Thin groups and superstrong approximation
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Preface

The workshop "Thin Groups and Super-strong Approximation" was held at MSRI from February 6th through 10th, 2012. The organizing committee consisted of Emmanuel Breuillard (Orsay), Jordan Ellenberg (Madison), Alex Gamburd (New York), Emmanuel Kowalski (Zurich), Hee Oh (New Haven), and, last but not least, Peter Sarnak (Princeton), whose initiative it was to organize this workshop in the first place as part of the MSRI Hot Topics series.

The present volume contains expanded versions of most of the invited lectures and represents quite faithfully the range of topics discussed at the workshop. The articles in this volume are almost entirely expository. The goal was to put together in a single volume a series of state-of-the-art surveys on the subject of superstrong approximation and its recent developments. Videos of the talks given at the workshop are available on the MSRI website and can constitute a valuable resource complementing the articles in this volume.

Let us explain briefly the two phrases in the title: *thin groups* and *superstrong approximation.* Discrete subgroups of Lie groups with finite covolume, also known as *lattices* (for example, $\mathrm{SL}(2, \mathbb{Z})$ inside $\mathrm{SL}(2, \mathbb{R})$), are central objects of mathematics, and they have been extensively studied in the second half of the twentieth century.

Let $\mathbf{G}$ be a connected semisimple real algebraic group. A celebrated result of Borel and Harish-Chandra in 1962 asserts that if $\mathbf{G}$ is defined over $\mathbb{Q}$, then its subgroup $\mathbf{G}(\mathbb{Z})$ of integer points, called *an arithmetic subgroup*, is a lattice in the Lie group $\mathbf{G}(\mathbb{R})$, providing many important examples of lattices.

By Borel's density theorem, any lattice Γ in $\mathbf{G}(\mathbb{R})$ is Zariski-dense, unless $\mathbf{G}(\mathbb{R})$ has a non-trivial compact connected normal subgroup. A discrete subgroup Γ of $\mathbf{G}(\mathbb{R})$ is called *thin* if it is Zariski-dense in $\mathbf{G}$ and yet $\mathbf{G}(\mathbb{R})/\,\Gamma$ is of infinite volume.

Although less obviously important than lattices, such groups do appear naturally in mathematics. And it is one of the themes of this volume to present a variety of situations where these groups are the natural objects to study. For instance, the symmetries of an Apollonian circle packing form a thin subgroup of the orthogonal group O(3, 1). Many classical properties of lattices continue to hold for thin subgroups. Non-abelian harmonic analysis and ergodic theory played a key role in the study of lattices, and not surprisingly they are again on the front line as far as thin groups are concerned. Yet many new ideas and techniques had to be developed. For example, group combinatorics and the study of *approximate groups* have played a major role, as is reflected in this volume.

Arithmetically defined thin groups, i.e., subgroups of an arithmetic subgroup $\mathbf{G}(\mathbb{Z})$ of infinite index, also share many of the known properties of arithmetic subgroups. For example, it is well known and easy to prove that the reduction mod p homomorphism from $\mathrm{SL}_d(\mathbb{Z})$ to $\mathrm{SL}_d(\mathbb{Z}/p\mathbb{Z})$ is surjective for every prime p. This is called *strong approximation* and holds for any arithmetic subgroup $\mathbf{G}(\mathbb{Z})$ as long as $\mathbf{G}$ is simply connected.

Remarkably, the strong approximation phenomenon continues to hold for arithmetically defined thin subgroups of $\mathbf{G}$. More precisely, the reduction mod p map remains surjective, for all but finitely many primes p, in restriction to any given thin subgroup Γ of $\mathbf{G}(\mathbb{Z})$, as proved by Matthews, Vaserstein, and Weisfeiler in the early 1980s. In other words, for a fixed finite generating subset S of Γ, the Cayley graph $\mathscr{C}(\mathbf{G}(\mathbb{Z}/p\mathbb{Z}), S)$ induced by the reduction mod p map is connected for almost all p. Due to the collective work of a number of mathematicians, pioneered by Helfgott and by Bourgain and Gamburd, we now know that these Cayley graphs are not only connected, but also form an expander family; we say that Γ has the *superstrong approximation* property. This result is a major achievement of recent years and the current volume is centered around it and its many applications. The property of being an expander is equivalent to the combinatorial spectral gap property on Cayley graphs that the smallest positive eigenvalue of the combinatorial Laplacian on these graphs is bigger than some positive number, which is independent of p.

Though the articles are organized alphabetically by last name, many readers will wish to turn first to the last contribution, by Peter Sarnak. It gives a panoramic overview of recent developments in connection with superstrong approximation for thin groups, and is therefore a good entry point for anyone willing to discover what this is all about.

Andrei Rapinchuk's article is a thorough introduction to strong-approximation for arithmetic groups as well as for their Zariski-dense subgroups.

The surveys by Alireza Salehi Golsefidy and Emmanuel Kowalski describe a number of recent applications of superstrong approximation to sieving and counting primes in orbits, applications that were one of the main motivations for establishing superstrong approximation in the first place.

The articles by Emmanuel Breuillard and by Laci Pyber and Endre Szabó focus on recent combinatorial advances which are key to the proof of superstrong approximation and concern approximate groups and the growth of finite subsets of large finite simple groups.

The article by Amir Mohammadi reports on his recent work with Oh on the ergodicity criterion for unipotent flows with respect to the Burger-Roblin measure on $\Gamma\backslash SO(n, 1)$ for geometrically finite subgroups Γ; this measure plays a key role in counting problems on orbits of Γ.

Michael Larsen's article is a survey of recent results regarding word maps in finite simple groups, their image and distribution.

The article by Hee Oh describes how to use ergodic theory and harmonic analysis to extend to thin subgroups many of the counting results known for lattices and its application to Apollonian circle packings and more general sphere packings.

The article by Gopal Prasad and Andrei Rapinchuk surveys their joint work, spread over the past few years and regarding arithmetic properties of Zariski-dense subgroups of semisimple algebraic groups, the notion of weak commensurability, the existence of regular elements with special properties, etc.

Jordan Ellenberg's paper introduces monodromy groups as a chief source of examples of thin groups and describes his recent work with Hall and Kowalski on how to apply superstrong approximation to prove gonality estimates and height bounds in certain situations arising in arithmetic geometry.

Elena Fuchs reports on recent investigations regarding a very explicit class of monodromy groups arising naturally from differential equations and various attempts to determine whether the given groups are thin or not.

Alan Reid's contribution, coauthored by Darren Long, focuses on various constructions of arithmetically defined subgroups of arithmetic groups (such as $SL_3(\mathbb{Z})$) which are isomorphic to the fundamental groups of closed surfaces.

Igor Rivin surveys some interesting results and questions regarding generic elements (in various senses) of geometrically interesting groups.

The articles by Jean Bourgain and Alex Kontorovich describe part of their work on various problems connected to local-global principles for thin (semi-)groups arising from Apollonian circle packings and the Zaremba conjecture.

We hope the reader will enjoy browsing and reading this volume. It is our pleasure to thank all the people involved in the making of this book for helping us assemble this nice collection of articles in a speedy manner: the referees for their prompt and professional advice, Silvio Levy and the editorial staff of Cambridge University Press for their support, and the authors for their contributions.

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