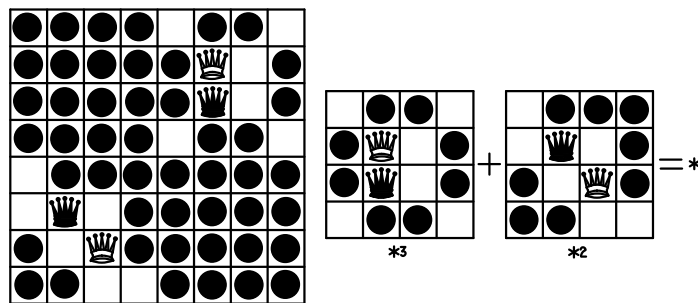
Using a *collage* idea we can join the two fundamental positions obtaining the following position (*G*):



With this construction it's possible to have all the results of the algebraic table under consideration:



For instance, if we want to obtain a * we can move like this:



With exhaustive analysis (with the help of [Siegel 2011]) it is possible to see that all the other possible moves are not winning moves in the game $G + *4$. So the exposed position is the first construction of a $*4$ in Amazons.

References

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