On some deformations of Fukaya categories

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A certain deformation of a mirror pair in Strominger–Yau–Zaslow mirror setting is discussed. We propose that the mirror dual of a deformation of a complex manifold by a certain (real) deformation quantization is a symplectic manifold with a foliation structure. In order to support our claim that these deformations of mirror pairs are mirror dual to each other, we construct categories associated to these deformations of complex and symplectic manifolds and discuss homological mirror symmetry between them.

1. Introduction

In this paper, we discuss a certain deformation of a mirror pair in Strominger–Yau–Zaslow mirror setting. In particular, we construct categories associated to these deformations of complex and symplectic manifolds and discuss homological mirror symmetry between them.

One of our hopes is to understand how to formulate deformations of categories. This is motivated by what the homological mirror symmetry [Kontsevich 1995] is expected to reproduce: (genus zero part of) the mirror symmetry isomorphism of Frobenius manifolds [Kontsevich 1995; Barannikov and Kontsevich 1998]. In this story, a category is believed to reproduce a Frobenius manifold as a space of deformations of the category with suitable structures on it. However, at present, there is no formulation of deformations of categories which reproduces Frobenius

manifolds. Actually, it is already unclear how to obtain a manifold of the space
of deformations even locally.

In order to figure out what we should do for this problem, we can study some
examples. For noncommutative two-tori, we can actually construct categories
associated to them, which are regarded as deformations of categories on two
tori, and discuss homological mirror symmetry (see for example [Kajiura 2002;
2004; Polishchuk and Schwarz 2003; Kim and Kim 2007], and also [Kajiura
2008] for an overview). This is one of the easiest examples. In this example, the
situation seems to be easier since it is of real dimension two. Thus, in [Kajiura
2007] we tried to consider categories on noncommutative higher dimensional tori
to discuss their homological mirror symmetry. There, we constructed (curved)
DG-categories which are deformations of holomorphic vector bundles on higher
dimensional complex tori by (real) Moyal star products. A Moyal star product
(giving a deformation quantization) is defined by a constant Poisson bivector. For
$T^{2n} = T^n \times T^n$ viewed as a trivial T^n bundle over $B = T^n$, denote by $x_1, \ldots, x_n$
and $y_1, \ldots, y_n$ the coordinates of the base space and the fiber, respectively. Then
a Poisson bivector is in general of the form

$$\sum_{i,j=1}^{n} \left(\tfrac{1}{2}(\theta_1)_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + (\theta_2)_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} + \tfrac{1}{2}(\theta_3)_{ij} \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \right).$$

These coefficients are combined together into the skew-symmetric $2n \times 2n$ matrix

$$\Theta := \begin{pmatrix} \theta_1 & \theta_2 \\ -\theta_2^t & \theta_3 \end{pmatrix}.$$

In [Kajiura 2007], noncommutative deformations corresponding to θ_1, θ_2 and θ_3
are discussed independently. We see that deformation by θ_1 is that on the base
space, and deformation by θ_3 is that on the fiber. In the present paper, we discuss
deformation corresponding to θ_2, which includes the case of noncommutative
two-tori. For this deformation of θ_2 type, almost all of the arguments can
be generalized to SYZ torus fibration set-up. Therefore, we construct non-
commutative deformation of this kind for those torus fibrations. The mirror
dual of this noncommutative deformation is then given in Section 5B, as some
deformation of symplectic torus fibration. As is for noncommutative two-tori
[Kajiura 2002], a foliation structure associated to θ_2 appears naturally in this
mirror dual symplectic side. Note that the foliations we treat are different
from Lagrangian foliations discussed in [Fukaya 1998]. For our purpose, we
hope to keep the objects (= Lagrangians = A-branes) as unchanged as we
can, and deform the composition structure and other data in the corresponding
category. Therefore, our Lagrangians are not foliations but still submanifolds

even after deformation. Our foliation structure corresponds to the configuration of open strings stretching between A-branes. This viewpoint gives an intuitive understanding of the composition ($=$ product) structure in the corresponding category. See [Kajiura 2002] for two-tori case.

From the viewpoint of generalized geometry [Gualtieri 2003] (see also [Ben-Bassat 2006a; 2006b]), the deformations we discuss in the present paper are explained as follows. We start from a torus fibration $\check{M} \to B$ as a complex manifold, but here, for simplicity, we let $\check{M}$ be the "canonical" complex n-tori T^{2n} whose complex structure J is defined by

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}, \qquad n = 1, \ldots, n.$$

The generalized complex structure $\mathcal{J} : \Gamma(T\check{M} \oplus T^*\check{M}) \to \Gamma(T\check{M} \oplus T^*\check{M})$ associated to this J is given in matrix description by

$$\mathcal{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

where each block is an n by n matrix. Namely, for each point $p \in \check{M}$, we express $T_p\check{M} \oplus T_p^*\check{M}$ as the direct sum

$$(T_p B \oplus T_p \check{F}_p) \oplus (T_p^* B \oplus T_p^* \check{F}_p), \tag{1}$$

where $\check{F}_p$ is the fiber T^n of $\check{M} \to B$ which includes the point p. Then, for a given skew-symmetric $2n \times 2n$ matrix Θ,

$$\mathcal{J}_\Theta := \begin{pmatrix} 1_{2n} & -\Theta \\ 0_{2n} & 1_{2n} \end{pmatrix} \mathcal{J} \begin{pmatrix} 1_{2n} & \Theta \\ 0_{2n} & 1_{2n} \end{pmatrix}$$

again defines a generalized complex structure. This is believed to correspond to the noncommutative deformation by Θ. By a direct calculation, $\mathcal{J}_\Theta$ turns out to be

$$\mathcal{J}_\Theta = \begin{pmatrix} 1 & 0 & -\theta_1 & -\theta_2 \\ 0 & 1 & \theta_2^t & -\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \theta_1 & \theta_2 \\ 0 & 1 & -\theta_2^t & \theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & \theta_2 - \theta_2^t & \theta_3 - \theta_1 \\ -1 & 0 & \theta_3 - \theta_1 & \theta_2^t - \theta_2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

On the other hand, taking mirror dual is T-dualizing the fiber T^n, which corresponds to exchanging $T_p \check{F}_p$ for $T_p^* \check{F}_p$ in (1). Thus, the mirror dual generalized complex structure $\check{\mathcal{J}}_\Theta$ to $\mathcal{J}_\Theta$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & \theta_2-\theta_2^t & \theta_3-\theta_1 \\ -1 & 0 & \theta_3-\theta_1 & \theta_2^t-\theta_2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \theta_3-\theta_1 & \theta_2-\theta_2^t & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & \theta_2^t-\theta_2 & \theta_3-\theta_1 & 0 \end{pmatrix}.$$

This shows that, if and only if $\theta_1 = \theta_3 = 0$, $\check{\mathcal{J}}_\Theta$ is the generalized complex structure corresponding to a symplectic structure ω_θ, $\theta := \theta_2$, where

$$\omega_\theta = \tfrac{1}{2}\begin{pmatrix} dx & dy \end{pmatrix} \begin{pmatrix} \theta_2 - \theta_2^t & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}. \tag{2}$$

In general, for the class of torus fibrations called T^n-invariant manifolds or semiflat torus fibrations, the argument above is similar [Ben-Bassat 2006a; 2006b]. We may just modify the 1 in the above matrix in (2) by using the metric g on the base space B; see (15).

This is the generalized geometric background of the geometry we shall discuss. Namely, generalized geometry suggests that the mirror dual of the noncommutative deformation of $\check{M}$ of θ_2 type is the symplectic manifold (M, ω_θ). We agree with this suggestion, but our assertion is that the mirror dual (M, ω_θ) possesses naturally an additional foliation structure $\mathcal{F}_\theta$. We support this assertion by constructing the corresponding categories and discussing homological mirror symmetry.

The present paper is organized as follows. In Section 2, we briefly recall the construction of a mirror pair $M \to B$ and $\check{M} \to B$ as SYZ torus fibrations over a base manifold B. In Section 3, we discuss Lagrangian submanifolds in M and holomorphic vector bundles on $\check{M}$ associated to sections of the torus bundle $M \to B$. In Section 4, we discuss the homological mirror symmetry in the SYZ set-up. This part should be mainly equivalent to that given in [Kontsevich and Soibelman 2001], but we include some generalizations in order to discuss their deformations. In Section 4A, we define a curved DG-category $DG_{\check{M}}$ consisting of line bundles on $\check{M}$ and the full subcategory $DG_{\check{M}}(0)$ where the line bundles are holomorphic. In Section 4B, we define a curved DG-category DG_M consisting of sections of $M \to B$ and the full subcategory $DG_M(0)$ of Lagrangian sections. As we explain in Section 4C, these two curved DG-categories $DG_{\check{M}}$ and DG_M are canonically isomorphic to each other, where $DG_{\check{M}}(0) \simeq DG_M(0)$. Thus, we can say that the homological mirror symmetry holds true if $DG_M(0)$ is A_∞-equivalent to the corresponding (full subcategory of the) Fukaya category Fuk(M). In Section 4E, we explain a rough idea to show

an A_∞-equivalence $DG_M(0) \overset{A_\infty}{\sim} \mathrm{Fuk}(M)$ following [Kontsevich and Soibelman 2001].

We then discuss some deformations. First, we consider a (real) noncommutative deformation $\check{M}_\theta = (\check{M}, \theta)$ of $\check{M}$ in Section 5A and propose that the mirror dual of $\check{M}_\theta$ is the triple $M_\theta := (M, \omega_\theta, \mathscr{F}_\theta)$ in Section 5B. To support the claim that they are mirror to each other, we construct curved DG-categories $\mathscr{DG}_{M_\theta}$ and $\mathscr{DG}_{\check{M}_\theta}$ on M_θ and $\check{M}_\theta$, respectively. The relation of $\mathscr{DG}_{\check{M}_\theta}$ with $DG_{\check{M}}$ is as follows. We consider a full subcategory of DG_M consisting of objects associated to affine sections. One can define deformations of these objects, as discussed in Section 6A. We further modify the space of morphisms, and define the curved DG-category $\mathscr{DG}_{\check{M}_\theta}$ consisting of these deformed objects. These curved DG-categories $\mathscr{DG}_{M_\theta}$ and $\mathscr{DG}_{\check{M}_\theta}$ are defined so that they are canonically isomorphic to each other. A geometric interpretation of these deformed objects in $\mathscr{DG}_{M_\theta}$ is given in Section 6B. Finally we discuss the relation of $\mathscr{DG}_{M_\theta}$ with the Fukaya category associated to M_θ in Section 6E.

2. T^n-invariant manifolds

In this section, we briefly review the SYZ torus fibration set-up [Strominger et al. 1996]. For more details see [Leung et al. 2000; Leung 2005].

Throughout this paper, we consider an n-dimensional tropical Hessian manifold B, which we will define shortly, as the base space of a torus fibration. A smooth manifold B is called *affine* if B has an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ such that the coordinate transformation is affine. This means that, for any U_λ and U_μ such that $U_\lambda \cap U_\mu \neq \varnothing$, the coordinate systems $x_{(\lambda)} := (x_{(\lambda)}^1, \ldots, x_{(\lambda)}^n)^t$ and $x_{(\mu)} := (x_{(\mu)}^1, \ldots, x_{(\mu)}^n)^t$ are related to each other by

$$x_{(\mu)} = \varphi_{\lambda\mu} x_{(\lambda)} + \psi_{\lambda\mu}, \tag{3}$$

with some $\varphi_{\lambda\mu} \in \mathrm{GL}(n; \mathbb{R})$ and $\psi_{\lambda\mu} \in \mathbb{R}^n$. If in particular $\varphi_{\lambda\mu} \in \mathrm{GL}(n, \mathbb{Z})$ for any $U_\lambda \cap U_\mu$, then B is called *tropical affine*. (If in addition $\psi_{\lambda\mu} \in \mathbb{Z}^n$, B is called *integral affine*.) See [Gross 2011] for these materials.

For simplicity, we take such an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ so that the open sets U_λ and their intersections are all contractible. It is known that B is an affine manifold if and only if the tangent bundle TB is equipped with a torsion free flat connection. When B is affine, then its tangent bundle TB forms a complex manifold. This fact is clear as follows. For each open set $U = U_\lambda$, let us denote by $(x^1, \ldots, x^n; y^1, \ldots, y^n)$ the coordinates of $U \times \mathbb{R}^n \simeq TB|_U$ so that a point

$$\sum_{i=1}^n y^i \frac{\partial}{\partial x^i}\Big|_x \in T_x B \subset TB$$

corresponds to $(x^1, \ldots, x^n; y^1, \ldots, y^n) \in U \times \mathbb{R}^n$. We locally define the complex coordinate system by $(z^1, \ldots, z^n)$, where $z^i := x^i + \mathbf{i}y^i$ with $i = 1, \ldots, n$. By the coordinate transformation (3), the bases are transformed by

$$\frac{\partial}{\partial x_{(\mu)}} = \varphi_{\lambda\mu}^{t,-1} \frac{\partial}{\partial x_{(\lambda)}}, \quad \frac{\partial}{\partial x} := \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right)^t,$$

and hence the corresponding coordinates are transformed by

$$y_{(\mu)} = \varphi_{\lambda\mu} y_{(\lambda)}, \qquad y := (y^1, \ldots, y^n)^t$$

so that the combination $\sum_i y^i (\partial/\partial x^i)$ is independent of the coordinate systems. This shows that the transition functions for the manifold TB are given by

$$\begin{pmatrix} x_{(\mu)} \\ y_{(\mu)} \end{pmatrix} = \begin{pmatrix} \varphi_{\lambda\mu} & 0 \\ 0 & \varphi_{\lambda\mu} \end{pmatrix} \begin{pmatrix} x_{(\lambda)} \\ y_{(\lambda)} \end{pmatrix} + \begin{pmatrix} \psi_{\lambda\mu} \\ 0 \end{pmatrix},$$

and hence the complex coordinate systems are transformed holomorphically:

$$z_{(\mu)} = \varphi_{\lambda\mu} z_{(\lambda)} + \psi_{\lambda\mu}.$$

On the other hand, for any smooth manifold B, the cotangent bundle T^*B has a (canonical) symplectic form ω_{T^*B}. For each $U_\lambda = U$, when we denote the coordinates of $T^*B|_U \simeq U \times \mathbb{R}^n$ by $(x^1, \ldots, x^n; y_1, \ldots, y_n)$, ω_{T^*B} is given by

$$\omega_{T^*B} := d \left(\sum_{i=1}^n y_i \, dx^i \right) = \sum_{i=1}^n dx^i \wedge dy_i.$$

This is defined globally since the coordinate transformations on T^*B are induced from the coordinate transformations of $\{U_\lambda\}_{\lambda \in \Lambda}$. Actually, one has

$$dx_{(\mu)} = \varphi_{\lambda\mu} dx_{(\lambda)}$$

and the corresponding coordinates are transformed by

$$\check{y}_{(\mu)} = \varphi_{\lambda\mu}^{t,-1} \check{y}_{(\lambda)}, \quad \check{y} := (y_1, \ldots, y_n)^t, \tag{4}$$

so the combination $\sum_{i=1}^n y_i \, dx^i \in T^*B$ is independent of the coordinates. This implies that the symplectic form $\omega_{T^*B} = d \left(\sum_{i=1}^n y_i \, dx^i \right)$ is defined globally.

By choosing a metric g on a smooth manifold B, one obtains a bundle isomorphism between TB and T^*B (sometimes called a *musical isomorphism*). For each $b \in B$, this isomorphism $TB \to T^*B$ is defined by $\xi \mapsto g(\xi,)$ for $\xi \in T_b B$. This actually defines a bundle isomorphism since g is nondegenerate at each point $b \in B$. This bundle isomorphism also induces a diffeomorphism

from TB to T^*B. In this sense, hereafter we sometimes identify TB and T^*B. By this identification, y^i and y_i is related by

$$y_i = \sum_{j=1}^{n} g_{ij} y^j, \quad g_{ij} := g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

When an affine manifold B is equipped with a metric g that is expressed locally as

$$g_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}$$

for some local smooth function ϕ, we call (B, g) a *Hessian manifold*. When B is a Hessian manifold, $TB \simeq T^*B$ is equipped with the structure of Kähler manifold as we explain below. In this sense, a Hessian manifold is also called an affine Kähler manifold.

First, when B is affine, then TB is already equipped with the complex structure J_{TB}. We fix a metric g and set a two-form ω_{TB} on TB as

$$\omega_{TB} := \sum_{i,j=1}^{n} g_{ij}\, dx^i \wedge dy^j.$$

This ω_{TB} is nondegenerate since g is nondegenerate. Furthermore, ω_{TB} is closed if and only if (B, g) is Hessian, where ω_{TB} coincides with the pullback of ω_{T^*B} by the diffeomorphism $TB \to T^*B$. This is shown by direct calculations as follows. The closedness of ω_{TB} implies that

$$d(g_{ij}\, dx^i) = 0 \tag{5}$$

for each j. In this situation, for each i there exists a function ϕ_i of x such that

$$\frac{\partial}{\partial x^i} \phi_j = g_{ij},$$

since the one-form $\sum_{i=1}^{n} g_{ji}\, dx^i$ is closed and hence exact locally. Furthermore, $\sum_{i=1}^{n} \phi_i\, dx^i$ is closed:

$$d\left(\sum_{i=1}^{n} \phi_i\, dx^i\right) = \sum_{i=1}^{n} d(\phi_i)\, dx^i = \sum_{i,j=1}^{n} g_{ij}\, dx^j \wedge dx^i,$$

where g_{ij} is symmetric with respect to i, j. Thus, locally, there exists a function ϕ such that $d(\phi) = \sum_{i=1}^{n} \phi_i\, dx^i$, which implies that

$$g_{ij} = \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \phi.$$

Conversely, if (B, g) is Hessian, then we see that ω_{TB} is the pullback of the symplectic form ω_{T^*B} and hence is closed.

So a Hessian manifold (B, g) is equipped with the complex structure J_{TB} and the symplectic structure ω_{TB}. A metric g_{TB} on TB is then given by

$$g_{TB}(X, Y) := \omega_{TB}(X, J_{TB}(Y))$$

for $X, Y \in \Gamma(T(TB))$. This is locally expressed as

$$g_{TB} = \sum_{i,j=1}^{n} (g_{ij}\, dx^i\, dx^j + g_{ij}\, dy^i\, dy^j).$$

This shows that g_{TB} is positive definite. To summarize, for a Hessian manifold (B, g), $(TB, J_{TB}, \omega_{TB})$ forms a Kähler manifold, where g_{TB} is the Kähler metric.

In order to define a Kähler structure on T^*B, we employ the dual affine coordinates on B. As we saw in (5), $\sum_{j=1}^{n} g_{ij}\, dx^j$ is closed if (B, g) is Hessian. Thus, for each i, there exists a function $x_i := \phi_i$ of x such that

$$dx_i = \sum_{i=1}^{n} g_{ij}\, dx^j.$$

This again defines an affine structure on B. Actually, the local description of the metric is changed by

$$g_{(\mu)} = \left\{ (g_{(\mu)})_{ij} \right\}_{i,j=1,\ldots,n} = \varphi_{\lambda\mu}^{t,-1}\, g_{(\lambda)}\, \varphi_{\lambda\mu}^{-1},$$

so one has $d\check{x}_{(\mu)} = \varphi_{\lambda\mu}^{t,-1}\, d\check{x}_{(\lambda)}$ for $\check{x} := (x_1, \ldots, x_n)^t$ and then

$$\check{x}_{(\mu)} = \varphi_{\lambda\mu}^{t,-1}\check{x}_{(\lambda)} + \check{\psi}_{\lambda\mu} \tag{6}$$

for some $\check{\psi}_{\lambda\mu} \in \mathbb{R}^n$. Thus, the combinations $z_i := x_i + \mathbf{i}y_i$, $i = 1, \ldots, n$, form a complex coordinate system on T^*B, and T^*B forms a complex manifold. Actually, by (4) and (6), one has the holomorphic coordinate transformation

$$\check{z}_{(\mu)} = \varphi_{\lambda\mu}^{t,-1}\check{z}_{(\lambda)} + \check{\psi}_{\lambda\mu}, \quad \check{z} := (z_1, \ldots, z_n)^t.$$

Using these dual coordinates, the symplectic form ω_{T^*B} is expressed locally as

$$\omega_{T^*B} = \sum_{i,j=1}^{n} g^{ij}\, dx_i \wedge dy_j,$$

where g^{ij} is the (i, j) element of the inverse matrix of $\{g_{ij}\}$. Then, we set a metric on T^*B by

$$g_{T^*B}(X, Y) := \omega_{T^*B}(X, J_{T^*B}(Y))$$

for $X, Y \in \Gamma(T(T^*B))$, which is locally expressed as

$$g_{T^*B} = \sum_{i,j=1}^{n} (g^{ij} dx_i dx_j + g^{ij} dy_i dy_j).$$

These structures define a Kähler structure on T^*B.

For a tropical Hessian manifold B, we consider two T^n-fibrations over B obtained by a quotient M of TB and a quotient $\check{M}$ of T^*B by fiberwise $\mathbb{Z}^n$ action as follows.

For TB, we locally consider $TB|_U$ and define a $\mathbb{Z}^n$-action generated by $y^i \mapsto y^i + 2\pi$ for each $i = 1, \ldots, n$. For T^*B, we again locally consider $T^*B|_U$ and define a $\mathbb{Z}^n$-action generated by $y_i \mapsto y_i + 2\pi$ for each $i = 1, \ldots, n$. Both $\mathbb{Z}^n$-actions are well-defined globally since B is tropical affine, i.e., the transition functions of n-dimensional vector bundles TB and T^*B belong to $GL(n; \mathbb{Z})$. Then

$$M := TB/\mathbb{Z}^n$$

is a Kähler manifold whose symplectic structure ω_M and complex structure J_M are those naturally induced from ω_{TB} and J_{TB} on TB. Similarly,

$$\check{M} := T^*B/\mathbb{Z}^n$$

is a Kähler manifold whose symplectic structure $\omega_{\check{M}}$ and complex structure $J_{\check{M}}$ are those induced from ω_{T^*B} and J_{T^*B}, respectively. These M and $\check{M}$ are often called *semiflat torus fibrations* or T^n-*invariant manifolds*. See [Leung et al. 2000; Leung 2005; Fukaya 2005].

Strominger, Yau and Zalow's approach [Strominger et al. 1996] to the mirror symmetry is based on a pair $(M, \check{M})$ of Kähler manifolds as above. For a given Calabi–Yau manifold X, we first try to describe X as a semiflat torus fibration. To do so, unfortunately we need to include singular fibers in general. Suppose now that X is described as a semiflat torus fibration with singular fibers, where the total space of the general fibers M is a dense subset of X. Then, the mirror $\check{X}$ of X is defined by modifying $\check{M}$ by what is called the instanton corrections. The construction of the mirror pairs of this kind is now extended to more general Kähler manifolds instead of Calabi–Yau manifolds (see [Auroux 2007]). When X is a Kähler manifold which is not Calabi–Yau, one may again describe it as a semiflat torus fibration with singular fibers and consider the dual fibration $\check{M}$ of the generic fiber M of X, but one should further add contributions of some holomorphic disks associated to the singular fibers as a "(Landau–Ginzburg) superpotential" (see [Cho and Oh 2006; Auroux 2007; Fukaya 2005]). In the present paper, we do not include singular fibers and treat a pair $(M, \check{M})$ as a mirror

pair. More precisely, we will discuss a (mirror) duality between the symplectic manifold (M, ω_M) and the complex manifold $(\check{M}, J_{\check{M}})$, and its deformations.

3. Lagrangian submanifolds and holomorphic vector bundles

In this section, we discuss Lagrangian submanifolds in M and holomorphic vector bundles on $\check{M}$ associated to sections of the torus bundle $M \to B$. These are discussed in [Leung et al. 2000; Leung 2005]. See also [Fukaya 2005].

3A. *Lagrangian submanifolds in M.* We fix a tropical affine open covering $\{U_\lambda\}_{\lambda \in \Lambda}$. Let $\underline{s} : B \to M$ be a section of $M \to B$. Locally, we may regard $\underline{s}$ as a section of $TB \simeq T^*B$ and describe it by a collection of functions as

$$y^i_{(\lambda)} = s^i_{(\lambda)}(x)$$

on each U_λ.

On $U_\lambda \cap U_\mu$, these local expressions are related to each other by

$$s_{(\mu)}(x) = s_{(\lambda)}(x) + I_{\lambda\mu} \tag{7}$$

for some $I_{\lambda\mu} \in \mathbb{Z}^n$. Here, x may be identified with either $x_{(\lambda)}$ or $x_{(\mu)}$. Also, $s_{(\lambda)}(x)$ and $s_{(\mu)}(x)$ are expressed by the common coordinates $y_{(\lambda)}$ or $y_{(\mu)}$. This transformation rule automatically satisfies the cocycle condition

$$I_{\lambda\mu} + I_{\mu\nu} + I_{\nu\lambda} = 0 \tag{8}$$

for $U_\lambda \cap U_\mu \cap U_\nu \neq \varnothing$. We denote by s such a collection $\{s_{(\lambda)} : U_\lambda \to TB|_{U_\lambda}\}_{\lambda \in \Lambda}$ which is equipped with the transformation rule (7) satisfying the cocycle condition (8).

Now we discuss when the graph of $\underline{s}$ forms a Lagrangian submanifold in M. By definition, an n-dimensional submanifold L in a $2n$-dimensional symplectic manifold (M, ω_M) is *Lagrangian* if and only if $\omega_M|_L = 0$. This is a local condition. Thus, in order to discuss whether the graph of a section $\underline{s} : B \to M$ is Lagrangian or not, we may check the condition locally and in particular in T^*B.

It is known (as shown easily by taking the basis) that the graph of $\sum_{i=1}^n y_i \, dx^i$ with local functions y_i is Lagrangian in T^*B if and only if there exists a local function f such that $\sum_{i=1}^n y_i \, dx^i = df$. Now, a section $\underline{s} : B \to M$ is locally regarded as a section of T^*B by setting $y_i = \sum_{j=1}^n g_{ij} y^j = \sum_{j=1}^n g_{ij} s^j$, from which one has

$$\sum_{i=1}^n y_i \, dx^i = \sum_{i=1}^n \left(\sum_{j=1}^n g_{ij} s^j \right) dx^i = \sum_{j=1}^n s^j \, dx_j.$$

Thus, the graph of the section $\underline{s} : B \to M$ is Lagrangian if and only if there exists a local function f such that $\sum_{j=1}^n s^j \, dx_j = df$.

The easiest example of these Lagrangian sections is the zero section s_0. Namely, $y = s_0(x)$ is the zero function on any U_λ. The corresponding section $\underline{s_0} : B \to M$ is called the zero section of $M \to B$.

If the tropical Hessian manifold B is oriented, then there exists a holomorphic n-form Ω which is locally defined as

$$\Omega = dz^1 \wedge \cdots \wedge dz^n.$$

Thus, TB and M are almost Calabi–Yau manifolds.[1] For an almost Calabi–Yau manifold, one can define special Lagrangian manifolds. Let us briefly discuss when a section $\underline{s}$ defines a special Lagrangian submanifold, though it is not needed in this paper. A special Lagrangian submanifold L is by definition a Lagrangian submanifold satisfying

$$\mathrm{Im}(e^{\mathbf{i}\alpha}\Omega)|_L = 0 \tag{9}$$

for some $\alpha \in \mathbb{R}$. For a section $\underline{s} : B \to M$, a basis of the tangent vector space of the graph of $\underline{s}$ at $(x, s(x)) \in M$ is given by

$$\xi_i := \frac{\partial}{\partial x^i} + \sum_{j=1}^{n} \frac{\partial s^j(x)}{\partial x^i} \frac{\partial}{\partial y^j}, \quad i = 1, \ldots, n.$$

Thus, the condition (9) turns out to be

$$0 = \mathrm{Im}\big(e^{\mathbf{i}\alpha}(dz^1 \wedge \cdots \wedge dz^n)(\xi_1, \ldots, \xi_n)\big)$$

$$= \mathrm{Im}\left(e^{\mathbf{i}\alpha}\det\left(1 + \left\{\frac{\partial s^j(x)}{\partial x^i}\right\}_{i,j}\right)\right).$$

This implies that $y = s(x)$ satisfies this equation with some α if s is affine with respect to x^i. (Thus, the zero section of $M \to B$ is a special Lagrangian submanifold.) On the other hand, in this paper, sections $\underline{s} : B \to M$ which are affine with respect to $\check{x}$ play a central role as we see in Section 6C. They are characterized as objects having constant curvatures. Unfortunately, sections which are affine with respect to $\check{x}$ do not correspond to special Lagrangians even if $\underline{s}$ defines a Lagrangian.

3B. *Holomorphic vector bundles on $\check{M}$.*

Consider a section $\underline{s} : B \to M$ and express it as a collection $s = \{s_{(\lambda)}\}_{\lambda \in \Lambda}$ of local functions. We define a line bundle V with a $U(1)$-connection on the mirror manifold $\check{M}$ associated to s. We

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[1] A Kähler manifold equipped with a nowhere vanishing holomorphic top form is called an almost Calabi–Yau manifold.

set the covariant derivative locally as

$$D := d + \mathbf{i} \sum_{i=1}^{n} s^i(x) dy_i,$$

whose curvature is

$$D^2 = \mathbf{i} \sum_{i,j=1}^{n} \frac{\partial s^i}{\partial x_j} dx_j \wedge dy_i.$$

The $(0, 2)$-part vanishes if and only if the matrix $(\partial s^i / \partial x_j)$ is symmetric, which is the case when there exists a function f locally such that $df = \sum_{i=1}^{n} s^i dx_i$. Thus, the condition that D defines a holomorphic line bundle on $\check{M}$ is equivalent to that the graph of $\underline{s}$ is Lagrangian in M.

This covariant derivative D is in fact defined globally. Suppose that D is given locally on each $\check{M}|_{U_\lambda}$ of the T^n-fibration $\check{M} \to B$ with a fixed tropical affine open covering $\{U_\lambda\}_{\lambda \in \Lambda}$. Namely, we continue to employ $\{U_\lambda\}_{\lambda \in \Lambda}$ for local trivializations of the line bundle associated to a section $\underline{s} : B \to M$. The transition functions for (V, D) are defined as follows. Recall that the section $\underline{s} : B \to M$ is expressed locally as

$$y_{(\lambda)}^i = s_{(\lambda)}^i(x)$$

on each U_λ, where, on $U_\lambda \cap U_\mu$, the local expression is related to each other by

$$s_{(\mu)}(x) = s_{(\lambda)}(x) + I_{\lambda\mu}$$

for some $I_{\lambda\mu} \in \mathbb{Z}^n$ (see (7)). Correspondingly, the transition function for the line bundle V with the connection D is given by

$$\psi_{(\mu)} = e^{-\mathbf{i} I_{\lambda\mu} \cdot \check{y}} \psi_{(\lambda)}$$

for local expressions $\psi_{(\lambda)}$, $\psi_{(\mu)}$ of a smooth section ψ of V, where

$$I_{\lambda\mu} \cdot \check{y} := \sum_{j=1}^{n} i_j y_j$$

for $I_{\lambda\mu} = (i_1, \ldots, i_n)$. We see the compatibility

$$(D\psi_{(\lambda)})_{(\mu)} = D(\psi_{(\mu)})$$

holds true since the left-hand side turns out to be

$$e^{-\mathbf{i} I_{\lambda\mu} \cdot \check{y}} ((d + \mathbf{i} s_{(\lambda)}(x) \cdot dy) e^{\mathbf{i} I_{\lambda\mu} \cdot \check{y}} \psi_{(\mu)})$$
$$= e^{-\mathbf{i} I_{\lambda\mu} \cdot \check{y}} e^{\mathbf{i} I_{\lambda\mu} \cdot \check{y}} ((d + \mathbf{i}(s_{(\lambda)}(x) + I_{\lambda\mu}) \cdot dy) \psi_{(\mu)})$$
$$= (d + \mathbf{i} s_{(\mu)}(x) \cdot dy) \psi_{(\mu)}.$$

Since (V, D) is locally-trivialized by $\{\check{M}|_{U_\lambda}\}_{\lambda \in \Lambda}$, for each $x \in B$, $\psi(x, \cdot)$ gives a smooth function on the fiber T^n. Thus, on each U_λ, $\psi(x, y)$ can be Fourier-expanded as

$$\psi(x, y)|_{U_\lambda} = \sum_{I \in \mathbb{Z}^n} \psi_{\lambda, I}(x) e^{\mathbf{i} I \cdot \check{y}},$$

where $I \cdot \check{y} := \sum_{j=1}^n i_j y_j$ for $I = (i_1, \ldots, i_n)$. Note that each coefficient $\psi_{\lambda, I}$ is a smooth function on U_λ. In this expression, the transition function acts to each $\psi_{\lambda, I}$ as

$$\sum_{I \in \mathbb{Z}^n} \psi_{\mu, I} e^{\mathbf{i} I \cdot \check{y}} = e^{-\mathbf{i} I_{\lambda\mu} \cdot \check{y}} \sum_{I \in \mathbb{Z}^n} \psi_{\lambda, I} e^{\mathbf{i} I \cdot \check{y}}$$

$$= \sum_{I \in \mathbb{Z}^n} \psi_{\lambda, I} e^{\mathbf{i}(I - I_{\lambda\mu}) \cdot \check{y}}$$

$$= \sum_{I \in \mathbb{Z}^n} \psi_{\lambda, I + I_{\lambda\mu}} e^{\mathbf{i} I \cdot \check{y}},$$

and hence $\psi_{\mu, I} = \psi_{\lambda, I + I_{\lambda\mu}}$.

4. Two (curved) DG-categories

4A. *Curved DG-category $DG_{\check{M}}$ associated to $\check{M}$.* We define a curved DG-category $DG_{\check{M}}$ as follows. The objects are line bundles V with $U(1)$-connections D associated to lifts s of sections as we defined in Section 3B. We often label these objects as s instead of (V, D). For any two objects $s_a = (V_a, D_a)$, $s_b = (V_b, D_b) \in DG_{\check{M}}$, the space $DG_{\check{M}}(s_a, s_b)$ of morphisms is defined by

$$DG_{\check{M}}(s_a, s_b) := \Gamma(V_b, V_a) \otimes_{C^\infty(\check{M})} \Omega^{0,*}(\check{M}),$$

where $\Omega^{0,*}(\check{M})$ is the space of antiholomorphic differential forms, and $\Gamma(V_b, V_a)$ is the space of homomorphisms from V_b to V_a. The space $DG_{\check{M}}(s_a, s_b)$ is a $\mathbb{Z}$-graded vector space, where the grading is defined as the degree of the antiholomorphic differential forms. The degree r part is denoted $DG_{\check{M}}^r(s_a, s_b)$. We define a linear map

$$d_{ab} : DG_{\check{M}}^r(s_a, s_b) \to DG_{\check{M}}^{r+1}(s_a, s_b)$$

as follows. We decompose D_a into its holomorphic part and antiholomorphic part $D_a = D_a^{(1,0)} + D_a^{(0,1)}$, and set $2D_a^{(0,1)} =: d_a$. Then, for $\psi \in DG_{\check{M}}^r(s_a, s_b)$, we set

$$d_{ab}(\psi) := d_a \psi - (-1)^r \psi d_b \in DG_{\check{M}}^{r+1}(s_a, s_b).$$

In particular, when s_b is the zero section, $s_b = s_0$, the differential

$$d_{a0} : DG_M^r(s_a, s_0) \to DG_M^{r+1}(s_a, s_0)$$

is also denoted $d_{a0} =: d_a$. We call the two-form W_a defined by

$$(d_a)^2 = W_a \wedge \; = \left(\frac{1}{2} \sum_{i,j=1}^{n} W_a^{ij} \, d\bar{z}_i \wedge d\bar{z}_j \right) \wedge$$

the *curvature* of the object $s_a \in DG_{\check{M}}$.

The product structure $m : DG_{\check{M}}(s_a, s_b) \otimes DG_{\check{M}}(s_b, s_c) \to DG_{\check{M}}(s_a, s_c)$ is defined by the composition of homomorphisms of line bundles together with the wedge product for the antiholomorphic differential forms. Then, $DG_{\check{M}}$ forms a curved DG-category.[2]

In particular, the full subcategory $DG_{\check{M}}(0)$ consisting of holomorphic line bundles forms a DG-category since $(d_a)^2 = 0$ for a holomorphic line bundle (V_a, D_a).

In order to construct another equivalent curved DG-category, we rewrite this curved DG-category $DG_{\check{M}}$ more explicitly. We Fourier-expand morphisms as we did for a section of a line bundle in Section 3B. For two objects $s_a, s_b \in DG_{\check{M}}$, $DG^0_{\check{M}}(s_a, s_b)$ is the space of sections of V_a if s_b is the zero-section, $s_b = s_0$, in TB. Now, we discuss $DG^r_{\check{M}}(s_a, s_b)$ with general s_a, s_b and r. For an element $\psi \in DG^r_{\check{M}}(s_a, s_b)$, we express this locally as

$$\psi(x, y) = \sum_{I \in \mathbb{Z}^n} \psi_I(x) e^{\mathbf{i} I \cdot \check{y}},$$

where ψ_I is locally a smooth antiholomorphic differential form of degree r. Namely, it is expressed as

$$\psi_I = \sum_{i_1, \ldots, i_r} \psi_{I; i_1 \cdots i_r} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_r},$$

with smooth functions $\psi_{I; i_1 \cdots i_r}$. Let us express the transformation rules for s_a and s_b as

$$(s_a)_{(\mu)} = (s_a)_{(\lambda)} + I_a, \quad (s_b)_{(\mu)} = (s_b)_{(\lambda)} + I_b$$

with $I_a = I_{a; \lambda \mu} \in \mathbb{Z}^n$, $I_b = I_{b; \lambda \mu} \in \mathbb{Z}^n$. Then, the transition function is given by $\psi_{(\mu)} = e^{-\mathbf{i}(I_a - I_b) \cdot \check{y}} \psi_{(\lambda)}$, and hence

$$\psi_{(\mu), I} = \psi_{(\lambda), I + I_a - I_b}.$$

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[2] A curved DG-category is a generalization of a DG-category such that the differentials are replaced by "curved differentials", i.e., their squares are in general nonzero and are regarded as curvatures. The precise definition can be found in [Kajiura 2008] and references therein.)

The differential d_{ab} is expressed locally as follows. Since

$$D_a = d + \mathbf{i} \sum_{j=1}^{n} s_a^j(x) dy_j$$

$$= \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} dx_j + \left(\frac{\partial}{\partial y_j} + \mathbf{i} s_a^j \right) dy_j \right)$$

$$= \frac{1}{2} \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} - \mathbf{i} \left(\frac{\partial}{\partial y_j} + \mathbf{i} s_a^j \right) \right) dz_j + \frac{1}{2} \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} + \mathbf{i} \left(\frac{\partial}{\partial y_j} + \mathbf{i} s_a^j \right) \right) d\bar{z}_j,$$

one has

$$d_a = 2 D_a^{(0,1)} = \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} - s_a^j + \mathbf{i} \frac{\partial}{\partial y_j} \right) d\bar{z}_j$$

and then

$$d_{ab}(\psi) = 2\bar{\partial}(\psi) - \sum_{i=1}^{n} (s_a - s_b)^i \, d\bar{z}_i \wedge \psi. \qquad (10)$$

4B. *Curved DG-category DG_M associated to M.* We define a curved DG-category DG_M as follows. As we shall see, we construct it so that it is canonically isomorphic to the previous curved DG-category $DG_{\check{M}}$. We fix a tropical affine open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of B.

The objects are the same as those in $DG_{\check{M}}$, that is, lifts s of sections of $M \to B$. For any two objects $s_a, s_b \in DG_M$, we express the transformation rules for s_a and s_b as

$$(s_a)_{(\mu)} = (s_a)_{(\lambda)} + I_a, \quad (s_b)_{(\mu)} = (s_b)_{(\lambda)} + I_b,$$

as we did in the previous subsection. For each $\lambda \in \Lambda$ and $I \in \mathbb{Z}^n$, let $\Omega_{\lambda,I}(s_a, s_b)$ be the space of complex-valued smooth differential forms on U_λ. The space $DG_M(s_a, s_b)$ is then the subspace of

$$\prod_{\lambda \in \Lambda} \prod_{I \in \mathbb{Z}^n} \Omega_{\lambda,I}(s_a, s_b)$$

consisting of elements with the following properties:

- $\phi_{\lambda,I} \in \Omega_{\lambda,I}(s_a, s_b)$ satisfies

$$\phi_{\mu,I}|_{U_\lambda \cap U_\mu} = \phi_{\lambda, I+I_a-I_b}|_{U_\lambda \cap U_\mu}$$

  whenever $U_\lambda \cap U_\mu \neq \varnothing$.

- The sum $\sum_{I \in \mathbb{Z}^n} \phi_{\lambda,I} e^{\mathbf{i} I \cdot \check{y}}$ converges as smooth differential forms on each $M|_{U_\lambda}$.

The space $DG_M(s_a, s_b)$ is a $\mathbb{Z}$-graded vector space, where the grading is defined as the degree of the differential forms. The degree r part is denoted $DG_M^r(s_a, s_b)$. We define a linear map $d_{ab} : DG_M^r(s_a, s_b) \to DG_M^{r+1}(s_a, s_b)$ which is expressed locally as

$$d_{ab}(\phi_{\lambda,I}) := d(\phi_{\lambda,I}) - \sum_{j=1}^{n} (s_a^j - s_b^j + i_j) \, dx_j \wedge \phi_{\lambda,I}$$

for $\phi_{\lambda,I} \in \Omega_{\lambda,I}(s_a, s_b)$ with $I := (i_1, \dots, i_n) \in \mathbb{Z}^n$, where d is the exterior differential on B. In particular, when $s_b = s_0$, the differential

$$d_{a0} : DG_M^r(s_a, s_0) \to DG_M^{r+1}(s_a, s_0)$$

is denoted by $d_{a0} =: d_a$. We call the two-form W_a defined by

$$(d_a)^2 = W_a \wedge \; = \left(\frac{1}{2} \sum_{i,j=1}^{n} W_a^{ij} \, dx_i \wedge dx_j \right) \wedge$$

the *curvature* of the object $a \in DG_M$. One has

$$(d_{ab})^2 = \sum_{i,j=1}^{n} - \left(\frac{\partial s_a^j}{\partial x_i} - \frac{\partial s_b^j}{\partial x_i} \right) dx_i \wedge dx_j \wedge = (W_a - W_b) \wedge .$$

Therefore, d_{ab} defines a differential if and only if s_a and s_b have the same curvature $W_a = W = W_b$. In particular, if s_a and s_b are both Lagrangian submanifolds, then $W_a = 0 = W_b$ and hence d_{ab} is a differential. Note that the curvature W_a is regarded as an element in $DG_M^2(s_a, s_a)$ such that $W_{a;\lambda,I} = W_a|_{U_\lambda}$ if $I = 0$ and otherwise $W_{a;\lambda,I} = 0$.

The composition of morphisms

$$m : DG_M(s_a, s_b) \otimes DG_M(s_b, s_c) \to DG_M(s_a, s_c)$$

is defined by

$$m(\phi_{ab;\lambda,I}, \phi_{bc;\lambda,J}) := \phi_{ab;\lambda,I} \wedge \phi_{bc;\lambda,J} \; \in \Omega_{\lambda,I+J}(s_a, s_c)$$

for $\phi_{ab;\lambda,I} \in \Omega_{\lambda,I}(s_a, s_b)$ and $\phi_{bc;\lambda,J} \in \Omega_{\lambda,J}(s_b, s_c)$. These structures define a curved DG-category DG_M. In particular, the full subcategory $DG_M(0)$ consisting of (lifts of) Lagrangian submanifolds is a DG category since $W_a = 0$ for s_a such that $\underline{s}_a$ is a Lagrangian submanifold. Note that this $DG_M(0)$ is believed to be A_∞-equivalent to the corresponding full subcategory of the Fukaya category $\mathrm{Fuk}(M)$. (Compare this $DG_M(0)$ with what is called the de Rham model for the Fukaya category in [Kontsevich and Soibelman 2001], in particular a construction

in the Appendix (Section 9.2).) In Section 4E, we shall explain the outline of how to compare $DG_M(0)$ with the Fukaya category.

For a fixed two form W, the full subcategory $DG_M(W)$ consisting of objects s_a with curvature $W_a = W$ also forms a DG-category. Later (in Section 6E) we shall discuss generalizations of $DG_M(0)$ of this kind.

4C. *Equivalence between DG_M and $DG_{\check{M}}$.* The curved DG-category DG_M is canonically isomorphic to the curved DG-category $DG_{\check{M}}$. In fact, we see that the objects in DG_M are the same as those in $DG_{\check{M}}$. The spaces of morphisms in DG_M and in $DG_{\check{M}}$ are also identified canonically as follows. For a morphism $\phi_{ab} = \{\phi_{ab;\lambda,I}\} \in DG_{M,I}^r(s_a, s_b)$, each $\phi_{ab;\lambda,I}$ is expressed as

$$\phi_{ab;\lambda,I} = \sum_{i_1,\dots,i_r} \phi_{ab;\lambda,I;i_1\cdots i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r}.$$

To this, we correspond an element in $DG_{\check{M}}^r(s_a, s_b)$ which is locally given as

$$\sum_{i_1,\dots,i_r} (\phi_{ab;\lambda,I;i_1\cdots i_r} e^{\mathbf{i}I\cdot\check{y}}) \, d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_r}$$

on U_λ. Let us denote this correspondence by

$$\mathfrak{f}: DG_M \to DG_{\check{M}}, \quad \mathrm{id}: \mathrm{Ob}(DG_M) \to \mathrm{Ob}(DG_{\check{M}})$$
$$f_1: DG_M(s_a, s_b) \to DG_{\check{M}}(s_a, s_b).$$

Proposition 4.1. *The functor $\mathfrak{f}: DG_M \to DG_{\check{M}}$ is a curved DG-isomorphism.*

Proof. It is obvious that $\mathfrak{f}$ preserves the product structure in these curved DG-categories. It is also clear that $f_1(W_a) = W_a$ for any $s_a \in DG_M$. The remaining thing to be checked is the compatibility of this correspondence $\mathfrak{f}$ with the operations d in both sides. Namely, we now check that

$$f_1(d_{ab}(\phi_I)) = d_{ab}(f_1(\phi_I)) \tag{11}$$

holds true. Since d_{ab} is expressed locally as

$$d_{ab} = 2\bar{\partial}(\psi) - \sum_{i=1}^n (s_a - s_b)^i \, d\bar{z}_i,$$

(see (10)), where $2\bar{\partial} = \sum_{j=1}^n d\bar{z}_j \left(\dfrac{\partial}{\partial x_j} + \mathbf{i}\dfrac{\partial}{\partial y_j} \right)$, the right-hand side of (11) becomes

$$d_{ab}(\psi) = \sum_{j=1}^n d\bar{z}_j \left(\frac{\partial}{\partial x_j} - \left(s_a^j - s_b^j - \mathbf{i}\frac{\partial}{\partial y_j} \right) \right)(\psi)$$

for $\psi = \sum_{i_1,\dots,i_r} (\phi_{I;i_1\cdots i_r} e^{iI\cdot\check{y}})\, d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_r}$. It is clear that this coincides with the left-hand side of (11). $\square$

4D. *On isomorphisms between objects.* Since objects in DG_M or $DG_{\check{M}}$ are the lifts $\underline{s}$ of sections of $M \to B$, different objects s and s' defining the same section $\underline{s} = \underline{s}' : B \to M$ should be isomorphic to each other. We shall confirm here that this actually holds true.

Given a section $\underline{s} : B \to M$, let $s \in DG_M$ be an object defined as $y = s_{(\lambda)}(x)$ on each U_λ. The identity morphism on $s \in DG_M$ is given by $\{\phi_{\lambda,I}\} \in DG_M(s,s)$ with

$$\phi_{\lambda,I} = \begin{cases} 1 & I = 0, \\ 0 & I \neq 0. \end{cases}$$

Now, let us consider another object $s' \in DG_M$ which is defined as

$$y = s'_{(\lambda)}(x) = s_{(\lambda)}(x) + J_{(\lambda)},$$

with some $J_{(\lambda)} \in \mathbb{Z}^n$ for each $\lambda \in \Lambda$. One sees that this together with the appropriate transformation rule actually gives an object $s' \in DG_M$ which corresponds to the same section $\underline{s} : B \to M$. Then, a morphism $\phi = \{\phi_{\lambda,I}\} \in DG_M(s,s')$ defined by

$$\phi_{\lambda,I} = \begin{cases} 1 & I = -J_{(\lambda)}, \\ 0 & \text{otherwise}, \end{cases}$$

is a closed morphism. Similarly, a morphism $\phi' = \{\phi'_{\lambda,I}\} \in DG_M(s',s)$ defined by $\phi'_{\lambda,J_{(\lambda)}} = 1$ and zero otherwise is a closed morphism. These ϕ and ϕ' give isomorphisms between s and s'.

Since $DG_{\check{M}}$ is canonically isomorphic to DG_M, there exist similar isomorphisms in $DG_{\check{M}}$. Namely, s and s' are isomorphic to each other in $DG_{\check{M}}$ if $\underline{s} = \underline{s}'$. Note that this implies that s and s' are in fact isomorphic line bundles.

4E. *The DG-category $DG_M(0)$ and the Fukaya category* Fuk(M). In this subsection, we discuss the relation of the DG-category $DG_M(0)$ with the Fukaya category Fuk(M) introduced in [Fukaya 1993]. The idea to relate them is to apply homological perturbation theory to the DG-category $DG_M(0)$ (as an A_∞-category) in an appropriate way so that the induced A_∞-category coincides with the full subcategory of the Fukaya category Fuk(M). More precisely, what should be induced directly from $DG_M(0)$ is the Fukaya–Oh category for the torus fibration $M \to B$ introduced in Section 5.2 of [Kontsevich and Soibelman 2001]. Here, the Fukaya–Oh category means the A_∞-category of Morse homotopy on B introduced in [Fukaya 1993]. It is shown in [Fukaya and Oh 1997] that the Fukaya–Oh category is equivalent to (a full subcategory of) the Fukaya category Fuk(T^*B). The Fukaya–Oh category for the torus fibration $M \to B$ is

a generalization of the Fukaya–Oh category on B so that it corresponds to the Fukaya category $\mathrm{Fuk}(M)$ instead of $\mathrm{Fuk}(T^*B)$. Below, we shall relate $DG_M(0)$ to the Fukaya category $\mathrm{Fuk}(M)$ in this way.

For a given DG-category or more generally an A_∞-category $\mathscr{C}$, assume that a deformation retract is given for each space of morphisms. Here, we say that a *deformation retract* is given for a chain complex (V, d) when we are given a degree minus one map $h : V^r \to V^{r-1}$ such that $P := \mathrm{id} - (dh + hd)$ is an idempotent. Then, the restriction of d onto PV still defines a differential. Namely, we obtain a subcomplex (PV, d_P) of (V, d). If in particular $d_P = 0$, then PV is isomorphic to the cohomology of (V, d). In this case, we call the deformation retract a *Hodge decomposition*. Then, the homological perturbation theory gives us another A_∞-category $\mathscr{C}'$ and an A_∞-quasiisomorphism $\mathscr{C}' \to \mathscr{C}$. Here, an A_∞-quasiisomorphism $\mathfrak{f} := \{f; f_1, f_2, \dots\}$ is an A_∞-functor such that

- $f : \mathrm{Ob}(\mathscr{C}') \to \mathrm{Ob}(\mathscr{C})$ is bijective and

- $f_1 : \mathscr{C}'(a, b) \to \mathscr{C}(f(a), f(b))$ is a quasiisomorphism for each pair of objects $a, b \in \mathscr{C}'$.

Since f is bijective, hereafter we drop f and identify objects of $\mathscr{C}'$ with those of $\mathscr{C}$. Then, the space $\mathscr{C}'(a, b)$ of morphisms which the homological perturbation theory gives is the subcomplex of $\mathscr{C}(a, b)$ given by the deformation retract.

Following the idea in [Kontsevich and Soibelman 2001], our plan is to adjust these homotopy operators h on the space of morphisms in $\mathscr{C} = DG_M$ so that the resulting A_∞-category $\mathscr{C}'$ coincides with (a full subcategory of) the Fukaya category $\mathrm{Fuk}(M)$. As we shall see below, there are many difficulties in proceeding this plan. However, we shall show some evidence implying that the plan can be accomplished. We hope to come back to presenting the details elsewhere.

First, in this story we assume that B is compact. Otherwise in general we need to modify the space of morphisms by imposing some conditions for the asymptotic behavior near the 'boundaries'. Given two objects $s_a, s_b \in DG_M(0)$, suppose that the corresponding graphs are transversal to each other. In this case we say that s_a and s_b are transversal. For $d_{ab} : DG_M^r(s_a, s_b) \to DG_M^{r+1}(s_a, s_b)$, we consider the following one-parameter family

$$d_{\epsilon;ab}^\dagger : DG_M^{r+1}(s_a, s_b) \to DG_M^r(s_a, s_b)$$

of operators:

$$d_{ab,\epsilon}^\dagger(\phi_{ab;\lambda,I}) := \left(\epsilon d^* - \sum_{i,j=1}^n (s_a - s_b + I)^i g_{ij} \, \iota_{\partial/\partial x_j} \right) \phi_{ab;\lambda,I},$$

where d^* is the adjoint of d and $\iota_{\partial/\partial x_i}$ is the inner derivation by $\partial/\partial x_i$. Then, a

one-parameter family $h_{\epsilon;ab}$ of homotopy operators is defined as

$$h_{\epsilon;ab} := d^{\dagger}_{\epsilon;ab} \int_0^{\infty} dt \, \exp\big(-(d_{ab}d^{\dagger}_{\epsilon;ab} + d^{\dagger}_{\epsilon;ab}d_{ab})\,t\big).$$

For a pair (s_a, s_b) which are not transversal to each other, we (tentatively) set $h_{ab,\epsilon} = 0$. These homotopy operators $h_{\epsilon;ab}$ define deformation retracts of $DG_M(s_a, s_b)$, where

$$P_{\epsilon;ab} := \mathrm{id} - (d_{ab}h_{\epsilon;ab} + h_{\epsilon;ab}d_{ab})$$

is the idempotent defining the space

$$\mathscr{C}'_{\epsilon}(s_a, s_b) := P_{\epsilon;ab} DG_M(s_a, s_b).$$

In particular, this is a Hodge decomposition when s_a and s_b are transversal to each other. Then, the family $\mathscr{C}'_{\epsilon}$ of A_{∞}-categories is obtained via the homological perturbation theory.

For each transversal pair s_a, s_b, the operators $d_{ab}, d^{\dagger}_{\epsilon;ab}$ and then $h_{\epsilon;ab}$ can also be regarded as operators on a covering space of B as follows. Since s_a and s_b define Lagrangians, locally there exist functions f_a and f_b such that $\sum_{i=1}^{n} s_a^i dx_i = df_a$ and $\sum_{i=1}^{n} s_b^i dx_i = df_b$. Then,

$$d_{ab} : DG_{M,I}(s_a, s_b) \to DG_{M,I}(s_a, s_b)$$

is expressed as

$$d_{ab} = d - (s_a - s_b + I) \cdot d\check{x} = d - d(f_a - f_b + I \cdot \check{x}) \wedge .$$

We formally introduce $\mathbb{Z}^n$ copies $\{U_{\lambda,I}\}_{(\lambda,I)\in\tilde{\Lambda}}$ of the tropical affine open covering $\{U_{\lambda}\}_{\lambda\in\Lambda}$. If $U_{\lambda} \cap U_{\mu} \neq \varnothing$, then we identify the corresponding subspaces of $U_{\lambda,I}$ with that of $U_{\mu,J}$ by the coordinate transformation, where J is determined naturally by the transformation rule of the lifted sections s_a and s_b. Repeating this, we obtain a tropical affine covering space $\tilde{B}_{ab} := \coprod_{(\lambda,I)\in\tilde{\Lambda}} U_{\lambda,I}/\sim$ of B. A morphism $\phi_{ab} = \{\phi_{ab;\lambda,I}\}_{(\lambda,I)\in\tilde{\Lambda}} \in DG_M(s_a, s_b)$ is then regarded as a complex-valued smooth differential form, which is also denoted by ϕ_{ab}, on $\tilde{B}_{ab}$. Form the locally defined Morse functions $\{f_a - f_b + I \cdot \check{x}\}$, one can define a globally defined Morse function F_{ab} on $\tilde{B}_{ab}$ (using the ambiguity of constant functions for each local Morse function). For each pair of objects s_a, s_b, we fix such a function F_{ab}. Then, d_{ab} and $d^{\dagger}_{\epsilon;ab}$ are expressed, as operations on $\tilde{B}_{ab}$, as

$$d_{ab} = d - dF_{ab}\wedge, \quad d^{\dagger}_{\epsilon;ab} = \epsilon d^* - \iota_{\mathrm{grad}(F_{ab})}.$$

For each transversal pair s_a, s_b, the space $\mathscr{C}'_\epsilon(s_a, s_b) = P_{\epsilon;ab} DG_M(s_a, s_b)$ consists of solutions $\phi_{ab} \in DG_M(s_a, s_b)$ of the equations

$$d_{ab}(\phi_{ab}) = 0, \quad d^\dagger_{\epsilon;ab}(\phi_{ab}) = 0. \tag{12}$$

A basis of the solution space $\mathscr{C}'_\epsilon(s_a, s_b)$ is given as follows. Let $p_{ab} \in B$ be a point such that the Lagrangian sections corresponding to s_a and s_b intersect in the fiber $M|_{p_{ab}}$ of p_{ab}. This is in one-to-one correspondence with a critical point $\tilde{p}_{ab} \in \tilde{B}_{ab}$ of the Morse function F_{ab} on $\tilde{B}_{ab}$. The connected component of $\tilde{B}_{ab}$ including the point $\tilde{p}_{ab}$ is denoted $\tilde{B}(p_{ab})$. Correspondingly, we define a subset $\tilde{\Lambda}(p_{ab}) \subset \tilde{\Lambda}$ so that

$$\tilde{B}(p_{ab}) = \Big(\coprod_{(\mu,J) \in \tilde{\Lambda}(p_{ab})} U_{\mu,J} \Big) \Big/ \sim .$$

This $\tilde{B}(p_{ab})$ is also a covering space of B (which is not compact in general). From a differential form ϕ on $\tilde{B}(p_{ab})$, a collection $\phi = \{\phi_{\lambda,I}\}$ is defined naturally by setting $\phi_J = 0$ on U_μ if (μ, J) does not belong to $\tilde{\Lambda}(p_{ab})$. This $\phi = \{\phi_{\lambda,I}\}$ defines an element in $DG_M(s_a, s_b)$ if the sum $\sum_{I \in \mathbb{Z}^n} \phi_{\lambda,I} e^{\mathrm{i}I \cdot \check{y}}$ converges on each U_λ. The gradient flow of F_{ab} is well-defined on $\tilde{B}(p_{ab})$, and let us denote by $\mathscr{U}(p_{ab})$ the unstable manifold associated to $p_{ab} \in \tilde{B}(p_{ab})$. Now, we define a solution $\phi_{ab} = e_\epsilon(p_{ab})$ of the equations (12). For each p_{ab}, there exists a differential form on $\tilde{B}(p_{ab})$

- whose degree coincides with n minus the Morse index of F_{ab} at $\tilde{p}_{ab}$, and

- which approaches to $e^{F_{ab}}[\mathscr{U}(p_{ab})]$ by the limit $\epsilon \to 0$.

We normalize this solution and set $e_\epsilon(p_{ab})$ so that

$$\lim_{\epsilon \to 0} e_\epsilon(p_{ab}) = e^{F_{ab}(x) - F_{ab}(\tilde{p}_{ab})}[\mathscr{U}(p_{ab})].$$

Then, the solution space $\mathscr{C}'_\epsilon(s_a, s_b)$ of the equations (12) is spanned by these bases $e_\epsilon(p_{ab})$ associated to the critical points p_{ab}.

Next, we discuss the A_∞-structure induced by the homological perturbation theory. Since we start from the DG-category $DG_M(0)$, the induced A_∞-structure $\mathfrak{m}^\epsilon$ is described in terms of trivalent rooted tree graphs as is done in [Kontsevich and Soibelman 2001]. Namely, for each trivalent rooted n-tree γ_n, we define $m^\epsilon_{\gamma_n}$ by assigning the product m in $DG_M(0)$ at the trivalent vertices, the homotopy operators $h_{\epsilon;**}$ at the internal edges and the projection $P_{\epsilon;**}$ at the root edge. The A_∞-product m^ϵ_n is then the sum over all those trivalent rooted n-trees:

$$m^\epsilon_n = \sum_{\gamma_n} m^\epsilon_{\gamma_n}.$$

The Fukaya category seems to correspond to the limit $\lim_{\epsilon\to 0}\mathscr{C}_\epsilon$. Unfortunately, in this limit, each $\lim_{\epsilon\to 0}e_\epsilon(p_{ab})$ is no more a differential form but a current. However, we can obtain "almost all of" the A_∞-products $m_n :=$ $\lim_{\epsilon\to 0}m_n^\epsilon$ by looking at the limit directly as follows. First, we consider a sequence $s_{a_1},s_{a_2},\ldots,s_{a_n},s_{a_{n+1}}$ of objects which are transversal to each other. For each i, we take the base $e_\epsilon(p_{a_ia_{i+1}}) \in DG_M(s_{a_i},s_{a_{i+1}})$ associated to a critical point $p_{a_ia_{i+1}}$ and set $e(p_{a_ia_{i+1}}) := \lim_{\epsilon\to 0}e_\epsilon(p_{a_ia_{i+1}})$. We discuss $m_n(e(p_{a_1a_2}),\ldots,e(p_{a_na_{n+1}}))$, which turns out to coincide with that in the Fukaya–Oh category for the torus fibration $M \to B$ in [Kontsevich and Soibelman 2001] and hence that in the Fukaya category when each $m_{\gamma_n}(e(p_{a_1a_2}),\ldots,e(p_{a_na_{n+1}}))$ can be well-defined. The term

$$m_{\gamma_n}(e(p_{a_1a_2}),\ldots,e(p_{a_na_{n+1}}))$$

is a linear combination of $e(p_{a_1a_{n+1}})$ with all critical points $p_{a_1a_{n+1}}$, so we may determine the coefficients $c_{p_{a_1a_2}\cdots p_{a_na_{n+1}}p_{a_{n+1}a_1}}(\gamma_n)$ of

$$
\begin{aligned}
&m_{\gamma_n}\big(e(p_{a_1a_2}),\ldots,e(p_{a_na_{n+1}})\big)\\
&\qquad\qquad = \sum_{p_{a_1a_{n+1}}}c_{p_{a_1a_2}\cdots p_{a_na_{n+1}}p_{a_{n+1}a_1}}(\gamma_n)\cdot e(p_{a_1a_{n+1}}).
\end{aligned}
$$

These coefficients are given as follows. First, for each $p_{a_ia_{i+1}}$, $i \in \mathbb{Z}$ with $i+(n+1)=i$, we fix λ such that $p_{a_ia_{i+1}} \in U_\lambda$, and consider $I_{a_ia_{i+1}} \in \mathbb{Z}^n$ such that $(\lambda, I_{a_ia_{i+1}}) \in \tilde{\Lambda}(p_{a_ia_{i+1}})$. Then, for each fixed collection

$$(p_{a_1a_2},I_{a_1a_2}),\ldots,(p_{a_na_{n+1}},I_{a_na_{n+1}}),(p_{a_{n+1}a_1},I_{a_{n+1}a_1})$$

with

$$I_{a_{n+1}a_1} = -(I_{a_1a_2}+\cdots+I_{a_na_{n+1}}),$$

we look for a tree graph in $\{U_{\lambda,I}\}_{(\lambda,I)\in\tilde{\Lambda}}$ whose external edges are gradient lines starting from $(p_{a_ia_{i+1}},I_{a_ia_{i+1}})$ and whose internal edges are on gradient lines of the corresponding Morse functions. Such a tree is called a *gradient tree*. We see that there exists only one or no gradient tree. Generically, a gradient tree obtained in such a way is a trivalent tree. We set

$$c_{p_{a_1a_2}\cdots p_{a_na_{n+1}}p_{a_{n+1}a_1}}^{I_{a_1a_2}\cdots I_{a_na_{n+1}}}(\gamma_n) := \pm\exp\!\Big(-\sum_{l:\,\text{edges}}S_l\Big), \qquad (13)$$

with an appropriate sign $\pm$ when there exists a gradient tree which is isomorphic to γ_n as rooted tree graphs. Here l is an (internal or external) edge of γ_n, which is identified with the gradient tree, and S_l is the symplectic area of a surface surrounded by the corresponding two Lagrangian sections on l in the

corresponding covering space $\tilde{B}_{**}$ of B and the fibers at the end points of l in $\tilde{B}_{**}$. If a gradient tree is not isomorphic to γ_n, then we set

$$c^{I_{a_1 a_2} \cdots I_{a_n a_{n+1}}}_{p_{a_1 a_2} \cdots p_{a_n a_{n+1}} p_{a_{n+1} a_1}}(\gamma_n) = 0.$$

The conclusion is that the coefficient $c_{p_{a_1 a_2} \cdots p_{a_n a_{n+1}} p_{a_{n+1} a_1}}(\gamma_n)$ given by the homological perturbation theory actually turns out to be the sum

$$\sum_{I_{a_1 a_2}, \ldots, I_{a_n a_{n+1}}} c^{I_{a_1 a_2} \cdots I_{a_n a_{n+1}}}_{p_{a_1 a_2} \cdots p_{a_n a_{n+1}} p_{a_{n+1} a_1}}(\gamma_n)$$

when all the operations in the homological perturbation theory formula are well-defined at the limit $\lim_{\epsilon \to 0}$. We can expect that this is the case when s_{a_i} are generic enough. This A_∞-structure is exactly the one in the Fukaya–Oh category for torus fibration given in [Kontsevich and Soibelman 2001] and hence coincides with the one in the Fukaya category $\mathrm{Fuk}(M)$.

To summarize, we can view the limit m_n when $s_{a_1}, \ldots s_{a_n}, s_{a_{n+1}}$ are objects such that:

- they are transversal to each other,

- any gradient tree is trivalent,

- the homological perturbation theory formula on $e(p_{a_1 a_2}), \ldots, e(p_{a_n a_{n+1}})$ is well-defined.

For each fixed $I_{a_1 a_2}, \ldots, I_{a_n a_{n+1}}$, the argument to derive the coefficient (13) via the homological perturbation theory reduces to that in the case of cotangent bundles instead of torus fibrations (see [Kajiura 2011]). In particular, when $B = \mathbb{R}$ (though $\mathbb{R}$ is noncompact), i.e., $T^*B = \mathbb{R}^2$, the A_∞-structure is calculated explicitly in [Kajiura 2009] along this story. This construction for $\mathbb{R}^2$ is directly applied to the construction for T^2, see the last section of [Kajiura 2011].

Fukaya [2005] introduced Morse functions on certain covering spaces of B to discuss mirror symmetry for torus fibrations *having singular fibers*. We do not still treat singular fibers in the present paper, so our Morse functions are not directly related to those in [Fukaya 2005], though the Morse functions in both constructions are related to the symplectic areas of holomorphic disks.

5. Deformations of torus fibrations

5A. *A noncommutative deformation $\check{M}_\theta$ of $\check{M}$.* We consider a deformation quantization of $\check{M}$ by a Poisson structure (Poisson bivector) θ which is locally defined by

$$\sum_{i,j=1}^{n} \theta_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j},$$

with a constant matrix $\{\theta\}_{i,j=1,\dots,n}$. We call such a Poisson structure *constant*. Here, for the torus fibration $\check{M} \to B$, we fix a tropical affine open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of B. As we explained in Section 2, $\{\check{M}|_{U_\lambda}\}_{\lambda\in\Lambda}$ forms an open covering of $\check{M}$ which locally-trivializes the tangent bundle $T\check{M}$ of $\check{M}$. In particular, the transition functions are given by constant matrices. Thus, the notion of constant Poisson structures is well-defined. In particular, if $T\check{M}$ is trivial, then we can always extend a locally defined constant Poisson structure globally. The pair $(\check{M}, \theta)$ is denoted $\check{M}_\theta$.

For a formal parameter $\hbar$, a (formal) deformation quantization of a Poisson manifold $(\check{M}, \theta)$ is a (noncommutative) algebra $(C^\infty(\check{M})[[\hbar]], \star)$, where $\star$ is an $\hbar$-bilinear associative product (called the *star product*) given by

$$f \star g = f \cdot g + \hbar\{f, g\} + \text{higher order terms in } \hbar$$

for $f, g \in C^\infty(\check{M}) \subset C^\infty(\check{M})[[\hbar]]$ [Bayen et al. 1978a; 1978b]. Here, $\{f, g\}$ is the Poisson bracket defined by the Poisson structure. A (formal) deformation quantization of a Poisson manifold $(\check{M}, \theta)$ actually exists and it is unique up to a natural equivalence relation. It is shown in [De Wilde and Lecomte 1983] when the Poisson structure is symplectic and in [Kontsevich 2003] for a general Poisson manifold. A geometric construction using a Weyl algebra bundle is also given in [Omori et al. 1991; Fedosov 1994] for θ being symplectic and in [Cattaneo et al. 2002] for a general Poisson structure θ.

In our case, since we treat a Poisson manifold $(\check{M}, \theta)$ such that θ is constant, there exists a canonical deformation quantization such that its star product is given locally by the following Moyal star product:

$$f \star g = f \exp\left(\frac{\hbar}{2} \sum_{i,j=1}^n \theta_{ij} \left(\frac{\overleftarrow{\partial}}{\partial x_i} \frac{\overrightarrow{\partial}}{\partial y_j} - \frac{\overleftarrow{\partial}}{\partial y_j} \frac{\overrightarrow{\partial}}{\partial x_i} \right) \right) g.$$

This star product is well-defined globally since the Poisson structure θ is well-defined globally and is constant.

If we discuss a nonformal deformation quantization, it corresponds to substituting $\hbar = 1$ in a formal deformation quantization. Instead, it is more popular (and natural from the viewpoint of physics) to consider a nonformal deformation quantization of complex-valued smooth functions with $\hbar = \mathbf{i}$ or $\hbar = -\mathbf{i}$. Namely, in our set-up, the star product is given locally by

$$f \star g = f \exp\left(-\frac{\mathbf{i}}{2} \sum_{i,j=1}^n \theta_{ij} \left(\frac{\overleftarrow{\partial}}{\partial x_i} \frac{\overrightarrow{\partial}}{\partial y_j} - \frac{\overleftarrow{\partial}}{\partial y_j} \frac{\overrightarrow{\partial}}{\partial x_i} \right) \right) g \qquad (14)$$

for complex-valued smooth functions f and g. For instance if $B = S^1$, then

$\check{M} = T^2$ and this nonformal deformation quantization is equivalent to a non-commutative torus such as in [Connes and Rieffel 1987]. However, for general $\check{M}$, the star product of Moyal type in (14) may not be well-defined since the exponential in (14) is defined as its infinite Taylor expansion. Here, we have three choices. One is to discuss a formal Moyal star product and its mirror dual. The other one is to discuss the case when the deformation quantization of a nonformal constant Poisson structure converges. The last one is to restrict the space of functions and differential forms on $\check{M}$ to a subspace where the Moyal star product of a constant Poisson structure is well-defined. We shall discuss the last case though the second case is enough for $\check{M}$ to be a real higher dimensional torus. We shall define the restricted subspaces of the space of morphisms in Section 6C.

5B. *Foliated symplectic manifold M_θ.* We claim that the mirror dual of the non-commutative complex manifold $\check{M}_\theta = (\check{M}, \theta)$ is M equipped with a symplectic form ω_θ and a foliation structure $\mathcal{F}_\theta$. The symplectic form ω_θ is defined as

$$\omega_\theta := \sum_{i,j=1}^{n} \left(g_{ij} dx^i \wedge dy^j + (\theta - \theta^t)_{ij} dy^i \wedge dy^j \right). \tag{15}$$

The foliation structure $\mathcal{F}_\theta$ is defined by

$$x_i - \sum_{j=1}^{n} \theta_{ij} y^j = \text{const.}$$

Namely, if $\theta = 0$, then the leaves of the foliation are the fibers. The triple $(M, \omega_\theta, \mathcal{F}_\theta)$ is denoted M_θ.

Now, it is clear that *there exists a one-to-one correspondence between non-commutative complex manifolds $\check{M}_\theta = (\check{M}, \theta)$ and foliated symplectic manifolds $M_\theta = (M, \omega_\theta, \mathcal{F}_\theta)$.* In order to support the claim that the correspondence between $\check{M}_\theta$ and M_θ is mirror, in the next section we construct categories on M_θ and $\check{M}_\theta$ as deformations of certain subcategories of DG_M and $DG_{\check{M}}$, respectively, and discuss the homological mirror symmetry between them.

6. Deformations of the DG-categories

Now, we would like to discuss a deformation of the curved DG-category $DG_{\check{M}}$ by a constant Poisson structure θ and the mirror dual. Though not all of the objects seem to behave well under the deformation, we can fortunately define natural deformations of line bundles having *constant* curvature connections. In the next subsection, we discuss how to define deformation of those objects. After that, we construct (curved) DG-categories deformed by θ.

6A. *Deformation of objects.* Let us start from the case where $\theta = 0$ (commutative case). For an object $s = (V, D) \in DG_{\check{M}}$, recall that D is the covariant derivation expressed locally as

$$D = d + \mathbf{i} \sum_{i=1}^{n} s^i \, dy_i \wedge .$$

For the transformation rule $s_{(\mu)} = s_{(\lambda)} + I_{\lambda\mu}$, the transition function is given as $\psi_{(\mu)} = e^{-\mathbf{i} I_{\lambda\mu} \cdot \check{y}} \psi_{(\lambda)}$ as we saw in Section 3B. Let us denote by $\Omega^*(\check{M})$ the space of smooth complex-valued differential forms on $\check{M}$. Then, $(\Omega^*(\check{M}), d, \wedge)$ forms a DG (commutative) algebra and (V, D) has a structure of a (right) curved DG module over $\Omega^*(\check{M})$. In particular, D satisfies the Leibniz rule

$$D(\phi \cdot a) = D(\phi) \cdot a + (-1)^{|\phi|} \phi \cdot d(a) \tag{16}$$

for any $a \in \Omega^*(\check{M})$ and $\phi \in \Gamma(V) \otimes_{C^\infty(\check{M})} \Omega^{|\phi|}(\check{M})$. The curved DG-modules defined in this way naturally form a curved DG-category. By extracting the antiholomorphic part from this curved DG-structure, one obtains the curved DG-structure in $DG_{\check{M}}$. So, in order to define deformations of $DG_{\check{M}}$, we will study noncommutative deformations of the curved DG modules (V, D).

Now, we switch on the noncommutative parameter θ. We first replace the wedge product $\wedge$ in $\Omega^*(\check{M})$ by $\overset{\star}{\wedge}$, the wedge product with the Moyal star product (14). This operation $\overset{\star}{\wedge}$ is defined locally, where the product of coefficients for the bases of the differential forms are replaced by the Moyal star product. Since B is affine, $\overset{\star}{\wedge}$ is well-defined globally. Then, $(\Omega^*(\check{M}), d, \overset{\star}{\wedge})$ again forms a DG algebra (which is not commutative).

For a given line bundle (V, D) on $\check{M}$ as above, we construct its deformation, which is again denoted (V, D), as a curved DG module over the DG-algebra $(\Omega^*(\check{M}), d, \overset{\star}{\wedge})$. Our discussion of this kind is originally inspired by the noncommutative supergeometry à la A. Schwarz; see [Schwarz 2003]. First, the transition functions are set to be the same ones as the commutative case, but the product is replaced by the Moyal star product:

$$\psi_{(\mu)} = e^{-\mathbf{i} I_{\lambda\mu} \cdot \check{y}} \star \psi_{(\lambda)}.$$

Thus, V is the set of collections of local functions $\{\psi_\lambda\}_{\lambda \in \Lambda}$ which are related to each other by the transition functions above. The covariant derivation D is given locally as

$$D = d + \mathbf{i} \sum_{i=1}^{n} s^i \, dy_i \overset{\star}{\wedge} \tag{17}$$

for local functions s^i. Since we replaced the wedge product $\wedge$ by $\overset{\star}{\wedge}$, this again

satisfies the Leibniz rule. However, each local function s^i is different from the original one defining (V, D) on M. It is modified by θ so that the locally-defined D satisfies the compatibility

$$(D\psi_{(\lambda)})_{(\mu)} = D(\psi_{(\mu)}),$$

which turns out to be

$$e^{-\mathbf{i}I_{\lambda\mu}\cdot\check{y}} \star \big((d + \mathbf{i}s_{(\lambda)}(x)\cdot dy) \star e^{\mathbf{i}I_{\lambda\mu}\cdot\check{y}} * \psi_{(\mu)}\big)$$
$$= \big(d + \mathbf{i}(e^{-\mathbf{i}I_{\lambda\mu}\cdot\check{y}} \star s_{(\lambda)}(x) \star e^{\mathbf{i}I_{\lambda\mu}\cdot\check{y}} + I_{\lambda\mu})\cdot dy\big) \star \psi_{(\mu)}$$
$$= (d + \mathbf{i}s_{(\mu)}(x)\cdot dy) \star \psi_{(\mu)}.$$

Thus, if $s = \{s_{(\lambda)}\}_{\lambda\in\Lambda}$ satisfies the transformation rule

$$s_{(\mu)} = e^{-\mathbf{i}I_{\lambda\mu}\cdot\check{y}} \star s_{(\lambda)}(x) \star e^{\mathbf{i}I_{\lambda\mu}\cdot\check{y}} + I_{\lambda\mu}, \tag{18}$$

then such an $s = \{s_{(\lambda)}\}_{\lambda\in\Lambda}$ is regarded as a deformation of the original $s = (V, D)$ on M.

Now we have two problems. One is that, for a given object $s = (V, D) \in DG_{\check{M}}$, there does not necessarily exist its deformation in the sense above. The other one is, as explained below, that the structure of the resulting curved DG category is not what we want. Assume that the deformation of an object $s = (V, D) \in DG_{\check{M}}$ exists. If we decompose $D = D^{(1,0)} + D^{(0,1)}$ and set $d := 2D^{(0,1)}$, the two-form $W \in \Omega^2(\check{M})$ defined by $d^2 = W \overset{\star}{\wedge}$ should be regarded as the curvature of the deformed object (V, D). For $W = \frac{1}{2}\sum_{i,j=1}^n W^{ij}\, d\bar{z}_i \wedge d\bar{z}_j$, one has

$$W^{ij} = \frac{\partial s^i}{\partial x_j} - \frac{\partial s^j}{\partial x_i}.$$

A different point from the commutative case is that the two-form W no more belongs to the center of $(\Omega^*(\check{M}), d, \overset{\star}{\wedge})$ if W^{ij} is not constant for some i and j. This means that, if we construct the curved DG-category consisting of these deformed objects, the full subcategory consisting of objects with the same curvature W does not form a DG-category.

One can solve these two problems at the same time if we concentrate on affine sections only as follows. If we consider a collection $s = \{s_{(\lambda)}\}_{\lambda\in\Lambda}$ such that each $s_{(\lambda)}$ is an affine function with respect to $\check{x}$, then W^{ij} becomes constant. Furthermore, in this case, the transformation rule (18) becomes simpler. Let us express s locally as

$$s = \mathfrak{A}\check{x} + \mathfrak{c}$$

for $\mathfrak{A} \in Mat(n; \mathbb{R})$ and $\mathfrak{c} \in \mathbb{R}^n$. Then, The transformation rule (18) turns out to be

$$s_{(\mu)} = s_{(\lambda)} + (1 + \mathfrak{A}\theta)I_{\lambda\mu}. \tag{19}$$

In particular, for a given line bundle $s = (V, D)$ on $\check{M}$ such that s is affine with respect to $\check{x}$, there exists its deformation s on $\check{M}_\theta$ with the same $\{I_{\lambda\mu}\}$ when θ is generic. In fact, when we express s on $\check{M}$ locally by $s = A\check{x} + c$, its deformation s satisfies the transformation rule (19) if

$$\mathfrak{A}\check{x} + \mathfrak{c} = (1 + \mathfrak{A}\theta)(A\check{x} + c),$$

which can be rewritten as

$$(1 - A\theta)\mathfrak{A}\check{x} + \mathfrak{c} = A\check{x} + (1 + \mathfrak{A}\theta)c;$$

hence one obtains $\mathfrak{A} = (1 - A\theta)^{-1}A$ if the matrix $(1 - A\theta)$ is invertible. This leads $1 + \mathfrak{A}\theta = (1 - A\theta)^{-1}$, so $\mathfrak{c} = (1 - A\theta)^{-1}c$. Thus, for a given s locally expressed as $A\check{x} + c$ on $\check{M}$, there exists its deformation s locally expressed as $s = \mathfrak{A}\check{x} + \mathfrak{c}$, $\mathfrak{A} = (1 - A\theta)^{-1}A$, $\mathfrak{c} = (1 - A\theta)^{-1}c$ unless $\det(1 - A\theta) = 0$. In Section 6D we shall define the curved DG-category $\mathcal{DG}_{\check{M}_\theta}$ consisting of these objects which are affine with respect to $\check{x}$.

6B. *Geometric interpretation of these deformed objects.* We give a geometric interpretation of the deformed object s in the previous subsection.

For the foliated symplectic manifold $M_\theta = (M, \omega_\theta, \mathcal{F}_\theta)$, the foliation $\mathcal{F}_\theta$ is transversal to the zero section s_0. Consider another affine section $\underline{s} : B \to M$ which is transversal to $\mathcal{F}_\theta$ and its lift s. It is locally expressed as $s^i = A^{ij}x_j + c^i$. Let $\check{x}_0 \in B$ be a point, which is identified with a point in the graph of $\bar{s}_0$, and consider the leaf which includes $\check{x}_0$. The intersection point of this leaf and the graph of $\underline{s}$ is given by the two equations $y^i = A^{ij}x_j + c^i$ and $x_i - \theta_{ij}y^j = x_{0,i}$, where $\check{x}_0 = (x_{0,1}, \ldots, x_{0,n})^t$. One obtains

$$\check{x} = (1 - \theta A)^{-1}(\check{x}_0 + \theta c),$$

$$y = (A(1 - \theta A)^{-1})(\check{x}_0 + \theta c) + c$$

$$= (1 - A\theta)^{-1}(A\check{x}_0 + c).$$

Namely, using the coordinates $\check{x}_0$, the section $\underline{s}$ is described by the last equation, that is, $y = \mathfrak{A}\check{x}_0 + \mathfrak{c}$ with $\mathfrak{A} = (1 - A\theta)^{-1}A$ and $\mathfrak{c} = (1 - A\theta)^{-1}c$. The condition that the graph of $\underline{s}$ is transversal to the leaves is equivalent to that the matrix $1 - A\theta$ is invertible.

Hereafter we drop the lower index $_0$, writing x_i for $x_{0,i}$ or $\check{x}$ for $\check{x}_0$. Thus, in the deformed categories we shall discuss, the coordinates $\check{x}$ are interpreted geometrically as the coordinates $\check{x}_0$ above.

Next, we check the transformation rule for these local descriptions. For the transformation rule (7),

$$s_{(\mu)}(x) = s_{(\lambda)}(x) + I_{\lambda\mu},$$

of the lifted section s, one has

$$A_{(\mu)}\check{x} + c_\mu = (A_{(\lambda)}\check{x} + c_\lambda) + I_{\lambda\mu}$$

on $U_\lambda \cap U_\mu$, where $A_{(\lambda)} = A_{(\mu)} = A$, and hence $c_\mu = c_\lambda + I_{\lambda\mu}$. Therefore, one obtains

$$\mathfrak{A}\check{x} + \mathfrak{c}_\mu = (1 - A\theta)^{-1}(A\check{x} + c_\mu) = (1 - A\theta)^{-1}(A\check{x} + c_\lambda + I_{\lambda\mu})$$
$$= \mathfrak{A}\check{x} + \mathfrak{c}_\lambda + (1 - A\theta)^{-1}I_{\lambda\mu}. \tag{20}$$

Recall that one has the identity $(1 - A\theta)^{-1} = 1 + \mathfrak{A}\theta$. Thus, the transformation rule for this deformed object agrees with the one (19) discussed previously:

$$s_{(\mu)} = s_{(\lambda)} + (1 + \mathfrak{A}\theta)I_{\lambda\mu}.$$

In the next subsection, we employ this transformation rule (20) for the definition of the objects of the deformed curved DG-category which we shall construct. Note that, though we started from a section $\underline{s} : B \to M$ to discuss the deformed object s, the s will not need to come from a section of $M \to B$. For instance, when θ is general, then in the deformed category we can include an object which corresponds to the Lagrangian fiber T^n at each point in B. This is the case where $\mathfrak{A} = -\theta^{-1}$ and so $(1 + \mathfrak{A}\theta) = 0$.

6C. *Curved DG-category* $\mathscr{DG}_{M_\theta}$. Now, for a fixed constant Poisson structure θ, we define two curved DG-categories $\mathscr{DG}_{M_\theta}$ and $\mathscr{DG}_{\check{M}_\theta}$ associated to M_θ and $\check{M}_\theta$. The objects shall be defined according to the previous two subsections. As mentioned at the end of Section 5A, the spaces of morphisms (in $\mathscr{DG}_{\check{M}_\theta}$) shall be defined so that the corresponding Moyal star product is well-defined. The curved DG-categories $\mathscr{DG}_{M_\theta}$ and $\mathscr{DG}_{\check{M}_\theta}$ are defined so that they are canonically curved DG-isomorphic to each other. We first define the curved DG-category $\mathscr{DG}_{M_\theta}$ below.

Each object $s \in \mathscr{DG}_{M_\theta}$ is a collection $\{s_{(\lambda)}\}$ such that:

- Each $s_{(\lambda)}$ is an affine function on U_λ described as

$$s_{(\lambda)} = \mathfrak{A}_{(\lambda)}\check{x} + \mathfrak{c}_\lambda$$

  for some $\mathfrak{A}_{(\lambda)} \in Mat(n, \mathbb{R})$ and $\mathfrak{c}_\lambda \in \mathbb{R}^n$.

- The transformation rule

$$s_{(\mu)} = s_{(\lambda)} + (1 + \mathfrak{A}\theta)I_{\lambda\mu} \tag{21}$$

is satisfied for each $U_\lambda \cap U_\mu \neq \varnothing$. Here, $I_{\lambda\mu}$ satisfies the cocycle condition (8).

For two objects s_a and s_b, we express their transformation rules as

$$(s_a)_{(\mu)} = (s_a)_{(\lambda)} + I_a, \quad (s_b)_{(\mu)} = (s_b)_{(\lambda)} + I_b$$

with $I_a = I_{a;\lambda\mu}$ and $I_b = I_{b;\lambda\mu}$. We define the space $\mathscr{D}\mathscr{G}_{M_\theta}(s_a, s_b)$ of morphisms as the subspace of

$$\prod_{\lambda \in \Lambda} \prod_{I \in \mathbb{Z}^n} \Omega_{\lambda,I}(s_a, s_b)$$

such that:

- For each open set $U_\lambda \subset B$ and a point $p \in U_\lambda$, the Taylor expansion

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \frac{(x_1 - p_1)^{i_1}}{i_1!} \cdots \frac{(x_n - p_n)^{i_n}}{i_n!} \left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n} \phi_{\lambda,I}(p)$$

  converges at any $\check{x} \in \mathbb{R}^n$ and coincides with $\phi_{\lambda,I}$ on U_λ for any $I \in \mathbb{Z}^n$.

- $\phi_{\lambda,I} \in \Omega_{\lambda,I}(s_a, s_b)$ satisfies

$$\phi_{\mu,I}(\check{x}) = \phi_{\lambda,I+I_b-I_a}\left(\check{x} + \tfrac{1}{2}(I_a + I_b)\theta\right), \qquad \check{x} \in U_\lambda \cap U_\mu$$

  for any $U_\lambda \cap U_\mu \neq \varnothing$.

- $\{\phi_{\lambda,I}(\check{x})\}_{I \in \mathbb{Z}^n}$ goes to zero as $I \to \infty$ faster than any inverse power of I (meaning $|i_1|^{-j_1}|i_2|^{-j_2}\cdots|i_n|^{-j_n}$ for some $j_1,\ldots,j_n \in \mathbb{Z}_{\geq 0}$) and uniformly with respect to $\check{x} \in \mathbb{R}^n$.

The last two conditions are modifications of those for $DG_M(s_a, s_b)$ in Section 4B. In the first condition, by the Taylor expansion of $\phi_{\lambda,I}$, we mean the Taylor expansions of the coefficients of the differential form $\phi_{\lambda,I}$. Thus, a degree zero element in $\mathscr{D}\mathscr{G}^0_{M_\theta}(s_a, s_b)$ is a kind of an entire function on the real analytic manifold $\tilde{B}_{ab}$. In the second condition, the point $\check{x} + \frac{1}{2}(I_a + I_b)\theta$ may belong to neither U_λ nor U_μ even if $\check{x} \in U_\lambda \cap U_\mu$. However, the right-hand side makes sense as its Taylor expansion. The third condition then implies that (each coefficient of the differential form) $\phi_{\lambda,I}(\check{x})$ is a Schwartz function on $I \in \mathbb{Z}^n$ but its convergence is uniform with respect to $\check{x}$.

The differential d_{ab} is modified by θ as follows. We define

$$d_{ab} : \mathscr{D}\mathscr{G}^r_{M_\theta}(s_a, s_b) \to \mathscr{D}\mathscr{G}^{r+1}_{M_\theta}(s_a, s_b)$$

locally by

$$d_{ab}(\phi_{\lambda,I}) := d(\phi_{\lambda,I}) - \sum_{j=1}^{n} \left(s_a - s_b + \left(1 + \tfrac{1}{2}(\mathfrak{A}_a + \mathfrak{A}_b)\theta\right)I\right)^j dx_j \wedge \phi_{\lambda,I}.$$

In particular, when $s_b = s_0$, the differential is denoted $d_{a0} =: d_a$, where the two-form $W_a = \tfrac{1}{2} \sum_{i,j=1}^{n} W_a^{ij} dx_i \wedge dx_j$ defined by $(d_a)^2 = W_a \wedge$ is called the *curvature* of the object a. Then, for any s_a, s_b, one has $(d_{ab})^2 = W_a - W_b$, which turns out to be zero if and only if s_a and s_b have the same constant curvature $W_a = W_b = W$.

The (deformed) composition

$$m : \mathscr{DG}_{M_\theta}(s_a, s_b) \otimes \mathscr{DG}_{M_\theta}(s_b, s_c) \to \mathscr{DG}_{M_\theta}(s_a, s_c)$$

of morphisms is then defined by

$$m(\phi_{ab;\lambda,I}, \phi_{bc;\lambda,J})(\check{x})$$
$$:= \phi_{ab;\lambda,I}\left(\check{x} + \tfrac{1}{2}\theta J\right) \wedge \phi_{bc;\lambda,J}\left(\check{x} - \tfrac{1}{2}\theta I\right) \in \Omega_{\lambda,I+J}(s_a, s_c) \quad (22)$$

for $\phi_{ab;\lambda,I} \in \Omega_{\lambda,I}(s_a, s_b)$, $\phi_{bc;\lambda,J} \in \Omega_{\lambda,J}(s_b, s_c)$ and $\check{x} \in U_\lambda$. Here, the formula for m again makes sense by regarding $\phi_{ab;\lambda,I}$ and $\phi_{bc;\lambda,J}$ as their Taylor expansions even if $\check{x} + (\theta J/2)$ or $\check{x} - (\theta I/2)$ may not be in U_λ. This product m is in fact associative and satisfies the Leibniz rule with respect to the differential. Thus, $\mathscr{DG}_{M_\theta}$ forms a curved DG-category. In particular, the full subcategory $\mathscr{DG}_{M_\theta}(W)$ consisting of objects having the same constant curvature W forms a DG category.

6D. *Curved DG-category* $\mathscr{DG}_{\check{M}_\theta}$. As stated in the previous subsection, we define the curved DG-category $\mathscr{DG}_{\check{M}_\theta}$ which is canonically isomorphic to $\mathscr{DG}_{M_\theta}$. The objects are the same as those in $\mathscr{DG}_{M_\theta}$. For any two objects $s_a, s_b \in \mathscr{DG}_{M_\theta}$ and morphism $\phi_{ab} := \{\phi_{ab;\lambda,I}\} \in \mathscr{DG}_{M_\theta}(s_a, s_b)$, we set $f_1(\phi_{ab}) = \psi_{ab}$, which is locally defined by

$$\psi_{ab}|_{U_\lambda} := \sum_{I \in \mathbb{Z}^n} \psi_{ab;\lambda,I} e^{iI \cdot \check{y}},$$

where $\psi_{ab;\lambda,I}$ is obtained by replacing each dx_i in $\phi_{ab;\lambda,I}$ by $d\bar{z}_i$. By construction, $\psi_{ab}|_{U_\lambda}$ and $\psi_{ab}|_{U_\mu}$ are related to each other by

$$\psi_{ab}|_{U_\mu} = e^{-iI_a y} \overset{\star}{\wedge} (\psi_{ab}|_{U_\lambda}) \overset{\star}{\wedge} e^{iI_b y}$$

on $U_\lambda \cap U_\mu$, where $I_a = I_{a;\lambda\mu}$ and $I_b = I_{b;\lambda\mu}$. We set the space $\mathscr{DG}_{\check{M}_\theta}(s_a, s_b)$ as the space of all such elements ψ_{ab}. The differential

$$d_{ab} : \mathscr{DG}^r_{M_\theta}(s_a, s_b) \to \mathscr{DG}^{r+1}_{M_\theta}(s_a, s_b)$$

is set to be

$$d_{ab}(\psi) := 2\bar{\partial}\psi - \sum_{i=1}^{n}(s_a^i \overset{\star}{\wedge} \psi - \psi \overset{\star}{\wedge} s_b^i)d\bar{z}_i.$$

In particular, when $s_b = s_0$, we write $d_{a0} =: d_a$, and the antiholomorphic two-form $W_a = \frac{1}{2}\sum_{i,j=1}^{n} W_a^{ij} d\bar{z}_i \wedge d\bar{z}_j$ defined by $(d_a)^2 = W_a\wedge$ is called the *curvature* of s_a. Note that when the curvature of an object $s_a \in \mathscr{DG}_{M_\theta}$ is $W_a = \frac{1}{2}\sum_{i,j} W_a^{ij} dx_i \wedge dx_j$, the corresponding object $s_a \in \mathscr{DG}_{\check{M}_\theta}$ has the curvature $W_a = \frac{1}{2}\sum_{i,j} W_a^{ij} d\bar{z}_i \wedge d\bar{z}_j$ with the same $\{W_a^{ij}\}$. In this sense, we denoted the curvatures in both categories by the same letter W_a. Hereafter, we say $s_a \in \mathscr{DG}_{M_\theta}$ and $s_b \in \mathscr{DG}_{\check{M}_\theta}$ have the *same* curvature W if $W_a^{ij} = W_b^{ij}$ for any $i, j = 1, \ldots, n$.

The composition of morphisms is defined so that the maps f_1 gives an isomorphism of these curved DG-categories $\mathscr{DG}_{M_\theta}$ and $\mathscr{DG}_{\check{M}_\theta}$. Namely,

$$m(f_1(\phi_{ab}), f_1(\phi_{bc})) := f_1(m(\phi_{ab}, \phi_{bc}))$$

for any $\phi_{ab} \in \mathscr{DG}_{M_\theta}(s_a, s_b)$ and $\phi_{bc} \in \mathscr{DG}_{M_\theta}(s_b, s_c)$. Since

$$f_1(\phi_{ab})|_{U_\lambda} = \psi_{ab}|_{U_\lambda} = \sum_I \psi_{ab;\lambda,I} e^{iI \cdot \check{y}},$$

$$f_1(\phi_{bc})|_{U_\lambda} = \psi_{bc}|_{U_\lambda} = \sum_I \psi_{bc;\lambda,I} e^{iI \cdot \check{y}},$$

one has

$$m(f_1(\phi_{ab}), f_1(\phi_{bc}))|_{U_\lambda}(x, y)$$

$$= f_1(m(\phi_{ab}, \phi_{bc}))|_{U_\lambda}(x, y)$$

$$= \sum_{I,J} \psi_{ab;\lambda,I}\left(x + \tfrac{1}{2}\theta J\right) \wedge \psi_{bc;\lambda,J}\left(x - \tfrac{1}{2}\theta I\right) e^{i(I+J) \cdot y}$$

$$= (\psi_{ab} \overset{\star}{\wedge} \psi_{bc})|_{U_\lambda}(x, y).$$

Namely, the composition of morphisms in $\mathscr{DG}_{\check{M}_\theta}$ is given by $\overset{\star}{\wedge}$, the "deformation quantized wedge product". Thus, $\mathscr{DG}_{\check{M}_\theta}$ is defined quite naturally as a category on (the deformation quantization of) $\check{M}_\theta$.

6E. *Homological mirror symmetry.* By construction, the correspondence f_1 in the previous subsection gives the canonical isomorphism between the curved DG-categories:

$$\mathscr{DG}_{M_\theta} \simeq \mathscr{DG}_{\check{M}_\theta}.$$

This in particular induces a DG-isomorphism $\mathscr{DG}_{M_\theta}(W) \simeq \mathscr{DG}_{\check{M}_\theta}(W)$ for any W. As we mentioned before, in the commutative case, the DG-category

$DG_M(0)$ (or $\mathcal{DG}_{M_{\theta=0}}(0)$) is the de Rham model for (a full-subcategory of) the Fukaya category $\mathrm{Fuk}(M)$. Now, we claim that the DG-category $\mathcal{DG}_{M_\theta}(W)$ is the de Rham model for *the Fukaya category* $\mathrm{Fuk}(M, \omega_\theta)$ *with potential* W in the following sense.

Let $\pi_{\mathcal{F}_\theta} : TB \to B$ be the projection associated to the foliation structure. Here, we identify the base space B with the graph of the zero section s_0 of TB. In particular, if $\theta = 0$, then $\pi_{\mathcal{F}_\theta}$ coincides with the projection defining the tangent bundle $TB \to B$. For a fixed constant two-form W on B, the two form

$$\omega_\theta - (\pi_{\mathcal{F}_\theta})^* W$$

again defines a symplectic form on TB. Since this is a constant two form with respect to the coordinates $\check{x}$ and y, there exists a corresponding symplectic form on M. Then, submanifolds defined by the objects in $\mathcal{DG}_{M_\theta}(W)$ form Lagrangian submanifolds with respect to this symplectic form. We call this Fukaya category consisting of these Lagrangians the Fukaya category $\mathrm{Fuk}(M, \omega_\theta)$ with potential W. [3] In order to show that each object $s \in \mathcal{DG}_{M_\theta}(W)$ defines a Lagrangian, it is enough to discuss it locally. A key point is that, using the coordinates associated to the foliation structure $\mathcal{F}_\theta$, ω_θ (15) is described as

$$\omega_\theta = \sum_{i=1}^{n} dx_{0,i} \wedge dy^i$$

on TB, where $x_{0,i}$ are the coordinates in the sense of Section 6B. Thus, in this coordinate system, one has

$$\omega_\theta - (\pi_{\mathcal{F}_\theta})^* W = \sum_{i=1}^{n} dx_{0,i} \wedge dy^i - \frac{1}{2} \sum_{i,j=1}^{n} W^{ij} dx_{0,i} \wedge dx_{0,j}$$
$$= \sum_{i=1}^{n} dx_{0,i} \wedge \left(dy^i - \tfrac{1}{2} W^{ij} dx_{0,j} \right).$$

This shows, by setting $y_0^i := y^i - \frac{1}{2} W^{ij} x_{0,j}$, the symplectic form is of the form

$$\omega_\theta - (\pi_{\mathcal{F}_\theta})^* W = \sum_{i=1}^{n} dx_{0,i} \wedge dy_0^i.$$

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[3]Here we called it so just because W plays a similar role to a Landau–Ginzberg superpotential (or the corresponding object in the mirror dual side). We do not give any interpretation of this potential in this paper.

Then, each object $s \in \mathscr{DG}_{M_\theta}(W)$ expressed locally as $y^i = \mathfrak{A}^{ij} x_{0,j} + \mathfrak{c}^i$ is rewritten as

$$y_0^i = \sum_{j=1}^n \mathfrak{A}^{ij} x_{0,j} + \mathfrak{c}^i - \frac{1}{2} \sum_{j=1}^n W^{ij} x_{0,j} = \frac{1}{2} \sum_{j=1}^n (\mathfrak{A} + \mathfrak{A}^t)^{ij} x_{0,j} + \mathfrak{c}^i,$$

since $W^{ij} = \mathfrak{A} - \mathfrak{A}^t$. We see that s defines a Lagrangian since the coefficient $\frac{1}{2}(\mathfrak{A} + \mathfrak{A}^t)^{ij}$ is symmetric with respect to i and j.

The precise statement of our hope is then as follows.

Conjecture 6.1. *The DG-category $\mathscr{DG}_{M_\theta}(W)$ is A_∞-equivalent to a full subcategory of the Fukaya category $\mathrm{Fuk}(M, \omega_\theta)$ with potential W.*

Here, we say an A_∞-functor is an A_∞-*equivalence* when it induces an equivalence of the cohomology categories of the A_∞-categories.[4] This notion is a generalization of the notion of A_∞-quasiisomorphisms. In fact, roughly speaking, an A_∞-equivalence is an A_∞-quasiisomorphism up to isomorphisms of objects in the target A_∞-category.

Conjecture 6.1 actually holds true if we assume the de Rham model $DG_M(0)$ is A_∞-equivalent to the full subcategory of the Fukaya category $\mathrm{Fuk}(M)$ in the usual commutative case as discussed in Section 4E. We already saw that the objects in $\mathscr{DG}_{M_\theta}(W)$ define Lagrangians with respect to the symplectic form $\omega_\theta - \pi^*_{\mathscr{F}_\theta} W$. In particular, just as in the commutative case, we see that two objects defining the same Lagrangian are isomorphic to each other. Then, as in the commutative case in Section 4E, we apply homological perturbation theory to $\mathscr{DG}_{M_\theta}$ in order to obtain the corresponding Fukaya category. More precisely, we obtain an A_∞-category $\mathscr{C}' = \lim_{\epsilon \to 0} \mathscr{C}'_\epsilon$ which is A_∞-quasiisomorphic to $\mathscr{DG}_{M_\theta}$. Since (infinitely) many objects s in $\mathscr{C}'$ defining the same Lagrangian are isomorphic to each other, we can choose a representative corresponding to each Lagrangian. Our hope is that the full A_∞-subcategory of $\mathscr{C}'$ consisting of these representatives is regarded as the corresponding Fukaya category. This means that $\mathscr{DG}_{M_\theta}$ is A_∞-equivalent to the corresponding Fukaya category.

We end with explaining that why the A_∞-category $\mathscr{C}'$ is expected to correspond, in the sense above, to the Fukaya category. First, for d_{ab}, the homotopy operator h_{ab} is given similarly. Then, we consider a sequence

$$s_{a_1}, s_{a_2}, \dots, s_{a_n}, s_{a_{n+1}}$$

of objects which are transversal to each other, and calculate the A_∞-products m_n defined by the homological perturbation theory formula. As we mentioned

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[4]This is what is called a quasiequivalence in [Seidel 2008].

in Section 4E, each A_∞-product is given as the sum

$$m_n = \sum_{\gamma_n} m_{\gamma_n}$$

over all the trivalent rooted n-trees γ_n, where each m_{γ_n} is defined by attaching the product m in $\mathscr{D}\mathscr{G}_{M_\theta}$ to the trivalent vertices, the homotopy operators h_{**} to the internal edges, and the projection P_{**} to the root edge. Then, the only difference from the commutative case is that the product m is deformed; we see in the formula (22) that m depends on the indices I and J. However, we may calculate the product m in the formula for m_{γ_n} by using the isomorphisms of objects mentioned above. For instance, on each U_λ, the product $m : \Omega_{\lambda, I}(s_a, s_b) \otimes \Omega_{\lambda, J}(s_b, s_c) \to \Omega_{\lambda, I+J}(s_a, s_c)$ can be obtained by $m : \Omega_{\lambda, 0}(s_a, s_b') \otimes \Omega_{\lambda, 0}(s_b', s_c') \to \Omega_{\lambda, 0}(s_a, s_c')$, where s_b' and s_c' are the objects which are isomorphic to s_b and s_c, respectively, and are expressed as

$$s_b' = s_b + I, \quad s_c' = s_c + I + J$$

on U_λ. Since the formula for the product,

$$m : \Omega_{\lambda, 0}(s_a, s_b') \otimes \Omega_{\lambda, 0}(s_b', s_c') \to \Omega_{\lambda, 0}(s_a, s_c'),$$

is the same as in the commutative case, the A_∞-products m_n are obtained just in the same way as in the commutative case in Section 4E.

7. Concluding remarks

As the objects of our curved DG-categories, we consider only those which come from sections of $M \to B$. They are, in the curved DG-category $DG_{\check{M}}$, line bundles with $U(1)$-connections. One can also construct higher dimensional vector bundles (with $U(1)$-connections) by considering multisections of $M \to B$. For instance, see [Leung et al. 2000]. It is not difficult to include these as the objects in our set-up. In mirror symmetry, one generally needs to include $U(1)$ local systems on Lagrangian submanifolds. This inclusion of $U(1)$ local system corresponds to including pure imaginary constant terms to the connections on vector bundles. This generalization is also neglected in this paper for simplicity.

However, even if we include these generalizations in our set-up, it is not clear whether the resulting objects are enough to discuss homological mirror symmetry or not. For instance, it is not clear whether the DG-category $DG_{\check{M}}(0)$ generates the derived category $D^b(\mathrm{coh}(\check{M}))$ of all coherent sheaves on $\check{M}$ as the zero-th cohomology of the pretriangulated DG-category of twisted complexes (in the sense of [Bondal and Kapranov 1990]) in $DG_{\check{M}}(0)$. The answer is positive for

the case $M = T^2$ and its noncommutative deformation, but it may depend on the manifold M. We postpone this discussion for a future paper.

Though we did not include singular fibers, there are not many examples of Kähler manifolds which are described as semiflat torus fibrations without singular fibers. For example, one can enjoy the constructions in the present paper for the case $M = \mathbb{CP}^1 \setminus \{2 \text{ points}\}$, where $B = \mathbb{R}$. Since B has an open covering consisting of only one coordinate chart, it is automatically tropical affine. One can also discuss the case when M is the generic T^n-fibration in $\mathbb{CP}^n$ in a similar way, where $B \simeq \mathbb{R}^n$. However, in order to discuss the mirror symmetry for $\mathbb{CP}^n$ instead of $M \subset \mathbb{CP}^n$, we need to compactify M by singular fibers and also add some effects from holomorphic disks (see [Cho and Oh 2006]). It may be interesting to discuss how to compactify M_θ and $\check{M}_\theta$ in these examples.

The case M is the trivial T^n fibration over $B = T^n$ is of course the main example of the construction in this paper, where M and $\check{M}$ are actually mirror dual to each other. In this case, the (curved) DG-category $\mathcal{DG}_{M_\theta}$ coincides with a (curved) DG-category associated to a (higher dimensional) noncommutative torus constructed in [Polishchuk and Schwarz 2003; Kajiura 2004; 2007], except for some minor details. However, in this $M = T^{2n}$ case, there is another way of defining the mirror dual since the base space and the fiber are both T^n. Namely, one can T-dualize both the base T^n and the fiber T^n. Equivalently, one can deal with vector bundles on $\check{M}$ such that their sections can be Fourier expanded on the base T^n instead of the fiber T^n, though one T-dualizes the fiber T^n according to [Strominger et al. 1996]. Actually, the mirror duality for (commutative) T^{2n} explained in the reference [Kajiura 2006] is the one in the sense above, which is therefore different from the one in the present paper. Anyway, all these (curved) DG-categories are equivalent via T-dualizing in various directions.

Finally, though we discussed a particular deformation of the categories, we believe that similar deformations can be formulated for various other categories. In particular, we hope to try it for DG enhancements of categories of singularities elsewhere.

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