Inside Out II
MSRI Publications
Volume **60**, 2012

# The Calderón inverse problem in two dimensions

COLIN GUILLARMOU AND LEO TZOU

We review recent progress on the two-dimensional Calderón inverse problem, that is, the uniqueness of coefficients of an elliptic equation on a domain of $\mathbb{C}$ (or a surface with boundary) from Cauchy data at the boundary.

## 1. The Calderón problem

The global uniqueness for inverse boundary value problems of elliptic equations at fixed frequency in dimension $n = 2$ is quite particular and remained open for many years. Now these problems are well understood, with a variety of results appearing in the last 10 or 15 years, essentially all using the complex structure $\mathbb{R}^2 \simeq \mathbb{C}$ and $\bar{\partial}$-techniques. This is therefore a good time to write a short survey on the subject. Although we tried to cover as much as we can, we do not pretend to be exhaustive and we apologize in advance for any forgotten reference, which is not a decision made on purpose but rather a sign of our ignorance. We have decided to give more details about the proofs of recent results based on Bukhgeim's idea [2008], for there is already a survey by Uhlmann [2003] on the subject about older results. The results of Astala, Lassas, and Päivärinta using quasiconformal methods are the subject of a separate survey in this volume [Astala et al. 2013]. Finally, we do not discuss questions about stability and reconstruction, nor inverse scattering results.

**1A.** ***The inverse problem for the conductivity.*** Let $\Omega \subset \mathbb{C}$ be a bounded domain with boundary (say smooth boundary) and let $\gamma \in L^\infty(\Omega, S^2_+(\Omega))$ be a field of positive definite symmetric matrices on $\Omega$. The Dirichlet-to-Neumann map is the operator

$$\mathcal{N}_\gamma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$

defined by

$$\langle f_1, \mathcal{N}_\gamma f_2 \rangle := \int_\Omega \gamma \nabla u_1 . \nabla u_2,$$

where $f_1, f_2 \in H^{\frac{1}{2}}(\partial\Omega)$, $\langle\cdot,\cdot\rangle$ is the pairing between $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$, $u_2$ is the $H^1(\Omega)$ solution of the elliptic equation

$$\operatorname{div}(\gamma\nabla u) = 0, \quad u|_{\partial\Omega} = f_2 \tag{1}$$

and $u_1$ is any $H^1$ function with trace $f_1$ on $\partial\Omega$. Equivalently, $\mathcal{N}_\gamma f_2 = \gamma\nabla u_2.\nu$, where $u$ is the solution of (1) and $\nu$ is the normal outward pointing vector field to the boundary. The operator $\mathcal{N}_\gamma$ is a nonlocal operator, in fact it is an elliptic pseudodifferential operator of order 1 on $\partial\Omega$, at least when $\gamma$ is smooth. Its dependence on $\gamma$ is nonlinear. The problem asked by Calderón [1980] is the following:

$$\text{Is the map } \gamma \to \mathcal{N}_\gamma \text{ injective?} \tag{Q1}$$

The conductivity is called *isotropic* when $\gamma = \gamma(x)\mathrm{Id}$ for some function $\gamma(x)$. If $\Omega$ is an inhomogeneous body with conductivity $\gamma$, then $\mathcal{N}_\gamma f$ is the current flux at the boundary corresponding to a voltage potential $f$ on $\partial\Omega$. The Dirichlet-to-Neumann operator represents the information which can be obtained from static voltage and current measurements at the boundary, and (Q1) is a question about uniqueness of a media giving rise to a given (infinite) set of measurements. The graph of $\mathcal{N}_\gamma$ is called the *Cauchy data space*.

**1B.** ***The inverse problem for metrics.*** An alternative and quite similar problem is as follows. Let $M$ be a surface with boundary and $g$ is a Riemannian metric on $M$, one can define the Dirichlet-to-Neumann operator associated to $(M, g)$ by

$$\mathcal{N}_{(M,g)} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega), \quad f \mapsto \partial_\nu u|_{\partial M}$$

where $u$ is the unique solution of the elliptic equation

$$\Delta_g u = 0, \quad u|_{\partial M} = f,$$

here $\Delta_g = d^*d$ where $d$ is the exterior derivative and $d^*$ its adjoint for the Riemannian $L^2$ product $\langle u, v\rangle = \int_M u\bar{v}\mathrm{dv}_g$. Then we ask

$$\text{Is the map } (M, g) \to \mathcal{N}_{(M,g)} \text{ injective ?} \tag{Q2}$$

Here $M$ runs over the set of Riemannian surfaces with a given fixed boundary $\partial M = N$.

**1C.** ***Gauge invariance.*** The obvious answer one can give for both (Q1) and (Q2) is "No". Indeed, if $\psi : \Omega \to \Omega$ and $\varphi : M \to \varphi(M)$ are two diffeomorphisms which satisfy $\psi|_{\partial\Omega} = \mathrm{Id}$ and $\varphi|_{\partial M} = \mathrm{Id}$, then

$$\mathcal{N}_{\psi_*\gamma} = \mathcal{N}_\gamma, \quad \mathcal{N}_{(M,g)} = \mathcal{N}_{(\varphi(M),\varphi_* g)}$$

where $\varphi_* g$ is the pushforward of the metric $g$ by $\varphi$ and

$$\psi_*\gamma(x) := \left(\frac{d\psi^t \gamma d\psi}{|\det d\psi|}\right)(\psi^{-1}(x)).$$

In fact, for the metric case, there is another invariance, which comes from the conformal covariance of the Laplacian in 2 dimensions: since $\Delta_g = e^{2\omega}\Delta_{e^{2\omega}g}$ for all smooth function $\omega$, one easily deduces that for all function $\omega$ which satisfies $\omega|_{\partial M} = 0$, then

$$\mathcal{N}_{(M,g)} = \mathcal{N}_{(\varphi(M), e^{2\omega}\varphi_* g)}.$$

The good questions to ask are then

$$\text{Does } \mathcal{N}_{\gamma_1} = \mathcal{N}_{\gamma_2} \text{ imply } \exists\psi : \Omega \to \Omega \text{ (diffeo) s.t. } \psi|_{\partial\Omega} = \mathrm{Id},\ \psi_*\gamma_1 = \gamma_2? \quad (\mathrm{Q1}')$$

and

$$\text{Do } \mathcal{N}_{(M_1,g_1)} = \mathcal{N}_{(M_2,g_2)} \text{ and } \partial M_1 = \partial M_2 \text{ imply } \exists\psi M_1 \to M_2 \text{(diffeo)}$$
$$\text{and } \omega : M_2 \to \mathbb{R} \text{ s.t. } \psi|_{\partial M_1} = \mathrm{Id} \text{ and } \psi_* g_1 = e^{2\omega} g_2? \quad (\mathrm{Q2}')$$

**1D.** ***The inverse problem for potentials.*** We conclude by another similar problem for Schrödinger operators. If $(M, g)$ is a fixed compact Riemannian surface with boundary, and $V \in L^\infty(M)$ is a potential such that $\Delta_g + V$ has no element in its kernel vanishing at $\partial M$, then the Dirichlet-to-Neumann operator associated to $V$ is defined as before by

$$\mathcal{N}_V : H^{\frac{1}{2}}(\partial M) \to H^{-\frac{1}{2}}(\partial M), \quad f \mapsto \partial_\nu u|_{\partial M}$$

where $u$ is the unique solution of the elliptic equation

$$(\Delta_g + V)u = 0, \quad u|_{\partial M} = f.$$

The uniqueness question in this case is

$$\text{Does } \mathcal{N}_{V_1} = \mathcal{N}_{V_2} \text{ imply } V_2 = V_1? \quad (\mathrm{Q3})$$

**1E.** ***Relation between isotropic conductivity and potentials.*** There is an easy remark that one can do about the relation between the isotropic conductivity problem and the potential problem: indeed, setting $u = \gamma^{-1/2} v$ shows that $u$ is a solution of $\mathrm{div}(\gamma\nabla u) = 0$ if and only if $v$ is a solution of $(\Delta + V_\gamma)v = 0$ with

$$V_\gamma = -\frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}.$$

Therefore, if $\gamma \in W^{2,\infty}(\Omega)$ and if $\gamma$ is supposed to be known at the boundary, a resolution of the problem (Q3) implies the resolution of (Q1) if the conductivity is isotropic.

**1F.** ***Other related problems.*** Another natural problem is to identify up to gauge a magnetic field or a Hermitian connection $\nabla^X = d + iX$ on a complex line bundle $E$ over a surface with boundary (or more simply a domain) from the Cauchy data space of the connection Laplacian $L_X := (\nabla^X)^* \nabla^X$. More generally one can add a potential $V$ to $L_X$ and try to identify $X$ up to gauge and $V$ from the Cauchy data space of $L_{X,V} = L_X + V$. The question also makes sense for connections on complex vector bundles, where one has to deal with elliptic systems, and for Dirac type operators.

**1G.** ***Partial data problems.*** Practically, there are many situations where we have measurements of the currents on only a small piece $\Gamma \subset \partial M$ of the boundary, it is therefore important to see what can be obtained from the Dirichlet-to-Neumann operator acting on functions supported in $\Gamma$. For instance, a natural question is to take two open sets $\Gamma_+, \Gamma_-$ of the boundary, and see if the partial Cauchy data set

$$\{(u|_{\Gamma_+}, \partial_\nu u|_{\Gamma_-}); (\Delta + V)u = 0,\ u \in H^1(M),\ u|_{\partial M \setminus \Gamma_+} = 0\}$$

determines the potential.

**1H.** ***Why these problems are not simple.*** Let $H_h$ be a family of elliptic operators of order 2, depending on a small parameter $h \in (0, h_0)$, and of the form $H_h = h^2 H + V_h$ where $V_h$ is a family of real potentials depending smoothly in $h \in [0, h_0)$ and $H$ an elliptic self-adjoint operator of order 2 with principal symbol $p$. The semiclassical theory tells us that, when there is a characteristic set $\{(m,\xi) \in T^*\mathbb{R}^2 \setminus \{\xi = 0\}; p(m,\xi) + V_0(m) = 0\} \neq \varnothing$, the solutions of $H_h u = 0$ are microlocalized near this set and oscillating with frequency of order $1/h$ as $h \to 0$, moreover the microlocal concentration is characterized by the flow of the Hamiltonian vector field associated to the Hamiltonian $p + V_0$. In particular, if one know something about this concentration on the boundary of the domain, one can expect to propagate it through this flow to say something in the interior of the domain. A typical example would be if we know the Dirichlet-to-Neumann operators $\mathcal{N}_{(M,g)}(\lambda)$ for the equation $(\Delta_g - \lambda^2)u = 0$ for all $\lambda > 0$, since one could set $\lambda = 1/h$. In the Calderón problem, we only know an information at 0 (or fixed) frequency, which makes the problem much more complicated. In a way, the solutions of this problem are often based on complexifying the frequencies to see high frequencies phenomena.

***Notation.*** We shall use the complex variable $z = x + iy$ for $\mathbb{C}$ and the variable $w = (x, y)$ for $\mathbb{R}^2$ in what follows.

## 2. Local uniqueness

**2A.** ***Kohn–Vogelius local uniqueness.*** For all the problems above, the Dirichlet-to-Neumann operator is a nonlocal operator with singular integral kernel and the singularities of its kernel at the diagonal contain local information about the coefficients of the elliptic equation in the interior. Actually, it is shown in all the cases above that it determines the Taylor expansion of the coefficients at the boundary, say when those coefficients are smooth. This was apparently first observed by Kohn and Vogelius [1984]

**Theorem 2.1** [Kohn and Vogelius 1984]. *Let $\gamma_1, \gamma_2$ be two smooth isotropic conductivities on a smooth domain $\Omega$, and assume that $\mathcal{N}_{\gamma_1} = \mathcal{N}_{\gamma_2}$. Then for all $k \geq 0$, we have $\partial_\nu^k \gamma_1 = \partial_\nu^k \gamma_2$ everywhere on $\partial\Omega$.*

In fact, the proof is a local determination and the assumption that $\mathcal{N}_{\gamma_1} = \mathcal{N}_{\gamma_2}$ can be replaced by

$$\langle \mathcal{N}_{\gamma_1} f, f \rangle = \langle \mathcal{N}_{\gamma_2} f, f \rangle \quad \text{for all } f \in C_0^\infty(\Gamma)$$

where $\Gamma \subset \partial\Omega$ is an open set, and this would show that $\partial_\nu^k \gamma_1 = \partial_\nu^k \gamma_2$ on $\Gamma$. Notice that this allows to say that the Dirichlet-to-Neumann map determines real analytic isotropic conductivities by analytic continuation from the boundary.

*Idea of proof.* The idea is to construct solutions $u_h$ depending on a small parameter $h > 0$ such that their boundary values $f_h$ are supported in an $h$-neighborhood of a point $x_0 \in \partial\Omega$, and that

$$\|f_h\|_{H^{1/2+\ell}(\partial\Omega)} = O(h^{-\ell}) \quad \text{for any } \ell > -M$$

for some $M > 0$ chosen arbitrarily large. Then one can show that if $U \subset \Omega$ is an open set with $d(\partial\Omega, \partial U) > 0$ and $W$ is an open neighborhood of $x_0$,

$$\|\nabla u_h\|_{L^2(U)} = O(h^M) \quad \text{and} \quad \|\rho^m \nabla u_h\|_{L^2(W)} = O(h^{-(1+\epsilon)m}) \qquad (2)$$

for some small $\epsilon > 0$ if $\rho(x) = \text{dist}(x, \partial\Omega)$. Assuming that $\partial_\nu^m \gamma_1 \neq \partial_\nu^m \gamma_2$ near $x_0$ for some $m \in \mathbb{N}$, then by writing the Taylor expansion in normal coordinates to the boundary, this means that either $\gamma_1 - \gamma_2 \geq C\rho^m$ or $\gamma_2 - \gamma_1 \geq C\rho^m$ in a neighborhood of $x_0$, for some $C > 0$. Let us assume the first case. From the estimates (2) above and taking $M \gg m$ and $h$ very small, this gives

$$\int_\Omega \gamma_1 |\nabla u_h|^2 \geq \int_W \gamma_1 |\nabla u_h|^2 \geq \int_W \gamma_2 |\nabla u_h|^2 + O(h^{-(1+\epsilon)m}) \geq \int_\Omega \gamma_2 |\nabla u_h|^2$$

which contradicts $\langle \mathcal{N}_{\gamma_1} f_h, f_h \rangle = \langle \mathcal{N}_{\gamma_2} f, f \rangle$. ◻

**2B.** ***Further results.*** The argument of Kohn–Vogelius can be extended easily to recover $k$ derivatives of $\gamma_1 - \gamma_2$ when $\gamma_1 - \gamma_2$ has $1 + k$ derivatives (by some $L^2$-Sobolev embeddings). The identification of the boundary value of an isotropic conductivity has been improved (in terms of regularity) for smooth domains by Sylvester and Uhlmann [1988] to continuous conductivities with an estimate

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \le C \|\mathcal{N}_{\gamma_1} - \mathcal{N}_{\gamma_2}\|_{H^{\frac{1}{2}}(\partial\Omega)\to H^{-\frac{1}{2}}(\partial\Omega)}.$$

The uniqueness is also local in the sense that one only needs to know the Dirichlet-to-Neumann map in an open set to determine the conductivity at a point of this set. Alessandrini [1990] proved that for Lipschitz domains, if $\gamma_j$ are Lipschitz and $\gamma_1 - \gamma_2$ is $C^k$ in a neighborhood of $\partial\Omega$ then $\mathcal{N}_{\gamma_1} = \mathcal{N}_{\gamma_2}$ implies $\partial^\alpha_( \gamma_1 - \gamma_2) = 0$ on $\partial\Omega$ for all $|\alpha| \le k$. Brown [2001] proved a result for continuous conductivities on Lipschitz domains. For smooth metrics, Lee and Uhlmann [1989] proved that the full symbol of the Dirichlet-to-Neumann operator (as a classical pseudodifferential operator of order 1) determines the Taylor expansion to all order of the metric at the boundary.

## 3. The method of complex geometric optic solutions

The first approach to recover a conductivity from boundary data was to reduce the problem to the potential problem, as explained above and to use particular solutions of the Schrödinger equation $(\Delta + V)u = 0$ where $V$ is a real potential of the form $-\Delta\gamma^{\frac{1}{2}}/\gamma^{\frac{1}{2}}$. The advantage of reducing the problem to $\Delta + V$ is that one has to identify a term of order 0 in the equation while for the conductivity problem, $\gamma$ is contained in the principal symbol of the operator. The first observation one can make using this fact is the following: if $u_1$ and $u_2$ are solutions of

$$(\Delta + V_j)u_j = 0, \quad u_j|_{\partial\Omega} = f_j, \quad j = 1, 2$$

in a domain $\Omega \Subset \mathbb{R}^2$, then Green's formula yields the integral identity

$$\begin{aligned}\int_\Omega (V_2 - V_1)u_1u_2 &= \int_\Omega \Delta u_1.u_2 - u_1.\Delta u_2 \\ &= \int_{\partial\Omega} \partial_\nu u_1.u_2 - u_1.\partial_\nu u_2 = \int_{\partial\Omega} \mathcal{N}_{V_1} f_1.f_2 - f_1 \mathcal{N}_{V_2} f_2;\end{aligned}$$

that is,

$$\int_\Omega (V_2 - V_1)u_1u_2 = \int_{\partial\Omega} (\mathcal{N}_{V_1} - \mathcal{N}_{V_2}) f_1.f_2, \tag{3}$$

where we have used the symmetry of the Dirichlet-to-Neumann map when the potential is real, which is a consequence of Green's formula again: for any solutions $w_1, w_2$ of $(\Delta + V)w = 0$ with boundary values $f_1, f_2$, we have

$$0 = \int_\Omega (\Delta + V)w_1.u_2 - w_1.(\Delta + V)w_2 = \int_{\partial\Omega} \mathcal{N}_V f_1.f_2 - f_1.\mathcal{N}_V f_2 \ .$$

The integral identity (3) shows that if $\mathcal{N}_{V_1} = \mathcal{N}_{V_2}$, then $V_1 - V_2$ is orthogonal in $L^2(\Omega)$ to the product of solutions of $(\Delta + V_1)u = 0$ with solutions of $(\Delta + V_2)u = 0$. The idea initiated by Calderón was then to construct certain families of solutions with contain high oscillations to give enough information on $V_1 - V_2$ when one integrates against those.

**3A.** ***The linearized Calderón problem.*** Calderón [1980] considered the linearized problem at the potential $V = 0$ as follows: if $V_t$ is a one parameter family of potentials ($t \in (-\epsilon, \epsilon)$) such that $V_0 = 0$ and $\partial_t \mathcal{N}_{V_t}|_{t=0} = 0$, then one has, for all $u_t, v_t$ satisfying $(\Delta + V_t)u_t = (\Delta + V_t)v_t = 0$ with respective fixed boundary value $f, g$,

$$\int_\Omega \nabla u_t.\nabla v_t + V_t u_t.v_t = \int_{\partial\Omega} \mathcal{N}_t f.g$$

therefore differentiating at $t = 0$ (which we denote by a dot)

$$0 = \int_\Omega \nabla u_0.\nabla \dot{v}_0 + \dot{V}_0 u_0.v_0 = \int_{\partial\Omega} \partial_\nu f.\dot{v}_0|_{\partial\Omega} + \int_\Omega \dot{V}_0 u_0.v_0 = \int_\Omega \dot{V}_0 u_0.v_0.$$

The element $\dot{V}_0$ in the kernel of the linearization of $V \to \mathcal{N}_V$ is orthogonal to the product of harmonic functions. Calderón's idea was to use particular solutions, in fact exponentials of linear holomorphic functions (recall that $z = x + iy \in \mathbb{C}$ denotes the complex variable)

$$u_0(z) = e^{z\zeta}, \ v_0(z) = e^{\overline{z\zeta}} \quad \text{with } \zeta \in \mathbb{C}.$$

These are clearly harmonic, since holomorphic and antiholomorphic, and therefore one obtains (recall the notation $w = (x, y) \in \mathbb{R}^2$)

$$\int_\Omega e^{2i\,\mathrm{Im}(z\zeta)} \dot{V}_0(z) = 0 = \int_\Omega e^{2iw.\zeta^T} \dot{V}_0(w)$$

where $\zeta^T := (\mathrm{Im}\,\zeta, \mathrm{Re}\,\zeta) \in \mathbb{R}^2$ which implies that the Fourier transform of $\mathbb{1}_\Omega \dot{V}_0$ at $\zeta^T$ is 0 for all $\zeta \in \mathbb{C}$, and thus that $\dot{V}_0 = 0$. The linearized Dirichlet-to-Neumann operator at 0 is injective, but Calderón [1980] observed that its range is not closed and therefore we cannot use the local inverse theorem to consider the nonlinear problem.

**3B.** ***The nonlinear case.*** The construction of solutions $u_h$ of the Schrödinger equation $(\Delta + V)u_h = 0$ which grow exponentially as $h \to 0$ appeared first in the work of Faddeev [1965; 1974] and were later used to solve the Calderón problem in dimension 2 for isotropic conductivities by Sylvester and Uhlmann [1986], under the assumption that the conductivity $\gamma$ is close to 1, and then by Nachman [1996] to solve the problem unconditionally (except for regularity conditions on $\gamma$). For the inverse scattering problem in dimension 2, this was used by Novikov [1992] to solve the problem under smallness assumptions on the potential. These solutions with complex phases depending on a parameter are called *complex geometric optic solutions* (CGO in short) or *Faddeev type solutions*, the first terminology obviously arising from the analogy with geometric optic solutions with real phases used in the WKB approximation of solutions of hyperbolic partial differential equations.

**Definition 3.1.** More precisely, we will say that a family of solutions $u_h$ (with $h \in (0, h_0)$ small) of $(\Delta + V)u_h = 0$ are complex geometric optic solutions with phase $\Phi$ if there exists a complex-valued function $\Phi$ and some functions $a \in L^2$ independent of $h$ and $r_h \in L^2$ such that

$$u_h = e^{\Phi/h}(a + r_h), \quad \|r_h\|_{L^2} \to 0 \text{ as } h \to 0.$$

Practically, these solutions will have their maximum (of the modulus) localized on the boundary $\partial\Omega$ and pairing with a function $V$ will concentrate for small $h$ the value of $V$ at the maximum, roughly speaking. However they will have an oscillating phase (given by $e^{i\,\mathrm{Im}(\Phi)/h}$) and this term can provide us with information on $V$ in $\Omega$, as in the linearized case. The construction of CGO as defined above is in fact not a very complicated thing to do if one thinks in terms of Carleman estimates (this will be developed below), but there is a complication: indeed an observation of the integral identity (3) shows that if $V_1, V_2$ are bounded potentials on $\Omega$, say, and if $\mathcal{N}_{V_1} = \mathcal{N}_{V_2}$, then

$$\int_\Omega (V_1 - V_2)u_1 u_2 = 0 \quad \text{for all } u_1, u_2 \in H^2(\Omega) \text{ s.t. } (\Delta + V_1)u_1 = 0 = (\Delta + V_2)u_2. \tag{4}$$

In particular, we see that if we expect to obtain information on $V_1 - V_2$ in the interior $\Omega$ from plugging CGO $u_1 = u_h$ with phase $\Phi_1$ and $u_2 = v_h$ with phase $\Phi_2$ we should ask that $\mathrm{Re}(\Phi_1) = -\mathrm{Re}(\Phi_2)$, which turns out to be a much more restrictive condition.

The phases which appeared in Sylvester and Uhlmann [1986] are linear and the existence of CGO for the isotropic conductivity equation was proved under the assumption that $\|1-\gamma\|_{W^{3,\infty}} \leq \epsilon$ where $\epsilon > 0$ is small depending only on $\Omega$.

Let us instead give the main technical result[1] of [Sylvester and Uhlmann 1987], which consists in showing the existence of CGO with linear phases without smallness assumptions on $\gamma$ or $V$.

**Theorem 3.2** [Sylvester and Uhlmann 1987]. *Let $V \in C^\infty(\Omega)$ where $\Omega \subset \mathbb{C}$ is a domain with smooth boundary. For any $s > 1$, there exist constants $C_1, C_2$ such that if $h > 0$ and $\zeta \in \mathbb{C}^2$, $\frac{1}{2} \le |\zeta| \le 2$, possibly depending on $h$, are such that*

$$\zeta.\zeta = 0, \quad h^{-1} \ge C_1 \|V\|_{H^s(\Omega)}$$

*then there exists $u_h \in H^s(\Omega)$ satisfying $(\Delta + V)u_h = 0$ of the form*

$$u_h(w) = e^{\frac{w.\zeta}{h}}(1 + r_h(\zeta, w)), \quad \textit{with } \|r_h\|_{H^s(\Omega)} \le C_2 h \|V\|_{H^s(\Omega)}.$$

We will give an idea of how to construct these CGO a bit later. First, let us check what we can deduce from the existence of such solutions, in comparison to the linearized case where we obtained the Fourier transform of the difference of potential. If $u_h, v_h$ are solutions of $(\Delta + V_1)u_h = 0 = (\Delta + V_2)v_h$ of the form

$$\left.\begin{aligned} u_h(w) &= e^{\frac{w.\zeta}{h}}(1 + r_h(\zeta, w)) \\ v_h(w) &= e^{\frac{w.\eta}{h}}(1 + s_h(\eta, w)) \end{aligned}\right\} \quad \text{with } \|r_h\|_{L^2} + \|s_h\|_{L^2} = O(h),$$

$$\zeta = \alpha + i\mu,\ \eta = -\alpha + i\nu,\ \alpha, \mu, \nu, \in \mathbb{R}^2,\ \zeta.\zeta = \eta.\eta = 0,\ |\zeta| = |\eta| = 1,$$

given by Theorem 3.2, this implies that $\mu = \pm J\alpha$ and $\alpha = \pm J\nu$, where $J$ is the rotation of angle $\pi/2$ in $\mathbb{R}^2$. By (4) with $u_1 = u_h$ and $u_2 = v_h$, we deduce

$$\int_\Omega (V_1 - V_2) e^{2i\frac{w.\mu}{h}} + O(h) = 0, \quad \text{if } 0 < h^{-1} \le C \max_{i=1,2} \|V_i\|_{H^1}.$$

We see that from this identity, we *cannot* show that $V_1 = V_2$ since as $h \to 0$ this equality does not say anything if $V_1 - V_2$ has a bit of regularity. It could however say something about the singularity of $V_1 - V_2$, for instance if the potentials have conormal singularities somewhere in the domain.

**3C.** ***Comparison with higher dimensions.*** In higher dimensions, $n > 2$, it turns out that CGO with linear phases give enough information to identify a potential and thus an isotropic conductivity. Indeed, applying Theorem 3.2 (recall that $\zeta$ there can also depend on $h$) we obtain

$$\left.\begin{aligned} u_h(w) &= e^{\frac{w.\zeta}{h}}(1 + r_h(\zeta, w)) \\ v_h(w) &= e^{\frac{w.\eta}{h}}(1 + s_h(\eta, w)) \end{aligned}\right\} \quad \text{with } \|r_h\|_{L^2} + \|s_h\|_{L^2} = O(h),$$

$$\zeta = (\alpha + kh) + i(\mu + kh), \quad \eta = -(\alpha + kh) + i(-\mu + kh),$$

[1] The construction of the CGO in [Sylvester and Uhlmann 1987] holds in any dimension.

where $\alpha, \mu, k \in \mathbb{R}^n$ are chosen such that $\alpha.k = \alpha.\mu = k.\mu = 0$, and $\frac{1}{2} < |\alpha| = |\mu| < 2$, in order that $\zeta.\zeta = \eta.\eta = 0$; here $h > 0$ is taken very small. Of course, here we use that there are at least 3 orthogonal directions to define $\alpha, k, \mu$. Plugging those in the integral identity (4), this yields

$$0 = \int_\Omega e^{iw.k}(V_1 - V_2) + O(h)$$

and by letting $h \to 0$, we see that $V_1 = V_2$ since its Fourier transform is 0. The element which tells us information is somehow the leading term $e^{w.(\pm k+ik)}$ in the amplitudes of the CGO. This is summarized as follows:

**Theorem 3.3** [Sylvester and Uhlmann 1986]. *Let $V_1, V_2 \in C^\infty(\Omega)$, where $\Omega$ is a domain in $\mathbb{R}^n$ with smooth boundary, with* $\boldsymbol{n \geq 3}$. *If the Dirichlet-to-Neumann for the Schrödinger equations $(\Delta + V_i)u = 0$ agree, i.e., $\mathcal{N}_{V_1} = \mathcal{N}_{V_2}$, then $V_1 = V_2$.*

It can be noticed from the proof of their paper that the smoothness assumption on $V_1, V_2$ can be relaxed to $W^{2,\infty}(\Omega)$ regularity.

On the other hand, in dimension $n = 2$, Sylvester and Uhlmann [1986] were able to prove by using the CGO with linear phases that $\mathcal{N}_\gamma$ determine locally a conductivity close to 1. See Theorem 4.1.

## 3D. *Constructing CGO in dimension 2.*

**3D.1.** *For linear phases, a direct approach using Fourier transform.* Let us first explain the method used in [Sylvester and Uhlmann 1987] to construct CGO with linear phases for the Schrödinger equation $(\Delta + V)u = 0$ on a domain $\Omega \subset \mathbb{C}$. Of course, here the characteristic variety for the conjugated Laplacian is much simpler than in higher dimension, which makes the proof easier, as we shall now see. We search for solutions of the form

$$u(w) = e^{\frac{w.\zeta}{h}}(1 + r_h(\zeta, w)), \quad \zeta \in \mathbb{C}^2, \zeta.\zeta = 0$$

where $\zeta$ may depend on $h$ but $\frac{1}{2} \leq |\zeta| \leq 2$. Then $r_h$ needs to solve

$$(h^2\Delta + h^2 V - 2h\zeta\nabla)r_h = -h^2 V. \tag{5}$$

If we think in terms of semiclassical calculus, one has an operator $P_h = h^2\Delta + h^2 V - 2h\zeta\nabla$ to invert on the right, and its semiclassical principal symbol is $p_h(w, \xi) = \xi^2 - 2i\zeta.\xi$. Writing $\zeta = \mu + i\nu$, we have by an elementary calculation (splitting self-adjoint and anti self-adjoint components)

$$\|P_h u\|_{L^2} \geq \|(h^2\Delta - 2h\zeta.\nabla)u\|_{L^2} - h^2\|Vu\|_{L^2}$$

and

$$\|(h^2\Delta - 2h\zeta\nabla)u\|_{L^2}^2 = \|(h^2\Delta - 2hi\nu.\nabla)u\|_{L^2}^2 + \|h\mu.\nabla u\|_{L^2}^2.$$

Observe that $|\nu|^{-1}(\nu,\mu)$ is an orthonormal basis of $\mathbb{C}$ if $\zeta=\mu+i\nu$ solves $\zeta.\zeta=0$. Fourier transforming, we obtain

$$\begin{aligned}\|(h^2\Delta-2h\zeta\nabla)u\|_{L^2}^2&=\int_{\mathbb{R}^2}|h\xi|^2|h\xi-2\nu|^2|\hat{u}(\xi)|^2d\xi\\&=h^{-2}\int_{\mathbb{R}^2}|\xi|^2|\xi-2\nu|^2|\hat{u}(\xi/h)|^2d\xi.\end{aligned}$$

Let $\chi_1^2+\chi_2^2+\chi_3^2=1$ be a partition of unity on $(-1,\infty)$, with $\chi_1\in C_0^\infty(-1,\frac{1}{2})$, $\chi_2\in C_0^\infty(\frac{1}{4},3)$ and $\chi_3$ with support in $(2,\infty]$. We write

$$\hat{u}=\sum_{i=1}^{3}\hat{u}_i,$$

where $\hat{u}_i(\xi):=\chi_i(|\xi|)\hat{u}(\xi/h)$. Since $\chi_3(|\nu|)=0=\chi_3(0)$, we clearly have

$$h^{-2}\int_{\mathbb{R}^2}|\xi|^2|\xi-2\nu|^2|\hat{u}_3(\xi)|^2d\xi\geq Ch^{-2}\|\hat{u}_3\|^2. \tag{6}$$

Now, observe by integrating by parts, we have for any $v\in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned}2n\int_{\mathbb{R}^n}|v|^2&=-\int_{\mathbb{R}^n}\nabla|v|^2.\nabla|\xi|^2d\xi=-\int_{\mathbb{R}^n}\nabla|v|^2.\nabla|\xi|^2d\xi\\&=-4\,\mathrm{Re}\int_{\mathbb{R}^n}\bar{v}\nabla v.\xi d\xi\\&\leq\frac{4}{\epsilon}\int_{\mathbb{R}^n}|\xi|^2|v|^2d\xi+4\epsilon\int_{\mathbb{R}^n}|\nabla v|^2\end{aligned} \tag{7}$$

for any $\epsilon>0$. We apply this with $v=\hat{u}_1$ after observing that $u(w)$ is supported in $|w|\leq R/h$ for some $R>0$ and thus the $H^1$ norm of $\hat{u}_1$ is controlled by $\|u\|$ as follows:

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla\hat{u}_1|^2&=h^4\int_{\mathbb{R}^2}|w|^2|\hat{\chi}_1\star u(hw)|^2dw\\&\leq h^2\|u\|_{L^2}^2\Big(\int|\hat{\chi}_1(w)(|w|+R/h|)\,dw\Big)^2\leq C\|u\|_{L^2}^2.\end{aligned}$$

This implies, by taking $\epsilon=\delta h^2$ with $\delta$ small,

$$\begin{aligned}h^{-2}\int_{\mathbb{R}^2}|\xi|^2|\xi-2\nu|^2|\hat{u}_1(\xi)|^2d\xi&\geq Ch^{-2}\int_{\mathbb{R}^n}|\xi|^2|\hat{u}_1|^2d\xi\\&\geq C\delta(\|\hat{u}_1\|_{L^2}^2-C\delta h^2\|u\|_{L^2}^2).\end{aligned} \tag{8}$$

For dealing with $\hat{u}_2$, we use the same argument after the change of variables $\xi \to \xi + 2\nu$:

$$\begin{aligned} h^{-2}\int_{\mathbb{R}^2} |\xi|^2|\xi-2\nu|^2|\hat{u}_2(\xi)|^2 d\xi \\ &= h^{-2}\int_{\mathbb{R}^2} |\xi+2\nu|^2|\xi|^2|\hat{u}(\xi+2\nu)|^2\chi_2(|\xi+2\nu|)^2 d\xi \\ &\geq Ch^{-2}\int_{\mathbb{R}^2} |\xi|^2|\hat{u}(\xi+2\nu)|^2\chi_2(|\xi+2\nu|)^2 d\xi \\ &\geq C\delta(\|\hat{u}_2\|^2_{L^2} - C\delta h^2\|u\|^2_{L^2}) \qquad (9) \end{aligned}$$

for some small $\delta > 0$. We conclude by taking $\delta$ small enough and combining (8), (9) and (6) that

$$\|(h^2\Delta - 2h\zeta\nabla)u\|^2_{L^2} \geq C\delta\Big(\int_{\mathbb{R}^2} |u(\xi/h)|^2 d\xi - C\delta h^2\|u\|^2_{L^2}\Big) \geq C\delta h^2\|u\|^2_{L^2}$$

and thus (fixing $\delta$)

$$\|P_h u\|_{L^2} \geq Ch\|u\|_{L^2}.$$

By the Riesz representation theorem (or Lax–Milgram), it is clear that $P_h^*$ has a right bounded inverse mapping $L^2(\Omega')$ to $H^1_0(\Omega') \cap H^2(\Omega')$ with norm $\mathbb{O}_{L^2\to L^2}(h^{-1})$, but $P_h$ has exactly the same form as $P_h$ with $\overline{V}$ instead of $V$ and $-\bar{\zeta}$ instead of $\zeta$, we can then apply the same argument to say that $P_h$ has a right inverse $G_h = L^2(\Omega') \to H^1_0(\Omega') \cap H^2(\Omega')$ with norm $\mathbb{O}_{L^2\to L^2}(h^{-1})$. Equation (5) is solved by setting

$$r_h = -h^2 G_h V$$

which has norm $\|r_h\|_{L^2} = \mathbb{O}(h)$. The Sobolev norm $\|\nabla r_h\|_{L^2} = \mathbb{O}(1)$ can also be obtained easily from this proof above.

**3D.2.** *For holomorphic phases.* We will now give a more direct argument based on Carleman estimates for general holomorphic phases without critical points. This follows the method of Imanuvilov, Uhlmann, and Yamamoto [Imanuvilov et al. 2010b]; see also [Guillarmou and Tzou 2009].

**Lemma 3.4.** *Let $\varphi$ be a harmonic function on $\Omega$ and $V \in L^\infty$. The following estimate holds for all $u \in C_0^\infty(\Omega)$ and $h > 0$:*

$$\|e^{-\varphi/h}\Delta e^{\varphi/h}u\|_{L^2} \geq C\Big(\frac{1}{h}\|\,|\nabla\varphi|u\,\|_{L^2} + \|\nabla u\|_{L^2}\Big).$$

*In particular, if $\varphi$ has no critical points, then for small $h > 0$*

$$\|e^{-\varphi/h}(\Delta + V)e^{\varphi/h}u\|_{L^2} \geq C\Big(\frac{1}{h}\|u\|_{L^2} + \|\nabla u\|_{L^2}\Big).$$

*Proof.* It suffices to prove this for real-valued $u$. We use that $\Delta = -\partial_{\bar{z}}\partial_z$ where $z = x + iy$ is the complex coordinate, and assume $\Omega$ is simply connected so that there exists $\psi$ harmonic with $\Phi = \varphi + i\psi$ holomorphic,

$$e^{-\varphi/h}\Delta e^{\varphi/h} = -e^{i\psi/h}\partial_{\bar{z}}e^{-i\psi/h}e^{i\psi/h}\partial_z e^{i\psi/h}$$

and computing explicitly and integrating by parts ($u \in C_0^\infty(\Omega)$)

$$\begin{aligned}
&\|e^{-i\psi/h}\partial_z e^{i\psi/h}u\|_{L^2}^2 \\
&\quad= \left\|\partial_z u + iu\frac{\partial_z\psi}{h}\right\|_{L^2}^2 = \int_\Omega\left(\partial_x u + u\frac{\partial_y\psi}{h}\right)^2 + \left(\partial_y u - u\frac{\partial_x\psi}{h}\right)^2 \\
&\quad= \|\nabla u\|^2 + \frac{1}{h^2}\|u\nabla\psi\|^2 + \frac{1}{h}\int_\Omega \partial_x(u^2)\partial_y\psi - \partial_y(u^2)\partial_x\psi \\
&\quad= \|\nabla u\|^2 + \frac{1}{h^2}\|u\nabla\psi\|^2. \qquad (10)
\end{aligned}$$

We can now use the Poincaré inequality: if $v \in C_0^\infty(\Omega)$,

$$\|e^{-i\psi/h}\partial_{\bar{z}}(e^{i\psi/h}v)\|_{L^2} = \|\partial_{\bar{z}}(e^{i\psi/h}v)\|_{L^2} = C\|\nabla(e^{i\psi/h}v)\| \geq C\|v\|_{L^2}^2, \quad (11)$$

where the second equality uses integration by parts and the fact that $v$ is compactly supported. Combining (11) and (10), this proves the Lemma. If the domain is not simply connected, the proof works the same for harmonic functions with a harmonic conjugate, but in fact by using local Carleman estimates and convexification arguments (see [Guillarmou and Tzou 2009]) this even works for all harmonic functions without critical points. ☐

Again, using Riesz representation theorem, this construct a right inverse $G_h^\pm$ on $L^2(\Omega)$ for $e^{\mp\Phi/h}(\Delta + V)e^{\pm\Phi/h}$ with $L^2 \to L^2$ norm $\mathcal{O}(h)$ and allows to construct complex geometric optic solutions of $(\Delta + V)u = 0$ by setting

$$u_h = e^{\Phi/h}(a + r_h), \quad \partial_z a = 0, \quad r_h = -G_h^+(Va) = \mathcal{O}_{L^2}(h)$$

since $ae^{\Phi/h}$ is a solution of $\Delta(ae^{\Phi/h}) = 0$. The same obviously holds if we take antiholomorphic phases $\overline{\Phi}$ instead of $\Phi$. The proof we just described is simpler than the Fourier transform approach above, but it is very particular to dimension 2 while the other one can be adapted to higher dimensions (for linear phases).

As for linear phases in 2 dimensions, it seems difficult to get enough information from these CGO. Indeed, if $u_h = e^{\Phi/h}(a + r_h)$ is a solution of $(\Delta + V_1)u = 0$ and $v_h = e^{-\Phi/h}(b + s_h)$ a solutions of $(\Delta + V_2)v = 0$ with $a, b$ holomorphic and $\|r_h\|_{L^2} + \|s_h\|_{L^2} = \mathcal{O}(h)$, we deduce from the integral identity (3) by letting $h \to 0$ that

$$\int_\Omega (V_1 - V_2)ab = 0,$$

which means that $V_1 - V_2$ is orthogonal to antiholomorphic functions. Similarly, we can show it is orthogonal to holomorphic functions, but that does not show that $V_1 = V_2$. If instead we take $v_h = e^{-\bar{\Phi}/h}(\bar{b} + s_h)$, we obtain (with $\psi = \mathrm{Im}(\Phi)$)

$$\int_\Omega (V_1 - V_2) e^{2i\psi/h} a\bar{b} = \mathbb{O}(h),$$

but the oscillating term is decaying in $h$ by nonstationary phase (we know $V_1 = V_2$ on the boundary, by boundary uniqueness); thus we do not obtain anything interesting.

**Remark.** Carleman estimates have been used extensively to prove unique continuation for solutions of PDE (or differential inequalities), they turn out to be very powerful. In control theory, they are also a strong tool. We refer the interested reader to [Lebeau and Le Rousseau 2011]. In inverse problems, they have been apparently first used for the Calderón problem in [Bukhgeim and Uhlmann 2002], and then developed in [Kenig et al. 2007; Dos Santos Ferreira et al. 2009]. The property of holomorphic phases (without critical points) to be good weights for Carleman estimates was observed in [Dos Santos Ferreira et al. 2009] and [Bukhgeim 2008]. The first of these two papers studied in general what the authors call *limiting Carleman weights*: weights $\varphi$ such that a Carleman estimate holds for both $\varphi$ and $-\varphi$ with $d\varphi$ never vanishing. In dimension 2, they showed that harmonic functions with no critical points verify this. In [Bukhgeim 2008], the author used holomorphic phases with critical points to solve the inverse problem for a potential (this will be explained further down.)

## 4. The inverse problem for conductivities

**4A.** ***Local uniqueness near constant conductivities.*** We start with the first result obtained for conductivities in dimension 2:

**Theorem 4.1** [Sylvester and Uhlmann 1986]. *Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary. There exists $\epsilon > 0$ depending on $\Omega$ such that if $\mathcal{N}_{\gamma_1} = \mathcal{N}_{\gamma_2}$ and $\|1 - \gamma_j\|_{W^{3,\infty}} \le \epsilon$ for $j = 1, 2$, then $\gamma_1 = \gamma_2$.*

The method is based on the construction of CGO with linear phases; we refer the reader to the original paper for details.

**4B.** ***Global uniqueness in a particular case.***

**Theorem 4.2** [Sun 1990]. *If $\gamma_1, \gamma_2 \in C^4(\Omega)$ with $\Omega$ simply connected and $\Delta \log(\gamma_1) = 0$ or $\Delta \gamma_1^m = 0$ for some $m \neq 0$, then $\mathcal{N}_{\gamma_1} = \mathcal{N}_{\gamma_2}$ implies $\gamma_1 = \gamma_2$.*

**4C.** ***The theorem of Nachman: global uniqueness.*** Here is the first definitive result for conductivity in 2 dimensions:

**Theorem 4.3** [Nachman 1996]. *Let $\Omega \subset \mathbb{C}$ be a Lipschitz bounded domain and $\gamma_1, \gamma_2 \in W^{2,p}(\Omega)$ for $p > 1$ be two positive functions* $\inf_\Omega \gamma_i > 0$. *If $\mathcal{N}_{\gamma_1} = \mathcal{N}_{\gamma_2}$, then $\gamma_1 = \gamma_2$.*

*Idea of proof.* Nachman's approach is based on some sort of scattering theory for complex frequencies. This uses Faddeev Green's functions [Faddeev 1974] and some $\bar{\partial}$ methods, which appeared first for one-dimensional inverse scattering in the work of Beals and Coifman [1981; 1988] and later in 2 dimensions in [Ablowitz et al. 1983; Grinevich and Manakov 1986; Grinevich and Novikov 1988b; Novikov 1986; 1992]. We have seen in Theorem 3.2 that linear CGO for $\Delta + V$ can be constructed for large complex frequencies $\zeta$ (i.e., $h = |\zeta|^{-1}$ small); this is one of the difficulties to recover $\gamma$ and somehow this is what Nachman achieves.

By boundary uniqueness, we can always extend the conductivities outside $\Omega$, so that they agree outside $\Omega$ and are equal to 1 outside a large ball of $\mathbb{C}$. The problem of solving the equation $(\Delta + V)u = 0$ can then be considered in the whole complex plane $\mathbb{C}$. Nachman actually proves that if $V$ comes from a conductivity, that is $V_\gamma = -\Delta\gamma^{\frac{1}{2}}/\gamma^{\frac{1}{2}}$, then it possible to construct CGO for all complex frequencies, not only for large ones, and with uniqueness if one assumes some decay at infinity. More precisely he shows that if $V_\gamma \in L^p$ for $1 < p < 2$ for any $\zeta \in \{\zeta \in \mathbb{C}^2 \setminus \{0\}; \zeta.\zeta = 0\}$, there is a unique solution $u_\zeta$ of $(\Delta + V_\gamma)u = 0$ which satisfies as a function of $w \in \mathbb{R}^2$

$$r_\zeta(w) := e^{-i\zeta.w}u_\zeta(w) - 1 \in L^q(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2), \quad \tfrac{1}{p} - \tfrac{1}{q} = \tfrac{1}{2}.$$

Essentially, to solve this problem for large $\zeta$ we have seen that it amounts to invert on the right the operator $\Delta - 2\zeta.\nabla$ acting on functions compactly supported (or decaying at infinity), with a decay estimate $\mathcal{O}(|\zeta|^{-1})$ in $L^2$ for the operator norm when $|\zeta| \to \infty$. This can be done when $|\zeta|$ is large enough. Nachman manages to show that the solution is unique under the decay condition at infinity, and using Fredholm theory, he manages to deal with small $|\zeta|$. The estimate he obtains on the CGO is

$$\|r_\zeta\|_{L^q} \le C|\zeta|^{-1}\|V\|_{L^p} \quad \text{for large } |\zeta|.$$

Nachman shows that $\mathcal{N}_\gamma$ determines $r_\zeta|_{\partial\Omega}$ by using a sort of scattering operator $S_k$: he shows that $u_\zeta|_{\partial\Omega}$ is the solution of the integral equation (we use $z = x + iy$ and $w = (x, y)$ to identify $\mathbb{C}$ with $\mathbb{R}^2$)

$$u_\zeta(z) = e^{ikz} - (S_k(\mathcal{N}_\gamma - \mathcal{N}_1)u_\zeta)(z) \quad \text{on } \partial\Omega, \quad \text{if } \zeta = (k, ik) \text{ with } k \in \mathbb{C}$$

where $S_k$ is the operator with integral kernel given by the trace at the boundary of the Faddeev Green's function:

$$S_k f(w) := \int_{\partial\Omega} G_k(w-w')f(w')\,dw',$$
$$G_k(w) := \frac{e^{izk}}{4\pi^2}\int_{\mathbb{R}^2}\frac{e^{iw.\xi}}{|\xi|^2+2k(\xi_1+i\xi_2)}d\xi,$$

and $\mathcal{N}_1$ is the Dirichlet-to-Neumann operator for the conductivity $\gamma_0 = 1$. Defining the scattering transform $t(k) \in \mathbb{C}$ to be

$$t(k) := \int_{\partial\Omega} e^{i\overline{zk}}(\mathcal{N}_\gamma - \mathcal{N}_1)u_\zeta|_{\partial\Omega}, \quad \zeta = (k, ik)$$

one sees that $t(k)$ is determined by $\mathcal{N}_\gamma$. The crucial observation of Nachman is that $\mu_k(z) := e^{-ikz}u_\zeta(z)$ solves a $\bar{\partial}$ equation in the frequency $k$ parameter involving $t(k)$

$$\partial_{\bar{k}}\mu_k(z) = \frac{1}{4\pi\bar{k}}t(k)e^{-2i\mathrm{Re}(kz)}\overline{\mu_k(z)}.$$

in a certain weighted Sobolev space in $z$. Notice that this $\bar{\partial}$-type equation in the frequency was also previously in the works [Grinevich and Manakov 1986; Grinevich and Novikov 1988b; Novikov 1992; Beals and Coifman 1988]. Nachman then shows that such equation have unique solutions by using a sort of Liouville theorem for pseudo-analytic functions, at least if we know that $\mu_k$ is bounded for $k$ near 0 and $t(k)/\bar{k}$ is not too singular at $k = 0$. The boundedness in $k \to 0$ is shown in [Nachman 1996], using in particular the fact that Faddeev Green's function does not degenerate too much as $k \to 0$ (essentially by a $\log k$). The function $\mu_k$ turns out to be the solution of the integral equation

$$\mu_k(z) = 1 + \frac{1}{8\pi^2 i}\int_{\mathbb{C}}\frac{t(k')}{(k'-k)\bar{k}'}e^{-2i\mathrm{Re}(kz)}\overline{\mu_{k'}(z)}\,dk'\wedge d\bar{k}'$$

in the weighted Sobolev space and it also satisfies $\mu_k(z) \to \gamma^{1/2}$ as $k \to 0$. In particular $\mu_k$ is determined by the scattering transform $t(k)$ and thus by $\mathcal{N}_\gamma$. Letting $k \to 0$ in $\mu_k(z)$ determines $\gamma$. □

**4D.** ***The Brown–Uhlmann and Beals–Coifman results.*** The regularity assumption was weakened to $\gamma \in W^{1,p}(\Omega)$ by Brown and Uhlmann [1997], who modified and simplified Nachman's proof using the $\bar{\partial}$-method of [Beals and Coifman 1988]. It turns out that the result of Beals and Coifman for the Davey–Stewartson equation, when interpreted in the right way, proves Nachman's result if one assumes smooth conductivities. (Amusingly, it seems almost a decade elapsed before someone made this observation.)

**Theorem 4.4** [Brown and Uhlmann 1997]. *Let $\Omega \subset \mathbb{C}$ be a Lipschitz bounded domain and $\gamma_1, \gamma_2 \in W^{1,p}(\Omega)$ for $p > 2$ be two positive functions* $\inf_\Omega \gamma_i > 0$. *If $\mathcal{N}_{\gamma_1} = \mathcal{N}_{\gamma_2}$, then $\gamma_1 = \gamma_2$.*

*Idea of proof.* As mentioned, the idea is to use the $\bar\partial$-method of [Beals and Coifman 1988]: $u$ is a solution of $\operatorname{div}(\gamma\nabla u) = 0$ if and only if

$$\left[\begin{pmatrix} \bar\partial & 0 \\ 0 & \partial \end{pmatrix} - \begin{pmatrix} 0 & V \\ \overline{V} & 0 \end{pmatrix}\right] \cdot \begin{pmatrix} v \\ v' \end{pmatrix} = 0, \qquad \text{with } v := \gamma^{\frac{1}{2}}\partial u,\ v' = \gamma^{\frac{1}{2}}\bar\partial u$$

and the potential is $V(z) := -\frac{1}{2}\partial_z \log\gamma$ involves only one derivative of the conductivity. The operator above also acts on $2\times 2$ matrices and the CGO in this setting is given by a

$$u_k(z) = m_k(z)\begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar z k} \end{pmatrix}, \qquad k \in \mathbb{C}\setminus\{0\}$$

where $m_k(z)$ is a matrix-valued function such that $m_k(z) \to \mathrm{Id}$ as $|z| \to \infty$, and $m_k - \to \mathrm{Id} \in L^q$ for some $q > 2$. It can be also be shown that $\|m_k(z) - \mathrm{Id}\|_{L^q(k\in\mathbb{C})} \le C$ uniformly in $z$ for $q = p/(p-2)$. The argument is then similar to what we explained above in Nachman's result: one uses the fact (proved in [Beals and Coifman 1988]) that $m_k(z)$ satisfies a $\bar\partial$ equation in $k$:

$$\partial_{\bar k} m_k(z) = m_{\bar k}(z) E_k S_k, \qquad \text{with } E_k := \begin{pmatrix} e^{2i\mathrm{Re}(z\bar k)} & 0 \\ 0 & e^{-2i\mathrm{Re}(zk)} \end{pmatrix}$$

where $S_k$ is the scattering data (which we do not define but is analogous to the Nachman scattering transform), shown to be determined by $\mathcal{N}_\gamma$. Then certain linear combinations $\omega_k(z)$ of the coefficients of $m_k(z)$ satisfy a pseudo-analytic equation of the form $\partial_{\bar k}\omega_k(z) = r(k)\overline{\omega_k(z)}$ with $r \in L^2$ and Brown and Uhlmann showed the following Liouville-type result: if $\omega \in L^p \cap L^2_{\mathrm{loc}}(\mathbb{C})$ is a solution of $\partial_{\bar k}\omega = a\omega + b\overline{\omega}$ with $a, b \in L^2$, then $u = 0$. This implies that $m_k(z)$ is determined by $\mathcal{N}_\gamma$ as in Nachman's paper. Now to recover the potential $V$, it suffices to notice that $D_k m_k = V m_k$ where

$$D_k := D + \frac{k}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

and the potential $V$ can be recovered by the expression

$$V(z)\mathrm{Vol}\, B_R(0) = -\lim_{k_0\to\infty} \int_{|k-k_0|\le R} D_k m_k(z) \frac{dk \wedge d\bar k}{2i}, \qquad R > 0 \text{ fixed},$$

since $m_k(z) \to 1$ as $k \to \infty$ in a sufficiently uniform way. □

**4E.** ***The theorem of Astala–Päivärinta.*** Astala and Päivärinta found a new approach related to quasiconformal techniques to show the uniqueness of an $L^\infty$ conductivity from the Dirichlet-to-Neumann operator.

**Theorem 4.5** [Astala and Päivärinta 2006]. *Let* $\Omega \subset \mathbb{C}$ *be a simply connected bounded domain and* $\gamma_1, \gamma_2 \in L^\infty(\Omega)$ *be two positive functions* $\inf_\Omega \gamma_i > 0$. *If* $\mathcal{N}_{\gamma_1} = \mathcal{N}_{\gamma_2}$, *then* $\gamma_1 = \gamma_2$.

*Idea of proof.* To avoid using regularity on the conductivity, the idea is to transform the equation $\operatorname{div}(\gamma \nabla u) = 0$ into a Beltrami equation. They show that if $u \in H^1(\Omega)$ is a solution of $\operatorname{div}(\gamma \nabla u) = 0$, then there exists a unique function $v \in H^1(\Omega)$ such that $f = u + iv$ solves the Beltrami equation

$$\partial_{\bar{z}} f = \mu \overline{\partial_z f}, \quad \text{where } \mu := \frac{1-\gamma}{1+\gamma}. \tag{12}$$

Conversely, if $f = u + iv$ is a solution of the Beltrami equation with $\|\mu\|_{L^\infty} < 1 - \epsilon$ for some $\epsilon > 0$, then

$$\operatorname{div}(\gamma \nabla u) = 0 \quad \text{and} \quad \operatorname{div}(\gamma^{-1} \nabla v) = 0,$$

where $\gamma = (1-\mu)/(1+\mu)$. The map $\mathcal{H}_\mu : u|_{\partial\Omega} \to v|_{\partial\Omega}$ is the $\mu$-Hilbert transform, and Astala and Päivärinta show that the $\mathcal{N}_\gamma$ determine the $\mathcal{H}_\mu$ and conversely (they also determine the $\mathcal{H}_{-\mu}$ and the $\mathcal{N}_{\gamma^{-1}}$). Similarly to the results discussed in Sections 4C and 4D, the authors show the existence of $m_k(z)$ such that $f_k(z) = e^{izk} m_k(z)$ solves (12) and $m_k(z) - 1 = \mathcal{O}(1/|z|)$ as $|z| \to \infty$. If $g_k = e^{izk} m'_k(z)$ denotes a solution for the Beltrami coefficient $-\mu$ instead of $\mu$, then $h_+ := (f_k + f'_k)/2$ and $h_- := i(\bar{f}_k - \bar{f}'_k)/2$ solves pseudo-analytic equations in the frequency parameter

$$\partial_{\bar{k}} h_+(k) = \tau(k) \overline{h_-(k)}, \quad \partial_{\bar{k}} h_-(k) = \tau(k) \overline{h_+(k)}$$

where $\tau(k)$ is the scattering coefficient

$$\overline{\tau(k)} := \frac{i}{4\pi} \int_{\mathbb{C}} \partial_{\bar{z}} (m_k(z) - m'_k(z)) \, dz \wedge d\bar{z}.$$

Like their predecessors, Astala and Päivärinta show that $\mathcal{N}_\gamma$ determines the coefficient $\tau(k)$. The main difficulty in their proof is to study the behavior of $m_k(z) - 1$ as $k \to \infty$, and the decay of this function was a fundamental tool in [Nachman 1996; Brown and Uhlmann 1997] to prove that once we know the scattering coefficient, $m_k(z)$ is determined uniquely. The decay was $m_k(z) - 1$ was of order $\mathcal{O}(1/|k|)$ when the conductivity was regular enough, but in the $L^\infty$ case Astala–Päivärinta show that $f_k(z) = e^{ik\varphi_k(z)}$ with $\varphi_k(z) - z = o(1)$ as $|k| \to \infty$ uniformly in $z \in \mathbb{C}$, which implies that $m_k(z) - 1 \to 0$ as $|k| \to \infty$ uniformly in $z$. □

**Remark.** Astala, Lassas and Päivärinta [2011] posted recently a paper where they describe in a way the classes of conductivities which can be determined by the Dirichlet-to-Neumann map and find examples which are invisible (related to "cloaking"). The sharp class for isotropic conductivities that can be identified are those $\gamma$ with values in $[0,\infty]$ such that

$$\int_\Omega \exp(\exp(q\gamma(w)+q/\gamma(w)))\,dw<\infty \quad \text{for some } q>0.$$

They also have sharp criteria for anisotropic cases in terms of the regularity of $\mathrm{Tr}(\gamma),\mathrm{Tr}(\gamma^{-1})$ and $\det(\gamma),\det(\gamma^{-1})$.

**4F.** ***From isotropic to anisotropic conductivity.*** When the conductivity is anisotropic, there is a way in dimension 2 (for domains of $\mathbb{C}$) to reduce the problem to the isotropic case by using isothermal coordinates. For a metric

$$\gamma=E\,dx^2+G\,dy^2+2F\,dx\,dy=\lambda|dz+\mu\,d\bar{z}|^2$$

on a domain $\Omega\subset\mathbb{C}$, where

$$\lambda:=\tfrac{1}{4}\big(E+G+2\sqrt{EG-F^2}\big)\quad\text{and}\quad \mu:=\frac{E-G+2iF}{\lambda},$$

there is a diffeomorphism $\Phi:\Omega\to\Omega'$ such that $\Phi_*\gamma=e^{\omega}|dz|^2$ is conformal to the Euclidean metric ($\omega$ is some function), and $\Phi$ is a complex-valued function solving the Beltrami equation

$$\partial_{\bar{z}}\Phi=\mu\partial_z\Phi.$$

An anisotropic conductivity $\gamma$ is a positive definite symmetric (with respect to the Euclidean metric) endomorphism acting on 1-forms and the anisotropic conductivity equation is $d\gamma du=0$. The push forward by a diffeomorphism $\Phi$ is defined in that setting by $(\Phi_*\gamma)\alpha:=\Phi_*(\gamma(\Phi^*\alpha))$ if $\alpha$ is any 1-form. Using those isothermal coordinates, we have:

**Theorem 4.6** [Sylvester 1990]. *Let $\gamma_1,\gamma_2$ be two anisotropic $C^3$ conductivities, viewed as endomorphisms acting on* 1*-forms, where $\Omega\subset\mathbb{C}$ is a domain with $C^3$ boundary. Then if there exists a $C^3$ diffeomorphism $\phi:\partial\Omega\to\partial\Omega$ such that $\phi_*\mathcal{N}_{\gamma_1}=\mathcal{N}_{\gamma_2}$, then there exists a conformal diffeomorphism $\Phi:\Omega\to\Omega$ which satisfies*

$$\Phi_*\gamma_1=\left(\frac{\det(\gamma_1\circ\Phi^{-1})}{\det\gamma_2}\right)^{\frac{1}{2}}\gamma_2.$$

*Idea of proof.* The method is to extend the conductivities $\gamma_j$ outside $\Omega$ so that it is the identity outside a compact set of $\mathbb{C}$ and they agree outside $\Omega$ (which is possible by Kohn–Vogelius boundary uniqueness) and then use the isothermal coordinates on $\mathbb{C}$, with the condition that the diffeomorphism $\Phi_j$ pushing the conductivity $\gamma_j$ to an isotropic one is asymptotically equal to Id near infinity and is the unique solution of a Beltrami equation with this asymptotic condition. Then the final step is to show that $\Phi_1 = \Phi_2$ in $\mathbb{C} \setminus \Omega$ by using equality of Dirichlet-to-Neumann operators. To that end, Sylvester [1990] uses CGO of the form

$$u_j(z;k) = e^{ik\Phi_j(z)} \det(\gamma_j)^{-1/4}(1 + r^j(z;k)), \quad r_j(z;k) \to 0 \text{ as } k \to \infty,$$

which are uniquely determined by their asymptotics

$$u_j(z;k) \sim e^{ik\Phi_j(z)} \det(\gamma_j)^{-1/4}$$

as $|z| \to \infty$. Then he uses that $\lim_{|k|\to\infty} \log(u(z;k))/k = \Phi_j(z)$ and the fact that $\mathcal{N}_{\gamma_1} = \mathcal{N}_{\gamma_2}$ implies that $u_1(z;k) = u_2(z;k)$ in $\mathbb{C}\setminus\Omega$ (by unique continuation in that set) to conclude that $\Phi_1(z) = \Phi_2(z)$ in $\mathbb{C} \setminus \Omega$. □

**Remarks.** Now, since $\gamma_2$ in the theorem can be pushed forward into an isotropic conductivity $e^{\omega}\text{Id}$ for some function $\omega$ using isothermal coordinates, we can use Nachman's theorem to deduce directly that $\mathcal{N}_{\gamma_1} = \mathcal{N}_{\gamma_2}$ implies $\gamma_1 = \Phi_*\gamma_2$ for some diffeomorphism $\Phi$ which is the identity on the boundary. The regularity was improved to $\gamma_j \in W^{1,p}$ with $p > 2$ in [Sun and Uhlmann 2003] and then to $\gamma_j \in L^\infty$ (with the condition that $C\text{Id} \ge \gamma_j \ge C^{-1}\text{Id}$ on $\Omega$ for some $C > 0$) in [Astala et al. 2005].

## 5. The inverse problem for potentials and magnetic field in domains of $\mathbb{C}$

**5A.** ***The case with a potential.*** As we have seen before, the CGO with linear phase do not provide enough information to be able to recover a general potential in 2 dimensions. But we have the following result for generic potentials:

**Theorem 5.1** [Sun and Uhlmann 1991]. *Let $\Omega \subset \mathbb{R}^2$ be domain with smooth boundary. There exists an open dense set $\mathcal{O} \subset W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$ such that if $\mathcal{N}_{V_1} = \mathcal{N}_{V_2}$ and $(V_1, V_2) \in \mathcal{O}$, then $V_1 = V_2$.*

The proof is based on construction of CGO with linear phases, combined with analytic Fredholm theory. We do not discuss it further and refer the interested reader to the original paper.

Grinevich and Novikov also showed local uniqueness for potentials close to positive constants, and later Novikov extended this to potentials close to nonzero constants:[2]

[2] The proofs dealt with the scattering problem, but the result for the bounded domain setting follows directly by using boundary uniqueness and extending the potentials to $\mathbb{R}^2$.

**Theorem 5.2** [Grinevich and Novikov 1988a; Novikov 1992]. *Let $\Omega \subset \mathbb{R}^2$ be domain with smooth boundary and $E \in \mathbb{R} \setminus \{0\}$. Then there exists $C_E > 0$ depending on $|E|$ such that for $V_1, V_2$ satisfying $\|V_j - E\|_{L^\infty} \leq C_E$, then $\mathcal{N}_{V_1} = \mathcal{N}_{V_2}$ implies $V_1 = V_2$. The constant $C_E$ tend to 0 as $|E| \to 0$ and to $+\infty$ as $|E| \to \infty$.*

The general case has been recently tackled by Bukhgeim. His new idea is to use Morse holomorphic phases with a critical point where one wants to identify the potential:[3]

**Theorem 5.3** [Bukhgeim 2008]. *Let $\Omega$ be a domain in $\mathbb{C}$ and $V_1, V_2 \in W^{1,p}(\Omega)$ for $p > 2$. If $\mathcal{N}_{V_1} = \mathcal{N}_{V_2}$, then $V_1 = V_2$.*

*Proof.* As we have seen in Lemma 3.4, holomorphic and antiholomorphic functions are Carleman weights in dimension 2. Before we start the proof, let us recall that if for a holomorphic function $\Phi$, one can construct $u_1 = e^{\Phi/h}(1 + r_h)$ and $u_2 = e^{-\bar{\Phi}/h}(1 + s_h)$ some CGO which solve $(\Delta + V_j)u_j = 0$ with $\|s_h\|_{L^2} + \|r_h\|_{L^2} = o(1)$ as $h \to 0$, the integral identity (4) tells us as $h \to 0$ that

$$\int_\Omega (V_1 - V_2)e^{2i\mathrm{Im}(\Phi)} + \mathcal{O}(\|r_h\|_{L^2} + \|s_h\|_{L^2}) = 0. \tag{13}$$

Of course, the oscillating term will be decreasing very fast as $h \to 0$ if the phase is nonstationary, and therefore we won't get any good information; we are instead tempted to take $\Phi$ with a nondegenerate critical point $z_0 \in \Omega$ and apply stationary phase to deduce the value of $V_1 - V_2$ at $z_0$. This is essentially the main idea of the proof. However, by inspecting Lemma 3.4, the remainder terms $r_h, s_h$ cannot reasonably be smaller than $\mathcal{O}_{L^2}(h)$ and the terms obtained by stationary phase in (13) is also of order $h$, which makes the recovery of $(V_1 - V_2)(z_0)$ quite tricky.

For constructing CGO, Bukhgeim makes a reduction of the problem to a $(\partial, \bar{\partial})$-system. Let $\Phi = \phi + i\psi$ be a holomorphic function on $\Omega$ with a unique critical point at $z_0$, which is nondegenerate in the sense $\partial_z^2\Phi(z_0) \neq 0$. Although here 1-forms and functions are easily identified on a domain $\Omega$, we prefer to keep in mind that the operator $\bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ maps functions $f(z)$ to $(0, 1)$-forms $(\partial_{\bar{z}} f)\, d\bar{z}$, since we will later discuss the same problems for Riemann surfaces. We denote by $\Lambda^k(\Omega)$ the bundle of $k$-forms and by $\Lambda^{0,1}(\Omega)$ (resp. $\Lambda^{1,0}(\Omega)$) the bundle whose sections are of the form $f(z)\, d\bar{z}$, or equivalently in $(T^*\Omega)^{0,1}$ (resp. of the form $f(z)\, dz$, or equivalently in $(T^*\Omega)^{1,0}$). The operator $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$ and is given by $\bar{\partial}^* = -2i * \partial$ where $*$ is the Hodge star operator mapping $\Lambda^{1,1}(\Omega)$ to $\Lambda^0(\Omega)$ by $*(dz \wedge d\bar{z}) = -2i$, $\partial$ maps $\Lambda^{(0,1)}(\Omega)$

[3] In [Bukhgeim 2008], it is claimed that a potential in $L^p(\Omega)$ with $p > 2$ can be identified with this method, but the argument of the paper does not seem to imply directly that such regularity can be dealt with.

to $\Lambda^{(1,1)}(\Omega)$ by $\partial(f(z)\,d\bar{z}) = (\partial_z f(z))\,dz\wedge d\bar{z}$ if $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$. Recall that the Laplace operator is $\Delta = \bar{\partial}^*\bar{\partial}$, therefore setting $\bar{\partial}u = w$, we can reduce the equation $(\Delta + V)u = 0$ to the equivalent first order system

$$\begin{pmatrix} V & \bar{\partial}^* \\ \bar{\partial} & -1 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix} = 0.$$

and $\mathcal{N}_V$ determine the boundary values of this system. In fact, as we shall see, the Cauchy data at the boundary

$$\mathscr{C}_Q := \{F|_{\partial\Omega};\,(D+Q)F = 0\}$$

for first order systems of the form $(D+Q)F = 0$ (with the notation (14)) determine any matrix potential $Q \in W^{1,p}(\Omega)$ with $p > 2$, where

$$D = \begin{pmatrix} 0 & \bar{\partial}^* \\ \bar{\partial} & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} q & 0 \\ 0 & q' \end{pmatrix} \tag{14}$$

are the Dirac type operator and the matrix potential, $q, q' \in W^{1,p}(M_0)$ (with $p > 2$) being complex-valued (both acting on sections of $\Sigma := \Lambda^0(\Omega) \oplus \Lambda^{0,1}(\Omega)$ over $\Omega$).

(i) *Construction of CGO*. The goal is to construct complex geometrical optics $F \in W^{1,p}(\Omega)$ that solve the equation

$$(D+Q)F = 0$$

on $\Omega$. It is clear that

$$D = \begin{pmatrix} e^{-\bar{\Phi}/h} & 0 \\ 0 & e^{-\Phi/h} \end{pmatrix} D \begin{pmatrix} e^{\Phi/h} & 0 \\ 0 & e^{\bar{\Phi}/h} \end{pmatrix}$$

and thus

$$\begin{pmatrix} e^{-\bar{\Phi}/h} & 0 \\ 0 & e^{-\Phi/h} \end{pmatrix} (D+Q) \begin{pmatrix} e^{\Phi/h} & 0 \\ 0 & e^{\bar{\Phi}/h} \end{pmatrix} = D + Q_h\,,$$

where

$$Q_h = \begin{pmatrix} e^{2i\psi/h}q & 0 \\ 0 & e^{-2i\psi/h}q' \end{pmatrix}.$$

We want to construct solutions $F_h$ of $(D+Q_h)F_h = 0$ having the form

$$F_h = \begin{pmatrix} a + r_h \\ b + s_h \end{pmatrix} =: A + R_h, \tag{15}$$

where $a$ is some holomorphic function on $\Omega$, $b$ some antiholomorphic 1-form, and $R_h = (r_h, s_h)$ an element of $W^{1,p}(\Omega)$ that decays appropriately as $h \to 0$.

In particular, we need to solve the system

$$(D+V_h)R_h = -Q_h A = -\begin{pmatrix} e^{2i\psi/h}qa \\ e^{-2i\psi/h}q'b \end{pmatrix}.$$

One can now use a right inverse for the operator $D$, by taking the Cauchy integral kernel:

$$\bar\partial^{-1} : f(z)\,dz \mapsto \left(z \to \frac{1}{\pi}\int_\Omega \frac{f(z')}{z-z'}\,dx'\,dy'\right), \qquad z' = x' + iy',$$
$$\partial^{-1} : f(z)\,dz\wedge d\bar z \mapsto \left(z \to \frac{d\bar z}{\pi}\int_\Omega \frac{f(z')}{\bar z-\bar z'}\,dx'\,dy'\right), \quad z' = x' + iy'.$$

We next define the operators $D^{-1}$

$$D^{-1} := \begin{pmatrix} 0 & \bar\partial^{-1} \\ \bar\partial^{*-1} & 0 \end{pmatrix}, \qquad \text{with } \bar\partial^{*-1} = -(2i)^{-1}\partial^{-1}*$$

which satisfy $DD^{-1} = \mathrm{Id}$ on $L^q(\Omega)$ for all $q \in (1,\infty)$. Similarly,

$$D_h^{-1} := \begin{pmatrix} 0 & \bar\partial_h^{-1} \\ \bar\partial_h^{*-1} & 0 \end{pmatrix}, \qquad \text{with } \bar\partial_h^{-1} = \bar\partial^{-1}e^{-2i\psi/h},\ \bar\partial_h^{*-1} = \bar\partial^{*-1}e^{2i\psi/h}.$$

To construct $R_h$ solving $(D+V_h)R_h = -Q_h A$ in $\Omega$, it then suffices to solve

$$(\mathrm{Id} + D_h^{-1}Q)R_h = -D_h^{-1}QA.$$

Writing the components of this system explicitly we get

$$\begin{aligned} r_h + \bar\partial_h^{-1}(q's_h) &= -\bar\partial_h^{-1}(q'b), \\ s_h + \bar\partial_h^{*-1}(qr_h) &= -\bar\partial_h^{*-1}(qa) \end{aligned} \tag{16}$$

Since we are allowed to choose any holomorphic function $a$ and antiholomorphic 1-form $b$, we may set $a = 0$ in (16) and solve for $r_h$ to get

$$(I - S_h)r_h = -\bar\partial_h^{-1}(q'b) \quad \text{with } S_h := \bar\partial_h^{-1}q'\bar\partial_h^{*-1}q. \tag{17}$$

where $q, q'$ are viewed as multiplication operators. Now we want to estimate the norm of $S_h$, and in that aim we can use the following crucial operator bound whose proof we give in detail since it is the main technical point of Bukhgeim's paper.[4]

**Lemma 5.4** (The key estimate). *There exist $\epsilon > 0$, $h_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$ and all $u \in W^{1,p}(\Omega, \Lambda^1(\Omega))$ and $v \in W^{1,p}(\Omega)$*

$$\|\bar\partial_h^{-1}u\|_{L^2(\Omega)} \le Ch^{\frac12+\epsilon}\|u\|_{W^{1,p}(\Omega)}, \quad \|\bar\partial_h^{*-1}v\|_{L^2(\Omega)} \le Ch^{\frac12+\epsilon}\|v\|_{W^{1,p}(\Omega)}.$$

[4] The proof presented here is not exactly Bukhgeim's proof but the idea essentially the same; we took it from [Guillarmou and Tzou 2011b].

*Proof of the lemma.* We only give a proof for $\bar\partial_h^{-1}$, the other case is exactly the same. The $L^2$ estimate will be obtained by interpolation between $L^q, q<2$ estimates and $L^q, q>2$ estimates. By standard Calderón–Zygmund theory for singular integral kernels, the operators $\bar\partial^{-1}$ and $\bar\partial^{*-1}$ map $L^q(\Omega)\to W^{1,q}(\Omega)$ for all $q\in(1,\infty)$.

(i) Case $q<2$. Let $\varphi\in C^\infty(\overline{\Omega})$ be a function which is equal to 1 for $|z-z_0|>2\delta$ and to 0 in $|z-z_0|\le\delta$, where $\delta>0$ is a parameter that will be chosen later (it will depend on $h$). Using the Minkowski inequality, one can write (with $z'=x'+iy'$)

$$\begin{aligned}\|\bar\partial^{-1}((1-\varphi)e^{-2i\psi/h}f)\|_{L^q(\Omega)}\\ &\le\int_\Omega\left\|\frac{1}{|\cdot-z'|}\right\|_{L^q(\Omega)}|(1-\varphi(z'))f(z')|\,dx'dy'\\ &\le C\|f\|_{L^\infty}\int_\Omega|(1-\varphi(z'))|\,dx'\,dy'\le C\delta^2\|f\|_{L^\infty}\end{aligned}\tag{18}$$

and we know by Sobolev embedding that $\|f\|_{L^\infty}\le C\|f\|_{W^{1,p}}$. On the support of $\varphi$, we observe that since $\varphi=0$ near $z_0$, we can use

$$\bar\partial^{-1}(e^{-2i\psi/h}\varphi f)=\frac{1}{2}ih\left[e^{-2i\psi/h}\frac{\varphi f}{\bar\partial\psi}-\bar\partial^{-1}\left(e^{-2i\psi/h}\bar\partial\left(\frac{\varphi f}{\bar\partial\psi}\right)\right)\right]$$

and the boundedness of $\bar\partial^{-1}$ on $L^q$ to deduce that for any $q<2$

$$\begin{aligned}\|\bar\partial^{-1}(\varphi e^{-2i\psi/h}f)\|_{L^q(\Omega)}\\ &\le Ch\left(\|\frac{\varphi f}{\bar\partial\psi}\|_{L^q}+\|\frac{f\bar\partial\varphi}{\bar\partial\psi}\|_{L^q}+\|\frac{\varphi\bar\partial f}{\bar\partial\psi}\|_{L^q}+\|\frac{f\varphi}{(\bar\partial\psi)^2}\|_{L^q}\right).\end{aligned}\tag{19}$$

The first term is clearly bounded by $\delta^{-1}\|f\|_{L^\infty}$ due to the fact that $\psi$ is Morse. For the last term, observe that since $\psi$ is Morse, we have $1/|\partial\psi|\le c/|z-z_0|$ near $z_0$; therefore

$$\left\|\frac{f\varphi}{(\bar\partial\psi)^2}\right\|_{L^q}\le C\|f\|_{L^\infty}\left(\int_\delta^1 r^{1-2q}dr\right)^{1/q}\le C\delta^{\frac{2}{q}-2}\|f\|_{L^\infty}.$$

The second term can be bounded by

$$\left\|\frac{f\bar\partial\varphi}{\bar\partial\psi}\right\|_{L^q}\le\|f\|_{L^\infty}\left\|\frac{\bar\partial\varphi}{\bar\partial\psi}\right\|_{L^q}.$$

Observe that while $\|\bar\partial\varphi/\bar\partial\psi\|_{L^\infty}$ grows as $\delta^{-2}$, $\bar\partial\varphi$ is only supported in a neighborhood of radius $2\delta$. Therefore we obtain

$$\left\|\frac{f\bar\partial\varphi}{\bar\partial\psi}\right\|_{L^q} \le \delta^{2/q-2}\|f\|_{L^\infty}.$$

The third term can be estimated by

$$\left\|\frac{\varphi\bar\partial f}{\bar\partial\psi}\right\|_{L^q} \le C\|\bar\partial f\|_{L^p}\left\|\frac{\varphi}{\bar\partial\psi}\right\|_{L^\infty} \le C\delta^{-1}\|\bar\partial f\|_{L^p}.$$

Combining these four estimates with (19) we obtain

$$\left\|\bar\partial^{-1}(\varphi e^{-2i\psi/h}f)\right\|_{L^q(\Omega)} \le h\|f\|_{W^{1,p}}(\delta^{-1}+\delta^{2/q-2}).$$

Combining this and (18) and optimizing by taking $\delta = h^{1/3}$, we deduce that

$$\|\bar\partial^{-1}(e^{-2i\psi/h}f)\|_{L^q(\Omega)} \le h^{2/3}\|f\|_{W^{1,p}} \tag{20}$$

(ii) Case $q \ge 2$. One can use the boundedness of $\bar\partial^{-1}$ on $L^q$ to obtain

$$\begin{aligned}\|\bar\partial^{-1}((1-\varphi)e^{-2i\psi/h}f)\|_{L^q(\Omega)} &\le \|(1-\varphi)e^{-2i\psi/h}f\|_{L^q(\Omega)}\\ &\le C\delta^{\frac{2}{q}}\|f\|_{L^\infty}.\end{aligned} \tag{21}$$

Now since $\varphi = 0$ near $z_0$, we can use the identity

$$\bar\partial^{-1}(e^{-2i\psi/h}\varphi f) = \tfrac{1}{2}ih\left[e^{-2i\psi/h}\frac{\varphi f}{\partial_{\bar z}\psi} - \bar\partial^{-1}\left(e^{-2i\psi/h}\partial_{\bar z}\left(\frac{\varphi f}{\partial_{\bar z}\psi}\right)\right)\right]$$

and the boundedness of $\bar\partial^{-1}$ on $L^q$ to deduce that for any $q \le p$, (19) holds again with all the terms satisfying the same estimates as before, so that

$$\left\|\bar\partial^{-1}(e^{-2i\psi/h}\varphi f)\right\|_{L^q} \le Ch\|f\|_{W^{1,p}}(\delta^{2/q-2}+\delta^{-1}) \le Ch\delta^{2/q-2}\|f\|_{W^{1,p}}$$

since now $q \ge 2$. Now combine the above estimate with (21) and take $\delta = h^{\frac{1}{2}}$ we get

$$\left\|\bar\partial^{-1}(e^{-2i\psi/h}f)\right\|_{L^q} \le h^{1/q}\|f\|_{W^{1,p}}$$

for $2 \le q \le p$. The estimate claimed in the lemma is obtained by interpolating the case $q < 2$ with $q > 2$. □

From this lemma, we see directly that

$$\|S_h\|_{L^2\to L^2} \le Ch^{1/2+\epsilon}$$

for some $\epsilon > 0$ when the potential $Q$ is $W^{1,p}(\Omega)$ with $p > 2$. Therefore (17) can be solved by using Neumann series by setting (for small $h > 0$)

$$r_h := -\sum_{j=0}^{\infty} S_h^j \bar{\partial}_h^{-1}(q'b) \tag{22}$$

as an element of any $L^2(\Omega)$. Substituting this expression for $r$ into (16) when $a = 0$, we get

$$s_h = -\bar{\partial}_h^{*-1}(qr_h). \tag{23}$$

(ii) Identification of the potential. Using boundary identification as in Section 2, one can assume that $Q_1 - Q_2 = 0$ at $\partial\Omega$ if the Cauchy data at $\partial\Omega$ agree. Let $F_h^1, F_h^2$ be some CGO solutions of the form (15) constructed as above for respectively the operators $(D + Q_1)$ and $(D + Q_2^*)$, where $Q_j$ are diagonal matrices defined as in(14) for some $q_j, q_j' \in W^{1,p}(\Omega)$. Assume that the boundary values of solutions of the equations $(D+Q_1)F = 0$ and $(D+Q_2)F = 0$ coincide. Then there exists a solution $F_h$ of $(D + V_2)F = 0$ such that $F_h|_{\partial\Omega} = F_h^1|_{\partial\Omega}$; therefore $(D + V_2)(F_h^1 - F_h) = (Q_2 - Q_1)F_h^1$, and using Green's formula and the vanishing of $F_h^1 - F_h$ on the boundary,

$$0 = \int_\Omega \langle (D + Q_2)(F_h^1 - F_h), F_h^2 \rangle = \int_\Omega \langle (Q_2 - Q_1)F_h^1, F_h^2 \rangle. \tag{24}$$

This gives

$$0 = \int_\Omega (q_2' - q_1')e^{-2i\psi/h}\big(|b|^2 + \langle b, s_h^2\rangle + \langle s_h^1, b\rangle\big) + (q_2 - q_1)e^{2i\psi/h} r_h^1 \overline{r_h^2}. \tag{25}$$

The last term is $\mathbb{O}(h^{1+2\epsilon})$ by Lemma 5.4. Using boundary identification as in Section 2, one has $Q_1 - Q_2 = 0$ at $\partial\Omega$ if the Cauchy data at $\partial\Omega$ agree. The stationary phase gives an asymptotic expansion as $h \to 0$

$$\int_\Omega (q_2' - q_1')e^{-2i\psi/h}|b|^2 = C_{z_0}|b(z_0)|^2(q_2'(z_0) - q_1'(z_0))h + o(h)$$

for some $C_{z_0} \neq 0$. Now for the remainder, we use (23) so that

$$\int_\Omega (q_2' - q_1')e^{-2i\psi/h}\langle b, s_h^2\rangle = -\int_\Omega e^{2i\psi/h}\bar{\partial}_h^{-1}(b(q_2' - q_1'))q_2\overline{r_h^2}$$

and by Lemma 5.4, this is a $\mathbb{O}(h^{1+2\epsilon})$. The same argument applies for the last term and this shows that $q_1' = q_2'$ since $z_0$ can be chosen arbitrarily in $\Omega$. One can prove the same thing for $q_1 = q_2$ with the same argument but taking $b = 0$ and $a(z_0) \neq 0$ in the CGO. ☐

**5B.** ***The case with a magnetic field.*** A natural problem to consider is the inverse problem for the magnetic Schrödinger Laplacian on a smooth domain $\Omega \subset \mathbb{C}$:

$$L_{X,V} = (d + iX)^*(d + iX) + V,$$

where $V \in L^\infty(\Omega)$ is a potential and $X \in W^{1,\infty}(\Omega; \Lambda^1(\Omega))$ is a real-valued 1-form, called the magnetic potential, and $*$ denotes the adjoint with respect to the Euclidean $L^2$ product. The natural Cauchy data space associated to $(X, V)$ in this case is

$$\mathscr{C}_{X,V} := \big\{(u, \partial_\nu u + iX(\nu)u)|_{\partial\Omega};\, L_{X,V}u = 0, u \in H^1(\Omega)\big\}.$$

The magnetic field is the exact 2-form $dX$ and it is easily seen that there is a gauge invariance since

$$\mathscr{C}_{X,V} = \mathscr{C}_{X+d\varphi,V} \tag{26}$$

for any function $\varphi \in W^{2,\infty}(\Omega)$ such that $\varphi = 0$ on $\partial\Omega$, simply by observing that $d + i(X + d\varphi) = e^{-i\varphi}(d + iX)e^{i\varphi}$. In fact, if $\varphi$ is function in $\mathbb{R}/2\pi\mathbb{Z}$ with $e^{i\varphi}|_{\partial\Omega} = 1$, then (26) holds true as well and one can therefore identify at best $X + F^{-1}dF$ where $F$ is any $S^1$-valued functions which are equal to 1 on the boundary. When $\Omega$ is simply connected, this is the same as identifying the magnetic field $dX$ (or equivalently the curvature of the connection $d + iX$).

**Theorem 5.5** [Sun 1993]. *Let $\Omega \subset \mathbb{C}$ be a smooth domain, then there exists an open dense set $\mathbb{O} \subset W^{1,\infty}(\Omega)$ such that for any $V \in \mathbb{O}$, there is a neighborhood $\mathbb{O}_V$ of $V$ in $\mathbb{O}$ and a constant $C > 0$ such that if $X_1, X_2 \in W^{3,\infty}(\Omega, \Lambda^1(\Omega))$ are two magnetic potentials satisfying $\|dX_j\|_{W^{2,\infty}} \le C$ and if $V_1, V_2 \in \mathbb{O}_V$, then*

$$\mathscr{C}_{X_1,V_1} = \mathscr{C}_{X_2,V_2} \quad \Longrightarrow \quad d(X_1 - X_2) = 0 \text{ and } V_1 = V_2.$$

Another result in this direction, given for smooth data, is the following:

**Theorem 5.6** [Kang and Uhlmann 2004]. *Let $\Omega \subset \mathbb{C}$ be a smooth domain, and let $X_1, X_2$, and $V_1, V_2$ be smooth 1-forms and potentials. Let $p > 2$, then there exists $\epsilon$ such that if $\|V_1\|_{W^{1,p}} \le \epsilon$ and $\mathscr{C}_{X_1,V_1} = \mathscr{C}_{X_2,V_2}$, then $d(X_1 - X_2) = 0$ and $V_1 = V_2$.*

In particular, when $V_1 = V_2 = 0$, this allows one to identify a smooth magnetic potential up to gauge in a simply connected domain.

***Further results.*** For partial data measurement, Imanuvilov, Uhlmann and Yamamoto [Imanuvilov et al. 2010a] obtain identification of magnetic field (up to gauge) and potential in a domain. Guillarmou and Tzou [2011b] recover a connection (i.e., a magnetic potential) up to gauge and the potential on a Riemann surface from full Cauchy data. See below for those results.

## 6. Inverse problems on Riemann surfaces

**6A.** ***The metric problem.*** As we mentioned in Section 1B, the Dirichlet-to-Neumann operator $\mathcal{N}_{M,g}$ associated to a Riemannian metric $g$ on a surface $M$ does not determine the isometry class, but can at best determine the conformal class of $g$ (and $g$ on the boundary). Recall that a conformal class on an oriented surface $M$ is equivalent to a complex structure, that is a atlas with biholomorphic changes of charts in $\mathbb{C}$.

This problem has been solved by Lassas and Uhlmann [2001], with a simpler proof by Lassas, Taylor and Uhlmann, who were also able to determine the topology of $M$ from $\mathcal{N}_g$:

**Theorem 6.1** [Lassas et al. 2003]. *Let $(M_1, g_1), (M_2, g_1)$ be two Riemannian surfaces with the same boundaries $Z$. If $\mathcal{N}_{(M_1,g_1)} = \mathcal{N}_{(M_2,g_2)}$. There exists a diffeomorphism $\Phi : M_1 \to M_2$ such that $\Phi^* g_2$ is conformal to $g_1$ and $\Phi|_Z = \mathrm{Id}$.*

*Sketch of proof.* The idea is to use the Green's function and show that for analytic manifolds, the Green's function determines the manifold and the conformal class in dimension 2. Near a point $p \in Z$ of the boundary, there exists a neighborhood $U_i$ in $M_i$ and some diffeomorphisms $\psi_i : B \to U_i$ where $B$ are half balls in the upper half plane $B = B(0,1) \cap \{\mathrm{Im}(z) > 0\}$, such that $\psi_i(0) = p$ and $\psi_i^* g_i = e^{2f_i}|dz|^2$ for some functions $f_i$ (isothermal coordinates). The associated Laplacians in these coordinates are given by $e^{-2f_i}\Delta$ where $\Delta = -\partial_z\partial_{\bar{z}}$ is the flat Laplacian in $\mathbb{R}^2$. The Green's functions $G_i(m, m')$ with Dirichlet conditions on $M_i$ are the $L^1$ functions such that in the distribution sense $\Delta_{g_i} G_i(m, m') = \delta(m - m')$ is the distribution kernel of the Identity with $G_i(m, m') = 0$ if $m$ or $m'$ (but $m \neq m'$) is on the boundary $Z$. In general, we have following property as an application of Green's formula (this is standard, see for instance [Guillarmou and Sá Barreto 2009, Lemma 3.1]):

**Lemma 6.2.** *For a metric $g$ on a manifold $M$, the Schwartz kernel of $\mathcal{N}_{(M,g)}$ is a singular integral kernel given for $y \neq y' \in \partial M$ by*

$$\mathcal{N}_{(M,g)}(y, y') = \partial_\nu \partial_{\nu'} G(m, m')|_{(m,m')=(y,y')}$$

*where $G$ is the Green kernel with Dirichlet conditions and $\partial_\nu, \partial_{\nu'}$ denote the normal derivative to the boundary in the left and right variable.*

Therefore we can deduce that if $\mathcal{N}_{(M_1,g_1)} = \mathcal{N}_{(M_2,g_2)}$ then $G_1(m, m') = G_2(m, m')$ when viewed in the chart $B$, since they solve the same elliptic problem with the same local Cauchy data (Dirichlet and Neumann), this is by unique continuation for a scalar elliptic PDE. The idea of [Lassas et al. 2003] is to define an analytic embedding of $M$ to a Hilbert space through the Green's function:

they define

$$\mathcal{G}_i : M_i \to L^2(B), \quad \mathcal{G}_i(m) := G_i(m, \psi_i(\cdot)).$$

Notice by the remark above that $\mathcal{G}_1 \circ \psi_1 = \mathcal{G}_2 \circ \psi_2$. Now, the crucial fact is that each $M_i$ has a real analytic atlas (induced by the complex local coordinates $z$), the metric in these charts is conformal to the Euclidean metric $|dz|^2$, and since the Green's function solves $\partial_z \partial_{\bar{z}} G_i(z, z') = 0$ when $z \neq z'$, the functions $G_i$ are analytic outside the diagonal (with respect to the analytic structure on $M_i$). This implies that $\mathcal{G}_i$ are analytic maps and in fact it can be shown that they are embeddings. Indeed, assume that $\nabla \mathcal{G}_i(m) = 0$ at some $m \in M_i$, then $\nabla_m G_i(m; m') = 0$ for all $m' \in U_i$, which by analyticity gives $\nabla_m G_i(m; m') = 0$ for all $m' \neq m$, but that contradicts the asymptotic behavior of $G_i$ at the diagonal (in local complex coordinates $z$)

$$G_i(z, z') = \frac{1}{2\pi} \log(|z - z'|) + C(z) + o(1) \quad \text{as } |z - z'| \to 0$$

for some function $C(z)$, and thus $\mathcal{G}_i$ is a local analytic diffeomorphism; it is also injective since $G_i(m_1, m') = G_i(m_2, m')$ for all $m' \in U_i$ implies the same for all $m' \in M_i$ and the asymptotic at the diagonal implies $m_1 = m_2$. It remains to show that $\mathcal{G}_1(M_1) = \mathcal{G}_2(M_2)$ and $\mathcal{G}_2^{-1} \circ \mathcal{G}_1$ is a conformal map. We already know that $\mathcal{G}_1(U_1) = \mathcal{G}_2(U_2)$ and it can be proved by analytic continuation and the implicit function theorem that the set $\{m \in M_1; \mathcal{G}_1(m) \in \mathcal{G}_2(M_2)\}$ is the whole of $M_1$. Essentially, this amounts to say that two submanifolds with boundary of $L^2(B)$ obtained by analytic embeddings are the same if they are equal on an open set. The map $J := \mathcal{G}_2^{-1} \circ \mathcal{G}_1$ is analytic and is such that $G_1(m, m') = G_2(J(m), J(m'))$ for $m, m' \in U_1$ and $m \neq m'$ since $\mathcal{G}_1 \circ \psi_1 = \mathcal{G}_2 \circ \psi_2$. But this identity extends to $M_1 \times M_1 \setminus \{m = m'\}$ by analytic continuation. Then by looking at the asymptotic of the Green's kernels near a diagonal point $(m, m) \in M_1 \times M_1$, this implies that there exist functions $C_1(m), C_2(m)$ such that

$$\log(d_{g_1}(m, m_t)) + C_1(m) = \log(d_{g_2}(J(m), J(m_t))) + C_2(J(m)) + o(1)$$

as $t \to 0$, if $t \mapsto m_t$ is a smooth curve such that $m_0 = m$ and $d_{g_i}$ denotes the Riemannian distance. Writing this equation in terms of the metric (here we use the notation $\dot{m}_s = \partial_s m(s)$) we get

$$\log(|\dot{m}_0|_{g_1(m)}) + C_1(m) = \log(|\dot{m}_0|_{J^* g_2(m)}) + C_2(J(m)) + o(1) \quad \text{as } t \to 0,$$

and since $\dot{m}_0$ can be chosen arbitrarily, one deduces $g_1 = e^{\omega} J^* g_2$ for some function $\omega \in C^\infty(M_1)$. □

***Remarks and further results.*** In fact, the agreement of the Dirichlet-to-Neumann operators on only a open set $\Gamma \subset Z = \partial M_i$ is sufficient to run this argument, as was shown in [Lassas and Uhlmann 2001]. The identification for Riemann surfaces was later shown using a different approach by Belishev [2003]. Then Henkin and Michel [2007] gave another proof of this result by using embedding of the surface into $\mathbb{C}^n$, and we should point out moreover that the method in [Henkin and Michel 2007] gives a reconstruction procedure for the Riemann surface, which is not quite the case with the previous results. In fact, Henkin and Michel [2007] show that the action of the Dirichlet-to-Neumann map $\mathcal{N}_g$ on 3 generic functions $(u_0, u_1, u_2)$ on $\partial M$ is sufficient to determine the Riemann surface $(M, g)$ (as a surface with complex structure). More precisely:

**Theorem 6.3** [Henkin and Michel 2007]. *Let $(M_1, g_1)$, $(M_2, g_2)$ be two Riemann surfaces with the same boundary $Z$, such that there exist some real-valued smooth function $u = (u_0, u_1, u_2) : Z^3 \to \mathbb{R}$ with $\mathcal{N}_{g_1} u = \mathcal{N}_{g_2} u$ and such that*

$$\Phi : Z \to \mathbb{C}^2, \quad \Phi : m \mapsto \left( \frac{(\mathcal{N}_{g_1} - i\partial_\tau)u_1}{(\mathcal{N}_{g_1} - i\partial_\tau)u_0}, \frac{(\mathcal{N}_{g_1} - i\partial_\tau)u_2}{(\mathcal{N}_{g_1} - i\partial_\tau)u_0} \right)$$

*is an embedding, where $\partial_\tau$ is an positively oriented length one tangent vector field to the boundary $Z$. Then $(M_1, g_1)$ is isomorphic to $(M_2, g_2)$ as a Riemann surface.*

The condition about the embedding is claimed to be generic, in a way, by the authors. The proof of this theorem is based on complex geometric arguments.

Finally, we refer to [Salo 2013] in this volume for a survey on the inverse problem on manifolds in dimension $n > 2$.

**6B.** ***Identification of a conductivity or a potential on a fixed Riemann surface.*** The same type of problem can be considered when the background setting is not a domain of $\mathbb{C}$ but a Riemann surface with boundary. Thus, we fix a Riemann surface with boundary $(M, g)$, a conductivity is a positive symmetric endomorphism $\gamma$ of $TM$ and we consider the elliptic equation

$$\mathrm{div}_g(\gamma \nabla^g u) = 0$$

where $\nabla^g u$ is the gradient of $u$ defined by $g(\nabla^g u, X) = X(u)$ for all vector field $X$, and $\mathcal{L}_X(\mathrm{dvol}_g) = \mathrm{div}_g(X)\mathrm{dvol}_g$ if $\mathcal{L}_X$ is the Lie derivative with respect to $X$. The Dirichlet-to-Neumann operator associated to $\gamma$ is still denoted $\mathcal{N}_\gamma$ and defined by $\mathcal{N}_\gamma f := g(\gamma \nabla^g u, \nu)$ where $\nu$ is the normal outward pointing vector field at the boundary, and we want to see if $\gamma \to \mathcal{N}_\gamma$ is injective up to gauge. An equivalent question is to consider the elliptic equation

$$d(\gamma du) = 0, \quad u|_{\partial M} = f$$

where $\gamma : M \to \mathrm{End}(T^*M)$ is a section of positive symmetric (with respect to a given metric $g$) endomorphisms on 1-forms and the Dirichlet-to-Neumann map is $\mathcal{N}_\gamma f := (\gamma(du)(\nu))|_{\partial M} \in \Lambda^1(M)|_{\partial M}$.

Viewing $M_0$ as a subset of a closed Riemann surface $M$ of genus $g$, Henkin and Michel [2008] consider $g+2$ points $A = \{A_1, \dots, A_{g+2}\}$ in $M \setminus M_0$ and use an immersion $j = (f_1, f_2) : M \subset A \to \mathbb{C}^2$ of $M$ into $\mathbb{C}^2$ using 2 independent meromorphic functions $f_1, f_2$ on $M$ with poles at $A$ and they assume the complex curve $j(M)$ can be written under the form $j(M) = \{(z_1, z_2) \in \mathbb{C}^2; P(z_1, z_2) = 0\}$ with $P$ a homogeneous holomorphic polynomial such that $\nabla P \neq 0$ on $j(M)$, i.e., $M$ can be viewed as a regular complex algebraic curve. They proved:

**Theorem 6.4** [Henkin and Michel 2008]. *If $\gamma \in C^3(M_0)$ is an isotropic conductivity on a fixed Riemann surface $M_0$ with boundary, which can be embedded in $\mathbb{C}^2$ as a subset of a regular complex algebraic curve as described above, then the Dirichlet-to-Neumann map $\mathcal{N}_\gamma$ determines $\gamma$ and $\gamma$ can be reconstructed by an explicit procedure.*

This result has been extended to anisotropic conductivities:

**Theorem 6.5** [Henkin and Santacesaria 2010]. *Let $M$ be a $C^3$ surface with boundary, and let $\gamma_1, \gamma_2$ be two $C^3$ positive definite symmetric endomorphisms of $TM$ (i.e., two anisotropic conductivities on $M$). If the Dirichlet-to-Neumann operators agree, $\mathcal{N}_{\gamma_1} = \mathcal{N}_{\gamma_2}$, then there exists a $C^3$ diffeomorphism $F : M \to M$ such that $F|_{\partial M} = \mathrm{Id}$ and $F_*\gamma_1 = \gamma_2$.*

As for the flat case, the case where $\gamma$ is isotropic (i.e., when it is of the form $\gamma\mathrm{Id}$ for some function $\gamma$) can be reduced to the case of the equation $\Delta_g + V$ with $V = -\Delta_g \gamma^{1/2}/\gamma^{1/2}$. We have:

**Theorem 6.6** [Guillarmou and Tzou 2009]. *If $V_1, V_2$ are two potentials*[5] *in $W^{1,p}(M)$ with $p > 2$ on a Riemann surface $(M, g)$ with boundary, and the map $\mathcal{N}_{V_1} = \mathcal{N}_{V_2}$ agree, then $V_1 = V_2$.*

In particular this allows to identify isotropic conductivities in $W^{3,p}(M_0)$ on the Riemann surface and to identify a metric in its conformal class.

*Arguments in the proof.* The proof is based on the Bukhgeim method, as described above, but in this geometric setting one needs to find holomorphic Morse functions with prescribed critical points (the function $(z - z_0)^2$ does not quite make sense anymore). Let us discuss the construction of the phase in this setting.

[5]The regularity of the potential is stated to be $C^\infty$ in [Guillarmou and Tzou 2009] for convenience of exposition, but the $W^{1,p}(M)$ regularity result follows from [Guillarmou and Tzou 2011b].

*Riemann surfaces and complex structure.* A conformal class $[g]$ on an oriented closed surface $M$ makes $M$ into a closed Riemann surface, i.e., a closed surface equipped with a complex structure via holomorphic charts $z_\alpha : U_\alpha \to \mathbb{C}$. The Hodge star operator $*$ acts on the cotangent bundle $T^*M$, its eigenvalues are $\pm i$ and the respective eigenspace $T^*_{1,0}M := \ker(*+i\,\mathrm{Id})$ and $T^*_{0,1}M := \ker(*-i\,\mathrm{Id})$ are subbundles of the complexified cotangent bundle $\mathbb{C}T^*M$, and we have a splitting $\mathbb{C}T^*M = T^*_{1,0}M \oplus T^*_{0,1}M$. The Hodge $*$ operator is conformally invariant on 1-forms on $M$, the complex structure depends only on the conformal class of $g$ (and orientation). In holomorphic coordinates $z = x + iy$ in a chart $U_\alpha$, one has $\star(u\,dx + v\,dy) = -v\,dx + u\,dy$ and

$$T^*_{1,0}M|_{U_\alpha} \simeq \mathbb{C}\,dz, \quad T^*_{0,1}M|_{U_\alpha} \simeq \mathbb{C}\,d\bar{z}$$

where $dz = dx + i\,dy$ and $d\bar{z} = dx - i\,dy$. We define the natural projections induced by the splitting of $\mathbb{C}T^*M$

$$\pi_{1,0} : \mathbb{C}T^*M \to T^*_{1,0}M, \quad \pi_{0,1} : \mathbb{C}T^*M \to T^*_{0,1}M.$$

The exterior derivative $d$ defines the De Rham complex $0 \to \Lambda^0 \to \Lambda^1 \to \Lambda^2 \to 0$ where $\Lambda^k := \Lambda^k T^*M$ denotes the real bundle of $k$-forms on $M$. Let us denote $\mathbb{C}\Lambda^k$ the complexification of $\Lambda^k$, then the $\partial$ and $\bar\partial$ operators can be defined as differential operators $\partial : \mathbb{C}\Lambda^0 \to T^*_{1,0}M$ and $\bar\partial : \mathbb{C}\Lambda_0 \to T^*_{0,1}M$ by

$$\partial f := \pi_{1,0} df, \quad \bar\partial := \pi_{0,1} df,$$

they satisfy $d = \partial + \bar\partial$ and are expressed in holomorphic coordinates by

$$\partial f = \partial_z f\,dz, \quad \bar\partial f = \partial_{\bar z} f\,d\bar{z}.$$

with $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar z} := \frac{1}{2}(\partial_x + i\partial_y)$. Similarly, one can define the $\partial$ and $\bar\partial$ operators from $\mathbb{C}\Lambda^1$ to $\mathbb{C}\Lambda^2$ by setting

$$\partial(\omega_{1,0} + \omega_{0,1}) := d\omega_{0,1}, \quad \bar\partial(\omega_{1,0} + \omega_{0,1}) := d\omega_{1,0}$$

if $\omega_{0,1} \in T^*_{0,1}M$ and $\omega_{1,0} \in T^*_{1,0}M$. In coordinates this is simply

$$\partial(u\,dz + v\,d\bar{z}) = \partial v \wedge d\bar{z}, \quad \bar\partial(u\,dz + v\,d\bar{z}) = \bar\partial u \wedge dz.$$

The Laplacian acting on functions is defined by

$$\Delta f := -2i * \bar\partial\partial f = d^* d$$

where $d^*$ is the adjoint of $d$ through the metric $g$ and $*$ is the Hodge star operator mapping $\Lambda^2$ to $\Lambda^0$ and induced by $g$.

*The holomorphic phase.* To construct holomorphic functions with prescribed critical points, we use the Riemann–Roch theorem: a divisor $D$ on $M$ is an element

$$D = \big((p_1, n_1), \dots, (p_k, n_k)\big) \in (M \times \mathbb{Z})^k, \quad \text{where } k \in \mathbb{N}$$

which will also be denoted $D = \prod_{i=1}^k p_i^{n_i}$ or $D = \prod_{p \in M} p^{\alpha(p)}$ where $\alpha(p) = 0$ for all $p$ except $\alpha(p_i) = n_i$. The inverse divisor of $D$ is defined to be $D^{-1} := \prod_{p\in M} p^{-\alpha(p)}$ and the degree of the divisor $D$ is defined by $\deg(D) := \sum_{i=1}^k n_i = \sum_{p\in M} \alpha(p)$. A meromorphic function on $M$ is said to have divisor $D$ if $(f) := \prod_{p\in M} p^{\mathrm{ord}(p)}$ is equal to $D$, where $\mathrm{ord}(p)$ denotes the order of $p$ as a pole or zero of $f$ (with positive sign convention for zeros). Notice that in this case we have $\deg(f) = 0$. For divisors $D' = \prod_{p\in M} p^{\alpha'(p)}$ and $D = \prod_{p\in M} p^{\alpha(p)}$, we say that $D' \geq D$ if $\alpha'(p) \geq \alpha(p)$ for all $p \in M$. The same exact notions apply for meromorphic 1-forms on $M$. Then we define for a divisor $D$

$$r(D) := \dim(\{f \text{ meromorphic function on } M; (f) \geq D\} \cup \{0\}),$$
$$i(D) := \dim(\{u \text{ meromorphic 1 form on } M; (u) \geq D\} \cup \{0\}).$$

The Riemann–Roch theorem states the following identity: for any divisor $D$ on the closed Riemann surface $M$ of genus $g$,

$$r(D^{-1}) = i(D) + \deg(D) - g + 1. \tag{27}$$

Notice also that for any divisor $D$ with $\deg(D) > 0$, one has $r(D) = 0$ since $\deg(f) = 0$ for all $f$ meromorphic. By [Farkas and Kra 1992, p. 70, Theorem], let $D$ be a divisor, then for any nonzero meromorphic 1-form $\omega$ on $M$, one has

$$i(D) = r(D(\omega)^{-1}) \tag{28}$$

which is thus independent of $\omega$. For instance, if $D = 1$, we know that the only holomorphic function on $M$ is 1 and one has $1 = r(1) = r((\omega)^{-1}) - g + 1$ and thus $r((\omega)^{-1}) = g$ if $\omega$ is a nonzero meromorphic 1 form. Now if $D = (\omega)$, we obtain again from (27)

$$g = r((\omega)^{-1}) = 2 - g + \deg((\omega))$$

which gives $\deg((\omega)) = 2(g-1)$ for any nonzero meromorphic 1-form $\omega$. In particular, if $D$ is a divisor such that $\deg(D) > 2(g-1)$, then we get $\deg(D(\omega)^{-1}) = \deg(D) - 2(g-1) > 0$ and thus $i(D) = r(D(\omega)^{-1}) = 0$, which implies by (27)

$$\deg(D) > 2(g-1) \Longrightarrow r(D^{-1}) = \deg(D) - g + 1 \geq g. \tag{29}$$

In particular, taking $M_0$ as a subset of a closed Riemann surface $M$ with genus $g$ (for instance by doubling the surface along the boundary and extending the conformal class smoothly), we know that by assigning a pole at $p \in M \setminus M_0$ of large order $N > 2(g-1)$, there exists a vector space $V$ of dimension $N-2(g-1)$ of functions holomorphic on $M_0$ meromorphic in $M$ with a unique pole of order at most $N$ at $p$ (take the divisor $D = p^N$). This implies by dimension count that for all $z_0 \in M_0$

(i) there exists a vector subspace $V(z_0) \subset$ of dimension $\geq N-2$ of holomorphic functions with a zero of order 2 at $z_0$.

This creates critical points everywhere we want, but unfortunately the functions will not a priori be Morse. Similarly, taking the divisors $D_1 = z_0^{-1} p^N$ and $D_2 = z_0^{-2} p^N$ and since their degree is larger than $2(g-1)$, one has $r(D_1^{-1}) - r(D_2^{-1}) = 1$ and thus

(ii) $\exists f \in V$, holomorphic on $M_0$ with a zero of order exactly 1 at $z_0$.

Using (i) and (ii), we show:

**Lemma 6.7.** *There exists a dense set of points $z_0$ in $M_0$ for which there exists a Morse holomorphic function with a critical point at $z_0$.*

By Cauchy–Riemann equations, a holomorphic function is Morse if and only if its real part is Morse. To prove this, one can use some transversality arguments of Uhlenbeck [1976]. Take the real vector space $H = \mathrm{Re}(V)$ and define the map

$$F : H \times M_0 \to T^* M_0, \quad F : (u, m) \mapsto (m, du(m)).$$

A real function $u$ is Morse if $F_u := F(u, \cdot)$ is transverse to the zero section $T_0^* M_0 = \{(m, 0) \in T^* M_0\}$ in $T^* M_0$ in the sense that

$$\mathrm{Im}(dF_u(m)) + T_{F_u(m)} T_0^* M_0 = T^*_{F_u(p)} M_0$$

for all $m$ such that $F(u, m) = 0$. Now an application of Sard's theorem gives that if $F$ is transverse to $T_0^* M_0$ (in the sense above but with $\mathrm{Im}(dF)(u, m)$ instead of $\mathrm{Im}(dF_u(m))$) then

$$\{u \in H; F_u \text{ is transverse to } T_0^* M_0\}$$

has Lebesgue measure 0 in $H$. A little inspection shows that the transversality of $F$ with respect to $T_0^* M_0$ can be proved if we can show that at any $z_0$ then there exist a function $f \in V$ such that $\partial_{z_0} f \neq 0$. But this is insured by (ii), and we conclude that the set of Morse functions in $V$ has a complement in $V$ of 0 measure (thus is dense in $V$). Taking $z_0 \in M_0$ and a function $f$ in $V(z_0)$ of (i), then for any $\epsilon > 0$ there exists a function $\Phi$ in $V$ (thus Morse) with $|\Phi - f| \leq \epsilon$ with respect to any norm on the finite dimensional space $V$, and thus in particular by Rouché theorem its critical point is going to be in a neighborhood of size $o(1)$ of $z_0$ as $\epsilon \to 0$. This concludes the proof of the lemma and gives us a phase for constructing CGO.

*The rest of the proof.* This goes similarly to what we explained about the Bukhgeim theorem. The main technical differences here are that we need to construct a global right inverse for $\bar\partial$ and its adjoint and since $\Phi$ may have several critical points, we choose the amplitudes (which are holomorphic or antiholomorphic functions and 1-forms) to vanish at all critical points except the point $z_0$ where we want to identify the potential, so that those other points do not contribute to the stationary phase. We refer to [Guillarmou and Tzou 2009; 2011b] for more details. □

**Remark.** The regularity $W^{1,p}$ with $p > 2$ on surfaces was proved in [Guillarmou and Tzou 2011b] for the potential using Bukhgeim method and was improved later to $C^\alpha$ for all $\alpha > 0$ for domains of $\mathbb{C}$ by Imanuvilov and Yamamoto [2011]. See also [Blasten 2011] for the $W^{1,p}$ regularity for domains.

**6C.** ***Inverse problems for systems and magnetic Schrödinger operators.*** Let $\pi : E \to M$ be a Hermitian complex vector bundle on a Riemann surface with boundary and $\nabla$ be a Hermitian connection on $E$. Such a bundle is trivializable and the connection in a trivialization is of the form $\nabla = d + iX$ for self-adjoint matrix-valued 1-form $X$. We can define the magnetic Schrödinger operator $L_{\nabla,V}$ (or connection Laplacian) by

$$L_{\nabla,V} := \nabla^*\nabla + V$$

where $V$ is a section of the endomorphism on $E$ (i.e., a potential). This elliptic operator has Cauchy data defined by

$$\mathscr{C}_{\nabla,V} := \{(u|_{\partial M}, \nabla_\nu u|_{\partial M}); L_{\nabla,V}u = 0, u \in H^1(M, E)\}.$$

It is then natural to ask whether $\mathscr{C}_{\nabla,V}$ uniquely determines the connection $\nabla$ and the endomorphism $V$. The answer is no since there is a gauge invariance. Indeed, consider a section $F$ of $\mathrm{End}(E)$ satisfying $F^* = F^{-1}$ and $F|_{\partial M} = Id$. Then it is easy to see that $\mathscr{C}_{\nabla,V} = \mathscr{C}_{F^*\nabla F, F^*VF}$. Therefore we can at best hope to identify $\nabla$ and $V$ up to gauge. The following result is proved:

**Theorem 6.8** [Albin et al. 2011]. *Let $\nabla^1$ and $\nabla^2$ be two Hermitian connections on a smooth Hermitian bundle $E$, of complex dimension $n$ and let $V_1$, $V_2$ be two sections of the bundle* $\mathrm{End}(E)$. *We assume that the connection forms of $\nabla^j$ have the regularity $C^r \cap W^{s,p}(M)$ with $0 < r < s$, $p \in (1, \infty)$ satisfy $r + s > 1$, $r \in \mathbb{N}$, $sp > 2n + 2$ and that $V_j \in W^{1,q}(M)$ with $q > 2$. Let $L_j := (\nabla^j)^*\nabla^j + V_j$ and assume that the Cauchy data spaces agree $\mathscr{C}_{\nabla^1,V_1} = \mathscr{C}_{\nabla^2,V_2}$, then there exists a* *unitary endomorphism* $F \in C^1(M; \mathrm{End}(E))$, *satisfying* $F|_{\partial M} = Id$, *such that* $\nabla^1 = F^* \nabla^2 F$ *and* $V_1 = F^* V_2 F$.

*Sketch of proof.* One uses reduction to a $\bar\partial$-system and the Bukhgeim method [2008]. Indeed, if we denote by $A_j = \pi_{0,1} X_j$ and $A_j^* = \pi_{1,0} X_j$ the $(T^{0,1}M)^*$ and $(T^{1,0}M)^*$ component of the connection 1-form $X_j$ of $\nabla^j$, (in a fixed trivialization), we have

$$\nabla^j = d + i X_j = (\bar\partial + i A_j) + (\partial + i A_j^*).$$

Thus, if $u_j$ is a solution to $L_{\nabla^j, V_j} u_j = 0$ we have

$$\begin{pmatrix} 0 & (\bar\partial + i A_j)^* \\ (\bar\partial + i A_j) & 0 \end{pmatrix} \begin{pmatrix} u_j \\ \omega_j \end{pmatrix} + \begin{pmatrix} *(dX_j + X_j \wedge X_j) + V_j & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_j \\ \omega_j \end{pmatrix} = 0. \quad (30)$$

if we set $\omega_j := (\bar\partial + i A_j) u_j$ It is clear that knowledge of the Cauchy data for the second order equation is equivalent to knowledge of the Cauchy data for (30). We would like to transform this problem, via conjugation, to the type of system considered by Bukhgeim (and explained above for Riemann surfaces) where only the $\bar\partial$ and $\bar\partial^*$ operator appear on the off diagonal.

As shown in [Kobayashi 1987, Chapter 1, Proposition 3.7], the operator $\partial + i A_j^*$ induces a holomorphic structure on $E$ and this structure is holomorphically trivializable since $M$ has nonempty boundary; see [Forster 1991, Theorems 30.1 and 30.4]. This means that there exists an invertible section $F_j$ of $\mathrm{End}(E)$ that is annihilated by the operator $\partial + i A_j^*$. More precisely, $\partial F_j = -i A_j^* F_j$. Taking adjoint of both sides we get that $(F_j^*)^{-1} \bar\partial F_j^* = i A_j$. Therefore, $(F_j^*)^{-1} \bar\partial F_j^* u = (\bar\partial + i A_j) u$ for all smooth sections $u$ of $E$. We would like to remark that such endomorphisms are by no means unique.

We are now in a position to transform system (30) into a simplified $\bar\partial$ system. Set $(\tilde u_j, \tilde\omega_j) := (F_j^* u_j, F_j^{-1} \omega_j)$ then system (30) is equivalent to

$$\begin{pmatrix} 0 & \bar\partial^* \\ \bar\partial & 0 \end{pmatrix} \begin{pmatrix} \tilde u_j \\ \tilde\omega_j \end{pmatrix} + \begin{pmatrix} F_j^{-1}(*(dX_j + X_j \wedge X_j) + V_j)(F_j^*)^{-1} & 0 \\ 0 & -F_j^* F_j \end{pmatrix} \begin{pmatrix} \tilde u_j \\ \tilde\omega_j \end{pmatrix} = 0. \quad (31)$$

However, the fact that the systems (30) have identical Cauchy data does not a priori ensure that the systems (31) have the same Cauchy data for $j = 1, 2$ since the conjugation factors $F_j$ may not necessarily agree on the boundary. An important part of the resolution of the problem is to show that the $F_j$ can be chosen to agree on the boundary if the Cauchy data agree.

**Lemma 6.9.** *If the Cauchy data for the systems* (30) *agree for* $j = 1, 2$, *then there exist invertible sections* $F_j$ *of* $\mathrm{End}(E)$ *satisfying* $(F_j^*)^{-1}\bar{\partial} F_j^* = iA_j$ *and* $F_1|_{\partial M} = F_2|_{\partial M}$

*Idea of proof in the case of a line bundle.* (Things are similar for higher-rank bundles.) The idea is to use CGO for the system (30) of the form

$$U_h^1 = \begin{pmatrix} e^{\Phi/h}(F_1^*)^{-1}(a + r_h^1) \\ e^{\overline{\Phi}/h} F_1 s_h^1 \end{pmatrix}, \quad U_h^2 = \begin{pmatrix} e^{-\Phi/h}(F_2^*)^{-1} r_h^2 \\ e^{-\overline{\Phi}/h} F_2(b + s_h^2) \end{pmatrix}$$

with $\bar{\partial} a = 0$, $\bar{\partial}^* b = 0$, and writing this system as

$$(D + Q_j)U = 0, \quad \text{with } D = \begin{pmatrix} 0 & \bar{\partial}^* \\ \bar{\partial} & 0 \end{pmatrix},$$

the following integral identity follows from the equality of Cauchy data spaces:

$$\begin{aligned} 0 &= \int_M \langle (Q_2 - Q_1) U_h^1, U_h^2 \rangle \\ &= i \int_M \langle (A_2 - A_1)(F_1^*)^{-1}(a + r_h^1), F_2(b + s_h^2) \rangle + o(1) \\ &= i \int_M \langle F_2^*(A_2 - A_1)(F_1^*)^{-1} a, b \rangle + o(1) = \int_M \langle \bar{\partial}(F_2^*(F_1^*)^{-1} a), b \rangle + o(1) \end{aligned}$$

as $h \to 0$. Letting $h \to 0$, and applying Stokes one gets $0 = \int_{\partial M} F_2^*(F_1^*)^{-1} a i_{\partial M}^* \bar{b}$, for all $b$ antiholomorphic 1-form, but by Hodge theory [Guillarmou and Tzou 2011b, Lemma 4.1], this means exactly that $F_2^*(F_1^*)^{-1} a|_{\partial M}$ is the boundary value of a holomorphic function. Taking $a = 1$, we set $F$ the holomorphic function with restriction $F_2^*(F_1^*)^{-1}$ at $\partial M$, this is invertible since one can apply the same argument by switching the role of $j = 1, 2$ and this gives a holomorphic function which multiplied by $F$ is equal to 1 on $\partial M$, thus is equal to 1 on $M$. Now modify $F_1^*$ by multiplying it by $F$ so that $(F_1 F^*)^*$ and $F_2^*$ agree at $\partial M$ and $F_1 F^*$ and $F_2$ play the role of $F_1$ and $F_2$ in the statement of the lemma. □

This lemma allows us to choose the conjugation factors for $j = 1, 2$ such that the conjugated systems (31) for $j = 1$ and $j = 2$ has the same Cauchy data. We have now reduced the problem to one or the type considered in [Bukhgeim 2008], but with higher rank. The same techniques used in that paper and adapted to Riemann surfaces in [Guillarmou and Tzou 2011b] can then be applied to deduce that

$$F_1^{-1}(*(dX_1 + X_1 \wedge X_1) + V_1)(F_1^*)^{-1} = F_2^{-1}(*(dX_2 + X_2 \wedge X_2) + V_2)(F_2^*)^{-1} \tag{32}$$

and

$$F_1^* F_1 = F_2^* F_2$$

with the boundary condition $F_1 = F_2$ on $\partial M$. We then set $F = (F_1^*)^{-1} F_2^* \in \mathrm{End}(E)$ which by using $F = F_1 F_2^{-1}$ satisfies $F^* = F^{-1}$ and moreover $F = \mathrm{Id}$ on $\partial M$. Then it is easy to see that $F$ solves the homogeneous elliptic equation

$$\bar{\partial} F + i A_1 F - i F A_2 = 0,$$

and by taking adjoint and using $F^* = F^{-1}$ this implies the equality of the connections $F^{-1}(d + iX_1)F = d + iX_2$, and thus $F^{-1} V_1 F = V_2$ by (32). □

**Remark.** The proof just described also allows one to identify zeroth-order terms up to gauge in a first order system

$$\begin{pmatrix} 0 & \bar{\partial}^* \\ \bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} Q_+ & A' \\ A & Q_- \end{pmatrix}.$$

See [Albin et al. 2011, Proposition 3] for details and precise statements.

## 7. Inverse problems with partial data measurements

**7A.** ***Identification of a potential from partial measurements.*** An important question related to Calderón's problem is to identify a conductivity or a potential from measurements on an open subset of the boundary instead of the whole boundary. The first partial data result seems to be in dimension $n > 2$ by [Bukhgeim and Uhlmann 2002], where Carleman estimates were fundamental to approach this problem. In dimension $n = 2$, Imanuvilov, Uhlmann and Yamamoto gave a proof by combining the method of [Bukhgeim 2008] with Carleman estimates:

**Theorem 7.1** [Imanuvilov et al. 2010b]. *Let $\Omega \subset \mathbb{C}$ be a domain and let $V_1, V_2 \in C^{2,\alpha}(\Omega)$ be two potentials. Let $\Gamma \subset \partial\Omega$, and consider the partial Cauchy data on $\Gamma$,*

$$C_{V_i}^{\Gamma} := \{(u, \partial_\nu u)|_{\Gamma};\ (\Delta + V_i)u = 0,\ u \in H^1(\Omega),\ u = 0 \text{ in } \partial\Omega \setminus \Gamma\}. \quad (33)$$

*If $\mathscr{C}_{V_1}^{\Gamma} = \mathscr{C}_{V_2}^{\Gamma}$ then $V_1 = V_2$.*

We later extended this result to Riemann surfaces:

**Theorem 7.2** [Guillarmou and Tzou 2011a]. *Let $(M, g)$ be a Riemann surface with smooth boundary and let $\Gamma \subset \partial M$ be any open subset of the boundary. Suppose $V_1$ and $V_2$ are $C^{1,\alpha}(M)$ potentials, for some $\alpha > 0$, such that $\mathscr{C}_{V_1}^{\Gamma} = \mathscr{C}_{V_2}^{\Gamma}$ with the notation similar to* (33), *then $V_1 = V_2$.*

*Sketch of proof.* We follow the method of [Imanuvilov et al. 2010b], but use geometric arguments to construct the phases of the CGO and we make adaptations for constructing appropriate CGO on Riemann surfaces.

To motivate the proof, we observe that since $\mathscr{C}_{V_1,\Gamma} = \mathscr{C}_{V_2,\Gamma}$, for any two solutions of $(\Delta_g + V_j)u_j = 0$ such that $u_1|_{\partial M}, u_2|_{\partial M} \in C_0^\infty(\Gamma)$, we have the boundary integral identity

$$\int_M \overline{u_1}(V_1 - V_2)u_2 = 0. \tag{34}$$

We wish to extend Bukhgeim's idea of using stationary phase expansion discussed in Section 5. To this end, we look to construct solutions of the type discussed in Section 3 having the additional property that when restricted to the boundary they are compactly supported in $\Gamma$.

The idea of [Imanuvilov et al. 2010b] is to first construct harmonic functions of exponential type which vanish on $\partial M\setminus\Gamma$, and correct them using a Carleman estimate. As such, suppose we have a holomorphic and Morse function $\Phi = \phi + i\psi$ with prescribed critical points which is purely real on $\partial M\setminus\Gamma$, and let $a$ be a holomorphic function purely real on $\Gamma$. Then the harmonic function defined by $u_0 := e^{\Phi/h}a - e^{\overline{\Phi}/h}\overline{a}$ is a harmonic function which vanishes outside of $\Gamma$ and we will search for solutions of $(\Delta + V)u = 0$ of the form $u = u_0 + e^{\phi/h}r_h$ where $r_h$ will be small in $L^2$ as $h \to 0$. Therefore, the first key step in constructing CGO which vanish on $\partial M \setminus \Gamma$ is to construct such a holomorphic function $\Phi$ on a general Riemann surface. Observe that since the potentials $V_j$ are assumed to be continuous, it suffices to prove that $V_1(p) = V_2(p)$ for a dense subset of $p \in M$. In [Guillarmou and Tzou 2011a], we proved:

**Lemma 7.3.** *Let $(M, g)$ be a Riemann surface with boundaries with $\Gamma \subset \partial M$ be an open subset. Then there exists a dense subset $\mathscr{A} \subset M$ such that for all $p \in \mathscr{A}$ there exists a holomorphic Morse function $\Phi$ which is purely real on $\partial M\setminus\Gamma$ such that $\partial\Phi(p) = 0$.*

The existence of $\Phi$ is also proved in [Imanuvilov et al. 2010b] for all points when $M = \Omega$ is a domain of $\mathbb{C}$, using variational methods.

*Outline of proof of Lemma 7.3.* We discuss briefly the procedure for constructing such holomorphic phase functions. For a detailed outline, we refer to [Guillarmou and Tzou 2011a] for the geometric approach which works for general surfaces and [Imanuvilov et al. 2010b] for the variational approach which works for planar domains. In this survey we will describe the geometric approach. To this end, we view holomorphic functions as sections of the trivial line bundle $E := \mathbb{C}\times M$ which is annihilated by the linear elliptic operator $\bar{\partial}$. And the fact that $\Phi$ is purely real on $\partial M\setminus\Gamma$ will be interpreted as $\Phi|_{\partial M}$ being a section of the totally real rank 1 subbundle[6] $F \subset E|_{\partial M}$ over $\partial M$ such that

$$F|_{\partial M\setminus\Gamma} = \mathbb{R}\times\partial M\setminus\Gamma. \tag{35}$$

Note that while $F$ needs to be $\mathbb{R}\times\partial M\setminus\Gamma$ on $\partial M\setminus\Gamma$, we have the freedom to choose it as we wish on $\Gamma$ as long as $F$ remains a smooth bundle of real rank one. The question is, what choices of $F$ satisfying condition (35) makes the $\bar\partial$ operator acting on

$$H^k(M,F) = \{u \in H^k(M); u|_{\partial M} \in F\}$$

a Fredholm operator? And how does the winding number (half of Maslov index) of $F$ affect the Fredholm index of $\bar\partial$? This type of problem is well known in Floer homology and $J$-holomorphic curves. The following Riemann–Roch theorem is shown in [McDuff and Salamon 2004]:

**Theorem 7.4** (Riemann–Roch with boundary). *The operator $\bar\partial$ acting on $H^k(F)$, denoted $\bar\partial_F$, is Fredholm and its index is the sum of the Euler characteristic and twice the winding number of $F$. Furthermore, if the sum of the winding number of $F$ and the Euler characteristic of $M$ is larger than zero, then the operator $\bar\partial_F$ is surjective and consequently the dimension of* $\ker\bar\partial_F$ *is the sum of the Euler characteristic of $M$ plus twice the winding number of $F$.*

With this theorem we are now ready to construct $F$ so that it satisfies condition (35) and at the same time having $\bar\partial_F$ possess the desirable Fredholm properties. Assume for simplicity that $M$ has only a single boundary component, so $\partial M \simeq S^1$, which we parametrize by $\theta \in [0, 2\pi)$ so that $\Gamma = (0, \theta_0)$. Let $\phi_N$ be a smooth function in $[0, 2\pi)$ such that $\phi_N(0) = 0$, $\phi_N(\theta) = 2\pi N$ for $\theta > \theta_0$. We then define the boundary real rank 1 subbundle $F_N$ fiberwise by

$$F_N(\theta) := e^{i\phi_N(\theta)}\mathbb{R}.$$

By construction, the winding number of $F_N$ is $N$ and taking this parameter large enough we can apply Theorem 7.4 to conclude that

$$\dim\ker\bar\partial_{F_N} = 2N + \chi(M) \tag{36}$$

where $\chi(M)$ is the Euler characteristic of the surface.
It remains now to prescribe critical points to these holomorphic functions. Indeed, if $p \in M$ is an interior point we consider the map

$$\Phi \in \ker\bar\partial_{F_N} \mapsto \partial\Phi(p).$$

[6]We mean a subbundle of real rank 1 of the real rank 2 bundle $\mathbb{C}\times\partial M$ where $\mathbb{C}\simeq\mathbb{R}^2$ is viewed as a real vector space.

By (36) this is a linear map from an $2N+\chi(M)$ dimensional subspace to a real 2 dimensional subspace and therefore has a nontrivial kernel. Consequently, there exists an $N-2$ dimensional subspace of holomorphic functions in $H^k(M, F_N)$ with a critical point at $p$. The holomorphic functions with prescribed critical point and the desirable boundary conditions is however, not a priori Morse. To remedy this fact we use the same type of arguments as those of the proof of Theorem 6.6, based on a transversality property: this shows that Morse holomorphic functions are dense within the space of holomorphic functions. More precisely:

**Lemma 7.5** [Guillarmou and Tzou 2011a]. *Let $\Phi$ be a holomorphic function on $M$ which is purely real on $\partial M\setminus\Gamma$. Then there exists a sequence of Morse holomorphic functions $\Phi_j$ which are purely real on $\partial M\setminus\Gamma$ such that $\Phi_j\to\Phi$ in $C^k(\overline{M})$ for all $k\in\mathbb{N}$.*

Starting with a holomorphic function $\Phi$, purely real on $\Gamma$, with a critical point at $p$, one finds a Morse sequence $\Phi_j$ approaching $\Phi$, and by Cauchy's argument principle we deduce easily that the critical points of $\Phi_j$ approach $p$. This proves Lemma 7.3. □

*Carleman estimate and construction of remainder terms.* We have so far constructed harmonic functions $u_0:=e^{\Phi/h}a-\overline{e^{\Phi/h}a}$ of exponential type that vanish on $\partial M\setminus\Gamma$. It remains to show that we can construct the suitable remainder term $r_h$ so that $u=u_0+e^{\phi/h}r_h$ is a family of solutions to $(\Delta_g+V)u=0$ such that $u$ vanishes on $\partial M\setminus\Gamma$ with $r_h$ satisfying suitable decaying properties as $h\to 0$.

The method here was developed [Imanuvilov et al. 2010b]; it is a combination of ideas from earlier works on partial data [Bukhgeim and Uhlmann 2002; Kenig et al. 2007] (e.g., Carleman estimates) with the idea of Bukhgeim [2008] of using phases with critical points. We present a sketch of proof which adapts to Riemann surface (details are in [Guillarmou and Tzou 2011a]), but the original proof for domains can be found in [Imanuvilov et al. 2010b].

We will split $r_h$ into two parts, $r_h=r_1+r_2$. The first term $r_1$ will be constructed using a Cauchy–Riemann operator $\bar\partial^{-1}$ (a right inverse for $\bar\partial$) to solve the approximate equation

$$e^{-\Phi/h}(\Delta_g+V)e^{\Phi/h}(a+r_1)=\mathbb{O}_{L^2}(h|\log h|)$$

with the boundary condition $e^{\Phi/h}r_1\in\mathbb{R}$ on $\partial M\setminus\Gamma$. We then subtract its complex conjugate to obtain

$$e^{-\phi/h}(\Delta_g+V)(e^{\Phi/h}(a+r_1)-\overline{e^{\Phi/h}(a+r_1)})=\mathbb{O}_{L^2}(h\,|\log h|)\qquad(37)$$

with $(e^{\Phi/h}(a+r_1)-\overline{e^{\Phi/h}(a+r_1)})=0$ on $\partial M\setminus\Gamma$.

To go from an approximate solution to a full solution, we can use a Carleman estimate with boundary terms. This first appeared in [Imanuvilov et al. 2010b] on domains.

**Proposition 7.6.** *Let $(M, g)$ be a smooth Riemann surface with boundary, and let $\phi : M_0 \to \mathbb{R}$ be a $C^k(M)$ harmonic Morse function for $k$ large. Then for all $V \in L^\infty(M)$ there exists an $h_0 > 0$ such that for all $h \in (0, h_0)$ and $u \in C^\infty(M)$ with $u|_{\partial M} = 0$, we have*

$$\frac{1}{h}\|u\|^2_{L^2(M)} + \|u\|^2_{H^1(M)} + \|\partial_\nu u\|^2_{L^2(\Gamma)} \leq C\Big(\|e^{-\phi/h}(\Delta_g + V)e^{\phi/h}u\|^2_{L^2(M)} + \frac{1}{h}\|\partial_\nu u\|^2_{L^2(\partial M\setminus\Gamma)}\Big), \quad (38)$$

*where $\partial_\nu$ is the inward pointing normal along the boundary.*

The Carleman estimate of Lemma 3.4 is a simpler version of this estimate. Let us simply indicate why the exponent $h^{-1}\|u\|^2_{L^2}$ is coming in (38), instead of the term $h^{-2}\||\nabla\phi|u\|^2_{L^2}$ of Lemma 3.4: for $u \in C_0^\infty$ supported in a small neighborhood of a critical point (given by $z = 0$ in local coordinates), we have $|\nabla\phi(z)|^2 \geq C|z|^2$ for some $C$ and using (7) with $\epsilon = \alpha h$ for some small $\alpha$ independent of $h$, we obtain

$$\frac{1}{h^2}\big\| |\nabla\phi|u\big\|^2_{L^2} + \|\nabla u\|^2_{L^2} \geq \frac{C}{h}\|u\|^2_{L^2} + C\|\nabla u\|^2_{L^2}$$

for some $C > 0$. Gluing local estimates with convexified weight methods, one obtain something of the form (38) for $u \in C_0^\infty(M)$, and for functions supported near the boundary, an estimate like this is also available but with boundary terms coming from integrating by parts.

A direct application of the Riesz theorem and Proposition 7.6 gives the following solvability result:

**Proposition 7.7.** *Let $f \in L^2(M)$, then for all $h < h_0$ there exists a solution $r$ to*

$$e^{-\phi/h}(\Delta_g + V)e^{\phi/h}r = f$$

*with $r|_{\partial M\setminus\Gamma} = 0$ satisfying the estimate $\|r\|_{L^2} \leq \sqrt{h}\|f\|_{L^2}$.*

With this proposition we can solve for the $O_{L^2}(h\log h)$ remainder in (37) and obtain a solution to $(\Delta_g + V)u = 0$ of the form

$$u = (e^{\Phi/h}(a + r_1) - \overline{e^{\Phi/h}(a + r_1)}) + e^{\phi/h}r_2$$

satisfying $u\,|_{\partial M\setminus\Gamma} = 0$

*Identification of the potential.* Once we have constructed CGO $u_j$ for the equation $(\Delta + V_j)u_j = 0$, having the form

$$u_1 = e^{\Phi/h}(a + r_1) + \overline{e^{\Phi/h}(a + r_1)} + e^{\varphi/h} r_2,$$
$$u_2 = e^{-\Phi/h}(a + s_1) + \overline{e^{-\Phi/h}(a + s_1)} + e^{-\varphi/h} s_2,$$

with $u_1|_{\partial M \setminus \Gamma} = u_2|_{\partial M \setminus \Gamma} = 0$, we plug this into the integral identity (34) and obtain (with some work to deal with the crossed terms involving $r_1$ and $s_1$)

$$0 = \int_M (V_1 - V_2)(a^2 + \bar{a}^2) + 2\mathrm{Re} \int_M e^{2i\mathrm{Im}(\Phi)/h}(V_1 - V_2)|a|^2 + o(h).$$

and applying stationary phase (knowing that $(V_1 - V_2)|_{\partial M} = 0$ by boundary local uniqueness) we deduce that $V_1(p) = V_2(p)$ at the critical point $p$ of $\Phi$ as $h \to 0$. Notice that $\Phi$ can have several critical points, in which case we choose the amplitude $a$ to vanish at all critical points except the point $p$ where one wants to recover the potential. This ends the proof of Theorem 7.1. □

**7B.** ***Identification of first-order terms.*** Imanuvilov, Uhlmann and Yamamoto [2010a] studied the partial data problem for general second order elliptic operators on a domain $\Omega \subset \mathbb{C}$, i.e., operators of the form

$$L = \Delta_g + A\partial_z + B\partial_{\bar{z}} + V \tag{39}$$

where $A, B, V$ are complex-valued functions and $g$ is a metric. They have a general result we shall only state the case of $g = \mathrm{Id}$ since the general statement is quite complicated (see their paper for the general case).

**Theorem 7.8** [Imanuvilov et al. 2010a]. *Let*

$$(A_j, B_j, V_j) \in C^{5,\alpha}(\Omega) \times C^{5,\alpha}(\Omega) \times C^{4,\alpha}(\Omega)$$

*for* $j = 1, 2$ *and* $\alpha > 0$. *Assume there exists an open set* $\Gamma \subset \partial\Omega$ *such that*

$$\{(u, \partial_\nu u)|_\Gamma ; L_1 u = 0, u \in H^1, u|_{\partial\Omega\setminus\Gamma} = 0\}$$
$$= \{(u, \partial_\nu u)|_\Gamma ; L_1 u = 0, u \in H^1, u|_{\partial\Omega\setminus\Gamma} = 0\},$$

*where* $L_j$ *is defined as in* (39) *with* $(A, B, V) = (A_j, B_j, V_j)$ *and* $g = \mathrm{Id}$. *Then there exists a function* $\eta \in C^{6,\alpha}(\Omega)$ *such that* $\eta|_\gamma = \partial_\nu \eta|_\Gamma = 0$ *and*

$$L_1 = e^{-\eta} L_2 e^{\eta}.$$

The multiplication by $e^\eta$ is the gauge invariance in this setting (here there is no "topology" so the gauge is exactly conjugation by exponential of a function $\eta$ with value in $\mathbb{C}$, while in theorem [Albin et al. 2011] it is not always the case). The proof is long and technical, and builds on previous work [Imanuvilov et al. 2010b]. The results are summarized and announced in [Imanuvilov et al. 2011a].

**7C.** ***Disjoint measurements.*** We conclude with a recent result.

**Theorem 7.9** [Imanuvilov et al. 2011b]. *Let $\Omega \subset \mathbb{C}$ be a connected and simply connected domain with smooth boundary. Assume that $\partial\Omega = \overline{\Gamma_+ \cup \Gamma_- \cup \Gamma_0}$ where $\Gamma_\pm, \Gamma_0$ are open disjoint subsets of $\partial\Omega$ such that $\overline{\Gamma_+} \cap \overline{\Gamma_-} = \varnothing$, $\Gamma_\pm$ have two connected components $\Gamma_\pm^1, \Gamma_\pm^2$ and each of the 4 connected components of $\Gamma_0$ has its closure intersecting both $\Gamma_+$ and $\Gamma_-$. If $V_1, V_2 \in C^{2+\alpha}(\Omega)$ for some $\alpha > 0$ and if the partial Cauchy data*

$$\mathscr{C}_{V_j}^{\Gamma} := \{(u|_{\Gamma_+}, \partial_\nu u|_{\Gamma_-}); (\Delta + V_j)u = 0, u \in H^1(\Omega), u|_{\partial\Omega\setminus\Gamma_+} = 0\}$$

*agree, i.e., $\mathscr{C}_{V_1}^{\Gamma} = \mathscr{C}_{V_2}^{\Gamma}$, then $V_1 = V_2$.*

## 8. Open problems in 2 dimensions

(1) Prove the $L^\infty$ conductivity result of Astala–Päivärinta in the context of Riemann surfaces.

(2) Find a better regularity assumption for the potential in Bukhgeim's result.

(3) Recover a metric $g$ on a Riemann surface $M$ up to isometry equal to Id on the boundary from the Cauchy data space for the equation $\Delta_g - \lambda$, where $\lambda \neq 0$.

(4) More generally, see what can be recovered from the Cauchy data space for the equation $\Delta_g + V$ where $V$ is a potential and $g$ a metric. On the region where $V = 0$, only the conformal class can be obtained.

(5) Find a "good" partial data measurement for d-bar type elliptic systems $Pu = 0$ which allows one to recover the coefficients inside the surface. That is, we search for natural subspaces of the full Cauchy data space which determine the coefficients inside the surface, for instance in terms of measurements on pieces of the boundary of certain components of the vector-valued $u$ (as in [Salo and Tzou 2010], for instance).

(6) Solve the general disjoint partial data measurements for $\Delta_g + V$, that is, when the Dirichlet data is measured on $\Gamma_1$ and the Neumann data is measured on $\Gamma_2$, with $\Gamma_1 \cap \Gamma_2 = \varnothing$.

(7) Find a reconstruction method for partial data measurement, as in [Nachman and Street 2010], for higher dimensions.

(8) Solve the inverse problem for the elasticity system of [Nakamura and Uhlmann 1993].

(9) Solve the Calderón problem for $L^\infty$ complex-valued conductivities.

## References


[Ablowitz et al. 1983] M. J. Ablowitz, D. Bar Yaacov, and A. S. Fokas, "On the inverse scattering transform for the Kadomtsev–Petviashvili equation", *Stud. Appl. Math.* **69**:2 (1983), 135–143. MR 85h:35179 Zbl 0527.35080

[Albin et al. 2011] P. Albin, C. Guillarmou, L. Tzou, and G. Uhlmann, "Inverse boundary problems for systems in two dimensions", preprint, 2011. arXiv 1105.4565

[Alessandrini 1990] G. Alessandrini, "Singular solutions of elliptic equations and the determination of conductivity by boundary measurements", *J. Differential Equations* **84**:2 (1990), 252–272. MR 91e:35210 Zbl 0778.35109

[Astala and Päivärinta 2006] K. Astala and L. Päivärinta, "Calderón's inverse conductivity problem in the plane", *Ann. of Math.* (2) **163**:1 (2006), 265–299. MR 2007b:30019 Zbl 1111.35004

[Astala et al. 2005] K. Astala, L. Päivärinta, and M. Lassas, "Calderón's inverse problem for anisotropic conductivity in the plane", *Comm. Partial Differential Equations* **30**:1-3 (2005), 207–224. MR 2005k:35421

[Astala et al. 2011] K. Astala, M. Lassas, and L. Päivärinta, "The borderlines of the invisibility and visibility for Calderón's inverse problem", preprint, 2011. arXiv 1109.2749

[Astala et al. 2013] K. Astala, M. Lassas, and L. Päivärinta, "Calderón's inverse problem: imaging and invisibility", pp. 1–54 in *Inverse problems and applications: Inside out II*, edited by G. Uhlmann, Publ. Math. Sci. Res. Inst. **60**, Cambridge University Press, New York, 2013.

[Beals and Coifman 1981] R. Beals and R. Coifman, "Scattering, transformations spectrales et équations d'évolution non linéaires", Exp. No. XXII in *Goulaouic–Meyer–Schwartz Seminar, 1980–1981*, École Polytech., Palaiseau, 1981. MR 84d:35129

[Beals and Coifman 1988] R. Beals and R. Coifman, "The spectral problem for the Davey–Stewartson and Ishimori hierarchies", pp. pages 15–23 in *Nonlinear evolution equations: integrability and spectral methods*, edited by A. Degasperis et al., Manchester University Press, 1988.

[Belishev 2003] M. I. Belishev, "The Calderon problem for two-dimensional manifolds by the BC-method", *SIAM J. Math. Anal.* **35**:1 (2003), 172–182. MR 2004f:58029 Zbl 1048.58019

[Blasten 2011] E. Blasten, "The inverse problem of the Shrödinger equation in the plane: A dissection of Bukhgeim's result", preprint, 2011. arXiv 1103.6200

[Brown 2001] R. M. Brown, "Recovering the conductivity at the boundary from the Dirichlet to Neumann map: a pointwise result", *J. Inverse Ill-Posed Probl.* **9**:6 (2001), 567–574. MR 2003a:35196 Zbl 0991.35104

[Brown and Uhlmann 1997] R. M. Brown and G. A. Uhlmann, "Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions", *Comm. Partial Differential Equations* **22**:5-6 (1997), 1009–1027. MR 98f:35155 Zbl 0884.35167

[Bukhgeim 2008] A. L. Bukhgeim, "Recovering a potential from Cauchy data in the two-dimensional case", *J. Inverse Ill-Posed Probl.* **16**:1 (2008), 19–33. MR 2008m:30049

[Bukhgeim and Uhlmann 2002] A. L. Bukhgeim and G. Uhlmann, "Recovering a potential from partial Cauchy data", *Comm. Partial Differential Equations* **27**:3-4 (2002), 653–668. MR 2003d:35262 Zbl 0998.35063

[Calderón 1980] A.-P. Calderón, "On an inverse boundary value problem", pp. 65–73 in *Seminar on Numerical Analysis and its Applications to Continuum Physics* (Rio de Janeiro, 1980), Soc. Brasil. Mat., Rio de Janeiro, 1980. MR 81k:35160

[Dos Santos Ferreira et al. 2009] D. Dos Santos Ferreira, C. E. Kenig, M. Salo, and G. Uhlmann, "Limiting Carleman weights and anisotropic inverse problems", *Invent. Math.* **178**:1 (2009), 119–171. MR 2010h:58033 Zbl 1181.35327

[Faddeev 1965] L. Faddeev, "Growing solutions of the Schrödinger equation", *Dokl. Akad. Nauk SSSR* **165** (1965), 514–517. In Russian. Translated in *Soviet Phys. Dokl.*, **10** (1965), 1033–1035.

[Faddeev 1974] L. D. Faddeev, "The inverse problem in the quantum theory of scattering, II", pp. 93–180 in *Current problems in mathematics*, vol. 3, Akad. Nauk SSSR Vsesojuz. Inst. Naučn. i Tehn. Informacii, Moscow, 1974. In Russian. MR 58 #25585 Zbl 0299.35027

[Farkas and Kra 1992] H. M. Farkas and I. Kra, *Riemann surfaces*, 2nd ed., Graduate Texts in Mathematics **71**, Springer, New York, 1992. MR 93a:30047 Zbl 0764.30001

[Forster 1991] O. Forster, *Lectures on Riemann surfaces*, Graduate Texts in Mathematics **81**, Springer, New York, 1991. MR 93h:30061 Zbl 0475.30002

[Grinevich and Manakov 1986] P. G. Grinevich and S. V. Manakov, "Inverse problem of scattering theory for the two-dimensional Schrödinger operator, the $\bar{\partial}$-method and nonlinear equations", *Funktsional. Anal. i Prilozhen.* **20**:2 (1986), 14–24, 96. MR 88g:35197

[Grinevich and Novikov 1988a] P. G. Grinevich and S. P. Novikov, "Inverse scattering problem for the two-dimensional Schrödinger operator at a fixed negative energy and generalized analytic functions", pp. 58–85 in *Plasma theory and nonlinear and turbulent processes in physics* (Kiev, 1987), edited by V. G. Baryakhtar et al., World Sci. Publishing, Singapore, 1988. MR 90c:35199 Zbl 0704.35137

[Grinevich and Novikov 1988b] P. G. Grinevich and S. P. Novikov, "A two-dimensional "inverse scattering problem" for negative energies, and generalized-analytic functions, I: Energies lower than the ground state", *Funktsional. Anal. i Prilozhen.* **22**:1 (1988), 23–33, 96. MR 90a:35181

[Guillarmou and Sá Barreto 2009] C. Guillarmou and A. Sá Barreto, "Inverse problems for Einstein manifolds", *Inverse Probl. Imaging* **3**:1 (2009), 1–15. MR 2010m:58040 Zbl 1229.58025

[Guillarmou and Tzou 2009] C. Guillarmou and L. Tzou, "Calderón inverse problem for the Schrödinger operator on Riemann surfaces", pp. 129–142 in *The AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis*, Proc. Centre Math. Appl. **44**, Australian National Univ., Canberra, 2009.

[Guillarmou and Tzou 2011a] C. Guillarmou and L. Tzou, "Calderón inverse problem with partial data on Riemann surfaces", *Duke Math. J.* **158**:1 (2011), 83–120. MR 2012f:35574 Zbl 1222.35212

[Guillarmou and Tzou 2011b] C. Guillarmou and L. Tzou, "Identification of a connection from Cauchy data on a Riemann surface with boundary", *Geom. Funct. Anal.* **21**:2 (2011), 393–418. MR 2795512 Zbl 05902965

[Henkin and Michel 2007] G. Henkin and V. Michel, "On the explicit reconstruction of a Riemann surface from its Dirichlet–Neumann operator", *Geom. Funct. Anal.* **17**:1 (2007), 116–155. MR 2009b:58051 Zbl 1118.32009

[Henkin and Michel 2008] G. Henkin and V. Michel, "Inverse conductivity problem on Riemann surfaces", *J. Geom. Anal.* **18**:4 (2008), 1033–1052. MR 2010b:58031 Zbl 1151.35101

[Henkin and Santacesaria 2010] G. Henkin and M. Santacesaria, "Gel'fand–Calderón's inverse problem for anisotropic conductivities on bordered surfaces in $\mathbb{R}^3$", preprint, 2010. arXiv 1006.0647

[Imanuvilov and Yamamoto 2011] O. Imanuvilov and M. Yamamoto, "Inverse boundary value problem for Schrödinger equation in two dimensions", preprint, 2011. arXiv 1105.2850

[Imanuvilov et al. 2010a] O. Imanuvilov, G. Uhlmann, and M. Yamamoto, "Partial Cauchy data for general second-order elliptic operators in two dimensions", preprint, 2010. arXiv 1010.5791

[Imanuvilov et al. 2010b] O. Y. Imanuvilov, G. Uhlmann, and M. Yamamoto, "The Calderón problem with partial data in two dimensions", *J. Amer. Math. Soc.* **23**:3 (2010), 655–691. MR 2012c:35472 Zbl 1201.35183

[Imanuvilov et al. 2011a] O. Y. Imanuvilov, G. Uhlmann, and M. Yamamoto, "Determination of second-order elliptic operators in two dimensions from partial Cauchy data", *Proc. Natl. Acad. Sci. USA* **108**:2 (2011), 467–472. MR 2012a:35364

[Imanuvilov et al. 2011b] O. Y. Imanuvilov, G. Uhlmann, and M. Yamamoto, "Inverse boundary value problem by measuring Dirichlet data and Neumann data on disjoint sets", *Inverse Problems* **27**:8 (2011), 085007. MR 2012c:78002 Zbl 1222.35213

[Kang and Uhlmann 2004] H. Kang and G. Uhlmann, "Inverse problems for the Pauli Hamiltonian in two dimensions", *J. Fourier Anal. Appl.* **10**:2 (2004), 201–215. MR 2005d:81135 Zbl 1081.35141

[Kenig et al. 2007] C. E. Kenig, J. Sjöstrand, and G. Uhlmann, "The Calderón problem with partial data", *Ann. of Math.* (2) **165**:2 (2007), 567–591. MR 2008k:35498 Zbl 1127.35079

[Kobayashi 1987] S. Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan **15**, Princeton University Press, 1987. MR 89e:53100 Zbl 0708.53002

[Kohn and Vogelius 1984] R. Kohn and M. Vogelius, "Determining conductivity by boundary measurements", *Comm. Pure Appl. Math.* **37**:3 (1984), 289–298. MR 85f:80008 Zbl 0586.35089

[Lassas and Uhlmann 2001] M. Lassas and G. Uhlmann, "On determining a Riemannian manifold from the Dirichlet-to-Neumann map", *Ann. Sci. École Norm. Sup.* (4) **34**:5 (2001), 771–787. MR 2003e:58037 Zbl 0992.35120

[Lassas et al. 2003] M. Lassas, M. Taylor, and G. Uhlmann, "The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary", *Comm. Anal. Geom.* **11**:2 (2003), 207–221. MR 2004h:58033 Zbl 1077.58012

[Lebeau and Le Rousseau 2011] G. Lebeau and J. Le Rousseau, "Introduction aux inégalités de Carleman pour les opérateurs elliptiques et paraboliques. Applications au prolongement unique et au contrôle des équations paraboliques", *ESAIM Control Optim. Calc. Var.* (2011).

[Lee and Uhlmann 1989] J. M. Lee and G. Uhlmann, "Determining anisotropic real-analytic conductivities by boundary measurements", *Comm. Pure Appl. Math.* **42**:8 (1989), 1097–1112. MR 91a:35166 Zbl 0702.35036

[McDuff and Salamon 2004] D. McDuff and D. Salamon, *$J$-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications **52**, American Mathematical Society, Providence, RI, 2004. MR 2004m:53154 Zbl 1064.53051

[Nachman 1996] A. I. Nachman, "Global uniqueness for a two-dimensional inverse boundary value problem", *Ann. of Math.* (2) **143**:1 (1996), 71–96. MR 96k:35189 Zbl 0857.35135

[Nachman and Street 2010] A. Nachman and B. Street, "Reconstruction in the Calderón problem with partial data", *Comm. Partial Differential Equations* **35**:2 (2010), 375–390. MR 2012b:35368 Zbl 1186.35242

[Nakamura and Uhlmann 1993] G. Nakamura and G. Uhlmann, "Identification of Lamé parameters by boundary measurements", *Amer. J. Math.* **115**:5 (1993), 1161–1187. MR 94k:35328 Zbl 0803.35164

[Novikov 1986] R. G. Novikov, "Reconstruction of a two-dimensional Schrödinger operator from the scattering amplitude in the presence of fixed energy", *Funktsional. Anal. i Prilozhen.* **20**:3 (1986), 90–91. MR 88g:35198

[Novikov 1992] R. G. Novikov, "The inverse scattering problem on a fixed energy level for the two-dimensional Schrödinger operator", *J. Funct. Anal.* **103**:2 (1992), 409–463. MR 93e:35080 Zbl 0762.35077

[Salo 2013] M. Salo, "The Calderón problem on Riemannian manifolds", pp. 167–247 in *Inverse problems and applications: Inside out II*, edited by G. Uhlmann, Publ. Math. Sci. Res. Inst. **60**, Cambridge University Press, New York, 2013.

[Salo and Tzou 2010] M. Salo and L. Tzou, "Inverse problems with partial data for a Dirac system: a Carleman estimate approach", *Adv. Math.* **225**:1 (2010), 487–513. MR 2011g:35432 Zbl 1197.35329

[Sun 1990] Z. Q. Sun, "The inverse conductivity problem in two dimensions", *J. Differential Equations* **87**:2 (1990), 227–255. MR 92a:35163 Zbl 0716.35080

[Sun 1993] Z. Q. Sun, "An inverse boundary value problem for the Schrödinger operator with vector potentials in two dimensions", *Comm. Partial Differential Equations* **18**:1-2 (1993), 83–124. MR 94e:35145 Zbl 0781.35073

[Sun and Uhlmann 1991] Z. Q. Sun and G. Uhlmann, "Generic uniqueness for an inverse boundary value problem", *Duke Math. J.* **62**:1 (1991), 131–155. MR 92b:35172 Zbl 0728.35132

[Sun and Uhlmann 2003] Z. Sun and G. Uhlmann, "Anisotropic inverse problems in two dimensions", *Inverse Problems* **19**:5 (2003), 1001–1010. MR 2004k:35415 Zbl 1054.35139

[Sylvester 1990] J. Sylvester, "An anisotropic inverse boundary value problem", *Comm. Pure Appl. Math.* **43**:2 (1990), 201–232. MR 90m:35202 Zbl 0709.35102

[Sylvester and Uhlmann 1986] J. Sylvester and G. Uhlmann, "A uniqueness theorem for an inverse boundary value problem in electrical prospection", *Comm. Pure Appl. Math.* **39**:1 (1986), 91–112. MR 87j:35377 Zbl 0611.35088

[Sylvester and Uhlmann 1987] J. Sylvester and G. Uhlmann, "A global uniqueness theorem for an inverse boundary value problem", *Ann. of Math.* (2) **125**:1 (1987), 153–169. MR 88b:35205 Zbl 0625.35078

[Sylvester and Uhlmann 1988] J. Sylvester and G. Uhlmann, "Inverse boundary value problems at the boundary—continuous dependence", *Comm. Pure Appl. Math.* **41**:2 (1988), 197–219. MR 89f:35213

[Uhlenbeck 1976] K. Uhlenbeck, "Generic properties of eigenfunctions", *Amer. J. Math.* **98**:4 (1976), 1059–1078. MR 57 #4264 Zbl 0355.58017

[Uhlmann 2003] G. Uhlmann, "Inverse boundary problems in two dimensions", pp. 183–203 in *Function spaces, differential operators and nonlinear analysis* (Teistungen, 2001), Birkhäuser, Basel, 2003. MR 2004h:35225 Zbl 1054.35140

*DMA, U.M.R. 8553 CNRS, École Normale Supérieure, 45 rue d'Ulm, F-75230 Paris cedex 05, France*
cguillar@dma.ens.fr

*Department of Mathematics, University of Helsinki, PO Box 68, 00014 Helsinki, Finland*
ltzou@gmail.com