Games of No Chance 4
MSRI Publications
Volume **63**, 2015

A handicap strategy for Hex

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We give an $\lceil \frac{n+1}{6} \rceil$-cell handicap strategy for the game of Hex on an $n \times n$ board: the first player is guaranteed victory if she is allowed to colour $\lceil \frac{n+1}{6} \rceil$ cells on her first move. Our strategy exploits a new kind of inferior Hex cell.

1. Introduction

Hex was invented independently by Piet Hein [1942] and John Nash [1952]. The game is played by two players, Black and White, on a board with hexagonal cells. The players alternate turns, colouring any single uncoloured cell with their colour. The winner is the player who creates a path of her colour connecting her two opposing board sides. See Figure 1.

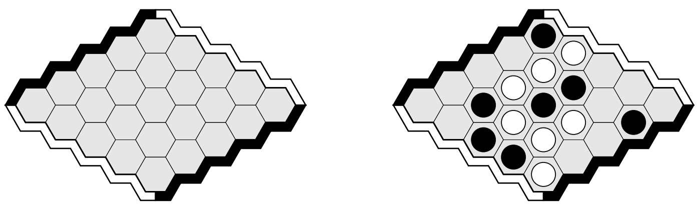

Figure 1. An empty 5×5 Hex board (left) and a completed game position (right).

Hein and Nash observed that Hex cannot end in a draw [Hein 1942; Nash 1952]: exactly one player has a winning path if all cells are coloured [Beck et al. 1969]. Also, an extra coloured cell is never disadvantageous for the player with that colour [Nash 1952]. For $n \times n$ boards, Nash showed the existence of a first-player winning strategy [1952]; however, his proof reveals nothing about the nature of such a strategy. For 8×8 and smaller boards, computer search can find all winning first moves [Hayward et al. 2004; Henderson et al. 2009]. For the 9×9 board, Yang found by human search that moving to the centre cell is a winning first move.

The authors gratefully acknowledge financial support from NSERC, Alberta Ingenuity Fund, and iCORE .

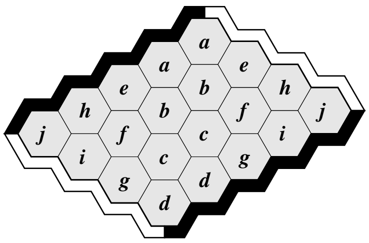

Figure 2. Shannon's pairing strategy on a 4×5 Hex board.

For $m \times n$ boards with $m \neq n$ (hereafter *irregular boards*), the player whose opposing board sides are closer together can win, even as the second player, via a pairing strategy due to Shannon [Gardner 1959]. Figure 2 shows this strategy for the 5×4 Hex board. By contrast, Reisch [1981] showed that solving arbitrary Hex positions is PSPACE-complete.

The problem of efficiently (i.e., in polynomial time) identifying a winning first move on an empty $n \times n$ Hex board has been unsolved for roughly 60 years. As a step towards solving this problem we ask the following:

For the empty $n \times n$ Hex board, let $h(n)$ represent the number of cells the first player needs to colour in order to reach a position with a known and polynomial-time winning strategy, where the placement of these handicap cells is specified. What is the minimum $h(n)$?

Solving the original problem would only require one handicap stone for all values of n, so it is possible that $h(n) = 1$ in the optimal case; for instance, if $\mathrm{P} = \mathrm{PSPACE}$. By colouring cells as in Figure 3, it is easy to see that $h(n) \leq \lceil \frac{n}{2} \rceil$. Before now, little else was known about $h(n)$.

In this paper, we show that $h(n) \leq \lceil \frac{n+1}{6} \rceil$. Our handicap strategy combines Shannon's $(n-1) \times n$ pairing strategy with the exploitation of a new kind of inferior cell: we colour handicap cells in the second row in a way that allows us

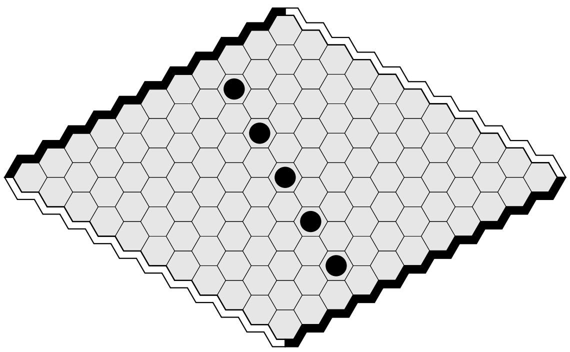

Figure 3. A winning Black handicap position.

to negate any opponent moves in the first row. The resulting $\lceil \frac{n+1}{6} \rceil$-cell handicap strategy is both explicit and efficient.

In Section 2 we review previous Hex inferior cell analysis, in Section 3 we introduce a new type of inferior cell, and in Section 4 we present our handicap strategy.

2. Inferior cell analysis

As discussed in [Berlekamp et al. 2001], inferior moves — such as dominated and reversible moves — can be pruned from consideration or bypassed; similar results have been proven in Hex. Following observations by Beck, Bleicher and Crowe [Beck et al. 1969] and Schensted and Titus [1975], Hayward and van Rijswijck defined a class of provably useless cells: with respect to a particular Hex position, an uncoloured cell c is *dead* if there does not exist some *completion* of the position (namely, a colouring of all uncoloured cells) in which changing c's colour changes the winner in the completion [Hayward and van Rijswijck 2006]. An equivalent definition is that a cell is dead if and only if it is not on any minimal winning path.

Although identifying dead cells is co-NP-complete on planar graphs [Björnsson et al. 2007], some dead cells can be recognized by matching patterns of neighbouring cells. For example, for each pattern in Figure 4 the uncoloured cell is dead [Hayward et al. 2004]. Since there is no completion in which a dead cell's colour matters, it follows that a dead cell can be assigned an arbitrary colour without changing the position's outcome (i.e., winner assuming optimal play) and that any move to a dead cell is useless.

An uncoloured cell c is *vulnerable* for player P if her opponent $\overline{P}$ has a move that makes c dead; this $\overline{P}$-move is c's *killer*. For example, in each pattern in Figure 5 the dotted cell is White-vulnerable to the shaded cell. Moves to vulnerable cells are reversible, but instead of being bypassed they can be pruned entirely from consideration; if a player has a winning strategy in a Hex position,

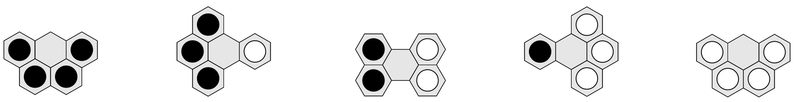

Figure 4. Dead patterns.

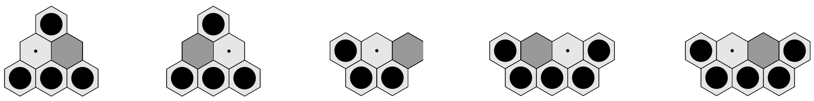

Figure 5. White-vulnerable patterns.

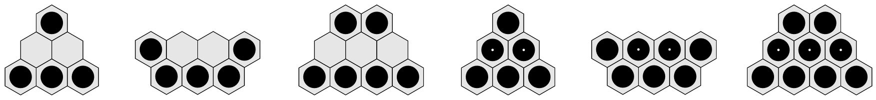

Figure 6. Black-captured patterns (left). Black-colouring a Black-captured set does not alter a position's value (right).

then they have a winning strategy that does not consider any dead or vulnerable cells [Hayward and van Rijswijck 2006].

A set S of uncoloured cells is *captured* by P if she has a second-player strategy to make all cells in S dead or her colour. P-colouring the cells of a P-captured set does not alter a position's outcome: the winning strategy in the original position is simply the combinatorial sum of the captured set strategy with the winning strategy on the filled-in board [Hayward 2003]. Figure 6 illustrates captured patterns relevant to our handicap strategy. See [Björnsson et al. 2007; Hayward 2003; Hayward et al. 2004; Hayward and van Rijswijck 2006] for more on Hex inferior cell analysis.

3. Permanently inferior cells

Before describing a new kind of inferior Hex cell, we first generalize the notion of vulnerability. A cell c is *vulnerable-by-capture* for player P if her opponent $\bar{P}$ has a move that captures a set S which when $\bar{P}$-coloured makes c dead; the $\bar{P}$-move is the *killer*, and the subsequent capturing strategy is the *killing strategy*. See Figure 7. S can be empty, so vulnerable-by-capture generalizes vulnerable. As with vulnerable cells, if player P has a winning strategy then P has a winning strategy that does not consider any P-vulnerable-by-capture cells.

We now introduce a new kind of inferior cell:

Definition 1. Given a Hex position with a set C of uncoloured cells and distinct cells $c_1, c_2 \in C$ such that c_1 is $\bar{P}$-vulnerable to c_2, and each cell in $C \setminus \{c_2\}$ is P-vulnerable-by-capture to a killing strategy using only cells in C, then we say that c_1 is *permanently inferior* for P and that C is the corresponding *permanently inferior region*.

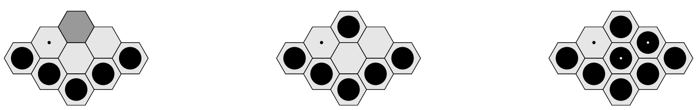

Figure 7. A White-vulnerable-by-capture cell (left, dotted cell). If Black plays the original shaded cell (middle), Black captures cells which in turn kill the original dotted cell (right).

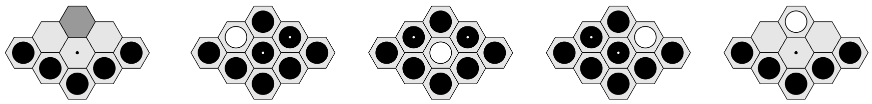

Figure 8. A White-permanently inferior pattern (left): the dotted cell is White-permanently inferior. The three unshaded cells are each White-vulnerable-by-capture to the shaded cell (middle three). Also, White-colouring the shaded cell kills the dotted cell (right).

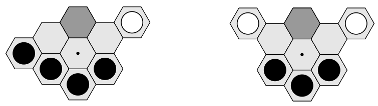

Figure 9. Two more White-permanently inferior patterns. Black-colouring a (dotted) White-permanently inferior cell does not alter a position's value by Lemma 1.

See Figure 8 for the White-permanently inferior pattern that will be applied in the handicap strategy. Figure 9 shows the only other known White-permanently inferior patterns.

Lemma 1. *Let c_1 be a P-permanently inferior cell in a Hex position S, and let T be the position obtained from S by $\overline{P}$-colouring c_1. Then the outcome of S equals the outcome of T.*

Proof. Extra $\overline{P}$-coloured cells are never disadvantageous for $\overline{P}$, so if $\overline{P}$ wins S then $\overline{P}$ wins T and we are done. Suppose then that P wins S. Thus P has a winning strategy for S in which she never plays a dead or P-vulnerable-by-capture cell. Let c_2 and C be as in Definition 1. Notice that the uncoloured cells of $C \setminus \{c_2\}$ remain P-inferior in any continuation from S in which P has not coloured any cell of $C \setminus \{c_2\}$: they will be either dead or P-vulnerable-by-capture. It follows that if P ever plays in C, then her first such move is to c_2, at which point c_1 is killed unless it is already $\overline{P}$-coloured. Thus P never plays at c_1, and her winning strategy also applies to position T. $\square$

As with captured cells, colouring a permanently inferior cell and finding a winning strategy on the filled-in board allows us to construct a winning strategy for the original position: if the opponent ever plays a vulnerable-by-capture cell in the permanently inferior region, follow the killing strategy; otherwise follow the filled-in board strategy.

4. Handicap strategy

We now describe Black's $\lceil \frac{n+1}{6} \rceil$-handicap strategy for $n \times n$ Hex, $n > 1$. The *handicap cells* are always in the second row, and located in column 4 for $n \geq 6$,

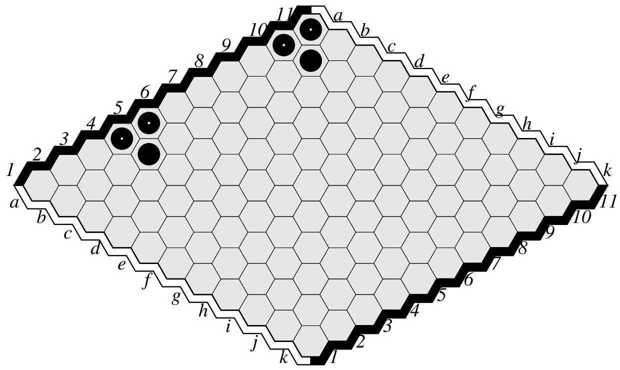

Figure 10. Handicap cell placement: handicap cells are solid black and primary cells are dotted black.

and in columns $n-1-6j$ for each j in $\{0, 1, \ldots, \max(0, \lfloor \frac{n}{6} \rfloor - 1)\}$. The *primary cells* are the first row cells adjacent to a handicap cell. See Figure 10.

Lemma 2. *Let S be an empty $n \times n$ Hex board with the addition of handicap cells as described above, and let T be the board obtained from S by filling-in the entire first row with Black cells. Then the outcome of S equals the outcome of T.*

Proof. The placement of handicap cells is such that two consecutive handicap cells are never separated by more than five uncoloured cells. Thus their respective primary cells are never separated by more than four uncoloured cells. By the captured patterns in Figure 6, it follows that all primary cells can be filled-in without altering the value of S; this completes the fill-in of all first row cells between two handicap cells that are adjacent or separated by a single uncoloured cell. See the two leftmost cases in Figure 11.

If the gap between two primary cells is of size one or two, then the first row cells in that gap are dead or captured. See the third and fourth cases in Figure 11, and the corresponding patterns in Figures 4 and 6. Thus in both these cases we can fill-in all first row cells between the two handicap cells.

If the gap between two primary cells is of size three or four, then the permanently inferior pattern in Figure 8 applies, filling-in the first row cells closest to the primary cells. See the rightmost case in Figure 11. Following this fill-in, the remaining first row gap is reduced to one or two uncoloured cells, which can once again be filled-in via dead or captured patterns as above. Similar arguments hold for any first row cells between a handicap cell and a White side. □

Theorem 1. *Black has a winning $\lceil \frac{n+1}{6} \rceil$-handicap strategy for $n \times n$ Hex.*

Proof. Placing the handicap cells as stated above requires $\lceil \frac{n+1}{6} \rceil$ cells for any $n \times n$ Hex board with $n > 1$. By Lemma 2 this initial handicap position is

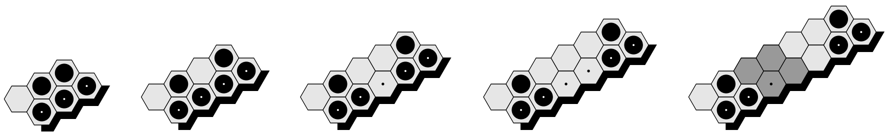

Figure 11. Strategies when the gap between two handicap cells is less than five. Primary cells (dotted black) are filled-in by captured patterns. Gaps of size one or two (unshaded dotted) are dead or captured as a consequence of capturing primary cells. Shaded regions indicate permanently inferior regions.

equivalent to an $(n-1) \times n$ board in Black's favour with $\lceil \frac{n+1}{6} \rceil$ first row Black cells on the board. Since additional Black cells cannot be detrimental to Black, it follows from Shannon's pairing strategy that this handicap position is a second-player Black win. □

Using the observations from Sections 2 and 3, we can construct a winning strategy for the original handicap position by combining the pairing strategy with the captured set and permanently inferior region strategies:

Black's handicap strategy. Colour the $\lceil \frac{n+1}{6} \rceil$ handicap cells. In response to each White move:

(1) if White colours a vulnerable-by-capture cell in a permanently inferior region (on the first two rows), then colour any killer;

(2) otherwise if White colours any Black-captured cell in the first two rows (i.e., primary cells, captured cells between primary cells, or cells in the killing strategy of a vulnerable-by-capture cell), then colour the corresponding reply in the capturing strategy;

(3) otherwise if White colours any cell with an uncoloured pair in the $(n-1) \times n$ Shannon strategy (obtained by ignoring the first row), then colour its pair;

(4) otherwise, colour any random cell.

As far as we know, this is the first two-stone handicap strategy for 11×11 Hex. The late Claude Berge, who was a Hex enthusiast [Berge 1977; 1981], would often give beginners three handicap stones on 11×11 Hex boards, suggesting that he did not expect them to find a winning strategy requiring fewer than four handicap stones. We would like to think that our result would have surprised him.

Acknowledgements

We thank the University of Alberta's Hex and GAMES group members for their feedback on this research, especially Broderick Arneson, Michael Johanson, and Morgan Kan.

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